FAIR DIVISION SURVEY

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1. Introduction

Fair division is a problem with roots in the Bible and in Greek mythology. When Abraham and Lot must divide the land of Canaan fairly among themselves, one divides the land into two parts (which they perceive to be equally valuable), and the other chooses their favorite piece. This problem has proven to be useful in practice. It has been used for international dispute resolution [1], distribution of food for food banks, and estate divisions [2–4]. This was formalized by [5] for both divisible continuous and indivisible discrete goods. Since then, several promising results have been published in both settings, with the existence of a bounded-time envy-free cake cutting procedure being resolved by [6]. However, the fair allocation of a finite set of indivisible goods to individuals "fairly" remains a fundamental open question within computer science and economics. Indeed, many interpretations of fairness exist. In this paper, we survey some common notions of fairness, their computability, and open questions within the field of Fair Division. We also present a small result connecting two notions of fairness (MNW and EQX).

2. Preliminaries

Let $\mathcal{N}$ be a set of agents with $|\mathcal{N}| = n$ and $\mathcal{M}$ be a set of goods with $|\mathcal{M}| = m$. Each agent $i \in \mathcal{N}$ possesses a valuation function $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$, representing their value of each bundle of goods within $S \subseteq \mathcal{M}$. It is assumed that for each agent $i \in \mathcal{N}$ that $v_i(\emptyset) = 0$. For simplicity, we write $v_i(\{g\})$ as $v_i(g)$ for individual items $g \in \mathcal{M}$. The valuation function is usually taken to be additive, meaning that an agent's value of a S is equal to the sum of the agent's values of the individual goods. Formally, $v_i(S) = \sum_{g \in S} v_i(g)$. For more general contexts, the valuation function is taken as monotone, i.e., that $v_i(S) \leq v_i(T)$ for all $S \subseteq T \subseteq \mathcal{M}$. In this survey, we focus on the additive context, however several of the results hold for monotone valuations as well.

An allocation X is a partition $\mathcal{M}$ into n disjoint sets and assign each partition to an agent $i \in \mathcal{N}$. The subset of $\mathcal{M}$ assigned to agent i (called i's bundle) is denoted X_i, and each agent i's valuation of their own bundle is i's "utility". The goal of fair division is to take a tuple $(\mathcal{N}, \mathcal{M}, \langle v_1, ..., v_n \rangle)$ and to compute a "fair" allocation efficiently. To this end, there are several notions of fairness. Computing fair allocations efficiently is not always known to be possible, or is hard, so the valuation functions are restricted. We primarily focus on two settings, that

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of identical valuations where $\forall_{i \in \mathcal{N}} v_i = v$, and binary valuations where $v_i(\{g\}) \in \{0, 1\}$. In these contexts, several notions become easier to compute.

3. Envy

Arguably, the gold standard of fair division is envy freeness (EF), where no agent would like another's bundle more than their own. Formally, $\forall_{i,j \in \mathcal{N}} v_i(X_i) \geq v_i(X_j)$. If this is not the case, (i.e. $v_i(X_i) < v_i(X_j)$), i is said to "envy" "j". It is clear that for discrete goods, this does not always exist, as seen in the following counterexample:

> *Proof*: Take two agents 1 and 2 and one positively valued good. Without loss of generality, allocate the good to agent 1. Agent 2 envies agent 1, as $v_2(X_2) = 0$ and $v_2(X_1) > 0$. ■

Worse still, computing if an arbitrary instance admits an EF allocation is NP-complete [7].

Thus, when allocating a finite set of goods, EF is relaxed to envy-freeness up to X (EFX) by allowing the removal of an arbitrary good from another agent's bundle such that $\forall_{i,j \in \mathcal{N}} v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$.

For two agents, EFX can be computed using a variation of the "cut-and-choose" procedure. The first agent divides $\mathcal{M}$ into two bundles, X_1 and X_2 such that they value each of them as near to $\frac{1}{2}$ as possible. Then, the second agents chooses the bundle which gives her higher utility, breaking ties arbitrarily. The second agent clearly does not envy the first, as they received the bundle with the highest valuation in their eyes. If the first agent prefers the bundle they received, we are done. Otherwise, the first agent cannot envy the second after the removal of a good, as otherwise this bring the valuation of the bundles closer to $\frac{1}{2}$, which would contradict that the first agent selected as near to $\frac{1}{2}$ as they could initially. For three agents, [8] showed that EFX can be computed as well in pseudo-polynomial time. [9] later gave a simpler algorithm for this using casework to iteratively improve the allocation.

Thus, we come to our first open question.

Open Problem 3.1: Do EFX allocations always exist? If so, can they be computed efficiently?

