SIAM J. NUMER. ANAL.
Vol. 22, No. 3, June 1985

003

A CONFORMING FINITE ELEMENT METHOD FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS*

CHRISTINE BERNARDI† AND GENEVIÈVE RAUGEL‡

Abstract. We study a spatial discretization of the Stokes problem in a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ by a finite element method, the unknowns being the velocity and the pressure; optimal error estimates are derived. Then, these results are extended to the Navier-Stokes equations.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$, $n=2$ or 3, with a Lipschitz-continuous boundary Γ; we consider the time-dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t}-\nu\,\Delta\mathbf{u}+\sum_{i=1}^{n} u_i\frac{\partial \mathbf{u}}{\partial x_i}+\mathbf{grad}\,p=\mathbf{f}\quad \text{in }\Omega,\quad t>0,\\ &\operatorname{div}\mathbf{u}=0\quad\text{in }\Omega,\quad t>0,\\ &\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma,\quad t>0,\\ &\mathbf{u}|_{t=0}=\mathbf{u}_0, \end{aligned}\tag{1.1}$$

where $\mathbf{u}=(u_1,\cdots,u_n)$ is the velocity, p is the pressure, $\mathbf{f}$ represents the density of body forces, $\nu>0$ is the viscosity and $\mathbf{u}_0$ is the initial velocity.

If we neglect the nonlinear terms in (1.1), we obtain the Stokes problem

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t}-\nu\,\Delta\mathbf{u}+\mathbf{grad}\,p=\mathbf{f}\quad \text{in }\Omega,\quad t>0,\\ &\operatorname{div}\mathbf{u}=0\quad\text{in }\Omega,\quad t>0,\\ &\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma,\quad t>0,\\ &\mathbf{u}|_{t=0}=\mathbf{u}_0. \end{aligned}\tag{1.2}$$

This paper is devoted to the analysis of a numerical approximation of equations (1.1) and (1.2): we introduce and study a spatial discretization of these problems by a conforming finite element method, the unknowns being the velocity and the pressure. In the stationary case, the discretization of the equations of Navier-Stokes in this formulation has been studied e.g. in [1], [9], [12], [14]; our aim is to extend some of their methods to the time-dependent problems (1.1) and (1.2). Indeed, we obtain error estimates of order 1 in the norms of several space-time Sobolev spaces, under minimal regularity assumptions on the data. The proofs use energy inequalities as well as fractional derivatives estimates. We consider first the Stokes problem (1.2), then extend the results to the nonlinear problem (1.1) by a discrete implicit function theorem.

The spatial discretization of the time-dependent Navier-Stokes equations in the velocity-pressure formulation has been studied previously in [17], [21], [22], [23], for instance. Using the semigroup techniques of [11], Okamoto [21], [22], analyses the special case of the finite element described in [1]. On the other hand, by using energy methods, Rannacher [23] derives several uniform-in-time error bounds for a large class

* Received by the editors May 13, 1983, and in revised form June 7, 1984.

† Analyse Numérique, Tour 55-65, Université Pierre et Marie Curie, 4 place Jussieu, 75230 Paris Cedex 05 France.

‡ Analyse Numérique, Université de Rennes, Rennes Beaulieu, 35042 Rennes Cedex France.

of conforming finite elements; however, as he works directly on the nonlinear problem, his proofs appear to be a little more technical than ours. Estimates of higher order are given in [17]. Other approximations of problems (1.1) and (1.2) can be found for instance in [4], [19].

An outline of the paper is as follows. Section 2 is devoted to the analysis of an abstract variational problem and its discretization. In § 3, we apply the previous results to the Stokes problem and give error estimates for the velocity and the pressure. In § 4, we give a theorem of existence and uniqueness for the discrete Navier-Stokes equations and prove similar error estimates.

In the following we shall use the classical Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, provided with the norm $\|\cdot\|_{s,\Omega}$. Let T be a real positive number. For any Banach space Z, we denote by $L^2(Z)$, $H^1(Z)$ and $H^{-1}(Z)$ the spaces $L^2(0, T; Z)$, $H^1(0, T; Z)$ and $H^{-1}(0, T; Z)$ respectively. For $0 \leqq s \leqq 1$, the space $H^s(Z)$ will be defined by interpolation between $L^2(Z)$ and $H^1(Z)$. We denote by $H_0^s(Z)$ the closure in $H^s(Z)$ of the space $\mathcal{D}(]0, T[; Z)$; it is equal to $H^s(Z)$ if $0 \leqq s \leqq \frac{1}{2}$. For any v in $H^s(Z)$, we consider the extension $\tilde{v}$ of v

$$\tilde{v}(t) = \begin{cases} v(t) & \text{in } (0, T), \\ v(2T-t) & \text{in } (T, 2T), \\ 0 & \text{elsewhere,} \end{cases}$$

and we introduce the Fourier transform $\hat{v}$ of $\tilde{v}$, with values in the natural complexified space $Z^c = Z + iZ$ of Z,

$$\hat{v}(\tau) = \int_{\mathbb{R}} e^{-it\tau} \tilde{v}(t)\, dt;$$

we recall that the mapping: $v \to \|(1+|\tau|^2)^{s/2} \hat{v}\|_{L^2(\mathbb{R}; Z^c)}$ is a norm on $H_0^s(Z)$ equivalent to the norm of $H^s(Z)$, for $0 \leqq s < \frac{1}{2}$, and we denote by $|v|_{H_0^s(Z)}$ the seminorm $\| |\tau|^s \hat{v} \|_{L^2(\mathbb{R}; Z^c)}$.

2. An abstract time-dependent problem.

2.1. The continuous problem. Let X and M denote two real reflexive Banach spaces. Let Y be a real Hilbert space such that X is contained in Y with a continuous and dense imbedding. If X' and Y' are the dual spaces of X and Y respectively, the duality pairing $\langle \cdot, \cdot \rangle$ between Y and Y' will be extended without change of notation to the duality pairing between X and X'.

We introduce three continuous bilinear forms $r(\cdot, \cdot)$, $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ on $Y \times Y$, $X \times X$ and $X \times M$ respectively. Next, we define the closed subset V of X by

$$V = \{v \in X; \forall \mu \in M, b(v, \mu) = 0\}$$

and we denote by H the closure of V in Y. We associate with the form r the continuous linear operator R from Y into Y', defined by

$$\forall u \in Y, \quad \forall v \in Y, \quad \langle Ru, v \rangle = r(u, v). \tag{2.1}$$

For a given function f in $L^2(X')$ and a given u_0 in H, we consider the problem (Q): find a pair (u, λ) in $L^2(X) \times \mathcal{D}'(0, T; M)$ such that

$$\forall v \in X, \quad \frac{d}{dt} r(u, v) + a(u, v) + b(v, \lambda) = \langle f, v \rangle \quad \text{in } \mathcal{D}'(0, T),$$

$$\forall \mu \in M, \quad b(u, \mu) = 0 \quad \text{a.e. in } (0, T),$$

$$\forall v \in Y, \quad r(u(0) - u_0, v) = 0,$$

and, with (Q), we associate the problem (P): find u in $L^2(V)$ such that

$$\forall v \in V, \quad \frac{d}{dt} r(u, v) + a(u, v) = \langle f, v \rangle \quad \text{a.e. in } (0, T),$$

$$\forall v \in H, \quad r(u(0) - u_0, v) = 0.$$

We see that, for any (u, λ) solution of (Q), u is a solution of (P).

We shall have to assume the following hypotheses:

(h.1) the space V is separable;

(h.2) the form a is symmetric and V-elliptic, i.e.

$$\forall v \in V, \quad a(v, v) \geqq \alpha \|v\|_X^2, \quad \alpha > 0;$$

(h.3) the form b satisfies the inf-sup condition (see [5])

$$\forall \mu \in M, \quad \sup_{v \in X} \frac{b(v, \mu)}{\|v\|_X} \geqq \beta \|\mu\|_M, \quad \beta > 0;$$

(h.4) the form r is symmetric on H and H-elliptic, i.e.

$$\forall v \in H, \quad r(v, v) \geqq \gamma \|v\|_Y^2, \quad \gamma > 0.$$

Let us denote by $\mathscr{W}_R$ the space of all functions v in $L^2(V)$ such that $(d/dt)Rv$ belongs to $L^2(V')$. The following result is well known when R is the Riesz isomorphism from H onto H' (cf. [12, Chap. V, Thm. 1.1], [24, Chap. III, Lemma 1.2]), and can be proved in the same way.

LEMMA 2.1. *Assume that the hypothesis* (h.4) *is satisfied. The space* $\mathscr{W}_R$ *is contained in* $\mathscr{C}^0([0, T]; H)$ *and in* $H^{1/2}(H)$ *with continuous imbeddings.*

THEOREM 2.1. *Assume that the hypotheses* (h.1) *to* (h.4) *are satisfied. Then, problem* (P) *has a unique solution* u *in* $\mathscr{W}_R$, *and*

$$\|u\|_{\mathscr{W}_R} \leqq c\{\|f\|_{L^2(V')} + \|u_0\|_H\}. \tag{2.2}$$

Moreover, problem (Q) *has a unique solution* (u, λ) *in* $\mathscr{W}_R \times H^{-1}(M)$ *where* u *is the solution of problem* (P).

Proof. The first part of the theorem is proved by a classical Faedo-Galerkin method (cf. [24, Chap. III, Thm 1.1]). The uniqueness of the solution of problem (Q) is an easy consequence of (h.3). To prove its existence, we follow the same way as in [24, Chap. III, Prop. 1.1]: if u is the solution of problem (P), for a.e. t in $(0, T)$, we define $L(t)$ in X' by

$$\forall v \in X, \quad \langle L(t), v \rangle = \int_0^t \{\langle f(s), v \rangle - a(u(s), v)\}\, ds + r(u_0, v) - r(u(t), v) \tag{2.3}$$

and notice that L belongs to $\mathscr{C}^0([0, T]; H)$ and satisfies: $\forall v \in V$, $\langle L(t), v \rangle = 0$. Then, by (h.3), we know (cf. [5], [12, Chap. I, Lemma 4.1]) that there exists a unique Λ in $\mathscr{C}^0([0, T]; M)$ such that, for every t in $[0, T]$,

$$\forall v \in X, \quad b(v, \Lambda(t)) = \langle L(t), v \rangle.$$

If λ denotes the derivative of Λ in $\mathscr{D}'(0, T)$, we obtain

$$\forall v \in X, \quad \frac{d}{dt} r(u, v) + a(u, v) + b(v, \lambda) = \langle f, v \rangle \quad \text{in } \mathscr{D}'(0, T).$$

We also have

$$\forall \mu \in M, \quad b(u, \mu)=0 \quad \text{a.e. in } (0, T).$$

Finally, (h.4) yields: $u(0)=u_0$. Then, (u, λ) is the solution of problem (Q).

