Introduction to Discrete Fair Division

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Premise

- How can we allocate a set of goods to a set of people fairly?
- This is surprisingly complicated
  - People may like different goods differently
  - What does fairness even mean?

Discrete Setting

- We define N to be the set of agents of cardinality n and M to be the set of goods of cardinality m.
- Intuitively, valuations are normalized so that $v_i(\emptyset) = 0$
- Each $i \in N$ is equipped with a valuation function v_i, which assigns a positive valuation to each subset of M

$$v_i : 2^M \to \mathbb{R}_{\geq 0}$$

- v_i is additive

$$v_i(A \cup B) = v_i(A) + v_i(B)$$

- Several papers have also relaxed this to be monotonic

$$v_i(A \cup \{g\}) \geq v_i(A)$$

Notions of Fairness

- Proportionality
  - Each agent believes they receive at least $\frac{1}{n}$ of the goods

$v_i(X_i) \geq \frac{1}{n} \times v_i(M)$

- Envy-Freeness (**EF**)
  - Each agent believes they receive weakly more than the other agents

$$\forall_{i,j \in N} v_i(X_i) \geq v_i(X_j)$$

Notions of Fairness

Example: In a discrete setting, EF allocations may not always exist

Proof: By counterexample, take 1 good and 2 agents. We arbitrarily give agent 1 the good

$$v_2(X_1) > 0 = v_2(X_2)$$

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A similar argument can be applied to proportionality

Relaxing EF

- Envy Free up to X (EFX)
  - Each agent believes they receive weakly more than the other agents without some good

$$\forall_{i,j} \forall_{g \in X_j} v_i(X_i) \geq v_i\left(X_j \setminus \{g\}\right) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i\left(X_j \setminus \min\left(X_j\right)\right)$$

- It is not clear if EFX allocations exist or can be computed in polynomial time in general
  - Several relaxations of EFX have been proposed

Known cases for EFX

- EFX can be computed efficiently for n=2
  - Cut and Choose
- EFX can be computed in pseudo polynomial time for n=3

Relaxing EFX

- Realized valuations
  - Shared valuation function v
- EF1

$$\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq v_i\left(X_j \setminus \{g\}\right) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i\left(X_j \setminus \max\left(X_j\right)\right)$$

- α-EFX
  - $$\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq \alpha \times v_i\left(X_j \setminus \{g\}\right)$$
  - If EFX allocations exist in general, then $\alpha = 1$
- EFkX

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Round Robin

Envy-Cycle Elimination

Maximin Share

Leximin