The leximin solution is one which maximizes the minimum utility, then maximizing the second least utility, and so on. The leximin++ is the leximin solution, which, after maximizing the value of the bundle with minimum utility, maximizes size of the bundle. Then, maximizes the size of the second least agent's utility subject to maximizing the size of the second bundle and so on.

Theorem 3.1. *For identical additive valuations, an EFX allocation can be computed by taking the leximin++ maximal allocation. [10]*

Proof*: For contradiction, let X be a leximin++ allocation which is not EFX $v(X_i) < v(X_j)$ for $i < j$. This means that there is a pair of agents with $i < j$, where $v(X_i) < v(X_j \setminus \{g\})$ for some $g \in X_j$. This means that the minimum utility of the bundle can be increased by giving that good g to X_i, contradicting that X is a leximin++ allocation.* ■

Although computing EFX for general additive valuations remains an open problem, several relaxations of EFX have been proposed which are easier to compute.

One natural easy-to-compute notion relaxation of EFX is EF1. Rather than allowing the removal of an arbitrary good, this is relaxed so that there must exist a good which can each be removed in order to resolve envy. In other words, $\forall_{i,j \in \mathcal{N}} v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for a good $g \in X_j$. This notion was first formally defined by [11]. Note that this is equivalent to removing the good which i values the most in j's bundle to enforce envy-freeness $\forall_{i,j \in \mathcal{N}} v_i(X_i) \geq v_i(X_j \setminus \{g^*\})$, with $g^* = \arg\max_{g \in X_j} v_i(g)$. For general valuations, EF1 can be computed efficiently using simple polynomial time algorithms.

One such well-known algorithm is the round robin algorithm, presented below: Informally, the algorithm iterate through agents in a fixed order, giving them their most valued good at each step (breaking ties arbitrarily) until no goods remain.

Round-Robin

Input: $(\mathcal{N}, \mathcal{M}, v)$
Output: An EF1 allocation X

$P \leftarrow \mathcal{M}$
$X =< \emptyset, ..., \emptyset >$
i := 1
while |P| > 0
  $X_i \leftarrow X_i \cup \{\arg\max_{g \in P} v_i(g)\}$ # Give i their favorite good.
  $i \leftarrow (i + 1 \bmod n) + 1$
$\mathcal{M} \leftarrow P$

Theorem 3.2. *[4] Round-Robin produces an EF1 allocation*

Proof*: Take two agents i and j. After the execution of fixed-order round-robin. First, we note that an agent i cannot envy only value another agent j with $(i \leq j)$. This is trivial for $j = i$, so let us take $j > i$. Here, in each iteration of the loop, j chooses their favorite good after i. Thus, i believes they received a weakly more valuable good than j. Thus, we get $v_i(X_i) \geq v_i(X_j)$ Note that in each iteration of the loop, for $i > j$, i receives weakly more than what j will receive in the following iteration of the loop according to i. We then only need to consider the first iteration of the loop, as envy cannot come from later in the execution of the algorithm. Since i may value X_j for $j < i$ by the item allocated in the first iteration of the loop, then the removal of the item will remove i's envy of X_j. Thus, the allocation is EF1.* ■

Another common algorithm for computing an EF1 is Envy-Cycle Elimination. Define the envy-graph G, with $V = N$ and a directed edge between agents, $i \to j$ iff $v_i(X_i) < v_i(X_j)$). The algorithm works as follows. Find an arbitrary unenvied agent i and give them their most valued good. If no such agent exists, then there must be a cycle $C = \langle c_i, c_{i+1}, ..., c_{i-1}, c_i \rangle$ where c_j envies c_{j+1} (as otherwise there would have to exist a sink, as it would be a DAG). To remove or "eliminate" this cycle (then for agent c_j, give them the bundle of agent c_{j+1}). Until an unenvied agent is found, repeatedly remove directed cycles. Then, give the agent their most favorite unallocated good.

Envy-Cycle Elimination

Input: $(\mathcal{N}, \mathcal{M}, v)$
Output: An EF1 and $\frac{1}{2}$-EFX allocation X

$X = \langle \emptyset, ..., \emptyset \rangle$
for $i = 1, ..., n$
  $X_i \leftarrow \emptyset$
for $\ell = 1...m$
  while $\nexists$ unenvied agent # Perform envy-cycle elimination
    Find an envy cycle $C = \langle c_i, c_{i+1}, ..., c_{i-1}, c_i \rangle$
    $C_i \leftarrow C_{i+1}$
  let i be an unenvied agent # Find an unenvied agent i
  let $g^* \in \arg\max_{g \in \mathcal{M}} v_i(g)$ # Give i their favorite good.
  $A_i \leftarrow A_i \cup \{g^*\}$
  $M \leftarrow M \setminus \{g^*\}$

Theorem 3.3. *[7] Envy-Cycle Elimination produces an EF1 allocation*

Proof*: Before the algorithm begins, the allocation is trivially EF (and EFX), as no good has been allocated and thus all bundles are valued at 0 by all agents. In each iteration of the loop, the algorithm allocates goods only to unenvied agents This implies that an agent may only envy another by a single good. This then implies that the algorithm is EF1 after each iteration of the loop, as the removal of a good* $g \in X_j$ *causing envy for any* i *who envies* j *will resolve envy.* ■

3.0.1. *α-EFX.*

Another relation of EFX is α-EFX, where $\forall_{i,j \in \mathcal{N}} v_i(X_i) \geq \alpha \times v_i(X_j \setminus \{g\})$ for $\alpha \in (0, 1]$.