We shall use the following regularity result, which can be proved by the same way as in [24, Chap. III, Prop. 1.2].

PROPOSITION 2.1. *Assume that the hypotheses* (h.1) *to* (h.4) *are satisfied. The mapping*: $(f, u_0) \to (u, \lambda)$ *is linear continuous from* $L^2(Y') \times V$ *into* $H^1(H) \times L^2(M)$.

Remark 2.1. Of course, if f belongs to $L^2(Y')$ and u_0 belongs to V, the first equation in problem (Q) is satisfied not only in $\mathscr{D}'(0, T)$ but also a.e. in $(0, T)$.

2.2. The discrete problem. From now on, h will be a real positive parameter tending to 0. We introduce two finite-dimensional subspaces X_h and M_h of X and M respectively and we define the subspace V_h of X_h by

$$V_h=\{v_h \in X_h\,;\, \forall \mu_h \in M_h,\, b(v_h, \mu_h)=0\}.$$

We consider the problem (Q_h): find (u_h, λ_h) in $H^1(X_h) \times L^2(M_h)$ such that

$$\forall v_h \in X_h \quad \frac{d}{dt} r(u_h, v_h)+a(u_h, v_h)+b(v_h, \lambda_h)=\langle f, v_h\rangle \quad \text{a.e. in } (0, T),$$

$$\forall \mu_h \in M_h, \quad b(u_h, \mu_h)=0 \quad \text{a.e. in } (0, T),$$

$$\forall v_h \in V_h, \quad r(u_h(0)-u_0, v_h)=0,$$

and, with (Q_h), we associate the problem (P_h): find u_h in $H^1(V_h)$ such that

$$\forall v_h \in V_h, \quad \frac{d}{dt} r(u_h, v_h)+a(u_h, v_h)=\langle f, v_h\rangle \quad \text{a.e. in } (0, T),$$

$$\forall v_h \in V_h, \quad r(u_h(0)-u_0, v_h)=0.$$

Of course, for any (u_h, λ_h) solution of (Q_h), u_h is a solution of (P_h).

We shall have to assume the following hypotheses:

(h.5) the form a is V_h-elliptic, i.e.

$$\forall v_h \in V_h, \quad a(v_h, v_h) \geqq \alpha_h \|v_h\|_X^2, \quad \alpha_h>0;$$

(h.6) the form b satisfies the inf-sup condition on X_h, i.e.

$$\forall \mu_h \in M_h, \quad \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geqq \beta_h \|\mu_h\|_M, \quad \beta_h>0;$$

(h.7) the form r is symmetric on V_h and satisfies

$$\forall v_h \in V_h, \quad r(v_h, v_h) \geqq \gamma_h \|v_h\|_Y^2, \quad \gamma_h>0.$$

Remark 2.2. Let us assume (h.7) and introduce the linear operators Π_h and Π_h^* from Y into V_h, defined by

$$\forall v \in Y, \quad \begin{cases} \Pi_h v \in V_h \quad \text{and} \quad \Pi_h^* v \in V_h, \\ \forall v_h \in V_h, \quad r(v-\Pi_h v, v_h)=r(v_h, v-\Pi_h^* v)=0. \end{cases} \tag{2.4}$$

Then, the initial condition in problems (P_h) and (Q_h) can be written:

$$u_h(0)=\Pi_h u_0.$$

THEOREM 2.2. *Assume that the hypotheses* (h.5) *to* (h.7) *are satisfied. Then, problem* (P_h) *has a unique solution* u_h *and problem* (Q_h) *has a unique solution* (u_h, λ_h), *where* u_h *is the solution of problem* (P_h).

Proof. By (h.5) and (h.7), problem (P_h) has a unique solution u_h; as V_h is finite-dimensional, u_h belongs to $H^1(V_h)$ and

$$\|u_h\|_{H^1(V_h)} \leqq c(h)\{\|f\|_{L^2(X')} + \|\Pi_h u_0\|_Y\}.$$

By (h.6), for a.e. t in $(0, T)$, there exists a unique $\lambda_h(t)$ in M_h such that

$$\forall v_h \in X_h, \quad b(v_h, \lambda_h(t)) = \langle f(t), v_h\rangle - \frac{d}{dt} r(u_h(t), v_h) - a(u_h(t), v_h)$$

and

$$\|\lambda_h(t)\|_M \leqq c(h)\left\{\|f(t)\|_{X'} + \|u_h(t)\|_X + \left\|\frac{d}{dt} u_h(t)\right\|_Y\right\};$$

then λ_h belongs to $L^2(M_h)$.

2.3. Error estimates. The basic error estimates are given by the following theorem.

THEOREM 2.3. *Assume that the hypotheses* (h.1) *to* (h.7) *are satisfied with the constants* α_h *and* γ_h *independent of* h. *We have the error estimates*

$$\|u - u_h\|_{L^2(X)} \leqq c\left\{\|u - \Pi_h u\|_{L^2(X)} + \left\|\sup_{v_h \in V_h} \frac{b(v_h, \lambda)}{\|v_h\|_X}\right\|_{L^2(\mathbb{R})}\right\} \tag{2.5}$$

and

$$\begin{aligned} &\|u - u_h\|_{\mathscr{C}^0([0,T];Y)} \\ &\quad \leqq c\left\{\|u - \Pi_h u\|_{L^2(X)} + \|u - \Pi_h u\|_{\mathscr{C}^0([0,T];Y)} + \left\|\sup_{v_h \in V_h} \frac{b(v_h, \lambda)}{\|v_h\|_X}\right\|_{L^2(\mathbb{R})}\right\}. \end{aligned} \tag{2.6}$$

Proof. Let us set

$$K(t) = \sup_{v_h \in V_h} \frac{b(v_h, \lambda(t))}{\|v_h\|_X}. \tag{2.7}$$

We also notice that, since (u, λ) and (u_h, λ_h) are the solutions of problems (Q) and (Q_h) respectively,

$$\forall v_h \in X_h, \quad \frac{d}{dt} r(u - u_h, v_h) + a(u - u_h, v_h) + b(v_h, \lambda - \lambda_h) = 0. \tag{2.8}$$

As

$$r\left(\frac{d}{dt}(u - \Pi_h u), u_h - \Pi_h u\right) = 0$$

and by (2.8), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} r(u_h - \Pi_h u, u_h - \Pi_h u) + a(u_h - \Pi_h u, u_h - \Pi_h u) \\ &\quad = -r\left(\frac{d}{dt}(u - u_h), u_h - \Pi_h u\right) + a(u_h - \Pi_h u, u_h - \Pi_h u) \\ &\quad = a(u - \Pi_h u, u_h - \Pi_h u) + b(u_h - \Pi_h u, \lambda), \end{aligned}$$

so that

$$\frac{1}{2}\frac{d}{dt}r(u_h-\Pi_h u, u_h-\Pi_h u)+\alpha\|u_h-\Pi_h u\|_X^2$$

$$\leqq\frac{\alpha_h}{4}\|u_h-\Pi_h u\|_X^2+\frac{c}{\alpha_h}\|u-\Pi_h u\|_X^2$$

$$+\frac{\alpha_h}{4}\|u_h-\Pi_h u\|_X^2+\frac{c}{\alpha_h}K(t)^2.$$

By integrating between 0 and t, we obtain

$$\frac{\gamma_h}{2}\|u_h(t)-\Pi_h u(t)\|_Y^2+\frac{\alpha_h}{2}\int_0^t\|u_h(s)-\Pi_h u(s)\|_X^2\,ds$$

$$\leqq\frac{c}{\alpha_h}\{\|u-\Pi_h u\|_{L^2(X)}^2+\|K\|_{L^2(\mathbb{R})}^2\},$$

which implies (2.5) and (2.6).

THEOREM 2.4. *Assume that the hypotheses* (h.1) *to* (h.7) *are satisfied with the constants α_h and γ_h independent of h. We have the error estimates*

$$(2.9)\quad |u-u_h|_{H_0^\alpha(Y)}\leqq c\left(\inf_{v_h\in H_0^\alpha(V_h)}\{\|u-v_h\|_{L^2(X)}+|u-v_h|_{H_0^\alpha(Y)}\}+\left\|\sup_{v_h\in V_h}\frac{b(v_h,\lambda)}{\|v_h\|_X}\right\|_{L^2(\mathbb{R})}\right),$$

for any $\alpha<\frac{1}{2}$, and, in the case $u_0=0$,

$$(2.10)\quad \begin{aligned}&\|u-u_h\|_{L^2(X)}+|u-u_h|_{H_0^{1/2}(Y)}\\&\leqq c\left(\inf_{v_h\in H_0^{1/2}(V_h)}\{\|u-v_h\|_{L^2(X)}+|u-v_h|_{H_0^{1/2}(Y)}\}+\left\|\sup_{v_h\in V_h}\frac{b(v_h,\lambda)}{\|v_h\|_X}\right\|_{L^2(\mathbb{R})}\right).\end{aligned}$$

Proof. All the bilinear forms will be extended into sesquilinear forms on the natural complexified spaces. We notice that, since (u,λ) and (u_h,λ_h) are the solutions of problems (Q) and (Q_h) respectively,

$$(2.11)\quad \forall v_h\in X_h,\quad i\tau\, r(\hat{u}-\hat{u}_h, v_h)+a(\hat{u}-\hat{u}_h, v_h)+b(v_h,\hat{\lambda}-\hat{\lambda}_h)=-r(u_0-\Pi_h u_0, v_h).$$

Let us choose $v_h(t)$ in V_h; by (2.11) we have

$$\begin{aligned}&i\tau\, r(\hat{u}_h-\hat{v}_h,\hat{u}_h-\hat{v}_h)+a(\hat{u}_h-\hat{v}_h,\hat{u}_h-\hat{v}_h)\\&\quad=i\tau\, r(\hat{u}-\hat{v}_h,\hat{u}_h-\hat{v}_h)+a(\hat{u}-\hat{v}_h,\hat{u}_h-\hat{v}_h)+b(\hat{u}-\hat{v}_h,\hat{\lambda}),\end{aligned}$$

so that, by separating the real and imaginary parts of the first member and using (h.5) and (h.7),

$$\frac{\gamma_h}{2}|\tau|\,\|\hat{u}_h-\hat{v}_h\|_Y^2+\frac{\alpha_h}{2}\|\hat{u}_h-\hat{v}_h\|_X^2$$

$$\leqq\frac{c}{\gamma_h}|\tau|\,\|\hat{u}-\hat{v}_h\|_Y^2+\frac{c}{\alpha_h}\{\|\hat{u}-\hat{v}_h\|_X^2+|\hat{K}(\tau)|^2\}.$$

In the case $u_0=0$, this inequality yields immediately (2.10). In the general case, we have:

1. For $|\tau|\geqq 1$,

$$|\tau|^{2\alpha}\|\hat{u}_h-\hat{v}_h\|_Y^2\leqq c\{|\tau|^{2\alpha}\|\hat{u}-\hat{v}_h\|_Y^2+|\tau|^{2\alpha-1}(\|\hat{u}-\hat{v}_h\|_X^2+|\hat{K}(\tau)|^2)\}$$
$$\leqq c\{|\tau|^{2\alpha}\|\hat{u}-\hat{v}_h\|_Y^2+\|\hat{u}-\hat{v}_h\|_X^2+|\hat{K}(\tau)|^2\},$$

2. For $|\tau|\leqq 1$,

$$|\tau|^{2\alpha}\|\hat{u}_h-\hat{v}_h\|_Y^2\leqq c\|\hat{u}_h-\hat{v}_h\|_X^2$$
$$\leqq c\{|\tau|\,\|\hat{u}-\hat{v}_h\|_Y^2+\|\hat{u}-\hat{v}_h\|_X^2+|\hat{K}(\tau)|^2\}$$
$$\leqq c\{\|\hat{u}-\hat{v}_h\|_X^2+|\hat{K}(\tau)|^2\};$$

the two last inequalities imply (2.9).

