Introduction to Discrete Fair Division

Table of Contents

Premise

- How can we allocate a set of goods to a set of people fairly?
- This is surprisingly complicated
  - People may like different goods differently
  - What does fairness even mean?

Discrete Setting

- We define N to be the set of agents of cardinality n and M to be the set of goods of cardinality m.
- Intuitively, valuations are normalized so that $v_i(\emptyset) = 0$
- Each $i \in N$ is equipped with a valuation function v_i, which assigns a positive valuation to each subset of M

$$v_i : 2^M \to \mathbb{R}_{\geq 0}$$

- An allocation is defined as a partition of M

$$X =< X_1, X_2, ...X_n >$$

- v_i is additive

$$v_i(A \cup B) = v_i(A) + v_i(B)$$

- Several papers have also relaxed this to be monotonic

$$v_i(A \cup \{g\}) \geq v_i(A)$$

Notions of Fairness

- Proportionality
  - Each agent believes they receive at least $\frac{1}{n}$ of the goods

$$v_i(X_i) \geq \frac{1}{n} \times v_i(M)$$

- Envy-Freeness (**EF**)
  - Each agent believes they receive weakly more than the other agents

$$\forall_{i,j \in N} v_i(X_i) \geq v_i(X_j)$$

Notions of Fairness

Example: In a discrete setting, EF allocations may not always exist

Proof: By counterexample, take 1 good and 2 agents. We arbitrarily give agent 1 the good

$$v_2(X_1) > 0 = v_2(X_2)$$

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A similar argument can be applied to proportionality

Lemma 0.1: EF $\rightarrow$ PROP for additive valuations

Proof: Assume that there is an envy free allocation and non proportional allocation. Every player believes that they have a piece of equal or better value to that of everyone else. Because the $v_i(X_j)$ may have value < 1/n, and there are n player, this means that $v_i(M)$ can be $< v_i(M)$ ∎

Relaxing EF

- Envy Free up to X (EFX)
  - Each agent believes they receive weakly more than the other agents without some good

$$\forall_{i,j} \forall_{g \in X_j} v_i(X_i) \geq v_i\left(X_j \setminus \{g\}\right) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i\left(X_j \setminus \min\left(X_j\right)\right)$$

- It is not clear if EFX allocations exist or can be computed in polynomial time in general
  - Several relaxations of EFX have been proposed

Known cases for EFX

- EFX can be computed efficiently for n=2
    - Cut and Choose
- EFX can be computed in pseudo polynomial time for n=3

Relaxing EFX

- Realized valuations
  - Shared valuation function v
- EF1

$$\forall_{i,j}\exists_{g\in X_j}v_i(X_i) \geq v_i\left(X_j \setminus \{g\}\right) \leftrightarrow \forall_{i,j}v_i(X_i) \geq v_i\left(X_j \setminus \max\left(X_j\right)\right)$$

- α-EFX
  - $$\forall_{i,j}\exists_{g\in X_j}v_i(X_i) \geq \alpha \times v_i\left(X_j \setminus \{g\}\right)$$
  - If EFX allocations exist in general, then $\alpha = 1$

Maximin Share

- Let $X_n(M)$ be the set of possible allocations of M goods to n agents
- $\mu_i^n(M)$ is one of the partitions which maximizes the least valuable bundle according to i

$$\mu_i^n(M) = \max_{B \in X_n(M)} \min_{S \in B} v_i(S)$$

- The MMS
- Maximin Share Fair (MMS) allocations do not always exist

Relaxing MMS

- α-MMS $v_i(X_i)$

Pareto Optimality

- An allocation is pareto optimal (PO) iff an agent would protest to another allocation
- The allocation A is PO if there is no allocation B such that $v_{i(B_i)} \geq v_i(A_i)$ for all i and one inequality is strict
- A is not *Pareto Dominated* by another allocation

MNW

- Maximize the geometric mean of values for agents with positive value

$$\left(\prod_{i}^{n} v_i(X_i)\right)^{\frac{1}{n}}$$

- MNW $\rightarrow$ EF $\wedge$ PO
- Can an EF1 and PO allocation be computed in polynomial time?

Round Robin

- Agents receive their most valued remaining good in a set order until no goods remain.

Round Robin

Theorem 0.1: Round Robin produces an EF1 Allocation

Proof:

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Envy-Cycle Elimination

- Let G be the envy graph on N
- $i \longrightarrow j$ in G iff $v_i(X_i) < v_i(X_j)$
- If a cycle is found, each agent can receive the bundle of the agent they envy in the cycle
- "Rotate" bundles around the cycle