Open Problem 3.0.1.1: What is the greatest α for which an α-EFX allocation exists?

It is important to note that if the best possible $\alpha = 1$, then EFX allocations can always exist, but otherwise, this question becomes meaningful as well.

[10] first attempt to compute approximate-EFX allocations by slightly modifying the Envy-Cycle elimination algorithm to compute $\frac{1}{2}$-EFX. [12] show that Envy-Cycle Elimination by itself can compute a $\frac{1}{2}$-EFX allocation, if ties are broken between unenvied agents in favor of those with empty bundles.

Theorem 3.0.1.4. *[12] Envy-Cycle Elimination produces an $\frac{1}{2}$-EFX allocation*

Proof*: Take two distinct agents, $i, j \in \mathcal{N}$. Assume that $v_i(X_i) \geq \frac{1}{2} \times v_i\big(X_j \setminus \{g\}\big)$ for all $g \in X_j$. This statement holds trivially for $|X_j| = 1$, so assume that $|X_j| = 2$.*

Let g^* be the good last allocated to j Since ties are broken in favor of empty bundles when picking an agent to allocate to, $X_i \neq \emptyset$, as otherwise g^* would have been allocated to i rather than j. Because $|X_i| \geq 1$, there must have been a round before g^* during which i was able to pick an item. Although X_i might be swapped with other agents' bundles, $v_i(X_i)$ is non-decreasing. Therefore $v_i(X_i) \geq v_i(g^*)$. Additionally, by the design of the algorithm, $v_i(X_i) \geq v_i\big(X_j \setminus \{g\}\big)$. Together, these imply that

$$2 \times v_i(X_i) \geq v_i\big(X_j\big) \geq v_i\big(X_j \setminus \{g\}\big), \forall g \in X_j$$

Thus, the allocation is $\frac{1}{2}$-EFX. ■

[13] improve this to a $\Phi - 1$-EFX allocation. Most recently, [14] compute a $\frac{2}{3}$-EFX allocation for general valuations.

4. Pareto Optimality

Take two allocations A and B. A is said to "Pareto Dominate" B iff $v_i(A_i) \geq v_i(B_i)$ for all $i \in \mathcal{N}$, and for at least one agent, $v_i(A_i) > v_{i(B_i)}$. This induces a partial ordering on the set of all feasible allocations. The maximal element(s) are said to be "Pareto Optimal".

Theorem 4.1. *[10] PO and EFX do not coincide for all fair division instances if zero marginal utility is allowed.*

Proof*:*

	a	***b***	***c***
agent 1	2	1	0
agent 2	2	0	1

$v_1(c) = 0$ and $v_2(c) > 0$ for a PO allocation, c must be given to agent 2. Similarly, b must be allocated to agent 1 as $v_1(b) > 0$ and $v_2(b) = 0$. Assume without loss of generality a is allocated to agent 1. Then $v_2(X_1 \setminus \{b\}) = 2 \geq v_2(X_2) = 1$. ■

Like Pareto dominating allocations, Lorenz dominating allocations induce a partial ordering over the set of allocations. Here, take two vectors of utilities which have been sorted A and B. A is said to "Lorenz dominate" B iff for every $k \in [n], \sum_{i=1}^{k}(A_i) \geq \sum_{i=1}^{k}(B_i)$ [15].

5. MMS

A Maximin-Share Fair allocation is one maximizing the "worst-off" agent according to each $i \in \mathcal{N}$. In other words, all agents $i \in \mathcal{N}$, $v_i(X_i) \geq \mu_i^n(\mathcal{M}) = \max_{X \in \Pi_n(\mathcal{M})} \min_{X_j \in X} v_i(X_j)$.

Unfortunately MMS may not always exist, [16], although it exists with high probability in general. Additionally, MMS is hard to compute [17]. Thus, several recent papers try to approximate it so that $v_i(X_i) \geq \alpha \times \mu_i^n(\mathcal{M})$, with $\alpha \in (0, 1]$. This is known as an α-MMS allocation.