Under some slightly stronger assumptions, we state two other error estimates.

PROPOSITION 2.2. *Assume that the hypotheses* (h.1) *to* (h.7) *are satisfied with the constants* α_h *and* γ_h *independent of* h, *and moreover that there exists a constant* δ *independent of* h *such that*

$$\forall v\in V,\quad \|\Pi_h^* v\|_X\leqq \delta\|v\|_X. \tag{2.12}$$

We have the error estimate

$$\left\|\frac{d}{dt}R(u-u_h)\right\|_{L^2(V')}\leqq c\left\{\|u-\Pi_h u\|_{L^2(X)}+\left\|\frac{d}{dt}R(u-\Pi_h u)\right\|_{L^2(V')}+\left\|\sup_{v_h\in V_h}\frac{b(v_h,\lambda)}{\|v_h\|_X}\right\|_{L^2(\mathbb{R})}\right\}. \tag{2.13}$$

Proof. We have

$$\left\|\frac{d}{dt}R(u_h-\Pi_h u)\right\|_{V'}=\sup_{v\in V}\frac{(d/dt)r(u_h-\Pi_h u,v)}{\|v\|_X}.$$

As $u_h-\Pi_h u$ belongs to V_h and by (2.8),

$$\left\|\frac{d}{dt}R(u_h-\Pi_h u)\right\|_{V'}=\sup_{v\in V}\frac{(d/dt)r(u_h-\Pi_h u,\Pi_h^* v)}{\|v\|_X}$$
$$=\sup_{v\in V}\frac{(d/dt)r(u-\Pi_h u,\Pi_h^* v)+a(u-u_h,\Pi_h^* v)+b(\Pi_h^* v,\lambda)}{\|v\|_X},$$

then, using (2.12), we obtain

$$\left\|\frac{d}{dt}R(u_h-\Pi_h u)\right\|_{V'}\leqq c\{\|u-u_h\|_X+|K|\},$$

which, with (2.5), implies (2.13).

PROPOSITION 2.3. *Assume that the hypotheses* (h.1) *to* (h.7) *are satisfied with the constant* β_h *independent of* h. *We have the error estimate*

$$\|\lambda-\lambda_h\|_{H^{-1}(M)}\leqq c\left\{\inf_{\mu_h\in H^{-1}(M_h)}\|\lambda-\mu_h\|_{H^{-1}(M)}+\|u-u_h\|_{H^{-1}(X)}\right.$$
$$\left.+\|R(u-u_h)\|_{L^2(X')}+\|R(u_0-\Pi_h u_0)\|_{X'}\right\}. \tag{2.14}$$

Proof. If we set: $\Lambda_h(t)=\int_0^t \lambda_h(s)\,ds$, since (u_h, λ_h) is the solution of problem (Q_h), we have

$$\forall v_h \in X_h,\ b(v_h, \Lambda_h(t))=\int_0^t \{\langle f(s), v_h\rangle - a(u_h(s), v_h)\}\,ds + r(u(0), v) - r(u(t), v),$$

which, with (2.3), yields

$$\begin{aligned}\forall v_h \in X_h,\quad b(v_h, \Lambda(t)-\Lambda_h(t)) &= -\int_0^t a(u(s)-u_h(s), v_h)\,ds\\ &\quad + r(u(0)-u_h(0), v_h) - r(u(t)-u_h(t), v_h).\end{aligned}\tag{2.15}$$

For any Ξ_h in $L^2(M_h)$, by (h.6) and (2.15), we have

$$\begin{aligned}&\beta_h\|\Lambda_h(t)-\Xi_h(t)\|_M\\ &\quad\leqq \sup_{v_h\in X_h}\frac{b(v_h, \Lambda_h(t)-\Xi_h(t))}{\|v_h\|_X}\\ &\quad\leqq c\sup_{v_h\in X_h}\left\{\frac{b(v_h, \Lambda(t)-\Xi_h(t))}{\|v_h\|_X}-\frac{\int_0^t a(u(s)-u_h(s), v_h)\,ds}{\|v_h\|_X}\right.\\ &\qquad\qquad\left.+\frac{r(u(0)-u_h(0), v_h)}{\|v_h\|_X}-\frac{r(u(t)-u_h(t), v_h)}{\|v_h\|_X}\right\}\\ &\quad\leqq c\left\{\|\Lambda(t)-\Xi_h(t)\|_M+\left\|\int_0^t (u(s)-u_h(s))\,ds\right\|_X\right.\\ &\qquad\qquad\left.+\|R(u(0)-u_h(0))\|_{X'}+\|R(u(t)-u_h(t))\|_{X'}\right\}.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\left\|\lambda_h-\frac{d\Xi_h}{dt}\right\|_{H^{-1}(M)}&\leqq c\|\Lambda_h-\Xi_h\|_{L^2(M)}\\ &\leqq c\left\{\|\Lambda-\Xi_h\|_{L^2(M)}+\left\|\int_0^t (u(s)-u_h(s))\,ds\right\|_{L^2(X)}\right.\\ &\qquad\left.+\|R(u_0-\Pi_h u_0)\|_{X'}+\|R(u-u_h)\|_{L^2(X')}\right\}\\ &\leqq c\left\{\left\|\lambda-\frac{d\Xi_h}{dt}\right\|_{H^{-1}(M)}+\|u-u_h\|_{H^{-1}(X)}+\|R(u-u_h)\|_{L^2(X')}\right.\\ &\qquad\left.+\|R(u_0-\Pi_h u_0)\|_{X'}\right\}.\end{aligned}$$

Remark 2.3. By Theorems 2.3 and 2.4, we know how to bound $\|u-u_h\|_{L^2(X)}$. Unfortunately, we are not able to obtain an improved estimate for $\|u-u_h\|_{H^{-1}(X)}$ or $\|R(u-u_h)\|_{L^2(X')}$.

Remark 2.4. In all the sequel, the form r that we are going to introduce will be symmetric on Y. Let us describe a situation where this property is not true: we consider for a while a simply-connected bounded domain Ω in $\mathbb{R}^2$; then a function $\mathbf{v}$ in $H_0^1(\Omega)^2$ is divergence-free if and only if there exists φ in $H_0^2(\Omega)$ such that: $\mathbf{v}=\mathbf{rot}\,\varphi$ in Ω. Choosing $X=H_0^2(\Omega)\times L^2(\Omega)$, $Y=H_0^1(\Omega)\times H^{-1}(\Omega)$ and $M=L^2(\Omega)$, we can write the

Stokes problem (1.2) in the following way: find $(u=(\psi, \omega), \lambda)$ in $L^2(X)\times \mathscr{D}'(0, T; M)$ such that

$$\begin{aligned}
&\forall v=(\varphi, \theta)\in X, \quad \frac{d}{dt} r(u, v)+\nu \int_\Omega \omega\theta\, dx+\int_\Omega (-\Delta\varphi-\theta, \lambda)\, dx \\
&\qquad\qquad = \int_\Omega \mathbf{f}\cdot \mathbf{rot}\, \varphi\, dx \quad \text{in } \mathscr{D}'(0, T), \\
&\forall \mu \in M, \quad \int_\Omega (-\Delta\psi-\omega)\mu\, dx=0 \quad \text{a.e. in } (0, T), \\
&\forall v\in X, \quad r(u(0)-u_0, v)=0,
\end{aligned} \tag{2.16}$$

where r is one of the three forms

$$r(u, v)=\int_\Omega \mathbf{rot}\, \psi\cdot \mathbf{rot}\, \varphi\, dx \quad \text{or} \quad \int_\Omega \omega\varphi\, dx \quad \text{or} \quad \int_\Omega \psi\theta\, dx$$

(the last two forms are not symmetric on Y, but only on $H=\{v=(\varphi, \theta=-\Delta\varphi)\in Y\}$). The previous framework can be used to study a finite element approximation of (2.16), in a slightly sharpened form so as to take finite elements only of class $\mathscr{C}^0$ (cf. [12] in the stationary case). For a detailed analysis of such methods, see [4].

3. Application to the time-dependent Stokes problem.

3.1. The continuous problem. We are going to study problem (1.2) by using the abstract framework given in § 2. To this end we set

$$X=H_0^1(\Omega)^n, \quad Y=L^2(\Omega)^n \quad \text{and} \quad M=L_0^2(\Omega)=\left\{q\in L^2(\Omega); \int_\Omega q\, dx=0\right\}.$$

We denote by $(\cdot,\cdot)$ the inner product on $L^2(\Omega)$ or $L^2(\Omega)^n$ and identify $L^2(\Omega)$ with its dual space. Hereafter the bilinear form $r(\cdot,\cdot)$ on $Y\times Y$ is the inner product $(\cdot,\cdot)$, so that the operator R is the identity on Y.

We define the continuous bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $X\times X$ and $X\times M$ respectively by

$$\forall \mathbf{u}\in X, \forall \mathbf{v}\in X, \quad a(\mathbf{u}, \mathbf{v})=\nu(\mathbf{grad}\, \mathbf{u}, \mathbf{grad}\, \mathbf{v}) \tag{3.1}$$

and

$$\forall \mathbf{v}\in X, \forall q\in M, \quad b(\mathbf{v}, q)=-(q, \operatorname{div} \mathbf{v}). \tag{3.2}$$

Clearly one has

$$\begin{aligned}
&V=(\mathbf{v}\in H_0^1(\Omega)^n; \operatorname{div} \mathbf{v}=0 \text{ in } \Omega\}, \\
&H=\{\mathbf{v}\in L^2(\Omega)^n; \operatorname{div} \mathbf{v}=0 \text{ in } \Omega \text{ and } \mathbf{v}\cdot \mathbf{n}=0 \text{ on } \Gamma\},
\end{aligned} \tag{3.3}$$

where $\mathbf{n}$ denotes the unit outward normal to Ω. Then problems (P) and (Q) are well defined.

