

Introduction to Discrete Fair Division

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Motivations

- People want stuff
- But there is only a finite amount of things
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Motivations

- People want stuff
- But there is only a finite amount of things
- This is ultimately the fundamental question of economics
 - How do we study the movement of resources across a society?

The Computational Problem

- It is a computationally difficult problem to distribute items among people
- How can we guarantee truthfulness in our algorithm? (People may lie about their value of a good)
 - This is more concerning when dividing continuous resources

Applications

Variations of this problem have been used for

1. Food distribution
 - Indy Hunger Network
 - Foodbank Australia
2. Computing resources
 - Memory
 - CPU Time
3. Estate Divisions
 - Spliddit



Premise

- How can we allocate a set of goods to a set of people fairly?
- This is surprisingly complicated
 - People may like different goods differently
 - What does fairness even mean?

Discrete Setting

- We define N (of cardinality n) to be the set of agents and M (of cardinality m) to be the set of goods.
- Each $i \in N$ is equipped with a valuation function v_i , which assigns a positive valuation to each subset of M

$$v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$$

- Intuitively, valuation $v_i(\emptyset) = 0$
- An allocation is a partition of M

$$X = \langle X_1, X_2, \dots, X_n \rangle$$

- v_i is additive $v_i(A \cup B) = v_i(A) + v_i(B)$

Notions of Fairness

- Proportionality
 - Each agent believes they receive at least $\frac{1}{n}$ of the goods

$$v_i(X_i) \geq \frac{1}{n} \times v_i(M)$$

- Envy-Freeness (**EF**)
 - Each agent believes they receive weakly more than the other agents

$$\forall_{i,j \in N} v_i(X_i) \geq v_i(X_j)$$

Notions of Fairness

Example: In a discrete setting, EF allocations may not always exist

Proof: By counterexample, take 1 good and 2 agents.
We arbitrarily give agent 1 the good

$$v_2(X_1) > 0 = v_2(X_2)$$



A similar argument can be applied to proportionality

Lemma 0.1: EF \rightarrow PROP for additive valuations

Proof: Assume that there is an envy free allocation and non proportional allocation. Every player believes that they have a piece of equal or better value to that of everyone else. Because the $v_i(X_j)$ may have value $< 1/n$, and there are n player, this means that $v_i(M)$ can be $< v_i(M)$ ■

Relaxing EF

- Envy Free up to X (EFX)
 - Each agent believes they receive weakly more than the other agents without some good

$$\forall_{i,j} \forall_{g \in X_j} v_i(X_i) \geq v_i(X_j \setminus \{g\}) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i(X_j \setminus \min(X_j))$$

- It is not clear if EFX allocations exist or can be computed in polynomial time in general
 - Several relaxations of EFX have been proposed

Known cases for EFX

- EFX can be computed efficiently for $n=2$
 - Cut and Choose
- An algorithm exists to compute an EFX allocation in pseudo polynomial time for $n=3$

Relaxing EFX

- Realized valuations
 - Shared valuation function v
 - EFX exists and a polynomial time algorithm is known
- EF1

$$\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq v_i(X_j \setminus \{g\}) \Leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i(X_j \setminus \max(X_j))$$

- α -EFX
 - $\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq \alpha \times v_i(X_j \setminus \{g\})$
 - If EFX allocations exist in general, then $\alpha = 1$

Break Time!

Let's Cut a Cake



Maximin Fair Share

- Let $X_n(M)$ be the set of possible allocations of M goods to n agents
- $\mu_i^n(M)$ is one of the partitions which maximizes the least valuable bundle according to i

$$\mu_i^n(M) = \max_{B \in X_n(M)} \min_{S \in B} v_i(S)$$

- Maximin Share Fair (MMS) allocations do not always exist

Relaxing MMS

- α -MMS $v_i(X_i)$

Pareto Optimality

- An allocation is Pareto Optimal (PO) iff an agent would protest to another allocation
- The allocation A is PO if there is no allocation B such that $v_i(B_i) \geq v_i(A_i)$ for all i and one inequality is strict
- A is not *Pareto Dominated* by another allocation

MNW

- Maximize the geometric mean of values for agents with positive value

$$\left(\prod_i^n v_i(X_i) \right)^{\frac{1}{n}}$$

- $\text{MNW} \rightarrow \text{EF} \wedge \text{PO}$

Proof:



- Can an EF1 and PO allocation be computed in polynomial time?

Round Robin

- Agents receive their most valued remaining good in a set order until no goods remain.

Envy-Cycle Elimination

- Let G be the envy graph on N
- $i \longrightarrow j$ in G iff $v_i(X_i) < v_i(X_j)$
- If a cycle is found, each agent can receive the bundle of the agent they envy in the cycle
- “Rotate” bundles around the cycle

Computing EF1

Round Robin and Envy-Cycle Elimination produce an EF1 Allocation

Proof:



Conclusion

- Fair division is a wide field with various applications