Oe common technique for producing $\frac{1}{2}$-MMS allocations is known as the bag-filling algorithm. First, the MMS values of all agents are normalized such that $\mu_i^n = 1$ for all agents $i \in \mathcal{N}$. Maintain a "bag", and keep adding arbitrary goods until an agent j values the bag at least $\frac{1}{2}$, breaking ties arbitrarily. Once such an agent is found, add all the goods in the bag to X_j and remove them from $\mathcal{M}$. When a single agent remains, give her the remaining unallocated goods.

Bag-Filling

Input: $(\mathcal{N}, \mathcal{M}, v)$
Output: A $\frac{1}{2}$-MMS allocation X

BAG $= \emptyset$
$X = \langle \emptyset, ..., \emptyset \rangle$
while $\exists i \in \mathcal{N} \wedge g \in \mathcal{M}$ such that $v_i(g) \geq \frac{1}{2}$
  $X_i \leftarrow \{g\}$ # Give i their favorite good.
  $\mathcal{N} \leftarrow \mathcal{N} \setminus \{i\}$
  $\mathcal{M} \leftarrow \mathcal{M} \setminus \{g\}$
while $|N| > 1$
  while $\nexists i \in \mathcal{N}$ with $v_i(\text{BAG}) < \frac{1}{2}$
    $g =$ arbitrary $g \in \mathcal{M}$
    BAG $=$ BAG $\cup \{g\}$
  Take $i \in \mathcal{N}$ where $v_i(\text{BAG}) \geq \frac{1}{2}$
  $\mathcal{N} \leftarrow \mathcal{N} \setminus \{i\}$
  $\mathcal{M} \leftarrow \mathcal{M} \setminus$ BAG
Let i be the remaining agent in $\mathcal{N}$
$X_i = \mathcal{M}$

Theorem 5.1. *[18] Removing an agent and removing a single good from a problem instance produces results in a weakly greater MMS value. In other words, $\mu_i^n(\mathcal{M}) \leq \mu_i^n(\mathcal{M} \setminus \{g\})$ for an arbitrary $g \in \mathcal{M}$.*

Proof*: Before the removal of the agent and good, $v_i(X_i) \geq \text{MMS}_i(n)$. Suppose w.l.o.g that $g \in X_n$, and reallocate all other $g' \in X_n$ to the other agents*

$1, ..., n-1$. This is clearly an allocation $\Pi_{n-1}(\mathcal{M} \setminus \{g\})$, denoted by $X' = (X'_1, ...X'_{n-1})$. Since each agent has weakly greater utility in the new allocation, $X'_i \geq X_i \geq \mu_i^n(\mathcal{M})$. ■

Theorem 5.2. *[18] The bag-filling algorithm produces a $\frac{1}{2}$-MMS allocation.*

Proof*: In each round of the algorithm, the bag is allocated with an agent with value at least $\frac{1}{2}$ for the bag. Note that the value of the other agents for the given bag is smaller than 1. Since, before the last good is added to the bag, the value of the bag for all agents is smaller than $\frac{1}{2}$ and greater than $\frac{1}{2}$ for at least one agent after. Thus, when an agent takes away a bag in some round, the remaining $n-1$ agents have total value at least $n-1$ of the remaining bundle. This can be repeated until 1 agent remains, who will then receive their entire MMS value.* ■

[19] first showed the existence of $\frac{2}{3}$-MMS approximation algorithms. [18] present a simple algorithm for computing such allocations by modifying the bag-filling algorithm to use low, medium, and highly valued goods to compute a $\frac{2}{3}$-MMS allocation. Several works attempt to [20] compute $\frac{3}{4} + O(f(n))$-MMS allocations, achieving up to an- $\frac{3}{4} + \min(\frac{1}{36}, \frac{3}{16n-4})$)-MMS allocation [21], but with no improvement on the constant approximation factor. Very recently, [22] improve this to a $\frac{3}{4} + \frac{3}{3836}$ -MMS.

Despite the progress made in computing approximations, the approximation of MMS is incomplete. [23] showed a counterexample, showing that MMS allocations cannot always be approximated higher than $\frac{39}{40}$, even for 3 agents.

Open Problem 5.1: What is the highest possible α for which α-MMS allocations exist?

Open Problem 5.2: Does a stronger approximation bound exist than $\frac{39}{40}$?

6. Maximum Nash Welfare

Maximum Nash Welfare (MNW) is a more "global" notion of welfare. Namely, it considers the utility of each agent rather than agents' value over others'. This first maximizes the number of agents receiving positive utility, then maximizes the geometric mean of utilities. Let $\Pi_k(S)$ represent the set of all partitions of a set S into k elements. The MNW allocation is given by $\arg\max_{X \in \Pi_k(\mathcal{M})} \prod_{i=1}^{n} v_i(X_i)$.