THEOREM 3.1. *For* $\mathbf{f}$ *given in* $L^2(H^{-1}(\Omega)^n)$ *and* $\mathbf{u}_0$ *in* H, *problem* (P) *has a unique solution* $\mathbf{u}$ *in* $L^2(V)$ *such that* $d\mathbf{u}/dt$ *belongs to* $L^2(V')$ *and*

$$\|\mathbf{u}\|_{L^2(H_0^1(\Omega)^n)}+\left\|\frac{d\mathbf{u}}{dt}\right\|_{L^2(V')}\leqq c\{\|\mathbf{f}\|_{L^2(V')}+\|\mathbf{u}_0\|_{0,\Omega}\}. \tag{3.4}$$

Moreover there exists p *in* $H^{-1}(L_0^2(\Omega))$ *such that* $(\mathbf{u}, p)$ *is the unique solution of problem* (Q).

Proof. This theorem is an immediate consequence of Theorem 2.1. Assumptions (h.1), (h.2) and (h.4) are obviously satisfied. Assumption (h.3) is proved in [12, Chap. I, Thm. 3.7].

We introduce now the stationary Stokes problem: for $\mathbf{g}$ in X', find $(\mathbf{v}, q)$ in $X \times M$ such that

$$(3.5)\qquad \begin{aligned} &-\nu\Delta\mathbf{v}+\mathbf{grad}\, q=\mathbf{g} \quad \text{in } \Omega,\\ &\operatorname{div}\mathbf{v}=0 \quad \text{in } \Omega,\\ &\mathbf{v}=\mathbf{0} \quad \text{on } \Gamma.\end{aligned}$$

Hereafter we assume that problem (3.5) is regular in Ω i.e. that the mapping: $(\mathbf{v}, q)\to -\nu\Delta\mathbf{v}+\mathbf{grad}\, q$ is an isomorphism from $[H^2(\Omega)^n \cap V]\times[H^1(\Omega)\cap L_0^2(\Omega)]$ onto $L^2(\Omega)^n$. For instance, problem (3.5) is regular if Γ is of class $\mathscr{C}^2$ or if Ω is a convex plane polygonal domain [15].

PROPOSITION 3.1. *The mapping*: $(\mathbf{f}, \mathbf{u}_0)\to(\mathbf{u}, p)$ *is linear continuous from* $L^2(L^2(\Omega)^n)\times V$ *into* $\{L^2(H^2(\Omega)^n)\cap H^1(L^2(\Omega)^n)\}\times L^2(H^1(\Omega))$.

Proof. It is a direct consequence of Proposition 2.1 and of the regularity of problem (3.5) (see [24, Chap. III, Prop. 1.2]).

Remark 3.1. If $(\mathbf{f}, \mathbf{u}_0)$ belongs to $L^2(L^2(\Omega)^n)\times V$, p belongs to $L^2(L_0^2(\Omega))$ and the first equation of problem (Q) is satisfied a.e. in $(0, T)$. Then problems (Q) and (1.2) coincide.

3.2. The discrete problem. For each h, let X_h and M_h be two finite-dimensional spaces such that:

$$X_h \subset H_0^1(\Omega)^n \quad \text{and} \quad M_h \subset L_0^2(\Omega).$$

We shall have to assume the following hypotheses:

(H.1) There exists a linear operator π_h from $H^2(\Omega)^n \cap H_0^1(\Omega)^n$ into X_h such that

$$\forall v \in H^2(\Omega)^n \cap H_0^1(\Omega)^n, \quad \begin{cases}\forall q_h \in M_h,\ (q_h, \operatorname{div}(\mathbf{v}-\pi_h\mathbf{v}))=0,\\ \|\mathbf{v}-\pi_h\mathbf{v}\|_{1,\Omega}\leqq Ch\|v\|_{2,\Omega};\end{cases}$$

(H.2) There exists a linear operator ρ_h from $H^1(\Omega)\cap L_0^2(\Omega)$ into M_h such that

$$\forall q \in H^1(\Omega)\cap L_0^2(\Omega), \quad \|q-\rho_h q\|_{0,\Omega}\leqq Ch\|q\|_{1,\Omega};$$

(H.3) For each q_h in M_h, there exists a function $\mathbf{v}_h$ in X_h such that

$$(\operatorname{div}\mathbf{v}_h, q_h)\geqq \beta_h\|q_h\|_{0,\Omega}\|\mathbf{v}_h\|_{1,\Omega};$$

(H.4) There exists a constant δ independent of h such that the operator Π_h defined by (2.4) satisfies the stability condition

$$\forall \mathbf{v}\in V, \quad \|\Pi_h\mathbf{v}\|_{1,\Omega}\leqq \delta\|\mathbf{v}\|_{1,\Omega}.$$

THEOREM 3.2. *Assume that the hypotheses* (H.1) *to* (H.3) *are satisfied. Then problem* (P_h) *has a unique solution* $\mathbf{u}_h$ *and problem* (Q_h) *has also a unique solution* $(\mathbf{u}_h, p_h)$ *where* $\mathbf{u}_h$ *is the solution of problem* (P_h).

Proof. It is a straightforward consequence of Theorem 2.2 because assumptions (h.5) to (h.7) are satisfied.

Let us now introduce the projection operator $\mathscr{P}_h$ from V onto V_h related to the discrete stationary Stokes problem, i.e.

$$(3.6)\qquad \forall \mathbf{v}\in V, \quad \begin{cases}\mathscr{P}_h\mathbf{v}\in V_h,\\ \forall \mathbf{v}_h\in V_h,\ a(\mathbf{v}-\mathscr{P}_h\mathbf{v}, \mathbf{v}_h)=0.\end{cases}$$

Of course, one has

$$\|\mathscr{P}_h\mathbf{v}\|_{1,\Omega} \leqq c\|\mathbf{v}\|_{1,\Omega}. \tag{3.7}$$

PROPOSITION 3.2. *Assume that the hypotheses* (H.1) *and* (H.2) *are satisfied. Then there exists a constant* $c>0$ *such that, for all* $\mathbf{v}$ *in* $V \cap H^2(\Omega)^n$,

$$\|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{1,\Omega} \leqq ch\|\mathbf{v}\|_{2,\Omega} \tag{3.8}$$

and, for all $\mathbf{v}$ *in* V,

$$\|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{0,\Omega} \leqq ch\|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{1,\Omega}. \tag{3.9}$$

This proposition is a direct consequence of [12, Chap. II, Thms. 2.1 and 2.2].

Then, we can give a sufficient condition in order to satisfy the hypothesis (H.4).

LEMMA 3.1. *Assume that the hypotheses* (H.1) *and* (H.2) *are satisfied, and that the following inverse inequality holds*

$$\forall \mathbf{v}_h \in X_h, \quad \|\mathbf{v}_h\|_{1,\Omega} \leqq Ch^{-1}\|\mathbf{v}_h\|_{0,\Omega}. \tag{3.10}$$

Then the hypothesis (H.4) *is satisfied.*

Proof. For all $\mathbf{v}$ in V, one has, by (3.10),

$$\begin{aligned}\|\mathbf{v}-\Pi_h\mathbf{v}\|_{1,\Omega} &\leqq \|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{1,\Omega}+\|\mathscr{P}_h\mathbf{v}-\Pi_h\mathbf{v}\|_{1,\Omega}\\ &\leqq \|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{1,\Omega}+ch^{-1}\|\Pi_h\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{0,\Omega}.\end{aligned}$$

Therefore, since Π_h is the projection operator onto V_h in $L^2(\Omega)^n$, we obtain

$$\|\mathbf{v}-\Pi_h\mathbf{v}\|_{1,\Omega} \leqq c(\|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{1,\Omega}+h^{-1}\|\mathbf{v}-\mathscr{P}_h\mathbf{v}\|_{0,\Omega}). \tag{3.11}$$

From (3.7) and (3.9), one deduces (H.4).

Now, we give some examples of X_h and M_h such that assumptions (H.1) to (H.4) are satisfied. Let Ω be a polyhedral domain in $\mathbb{R}^n$, $n=2$ or 3, and $(\mathscr{S}_h)_h$ a regular family of triangulations of $\bar{\Omega}$ where, for each h, $\mathscr{S}_h$ is made of n-simplices K with diameters bounded by h. For any integer l, $P_l(K)$ denotes the space of polynomials of degree $\leqq l$ on K.

Example 3.1. In the case $n=2$, we set

$$X_h=\{\mathbf{v}_h \in \mathscr{C}^0(\bar{\Omega})^2 \cap H_0^1(\Omega)^2;\ \forall K \in \mathscr{S}_h,\ \mathbf{v}_{h|K} \in P_2(K)^2\},$$

$$M_h=\{q_h \in L_0^2(\Omega);\ \forall K \in \mathscr{S}_h,\ q_{h|K} \in P_0(K)\}.$$

Example 3.2. For any K in $\mathscr{S}_h$ with vertices a_i, $1 \leqq i \leqq n+1$, we denote by λ_i the barycentric coordinate associated with a_i and by $\mathbf{n}_i$ the unit outward normal to the face which does not contain a_i. Then, we set

$$X_h=\{\mathbf{v}_h \in \mathscr{C}^0(\bar{\Omega})^n \cap H_0^1(\Omega)^n;\ \forall K \in \mathscr{S}_h,\ \mathbf{v}_{h|K} \in P_K\},$$

$$M_h=\{q_h \in L_0^2(\Omega);\ \forall K \in \mathscr{S}_h,\ q_{h|K} \in P_0(K)\},$$

where, for any K, P_K is spanned by the polynomials of $P_1(K)^n$ and $\mathbf{p}_i=\mathbf{n}((\lambda_1 \cdots \lambda_{n+1})/\lambda_i)$, $1 \leqq i \leqq n+1$.

In both examples, assumptions (H.1) to (H.3) are satisfied with the constant β_h independent of h. Moreover, if the family $(\mathscr{S}_h)_h$ is uniformly regular, the inverse inequality (3.10) holds, which yields (H.4). For proofs and other examples, see [3], [9], [10], [12], [20].

Remark 3.2. The condition (3.10) appears to be more drastic than (H.4). Indeed, in the two-dimensional case and for the particular finite element space

$$V_h=\{v_h\in\mathscr{C}^0(\bar\Omega)\cap H_0^1(\Omega);\forall K\in\mathscr{S}_h,\ v_{h|K}\in P_l(K)\},$$

the stability of the operator Π_h in the norm $\|\cdot\|_{1,\Omega}$ has been proven under a much weaker condition on $(\mathscr{S}_h)_h$ [8] (i.e. the diameters of the triangles K in $\mathscr{S}_h$ are not allowed to grow exponentially). In the three-dimensional case, as far as we know, this stability is proven only if the family $(\mathscr{S}_h)_h$ is uniformly regular.