In the notion of the geometric mean, under identical additive valuations, this is maximized when all agents with positive utilities have valuations which are as similar as possible (after maximizing the agents with positive utilities). This is quite useful, as MNW provides strong guarantees for other notions of fairness.

Theorem 6.1. *[4] A MNW allocation is PO and EF1*

Proof*: MNW implies PO, as a Pareto dominating allocation would have a higher product of utilities. It is left to show that MNW $\Rightarrow$ EF1.*
Assume for contradiction that MNW $\not\Rightarrow$ EF1. This implies that there exists $i, j \in \mathcal{N}$ where $\exists_{g \in X_j} v_i(X_i) < v_i(X_j \setminus \{g\})$.
Proving EF1 is slightly more involved. Assume for contradiction that there exists two agents i and j such that $v_i < v_{i(X_j \setminus \{g\})}$ for every $g \in X_j$. We must show that there is another allocation X' with Nash Welfare larger than that of X.
Because i envies j, there exists a $g^* = \arg\min_{g \in X_j, v_i(g) > 0} \left\{ \frac{v_j(g)}{v_i(g)} \right\}$. Let $X'_k = X_k$ for all $k \neq i, j$. Next, move the g^* from j's bundle to i's. In other words, $X'_i = X_i \cup \{g^*\}$ and $X'_j = X_j \setminus \{g^*\}$.
We must show that $v_i(X'_i) \times v_j(X'_j) > v_i(X_i) \times v_j(X_j)$
To begin, we know that $\frac{v_j(g^*)}{v_i(g^*)} \leq \frac{v_j(X_j)}{v_i(X_j)}$, which implies that $\frac{v_j(g^*)}{v_j(X_j)} \leq \frac{v_i(g^*)}{v_i(X_j)} < v_i \frac{g^*}{v_i(X_j) + v_i(g^*)}$, since i envies j even after the removal of an item g^* in X_j. Therefore,

$$v_i(X'_i) \times v_j(X'_j) = (v_i(X_j) + v_i(g^*)) \times (v_j(X_j) - v_j(g^*))$$

$$= \left(1 + \frac{v_i(g^*)}{v_i(X_i)}\right) \times \left(1 - \frac{v_j(g^*)}{v_j(X_j)}\right) \times v_i(X_j) \times v_i(X_i) \times v_j(X_j)$$

which, from the previous inequality, $> (1 - \frac{v_i(g^*)}{v_i(X_i) + v_i(g^*)} \times \left(\frac{v_i(X_i) + v_i(g^*)}{v_i(X_i)}\right) \times v_i(X_j) \times v_i(X_i) \times v_j(X_j)\ v_i(X_i) \times v_j(X_j)$ ■

[4] also showed that a MNW allocation is guaranteed to be $\frac{2}{1+\sqrt{4n-3}}$-MMS.

These guarantees however do not extend much further. [24] show that MNW does not give an EFX allocation in general, MNW is NP-hard even for identical additive valuations [25], although under binary and bi-valued (where $v_i(g) \in \{a, b\}$) instances it can be computed in polynomial time [26]. For 3-value instances, where $v_i(g) \in \{a, b, c\}$, the problem of computing a MNW allocation is NP-complete [27]. Thus, due to its hardness to compute, we seek to guarantee these properties without computing them through MNW.

Open Problem 6.1: Can an EF1 and PO allocation be computed in polynomial time?

As computing MNW allocations is APX-hard, thus prior research has sought to approximate MNW allocations for its desiderata. Can an EF1 and PO allocation be computed in polynomial time? For bounded valuations, [28] develop a polynomial time algorithm, and in pseudo-polynomial time for general valuations.

Open Problem 6.2: What is the best possible α for which α-MNW allocations exist?

7. EQX

Another notion of fairness based on utilities is EQX. An equitable (EQ) allocation is one in which all agent utilities are roughly the same. $v_i(X_i) > v_j(X_j)$, however like EFX, this clearly does not exist. This motivates a similar relaxation, by allowing the removal of a good $g \in X_j$. EQX is well studied in the literature for both goods and for chores, and unlike EFX, is fairly easy to compute.

For identical valuations, EFX coincides with EQX, both of which can be computed efficiently. For strictly positive valuations, an EQX + PO allocation always exists, and an EQ1 + PO allocation can be found in pseudo-polynomial time [29].

[30] shows that for all chores or all goods, EQX exists even for monotone allocations, and can be computed in polynomial time, although this is not true for arbitrary valuations (unless P=NP).

Theorem 7.1. MNW $\Rightarrow$ EQX *under identical additive valuations.*

Proof*: Assume for contradiction that MNW does not imply EQX under identical valuations. Start with an MNW allocation X. This means that there is an agent from which $v(X_i \setminus \{g\}) > v(X_j)$. Otherwise, if $v(X_i) \leq v(X_j)$ EQX holds trivially.*

If $v(X_i) > v(X_j)$, then it suffices to show that moving a good from i to j will result in a larger geometric mean.

$$\big(v(X_j) + v(g)\big)(v(X_i) - v(g))$$

$$= \big(v(X_j)v(X_i)\big) - \big(v(g)v(X_j)\big) + (v(g)v(X_i)) - v(g)^2$$

$$= \big(v(X_j)v(X_i)\big) + v(g)\big(v(X_i) - v(X_j) - v(g)\big)$$

By the assumption,

$$v(X_i \setminus \{g\}) > v(X_j)$$

and since valuations are additive $v(X_i) > v(X_j)$, so $v(X_i) - v(X_j) > v(g)$. Thus we have derived a contradiction, as $\big(v(X_j)v(X_i)\big)$ is increased by moving the good g. ■

8. Between EFX and EF1

EFL (Envy-Freeness up to a Less Preferred Good) is meant to be “in-between” EFX and EF1. For agents i and j, rather than allowing the removal of an arbitrary good from X_j, it only allows the removal of goods g which are less valuable to i than X_i. In other words $v_I(X_i) \geq v_j(X_j \setminus \{g\})$ for $g \in X_j$ and $v_i(g) < v_i(X_i)$. This can be computed in polynomial time using envy-cycle elimination.

8.1. **Randomness.**

Another line research focuses on randomness in fairness allocations.

8.1.1. *EFR.*

[31] introduced the notion of envy-freeness up to a random good (EFR), where the expected envy of i towards an agent j can be eliminated. [31] also show that a 0.73-EFR allocation can be computed in polynomial time. However, such a mechanism cannot be strategyproof and envy free even in expectation [32].

9. Improving the Maximin Share

9.1. **GMMS.**

EFX can be connected to to MMS through the stronger notion of (Groupwise MMS) GMMS.

[33] define an allocation as GMMS as if for all subsets of $\mathcal{N}$ and their bundle, their allocation is also MMS. $v_i(X_i) \geq \mathrm{GMMS}_i = \max_{S \subseteq \mathcal{N}: i \in S} \mu_i^{|S|}\left(\cup_{j \in S} X_j\right)$. It is clear that because MMS allocations do not exist, GMMS allocation allocations do not always exist.

9.2. **PMMS.**

PMMS is a slightly weaker form of GMMS, in which $|S| = 2$. In other words, for all $S \subseteq \mathcal{N}$, $v_i(X_i) \geq \mathrm{PMMS}_i = \max_{S \subseteq \mathcal{N}: i \in S} \mu_i^{|S|}\left(\cup_{j \in S} X_j\right)$.

A slightly easier notion is α-GMMS, where $v_i(X_i) \geq \alpha - \mathrm{GMMS}_i$. [33] show that $\frac{1}{2}$-GMMS allocations always exist, and can be computed efficiently. [13] show that improved this to a $\frac{2}{\Phi+2}$ approximation. The currently highest approximation is to $\frac{4}{7}$-[34].

A similar line of work has attempted to approximate PMMS, with the highest known approximation as an ≈ 0.781-PMMS allocation [35].

Open Problem 9.2.1: What is the highest α for which α-GMMS allocations exist?

GMMS connects MMS (and its related notions) to EF and its relaxations through the hierarchy:

$$\mathrm{EF} \Rightarrow \mathrm{GMMS} \Rightarrow \mathrm{EFX} \Rightarrow \mathrm{EFL} \Rightarrow \mathrm{EF1}$$

Theorem 9.2.1. *[33] GMMS strictly generalizes MMS and PMMS.*

Proof*: It is trivially true that* $\mathrm{GMMS} \Rightarrow \mathrm{MMS}$ *and that* $\mathrm{GMMS} \Rightarrow \mathrm{PMMS}$*, as MMS must hold for all subsets for* n *for* 2 *respectively by the definition of GMMS. For showing that* $\mathrm{MMS} \not\Rightarrow \mathrm{GMMS}$*, we provide the example given by [33].*

Take an instance with 9 *agents and* 8 *goods, where* $\mathcal{M} = \{g_1, ..g_8\}$*. This means that* $\mathrm{MMS}_i = 0$*, since some agents must receive nothing.* $v_1(g_1) = v_1(g_2) = v_1(g_3) = 5, v_1(g_4) = v_1(g_5) = 3,$ *and* $v_1(g_6) = v_1(g_7) = v_1(g_8) = 1$*. For* $z \in \{1, ..., 9\}$ *and all other agents* $i, v_i(g_z) = 1$*.*

Take an allocation $X_1 = \{g_1\}, X_2 = \{g_2, g_4\}, X_3 = \{g_3, g_5\}, X_4 = \{g_6, g_7, g_8\}$*, with all other agents receiving* $\emptyset$*. This is both MMS and PMMS, but is not GMMS.* ■

10. Other EFX Notions

10.1. **Epistemic EFX.**

[36] proposed the notion of Epistemic Envy-Freeness where for each agent i, the allocation can be made envy-free for i by redistributing $\mathcal{M} \setminus X_i$ among $\mathcal{N} \setminus i$. [37] propose the slightly weaker notion of Epistemic EFX, defined similarly, but only requiring that the allocation can be made envy-free up to X for i. [38] recently showed that EEFX allocations always exist and can be computed in polynomial time.