3.3. Error estimates.

THEOREM 3.3. *Assume that the hypotheses* (H.1) *to* (H.4) *are satisfied. Then we have the error estimates, for all* $(\mathbf{f},\mathbf{u}_0)$ *in* $L^2(L^2(\Omega)^n)\times V$,

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(H_0^1(\Omega)^n)}\leqq ch\{\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}\},\tag{3.12}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{C}^0([0,T];L^2(\Omega)^n)}\leqq ch\{\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}\},\tag{3.13}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(V')}\leqq ch\{\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}\},\tag{3.14}$$

and, for $\alpha<\frac12$,

$$|\mathbf{u}-\mathbf{u}_h|_{H_0^\alpha(L^2(\Omega)^n)}\leqq ch\{\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}\}.\tag{3.15}$$

Proof.

1) In order to prove estimate (3.12), we use Theorem 2.3. Hence we obtain

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(H_0^1(\Omega)^n)}\leqq c\left\{\|\mathbf{u}-\Pi_h\mathbf{u}\|_{L^2(H_0^1(\Omega)^n)}+\left\|\sup_{\mathbf{v}_h\in V_h}\frac{b(\mathbf{v}_h,p)}{\|\mathbf{v}_h\|_{1,\Omega}}\right\|_{L^2(\mathbb{R})}\right\}.$$

But

$$\sup_{\mathbf{v}_h\in V_h}\frac{b(\mathbf{v}_h,p)}{\|\mathbf{v}_h\|_{1,\Omega}}=\sup_{\mathbf{v}_h\in V_h}\frac{b(\mathbf{v}_h;p-\rho_hp)}{\|\mathbf{v}_h\|_{1,\Omega}}$$

and, by (H.2), we have

$$\left\|\sup_{\mathbf{v}_h\in V_h}\frac{b(\mathbf{v}_h,p)}{\|\mathbf{v}_h\|_{1,\Omega}}\right\|_{L^2(\mathbb{R})}\leqq ch\|p\|_{L^2(H^1(\Omega))}.\tag{3.16}$$

On the other hand, estimates (3.8), (3.9) and (3.11) yield

$$\|\mathbf{u}-\Pi_h\mathbf{u}\|_{L^2(H_0^1(\Omega)^n)}\leqq ch\|\mathbf{u}\|_{L^2(H^2(\Omega)^n)}.\tag{3.17}$$

Then estimate (3.12) follows from (3.16), (3.17) and Proposition 3.1.

2) As $L^2(H^2(\Omega)^n)\cap H^1(L^2(\Omega)^n)$ is contained in $\mathscr{C}^0([0,T];H^1(\Omega)^n)$ with a continuous imbedding, $\mathbf{u}$ belongs to $\mathscr{C}^0([0,T];H^1(\Omega)^n)$ by Proposition 3.1.

We have, by (3.7) and (3.9),

$$\begin{aligned}\|\mathbf{u}-\Pi_h\mathbf{u}\|_{\mathscr{C}^0([0,T];L^2(\Omega)^n)}&\leqq\|\mathbf{u}-\mathscr{P}_h\mathbf{u}\|_{\mathscr{C}^0([0,T];L^2(\Omega)^n)}\\&\leqq ch\|\mathbf{u}\|_{\mathscr{C}^0([0,T];H^1(\Omega)^n)}.\end{aligned}$$

Then (2.6), together with (3.16), (3.17) and the above inequality, yields (3.13).

3) Now we prove estimate (3.14). By Lemma 3.1, assumption (2.12) is fulfilled. Thus we may apply Proposition 2.2 and we have to estimate $\|(d/dt)(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^2(V')}$. But

$$\left\|\frac{d}{dt}(\mathbf{u}-\Pi_h\mathbf{u})\right\|_{V'}=\sup_{\mathbf{v}\in V}\frac{((d/dt)(\mathbf{u}-\Pi_h\mathbf{u}),\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}}=\sup_{\mathbf{v}\in V}\frac{((d/dt)(\mathbf{u}-\Pi_h\mathbf{u}),\mathbf{v}-\Pi_h\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}}$$

and

$$\left\|\frac{d}{dt}(\mathbf{u}-\Pi_h\mathbf{u})\right\|_{V'} \leqq \left\|\frac{d\mathbf{u}}{dt}\right\|_{0,\Omega} \sup_{\mathbf{v}\in V} \frac{\|\mathbf{v}-\Pi_h\mathbf{v}\|_{0,\Omega}}{\|\mathbf{v}\|_{1,\Omega}}. \tag{3.18}$$

Using now (3.7), (3.9) and (3.18) yields

$$\left\|\frac{d}{dt}(\mathbf{u}-\Pi_h\mathbf{u})\right\|_{L^2(V')} \leqq ch\left\|\frac{d\mathbf{u}}{dt}\right\|_{L^2(L^2(\Omega)^n)}. \tag{3.19}$$

4) As $L^2(H^2(\Omega)^n)\cap H^1(L^2(\Omega)^n)$ is contained in $\mathscr{C}^0([0,T];H^1(\Omega)^n)$ with a continuous imbedding, estimate (3.15) is a direct consequence of Theorem 2.4, Propositions 3.1 and 3.2 and of inequality (3.16).

THEOREM 3.4. *Assume that the hypotheses* (H.1) *to* (H.3) *are satisfied. If* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$, *we have the error estimate, for all* $\mathbf{f}$ *in* $L^2(L^2(\Omega)^n)$,

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(H_0^1(\Omega)^n)}+|\mathbf{u}-\mathbf{u}_h|_{H_0^{1/2}(L^2(\Omega)^n)}\leqq ch\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}. \tag{3.20}$$

Proof. Estimate (3.20) is a straightforward consequence of Theorem 2.4, Propositions 3.1 and 3.2 and of inequality (3.16).

Now we give an estimate of $\|p-p_h\|_{H^{-1}(L^2(\Omega))}$, which follows from Proposition 2.3 and Theorems 3.3 and 3.4.

PROPOSITION 3.3. *Assume that the hypotheses* (H.1) *to* (H.3) *are satisfied with the constant* β_h *independent of* h. *If* (H.4) *holds or if* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$, *we have the error estimate, for all* $(\mathbf{f},\mathbf{u}_0)$ *in* $L^2(L^2(\Omega)^n)\times V$,

$$\|p-p_h\|_{H^{-1}(L^2(\Omega))}\leqq ch\{\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}\}. \tag{3.21}$$

We obtain an improved error bound for $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(L^2(\Omega)^n)}$, by a duality method.

PROPOSITION 3.4. *Assume that the hypotheses* (H.1) *to* (H.3) *are satisfied. If* (H.4) *holds or if* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$, *we have the error estimate, for all* $(\mathbf{f},\mathbf{u}_0)$ *in* $L^2(L^2(\Omega)^n)\times V$,

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(L^2(\Omega)^n)}\leqq ch^2(\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}). \tag{3.22}$$

Proof. We shall prove estimate (3.22) only when (H.4) holds. We have of course

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(L^2(\Omega)^n)}=\sup_{\mathbf{g}\in L^2(L^2(\Omega)^n)}\frac{\int_0^T(\mathbf{u}-\mathbf{u}_h,\mathbf{g})\,dt}{\|\mathbf{g}\|_{L^2(L^2(\Omega)^n)}}.$$

Let $\mathbf{g}$ be any function in $L^2(L^2(\Omega)^n)$. The problem: find $(\mathbf{w},z)$ in $L^2(H_0^1(\Omega)^n)\times H^{-1}(L_0^2(\Omega))$ such that

$$\begin{aligned}
&\forall\mathbf{v}\in H_0^1(\Omega)^n,\quad \left(-\frac{d\mathbf{w}}{dt},\mathbf{v}\right)+a(\mathbf{w},\mathbf{v})+b(\mathbf{v},z)=(\mathbf{g},\mathbf{v})\quad\text{a.e. in }(0,T)\\
&\forall q\in L_0^2(\Omega),\quad b(\mathbf{w},q)=0\quad\text{a.e. in }(0,T),\\
&\mathbf{w}(T)=0
\end{aligned} \tag{3.23}$$

has a unique solution and, by Proposition 3.1,

$$\|\mathbf{w}\|_{L^2(H^2(\Omega)^n)\cap H^1(L^2(\Omega)^n)}+\|z\|_{L^2(H^1(\Omega))}\leqq c\|\mathbf{g}\|_{L^2(L^2(\Omega)^n)}. \tag{3.24}$$

Let us now compute $\int_0^T(\mathbf{u}-\mathbf{u}_h,\mathbf{g})\,dt$.

$$\begin{aligned}
\int_0^T(\mathbf{u}-\mathbf{u}_h,\mathbf{g})\,dt&=\int_0^T\left(\left(\frac{d}{dt}(\mathbf{u}-\mathbf{u}_h),\mathbf{w}\right)+a(\mathbf{w},\mathbf{u}-\mathbf{u}_h)+b(\mathbf{u}-\mathbf{u}_h,z)\right)dt\\
&\quad+(\mathbf{u}_0-\Pi_h\mathbf{u}_0,\mathbf{w}(0)).
\end{aligned}$$

By using (2.8) with $\mathbf{v}_h=\Pi_h\mathbf{w}$, we obtain

$$\int_0^T(\mathbf{u}-\mathbf{u}_h,\mathbf{g})\,dt=\int_0^T\left(\left(\frac{d}{dt}(\mathbf{u}-\mathbf{u}_h),\mathbf{w}-\Pi_h\mathbf{w}\right)+a(\mathbf{u}-\mathbf{u}_h,\mathbf{w}-\Pi_h\mathbf{w})+b(\mathbf{u}-\mathbf{u}_h,z)+b(\Pi_h\mathbf{w},p)\right)dt+(\mathbf{u}_0-\Pi_h\mathbf{u}_0,\mathbf{w}(0)-\Pi_h\mathbf{w}(0)). \tag{3.25}$$

It remains to bound all the terms in (3.25).

$$\left|\int_0^T\left(\frac{d}{dt}(\mathbf{u}-\mathbf{u}_h),\mathbf{w}-\Pi_h\mathbf{w}\right)dt\right|\leqq c\|\mathbf{u}-\mathcal{P}_h\mathbf{u}\|_{H^1(L^2(\Omega)^n)}\|\mathbf{w}-\Pi_h\mathbf{w}\|_{L^2(L^2(\Omega)^n)}, \tag{3.26}$$

$$\left|\int_0^T a(\mathbf{u}-\mathbf{u}_h,\mathbf{w}-\Pi_h\mathbf{w})\,dt\right|\leqq c\|\mathbf{u}-\mathbf{u}_h\|_{L^2(H^1(\Omega)^n)}\|\mathbf{w}-\Pi_h\mathbf{w}\|_{L^2(H^1(\Omega)^n)}. \tag{3.27}$$