Open Problem 10.1.1: Do Epistemic Envy-Free allocations exist in general?

10.2. **Charity.**

Although the existence of EX allocations is open, it always exists for a subset of goods. For example, trivially, the removal of all goods yields an EFX allocation. The meaningful form of this is the idea of "charity", where a set of goods $P \subseteq \mathcal{M}$ remain unallocated (i.e. "donated" to charity), and no agent should envy charity. [34] give a polynomial time algorithm which donates at most $n-1$ goods to charity. [39] improve this to donating $n-2$ goods, and give an algorithm which donates only 1 good in the case of 4 agents.

Open Problem 10.2.1: Does EFX exist while donating a sublinear number of goods to charity?

Note that if an EFX allocations with 0 goods may be donated to charity exists, this is simply an EFX allocation.

[40] propose a similar notion called EFkX, in which the removal of 2 goods from another agents' bundle rather than just one (i.e. EF1X is the same as EFX). They show the existence of EF2X allocations for restricted additive valuations (where $v_i\{g\} \in \{0, v(g)\}$). note that an EFX allocation with charity can be turned into an EF2X allocation by giving the $n-2$ goods to $n-2$ unique agents.

11. Different Setting

11.1. **Online Fair Division.**

While in the traditional model of fair division, items are assumed to be standard, agents and/or goods may arrive in an online manner. [41] survey online fair division in depth. Here, either agents, goods,or both arrive online at times $t = 1, 2, ..., T$. One approach to online fairness mechanisms is to asymptotically allocate goods such that envy the envy-per-good $\frac{\text{envy}_t}{t}$ tends to 0, where envy_t is the maximum envy one agent has. [42] show that randomly allocating goods removes the expected envy. More interestingly, it is impossible to do better than such a mechanism, which is optimal up to a logarithmic factor. Online fair division however is not unilaterally easy to compute. [43] show that an ex-ante envy free (i.e. envy-free in expectation) and PO allocation may not always exist.

Theorem 11.1.1. *[43] A simultaneously ex-ante envy-free and PO allocation does not always exist.*

Proof*:*

Agent	g_1	g_2
1	*1*	*2*
2	*2*	*1*

By the definition of ex-ante envy-freeness, each agent must have equal probability of receiving each item (i.e. 12*), however to ensure ex-ante Pareto optimality, each agent must only be given their favorite item.* ∎

11.2. **Chores and Mixed Manna.**

When all goods are chores, i.e. : $2^{\mathcal{M}} \to \mathbb{R}_{\leq 0}$, the problem of fair division becomes much more difficult, and leads to several open problems. EFX for chores does not exist, and indeed is NP-complete [44]. Should such an allocation exist, it is mutually exclusive from PO [45]. It also cannot have a finite approximation. Still, under the restricted context of bi-valued and binary valuations, EF1 and PO allocations exist [45, 46].

The division of goods $g \in \mathcal{M}$ with arbitrary valuations, where $v_i : 2^{\mathcal{M}} \to \mathbb{R}$ have shown several barriers for computation. For EQX allocations, [30] showed that EQX allocations are not guaranteed to exist, even under additive normalized valuations, and that determining if such an allocation exists is strongly NP-hard.

11.3. **Incentives.**

Consider a setting with strategic agents. Here, an agent may misreport their valuations of goods in order to receive a better allocation. A *strategyproof* allocation is one in which such misreporting is not beneficial for such an agent.

Strategyproof mechanisms may not always exist while fulfilling the desiderata. [47] show that in the context of strategic agents, an EFX allocation may not exist. [48] showed that no truthful mechanism could output an α-MMS allocation for $\alpha > \frac{2}{m}$. Indeed, overall truthful mechanisms [49] are incompatible allocations for general valuations.

For some restricted contexts, however, this can be done. Namely for binary valuations, computing an MNW and Lorenz-dominating allocation (and thus trivially a PO and EF1) allocation are strategyproof. [15, 50] both develop algorithms to compute such allocations. They further show that under binary valuations, and even submodular valuations, a truthful mechanism exists to compute EFX allocations.