As $b(\mathbf{u}-\mathbf{u}_h,z)=b(\mathbf{u}-\mathbf{u}_h,z-\rho_hz)$, we have

$$\left|\int_0^T b(\mathbf{u}-\mathbf{u}_h,z)\,dt\right|\leqq c\|\mathbf{u}-\mathbf{u}_h\|_{L^2(H^1(\Omega)^n)}\|z-\rho_hz\|_{L^2(L^2(\Omega))}. \tag{3.28}$$

In the same way,

$$\left|\int_0^T b(\Pi_h\mathbf{w},p)\,dt\right|\leqq c\|\mathbf{w}-\Pi_h\mathbf{w}\|_{L^2(H^1(\Omega)^n)}\|p-\rho_hp\|_{L^2(L^2(\Omega))}. \tag{3.29}$$

Finally

$$|(\mathbf{u}_0-\Pi_h\mathbf{u}_0,\mathbf{w}(0)-\Pi_h\mathbf{w}(0))|\leqq c\|\mathbf{u}_0-\Pi_h\mathbf{u}_0\|_{0,\Omega}\|\mathbf{w}(0)-\Pi_h\mathbf{w}(0)\|_{0,\Omega}. \tag{3.30}$$

Using estimates (3.8) to (3.12) and (3.24) to (3.30) yields

$$\left|\int_0^T(\mathbf{u}-\mathbf{u}_h,\mathbf{g})\,dt\right|\leqq ch^2\|\mathbf{g}\|_{L^2(L^2(\Omega)^n)}(\|\mathbf{f}\|_{L^2(L^2(\Omega)^n)}+\|\mathbf{u}_0\|_{1,\Omega}).$$

Remark 3.3. Of course, we can apply the same techniques to study higher-order approximations of the time-dependent Stokes problem. However, in order to obtain optimal error estimates, we are led to make regularity assumptions on the solution $(\mathbf{u},p)$ of problem (1.2), which are not true in the general case.

3.4. Convergence results. We define on $L^2(H^{-1}(\Omega)^n)\times H$ the operators

$$\mathcal{T}:(\mathbf{f},\mathbf{u}_0)\to\mathbf{u}\text{ solution of problem (P)},$$

$$\mathcal{T}_h:(\mathbf{f},\mathbf{u}_0)\to\mathbf{u}_h\text{ solution of problem }(\mathrm{P}_h).$$

If $\mathbf{u}_0$ is equal to $\mathbf{0}$, we define on $L^2(H^{-1}(\Omega)^n)$ the operators

$$\mathcal{T}_0:\mathbf{f}\to\mathbf{u}\text{ solution of problem (P)},$$

$$\mathcal{T}_{0h}:\mathbf{f}\to\mathbf{u}_h\text{ solution of problem }(\mathrm{P}_h).$$

We obtain the following theorem.

THEOREM 3.5. *Assume that the hypotheses* (H.1) *to* (H.3) *are satisfied.*

1) *If* (H.4) *holds, for all* $(\mathbf{f},\mathbf{u}_0)$ *in* $L^2(H^{-1}(\Omega)^n)\times H$, *for* $\alpha<\frac{1}{2}$, *we have*

$$\lim_{h\to0}\|(\mathcal{T}-\mathcal{T}_h)(\mathbf{f},\mathbf{u}_0)\|_{L^2(H_0^1(\Omega)^n)\cap H^\alpha(L^2(\Omega)^n)\cap\mathcal{C}^0([0,T];L^2(\Omega)^n)\cap H^1(V')}=0; \tag{3.31}$$

2) *For all* $\mathbf{f}$ *in* $L^2(H^{-1}(\Omega)^n)$, *we have*

$$\lim_{h\to 0} \|(\mathcal{T}_0 - \mathcal{T}_{0h})\mathbf{f}\|_{L^2(H_0^1(\Omega)^n)\cap H_0^{1/2}(L^2(\Omega)^n)} = 0. \tag{3.32}$$

Proof. By a classical way, one can show that, if $\mathbf{u}_h$ is the solution of problem (P_h)

$$\|\mathbf{u}_h\|_{L^2(H_0^1(\Omega)^n)\cap \mathscr{C}^0([0,T];\,L^2(\Omega)^n)} \leqq c\{\|\mathbf{f}\|_{L^2(H^{-1}(\Omega)^n)} + \|\mathbf{u}_0\|_{0,\Omega}\}. \tag{3.33}$$

Moreover, if (H.4) holds, one has

$$\|\mathbf{u}_h\|_{H^1(V')} \leqq c\{\|\mathbf{f}\|_{L^2(H^{-1}(\Omega)^n)} + \|\mathbf{u}_0\|_{0,\Omega}\}. \tag{3.34}$$

Using the Fourier transform we also prove that

$$\|\mathbf{u}_h\|_{H^\alpha(L^2(\Omega)^n)} \leqq c\{\|\mathbf{f}\|_{L^2(H^{-1}(\Omega)^n)} + \|\mathbf{u}_0\|_{0,\Omega}\}, \tag{3.35}$$

for $0<\alpha<\frac{1}{2}$, and $\alpha=\frac{1}{2}$ in the case $\mathbf{u}_0=\mathbf{0}$.

The stability results (3.33) to (3.35), the error estimates (3.12) to (3.15) and (3.20), together with a classical density argument, yield the theorem.

4. Analysis of the time-dependent Navier–Stokes equations.

4.1. Some properties of the continuous problem. From now on, we assume that Ω is a bounded domain in $\mathbb{R}^n$, $n=2$ or 3, such that problem (3.5) is regular.

Let us consider the nonlinear term in (1.1). We notice that

$$\forall \mathbf{u}\in H_0^1(\Omega)^n,\quad \forall \mathbf{v}\in H_0^1(\Omega)^n,$$

$$\int_\Omega \sum_{i=1}^n u_i \frac{\partial \mathbf{u}}{\partial x_i}\cdot \mathbf{v}\,dx = -\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\operatorname{div}\mathbf{u}\,dx - \int_\Omega \sum_{i,j=1}^n u_i u_j \frac{\partial v_j}{\partial x_i}\,dx. \tag{4.1}$$

We define the continuous trilinear form $a_1(\cdot,\cdot,\cdot)$ on $H_0^1(\Omega)^n\times H_0^1(\Omega)^n\times H_0^1(\Omega)^n$ and the continuous bilinear operator $A_1(\cdot,\cdot)$ from $H_0^1(\Omega)^n\times H_0^1(\Omega)^n$ into $H^{-1}(\Omega)^n$ by

$$\forall \mathbf{u}\in H_0^1(\Omega)^n,\quad \forall \mathbf{v}\in H_0^1(\Omega)^n,\quad \forall \mathbf{w}\in H_0^1(\Omega)^n,$$

$$a_1(\mathbf{w},\mathbf{u},\mathbf{v}) = \langle A_1(\mathbf{w},\mathbf{u}),\mathbf{v}\rangle = -\int_\Omega \sum_{i,j=1}^n w_i u_j \frac{\partial v_j}{\partial x_i}\,dx. \tag{4.2}$$

LEMMA 4.1. *In the case* $n=2$, *the operator* $A_1(\cdot,\cdot)$ *is bilinear continuous from* $\{L^2(H_0^1(\Omega)^2)\cap \mathscr{C}^0([0,T];\,L^2(\Omega)^2)\}^2$ *into* $L^2(H^{-1}(\Omega)^2)$ *and from* $\{L^2(H_0^1(\Omega)^2)\cap H^{1/2}(L^2(\Omega)^2)\}^2$ *into* $L^2(H^{-1}(\Omega)^2)$. *In the case* $n=3$, *the operators*: $(\mathbf{w},\mathbf{u})\to A_1(\mathbf{w},\mathbf{u})$ *and* $(\mathbf{w},\mathbf{u})\to A_1(\mathbf{u},\mathbf{w})$ *are bilinear continuous from* $L^2(H_0^1(\Omega)^3)\times \mathscr{C}^0([0,T];\,L^3(\Omega)^3)$ *into* $L^2(H^{-1}(\Omega)^3)$.

Proof. In the case $n=2$, let us study $a_1(\mathbf{w},\mathbf{u},\mathbf{v})$ for $\mathbf{u}$ and $\mathbf{w}$ in $L^2(H_0^1(\Omega)^2)\cap \mathscr{C}^0([0,T];\,L^2(\Omega)^2)$ or in $L^2(H_0^1(\Omega)^2)\cap H^{1/2}(L^2(\Omega)^2)$, for any $\mathbf{v}$ in $L^2(H_0^1(\Omega)^2)$; we have the Sobolev imbeddings

$$\left.\begin{array}{l} L^2(H^1(\Omega))\cap \mathscr{C}^0([0,T];\,L^2(\Omega)) \hookrightarrow L^4(H^{1/2}(\Omega)) \\ L^2(H^1(\Omega))\cap H^{1/2}(L^2(\Omega)) \hookrightarrow H^{1/4}(H^{1/2}(\Omega)) \end{array}\right\} \hookrightarrow L^4(L^4(\Omega))$$

and it is clear that

$$|a_1(\mathbf{w},\mathbf{u},\mathbf{v})| \leqq c\|\mathbf{w}\|_{L^4(L^4(\Omega)^2)}\|\mathbf{u}\|_{L^4(L^4(\Omega)^2)}\|\mathbf{v}\|_{L^2(H_0^1(\Omega)^2)}.$$

In the case $n=3$, by the Sobolev imbedding: $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$, we have for $\mathbf{w}$ in $L^2(H_0^1(\Omega)^3)$ and $\mathbf{u}$ in $\mathscr{C}^0([0,T];\,L^3(\Omega)^3)$, for any $\mathbf{v}$ in $L^2(H_0^1(\Omega)^3)$,

$$|a_1(\mathbf{w},\mathbf{u},\mathbf{v})| + |a_1(\mathbf{u},\mathbf{w},\mathbf{v})| \leqq c\|\mathbf{w}\|_{L^2(H_0^1(\Omega)^3)}\|\mathbf{u}\|_{\mathscr{C}^0([0,T];\,L^3(\Omega)^3)}\|\mathbf{v}\|_{L^2(H_0^1(\Omega)^3)}.$$

Remark 4.1. By (4.1), the two forms

$$\int_\Omega \sum_{i=1}^n u_i \frac{\partial \mathbf{u}}{\partial x_i} \cdot \mathbf{v}\, dx \quad \text{and} \quad -\int_\Omega \sum_{i,j=1}^n u_i u_j \frac{\partial v_j}{\partial x_i}\, dx$$

are equivalent for the continuous problem (but not for the discrete problem). However, in the case $n=2$, the mapping: $\mathbf{u} \to A_1(\mathbf{u}, \mathbf{u})$ is of class $\mathscr{C}^\infty$ from $L^2(H_0^1(\Omega)^2) \cap \mathscr{C}^0([0, T]; L^2(\Omega)^2)$ or from $L^2(H_0^1(\Omega)^2) \cap H^{1/2}(L^2(\Omega)^2)$ into $L^2(H^{-1}(\Omega)^2)$ only if $a_1(\cdot, \cdot, \cdot)$ is defined by (4.2).