12. Proportionality

One of the earliest notions of fairness it that of proportionality (PROP), where each good values their allocation at least $v_i(\frac{\mathcal{M}}{n})$. Clearly, like envy-freeness, it cannot be guaranteed in the case of discrete goods, so it is necessary to weaken it slightly by allowing the addition of goods. In proportionality up to 1 good (PROP1), there exists a good $g \in \mathcal{M} \setminus X_i$ where $v_i(X_i \cup \{g\}) \geq v_i(\frac{\mathcal{M}}{n})$. Similarly, in proportionality up to x (PROPx), all goods $g \in \mathcal{M} \setminus X_i$ where $v_i(X_i \cup \{g\} \geq v_i(\frac{\mathcal{M}}{n}))$.

[51] showed that a PROP1 and PO allocation always exists and can be computed in polynomial time.

Still proportionality can be disappointing, as [52] showed that even for simple cases, a PROPx allocation may not exist.

Theorem 12.1. *A PROPx allocation may not always exist.*

Proof*: Take 3 agents with identical valuations, and 5 goods. For all agents $i \in \mathcal{N}, v_i(g) = 3$ for $g \in \{g_1, ..., g_4\}$ and $v_i(g_5) = 1$. No allocation admits a PROPx allocation.* ■

An intermediary notion, PROPm has been shown to always exist and be computable in polynomial time. [53] defines the maximin good as $d(i) = \max_{i' \neq i} \min_{g \in X_i} v_i(g)$ is the largest among the smallest goods in other agent's bundles. If adding this good's value to i's utility is greater than or equal to $\frac{1}{n}$, i.e. $v_i(X_i) + d(i) \geq \frac{1}{n}$, then an allocation is PROPm.

13. Fisher Markets

Although Nash Welfare is compelling, for its simultaneous EF1 + PO, it is also NP-hard to compute. One line of work attempting to simultaneous achieve fairness and efficiency tries to use the computation of market equilibria of fisher markets. This has also been used to compute approximate MNW, as mentioned previously. In the fisher market, goods are given prices $p = \langle p_1, ..., p_m \rangle$, agents are given earnings ℓ, and monotone valuations, as defined above. These goods may be fractionally allocated with linear valuations, i.e. $v_i(X_i) = \sum_{g \in X_i} g \times x_{i,g}$ where $x_{i,g}$ is the fraction of g given to i.

A fisher market equilibrium is a fractional allocation in which

1. All goods $g \in \mathcal{M}$ allocated, i.e. that $\sum_{i \in \mathcal{N}} x_{i,g} = 1$
2. For all agents $i \in \mathcal{N}, \sum_{g \in \mathcal{M}} x_{i,g} \times p_g = \ell_i$
3. All agents only receive goods with maximum bang per buck, i.e. for all $i \in \mathcal{N}, g \in \mathcal{M}$, if $x_{i,g} > 0$, then $v_i\left(\frac{g}{p_g}\right) = \mathrm{MBB}_i$

When $\ell_i = 1$ for all agents i, this is called a Competitive Equilibrium for Equal Incomes (CEEI).

Open Problem 13.1: Can an exact CEEI be computed in polynomial time?

Although this remains an open question, prior work seeks to compute an approximate $(1 - \varepsilon)$-CEEI allocation. In such an allocation, agents may spend $\ell_i(1 \pm \varepsilon)$ for some $\varepsilon \in [0, 1]$.

Some work has been dedicated to computing approximations for CEEI. [54] computes a $1-\varepsilon$-CEEI allocation with running time proportional to $\frac{1}{\varepsilon}$. [55] gives an algorithm to compute a $(1-\varepsilon)$-CEEI in polynomial time for $\varepsilon = \frac{1}{5nm}$.

A fractionally Pareto optimal allocation (fPO) is a fractional allocation which is not Pareto-dominated by another fractional allocation. It is clear that an fPO allocation is PO, but not necessarily vice versa.

CEEI is compelling in computing fPO. [56] show that a CEEI is also fPO, and thus PO.

14. Empirical Results

We can see the practical usefulness of these results from prior experimental results. For discrete fair division, [57] collected a set of items, and computed an EF1 allocation among they agents. They then allowed agents to swap, and found that agents preferred to swap their bundles with other agents in 95% of allocations.

Another common approach to allowing measurement of goods is to give each agent some number of “points”, and allow them to distribute the points among the goods based on their preference, which for can still maintain the fairness properties of MNW and EF1. In Spliddit, for 1000 points and 10 agents, an MNW (EF1 and PO) allocation is computed in less than 3 seconds [4, 58].

15. Discussion

In this survey, we discussed some of the most common notions of fairness. Although there are some further related notions of fairness, these are the most well-studied among the literature. We also showed that, under identical additive valuations, MNW $\Rightarrow$ EQX.

Related to fair division, within economics and computation (EC), recent research has also studied computing market equilibria, beyond the Fisher Market. Other works in EC focus more on strategyproofness, studying voting and consensus procedures.

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