LEMMA 4.2. *For any* $\mathbf{u}$ *in* $\mathscr{C}^0([0, T]; L^q(\Omega)^n)$, $q > n$, *the operators* $A_1(\cdot, \mathbf{u})$ *and* $A_1(\mathbf{u}, \cdot)$ *are compact from* $L^2(H_0^1(\Omega)^n) \cap H^\alpha(L^2(\Omega)^n)$, $\alpha > 0$, *into* $L^2(H^{-1}(\Omega)^n)$.

Proof. We notice that, if $\mathbf{w}$ belongs to $L^2(L^{q'}(\Omega)^n)$, with $1/q + 1/q' = \frac{1}{2}$, we have, for any $\mathbf{v}$ in $L^2(H_0^1(\Omega)^n)$,

$$|a_1(\mathbf{w}, \mathbf{u}, \mathbf{v})| + |a_1(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leqq c \|\mathbf{w}\|_{L^2(L^{q'}(\Omega)^n)} \|\mathbf{u}\|_{\mathscr{C}^0([0, T]; L^q(\Omega)^n)} \|\mathbf{v}\|_{L^2(H_0^1(\Omega)^n)}.$$

But, as $q' < +\infty$ in the case $n=2$, and $q' < 6$ in the case $n=3$, the injection from $L^2(H_0^1(\Omega)) \cap H^\alpha(L^2(\Omega))$, $\alpha > 0$, into $L^2(L^{q'}(\Omega))$ is compact (cf. [24, Chap. III, Thm. 2.2]), so that $A_1(\cdot, \mathbf{u})$ and $A_1(\mathbf{u}, \cdot)$ are compact.

We consider now problem (1.1). We set

$$\mathscr{X} = L^2(H_0^1(\Omega)^n) \cap \mathscr{C}^0([0, T]; L^2(\Omega)^n) \cap H^\alpha(L^2(\Omega)^n) \cap H^1(V'), \quad 0 < \alpha < \tfrac{1}{2},$$

$$\mathscr{X}_0 = L^2(H_0^1(\Omega)^n) \cap H^{1/2}(L^2(\Omega)^n),$$

$$\mathscr{Y} = L^2(H^{-1}(\Omega)^n) \times H, \qquad \mathscr{Y}_0 = L^2(H^{-1}(\Omega)^n),$$

and define, for $\mathbf{f}$ in $L^2(H^{-1}(\Omega)^n)$ and $\mathbf{u}_0$ in H,

$$G(\mathbf{u}) = (A_1(\mathbf{u}, \mathbf{u}) - \mathbf{f}, -\mathbf{u}_0), \qquad G_0(\mathbf{u}) = A_1(\mathbf{u}, \mathbf{u}) - \mathbf{f}, \tag{4.3}$$

then introduce the problems: find $\mathbf{u}$ in $\mathscr{X}$ such that

$$\mathbf{u} + \mathscr{T} G(\mathbf{u}) = \mathbf{0} \tag{4.4}$$

and, in the case $\mathbf{u}_0 = \mathbf{0}$, find $\mathbf{u}$ in $\mathscr{X}_0$ such that

$$\mathbf{u} + \mathscr{T}_0 G_0(\mathbf{u}) = \mathbf{0}. \tag{4.5}$$

Of course, for any $(\mathbf{u}, p)$ solution of (1.1) (regular enough in the case $n=3$), $\mathbf{u}$ is solution of (4.4) and, in the case $\mathbf{u}_0 = \mathbf{0}$, of (4.5).

By Lemma 4.1, in the case $n=2$, G (resp. G_0) is of class $\mathscr{C}^\infty$ from $\mathscr{X}$ into $\mathscr{Y}$ (resp. $\mathscr{X}_0$ into $\mathscr{Y}_0$); in the case $n=3$, for $\mathbf{u}$ in $L^2(H_0^1(\Omega)^3) \cap \mathscr{C}^0([0, T]; L^3(\Omega)^3)$, $G(\mathbf{u})$ (resp. $G_0(\mathbf{u})$) belongs to $\mathscr{Y}$ (resp. $\mathscr{Y}_0$) and $DG(\mathbf{u})$ (resp. $DG_0(\mathbf{u})$) is linear continuous from $\mathscr{X}$ into $\mathscr{Y}$ (resp. $\mathscr{X}_0$ into $\mathscr{Y}_0$). The operator $\mathscr{T}$ (resp. $\mathscr{T}_0$) is defined from $\mathscr{Y}$ into $\mathscr{X}$ (resp. $\mathscr{Y}_0$ into $\mathscr{X}_0$). Moreover, we have the next lemma.

LEMMA 4.3. *For any* $\mathbf{u}$ *solution of* (4.4) (*resp.* (4.5)) *in* $L^2(H_0^1(\Omega)^n) \cap \mathscr{C}^0([0, T]; L^q(\Omega)^n)$, $q > n$, *satisfying moreover in the case* $n=3$

$$\|\mathbf{u}\|_{\mathscr{C}^0([0, T]; L^3(\Omega)^3)} \leqq c(\Omega)\nu, \tag{4.6}$$

the operator $I + \mathscr{T} DG(\mathbf{u})$ (*resp.* $I + \mathscr{T}_0 DG_0(\mathbf{u})$) *is an isomorphism in* $\mathscr{X}$ (*resp. in* $\mathscr{X}_0$).

Proof. By Lemma 4.2, $I + \mathscr{T} DG(\mathbf{u})$ (resp. $I + \mathscr{T}_0 DG_0(\mathbf{u})$) is an isomorphism in $\mathscr{X}$ (resp. $\mathscr{X}_0$) if and only if it is injective, i.e. if the solution of the linearized Stokes problem is unique, which one can prove by the same way as in [12, Chap. V, Thm. 1.5].

4.2. Statement of the discrete problem and error estimates. From now on, we assume that (H.1) to (H.3) are satisfied. Of course, we define our discrete problem by: find $\mathbf{u}_h$

in $\mathscr{X}_h = \mathscr{X} \cap L^2(V_h)$ such that

$$\mathbf{u}_h + \mathscr{T}_h G(\mathbf{u}_h) = \mathbf{0} \tag{4.7}$$

and, in the case $\mathbf{u}_0 = \mathbf{0}$, find $\mathbf{u}_h$ in $\mathscr{X}_{0h} = \mathscr{X}_0 \cap L^2(V_h)$ such that

$$\mathbf{u}_h + \mathscr{T}_{0h} G_0(\mathbf{u}_h) = \mathbf{0}. \tag{4.8}$$

Obviously, (4.7) and (4.8) are equivalent to find $\mathbf{u}_h$ in $H^1(V_h)$ such that

$$\begin{aligned} &\forall \mathbf{v}_h \in V_h, \quad \frac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \text{a.e. in } (0, T), \\ &\forall \mathbf{v}_h \in V_h, \quad (\mathbf{u}_h(0) - \mathbf{u}_0, \mathbf{v}_h) = 0; \end{aligned} \tag{4.9}$$

this problem has at most a solution in $H^1(V_h)$.

THEOREM 4.1. *In the case $n = 2$, let $(\mathbf{u}, p)$ be a solution of* (1.1) *such that* $\mathbf{u}$ *belongs to* $L^2(H_0^1(\Omega)^2) \cap \mathscr{C}^0([0, T]; L^q(\Omega)^2)$, $q > 2$. *Then, if* (H.4) *holds or if* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$, *for h small enough, problem* (4.9) *has a unique solution*; *moreover, we have the error estimates*

1) *for all* $(\mathbf{f}, \mathbf{u}_0)$ *in* $L^2(L^2(\Omega)^2) \times V$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{X}} \leqq c(\mathbf{f}, \mathbf{u}_0) h; \tag{4.10}$$

2) *for all* $\mathbf{f}$ *in* $L^2(L^2(\Omega)^2)$ *if* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{X}\mathrm{x}_0} \leqq c(\mathbf{f}) h. \tag{4.11}$$

In the case $n = 3$, let $(\mathbf{u}, p)$ be a solution of (1.1) *such that* $\mathbf{u}$ *belongs to* $L^2(H^2(\Omega)^3) \cap H^1(L^2(\Omega)^3)$ *and satisfies* (4.6). *Then, if the inverse inequality* (3.10) *holds, for h small enough, problem* (4.9) *has a unique solution*; *moreover, we have the error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{X}} \leqq c(\mathbf{u}) h. \tag{4.12}$$

Proof. We use a discrete implicit function theorem (see [6, Thm. 1]) in the sharpened form given in [7, Thm. 2.2]. First, by Theorem 3.5 and Lemma 4.3, $I + \mathscr{T}_h DG(\mathbf{u})$ (resp. $I + \mathscr{T}_{0h} DG_0(\mathbf{u})$) is an isomorphism of $\mathscr{X}$ (resp. $\mathscr{X}_0$), with its inverse uniformly bounded. Then, let us set

$$\varepsilon_h = \|(\mathscr{T} - \mathscr{T}_h) G(\mathbf{u})\|_{\mathscr{X}} \quad (\text{resp.} = (\|\mathscr{T}_0 - \mathscr{T}_{0h}) G_0(\mathbf{u})\|_{\mathscr{X}_0}),$$

$$L_h = \sup_{\mathbf{u}^* \in \mathscr{X}, \mathbf{u}^* \neq \mathbf{u}} \frac{\|DG(\mathbf{u}^*) - DG(\mathbf{u})\|_{\mathscr{L}(\mathscr{X}_h, \mathscr{Y})}}{\|\mathbf{u}^* - \mathbf{u}\|_{\mathscr{X}}}$$

$$\left(\text{resp.} \sup_{\mathbf{u}^* \in \mathscr{X}_0, \mathbf{u}^* \neq \mathbf{u}} \frac{\|DG_0(\mathbf{u}^*) - DG_0(\mathbf{u})\|_{\mathscr{L}(\mathscr{X}_{0h}, \mathscr{Y})}}{\|\mathbf{u}^* - \mathbf{u}\|_{\mathscr{X}_0}}\right).$$

1) In the case $n = 2$, we have by Theorem 3.5: $\lim_{h \to 0} \varepsilon_h = 0$, and by Lemma 4.1: $L_h \leqq c$. Hence, for h small enough, problem (4.9) has a solution, and

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{X}} \leqq c\|(\mathscr{T} - \mathscr{T}_h) G(\mathbf{u})\|_{\mathscr{X}} \quad (\text{resp. } \|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{X}_0} \leqq c\|(\mathscr{T}_0 - \mathscr{T}_{0h}) G_0(\mathbf{u})\|_{\mathscr{X}_0}).$$

By [24, Chap. III, Thm. 3.10], if $(\mathbf{f}, \mathbf{u}_0)$ belongs to $L^2(L^2(\Omega)^2) \times V$, $A_1(\mathbf{u}, \mathbf{u})$ belongs to $L^2(L^2(\Omega)^2)$. Then (4.10) and (4.11) follow from Theorems 3.3 and 3.4.

2) In the case $n = 3$, if $\mathbf{u}$ belongs to $L^2(H^2(\Omega)^3) \cap H^1(L^2(\Omega)^3)$, $(A_1(\mathbf{u}, \mathbf{u}) - \mathbf{f}, -\mathbf{u}_0)$ belongs to $L^2(L^2(\Omega)^3) \times V$, so that, by Theorems 3.3 and 3.4, one has: $\varepsilon_h \leqq c(\mathbf{u}) h$;

moreover, using (3.10), we see that, for any $\mathbf{w}_h$ in $\mathscr{X}_h$,

$$\begin{aligned}\|A_1(\mathbf{w}_h, \mathbf{u}-\mathbf{u}^*)+A_1(\mathbf{u}-\mathbf{u}^*, \mathbf{w}_h)\|_{L^2(H^{-1}(\Omega)^3)} &\leqq c\|\mathbf{w}_h\|_{\mathscr{C}^0([0,T];H^{1/2}(\Omega)^3)}\|\mathbf{u}^*-\mathbf{u}\|_{L^2(H_0^1(\Omega)^3)}\\ &\leqq ch^{-1/2}\|\mathbf{w}_h\|_{\mathscr{X}}\|\mathbf{u}^*-\mathbf{u}\|_{\mathscr{X}},\end{aligned}$$

so that: $L_h \leqq ch^{-1/2}$. Finally, we obtain: $\lim_{h\to 0}\varepsilon_h L_h=0$; hence, for h small enough, problem (4.9) has a solution, and

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{X}} \leqq c\|(\mathscr{T}-\mathscr{T}_h)G(\mathbf{u})\|_{\mathscr{X}} \leqq c(\mathbf{u})h.$$

Remark 4.2. In the case $n=3$, a sufficient condition for (1.1) to have a solution $(\mathbf{u},p)$ in $\{L^2(H_0^1(\Omega)^3)\cap H^1(L^2(\Omega)^3)\}\times L^2(H^1(\Omega))$ is that $(\mathbf{f},\mathbf{u}_0)$ belongs to $L^2(L^2(\Omega)^3)\times V$, and that T is small enough (see [16, Thm. 2′]).

We can also consider the discrete problem: find $(\mathbf{u}_h, p_h)$ in $H^1(V_h)\times L^2(M_h)$ such that

$$\text{(4.13)}\quad\begin{aligned}&\forall \mathbf{v}_h\in X_h,\quad \frac{d}{dt}(\mathbf{u}_h,\mathbf{v}_h)+a(\mathbf{u}_h,\mathbf{v}_h)+a_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)-(p_h,\operatorname{div}\mathbf{v}_h)\\ &\qquad\qquad =(\mathbf{f},\mathbf{v}_h)\quad \text{a.e. in } (0,T),\\ &\forall q_h\in M_h,\quad (q_h,\operatorname{div}\mathbf{u}_h)=0\quad \text{a.e. in } (0,T),\\ &\forall \mathbf{v}_h\in V_h,\quad (\mathbf{u}_h(0)-\mathbf{u}_0,\mathbf{v}_h)=0;\end{aligned}$$

by (H.3), for each $\mathbf{u}_h$ in $H^1(V_h)$ solution of (4.9), there exists a unique p_h in $L^2(M_h)$ such that $(\mathbf{u}_h, p_h)$ is a solution of (4.13).

We have the following proposition.

PROPOSITION 4.1. *Let* $(\mathbf{u},p)$ *be a solution of* (1.1) *in* $\{L^2(H^2(\Omega)^n)\cap H^1(L^2(\Omega)^n)\}\times L^2(H^1(\Omega))$, *such that* $\mathbf{u}$ *satisfies* (4.6) *in the case* $n=3$. *If* (H.3) *holds with the constant* β_h *independent of* h *and if moreover*:

1) (H.4) *holds or* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$ *in the case* $n=2$,

2) *the inverse inequality* (3.10) *holds in the case* $n=3$,

we have the error estimate

$$\text{(4.14)}\qquad \|p-p_h\|_{H^{-1}(L^2(\Omega))} \leqq c(\mathbf{u},p)h.$$

Proof. In the same way as in the linear case, we see that

$$\begin{aligned}\|p-p_h\|_{H^{-1}(L^2(\Omega))} \leqq c\{&\|p-\rho_h p\|_{H^{-1}(L^2(\Omega))}+\|\mathbf{u}-\mathbf{u}_h\|_{L^2(X')}\\ &+\|\mathbf{u}-\mathbf{u}_h\|_{H^{-1}(X)}+\|\mathbf{u}_0-\Pi_h\mathbf{u}_0\|_{X'}+\|A_1(\mathbf{u},\mathbf{u})-A_1(\mathbf{u}_h,\mathbf{u}_h)\|_{H^{-1}(X')}\}.\end{aligned}$$

Clearly, one has

$$\begin{aligned}\|A_1(\mathbf{u},\mathbf{u})-A_1(\mathbf{u}_h,\mathbf{u}_h)\|_{H^{-1}(X')} &\leqq c\|A_1(\mathbf{u},\mathbf{u}-\mathbf{u}_h)\|_{L^2(X')}+\|A_1(\mathbf{u}-\mathbf{u}_h,\mathbf{u}_h)\|_{L^{4/3}(X')}\\ &\leqq c\|u-u_h\|_{\mathscr{X}}\quad\text{or}\quad c\|u-u_h\|_{\mathscr{X}_0},\end{aligned}$$

so that we deduce (4.14) from (H.2) and Theorem 4.1.

By a duality method, we obtain the following result.

PROPOSITION 4.2. *In the case* $n=2$, *if* (H.4) *holds or if* $\mathbf{u}_0$ *is equal to* $\mathbf{0}$, *for all* $(\mathbf{f},\mathbf{u}_0)$ *in* $L^2(L^2(\Omega)^2)\times V$, *we have the error estimate*

$$\text{(4.15)}\qquad \|\mathbf{u}-\mathbf{u}_h\|_{L^2(L^2(\Omega)^2)} \leqq c(\mathbf{f},\mathbf{u}_0)h^2.$$

REFERENCES

[1] M. BERCOVIER AND O. PIRONNEAU, *Error estimates for the finite element method solution of the Stokes problem in the primitive variables*, Numer. Math., 33 (1979), pp. 211-224.

[2] C. BERNARDI AND G. RAUGEL, *Approximation numérique de certaines équations paraboliques non linéaires* RAIRO Anal. Numér., 18 (1984), pp. 237-285.

[3] ———, *Analysis of some finite elements for the Stokes problem*, Maths. Comp., 44 (1985), to appear.

[4] ———, *A mixed method for the time-dependent Stokes problem*, Rapport interne L.A. 189, Université Paris VI, 1983.

[5] F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers*, RAIRO Anal. Numér., 44 (1974), pp. 129-151.

[6] F. BREZZI, J. RAPPAZ AND P.-A. RAVIART, *Finite-dimensional approximation of nonlinear problems. Part* I: *branches of nonsingular solutions*, Numer. Math., 36 (1980), pp. 1-25.

[7] M. CROUZEIX, *Approximation de problèmes semi-linéaires*, Publications de l'Université de Rennes, to appear.

[8] ———, private communication.

[9] M. CROUZEIX AND P.-A. RAVIART, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, RAIRO Anal. Numér., 7 (1973), pp. 33-76.

[10] M. FORTIN, *Old and new elements for incompressible flows*, Int. J. Num. Meth. Fluids, 1 (1981), pp. 347-364.

[11] H. FUJITA AND A. MIZUTANI, *On the finite element method for parabolic equations*, I: *approximation of holomorphic semi-groups*, J. Math. Soc. Japan, 28 (1976), pp. 749-771.

[12] V. GIRAULT AND P.-A. RAVIART, *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Mathematics 749, Springer-Verlag, New York, 1979.

[13] R. GLOWINSKI, B. MANTEL AND J. PÉRIAUX, *Numerical solution of the time-dependent Navier-Stokes equations for incompressible viscous fluids by finite element and alternating direction methods*, to appear.

[14] R. GLOWINSKI AND O. PIRONNEAU, *On a mixed finite element approximation of the Stokes problem* (I); *convergence of the approximate solution*, Numer. Math., 33 (1979), pp. 397-424.

[15] P. GRISVARD, *Singularité des solutions du problème de Stokes dans un polygone*, Publications de l'Université de Nice, 1978.

[16] J. G. HEYWOOD, *The Navier-Stokes equations*: *on the existence, regularity and decay of solutions*, Indiana Univ. Math. J., 29 (1980), pp. 639-681.

[17] J. G. HEYWOOD AND R. RANNACHER, *Finite element approximation of the nonstationary Navier-Stokes problem*. I. *Regularity of solutions and second-order estimates for spatial discretization*, this Journal, 19 (1982), pp. 275-311.

[18] C. JOHNSON, *A mixed finite element method for Navier-Stokes equations*, RAIRO Anal. Numér., 12 (1978), pp. 333-348.

[19] C. JOHNSON AND V. THOMÉE, *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Anal. Numér., 15 (1981), pp. 41-78.

[20] L. MANSFIELD, *Finite element subspaces with optimal rates of convergence for the stationary Stokes problem*, RAIRO Anal. Numér., 16 (1982), pp. 49-66.

[21] H. OKAMOTO, *On the semi-discrete finite element approximation for the nonstationary Stokes equations*, J. Fac. Sci. Univ. Tokyo Sect. IA, 29 (1982), pp. 241-260.

[22] ———, *On the semi-discrete finite element approximation for the nonstationary Navier-Stokes equation*, J. Fac. Sci. Univ. Tokyo Sect. IA, 29 (1982), pp. 613-652.

[23] R. RANNACHER, *On the finite element approximation of the nonstationary Navier-Stokes problem*, in Approximation methods for Navier-Stokes problems, R. Rautmann ed., Lecture Notes in Mathematics 771, Springer-Verlag, New York, 1980, pp. 408-424.

[24] R. TEMAM, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1977.