Models for Stationary Time Series: Stationarity & Invertibility

Week VI: Video 17

STAT 485/685, Fall 2020, SFU

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Video 17 Learning Objectives

By the end of this video, we should be able to:

- Define the stationarity conditions for an AR process
- Use the roots of the characteristic equation to determine what the autocorrelation function of an AR process will look like
- Write the AR(1) process as a general linear process
- Define the invertibility conditions for an MA process, and identify why they are important

The AR(p) Process

Recall: The autoregressive process of order p, i.e. **AR(p)**, is:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Properties:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \gamma_0 & \text{for } k = 0 \\ \phi_1\,\gamma_{k-1} + \phi_2\,\gamma_{k-2} + \phi_3\,\gamma_{k-3} + \cdots + \phi_p\,\gamma_{k-p} & \text{for all } k \geq 1 \end{cases}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1\,\rho_{k-1} + \phi_2\,\rho_{k-2} + \phi_3\,\rho_{k-3} + \cdots + \phi_p \rho_{k-p} & \text{for all } k \geq 1 \end{cases}$$

$$\gamma_0 = Var(Y_t) = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}$$

Stationarity of the AR(p) Process

Definition: The **AR characteristic polynomial** for the AR(p) process is:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

The corresponding **AR characteristic equation** is:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Stationarity of the AR(p) Process

Definition: The **AR characteristic polynomial** for the AR(p) process is:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

The corresponding **AR characteristic equation** is:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

The AR characteristic equation has p roots.

It can be shown that the AR(p) process is stationary *if and only if* each of the p roots of the AR characteristic equation is > 1 (in absolute value).

Stationarity of the AR(p) Process: AR(1)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Recall: AR(1) process:

$$Y_t = \phi Y_{t-1} + e_t$$

Stationarity of the AR(p) Process: AR(1)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Recall: AR(1) process:

$$Y_t = \phi Y_{t-1} + e_t$$

For the AR(1) process, the characteristic equation becomes:

$$1 - \phi x = 0$$

Root:

$$x = \frac{1}{\phi}$$

Stationarity of the AR(p) Process: AR(1)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Recall: AR(1) process:

$$Y_t = \phi Y_{t-1} + e_t$$

For the AR(1) process, the characteristic equation becomes:

$$1 - \phi x = 0$$

Root:

$$x = \frac{1}{\phi}$$

So, the AR(1) process is stationary if and only if $|\frac{1}{\phi}| > 1$, i.e. $|\phi| < 1$.

This is referred to as the **stationarity condition**.

Stationarity of the AR(p) Process: AR(2)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Recall: AR(2) process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Stationarity of the AR(p) Process: AR(2)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

Recall: AR(2) process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

For the AR(2) process, the characteristic equation becomes:

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

Roots:

$$x = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \quad \& \quad x = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

Stationarity of the AR(p) Process: AR(2)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

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For the AR(2) process, the characteristic equation becomes:

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

Roots:

$$x = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \quad \& \quad x = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

So, the AR(2) process is stationary if and only if: (pf in Appendix B, pg. 84)

$$\phi_1 + \phi_2 < 1 \quad \& \quad \phi_2 - \phi_1 < 1 \quad \& \quad |\phi_2| < 1$$

(See Exhibit 4.17 on pg. 72 for a visualization of this space.)

Stationarity of the AR(p) Process: AR(p)

AR characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

It can be shown that the following two conditions are *necessary* (but not sufficient) for stationarity:

$$\phi_1 + \phi_2 + \cdots + \phi_p < 1$$

$$|\phi_p| < 1$$

Autocorrelation Function of the AR(2) Process

Recall: AR(2) process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Autocorrelation function for an AR(2) process:

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1/(1-\phi_2) & \text{for } k = 1 \\ [\phi_2(1-\phi_2) + \phi_1^2]/(1-\phi_2) & \text{for } k = 2 \\ \phi_1\,\rho_{k-1} + \phi_2\,\rho_{k-2} & \text{for all } k \geq 1 \end{cases}$$

What if we want a more explicit formula for ρ_k?

Autocorrelation Function of the AR(2) Process (cont'd)

The expression of ρ_k depends on the roots of the characteristic polynomial:

$$x = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

These roots may be:

a) Real and different from each other: if $\phi_1^2 + 4\phi_2 > 0$

b) Real and equal: if $\phi_1^2 + 4\phi_2 = 0$

c) Complex and different from each other: if $\phi_1^2 + 4\phi_2 < 0$

Autocorrelation Function of the AR(2) Process (cont'd)

a) If the roots of the characteristic polynomial are real and different from each other (i.e., $\phi_1^2 + 4\phi_2 > 0$):

$$\rho_k = \frac{(1 - G_2^2)G_1^{k+1} - (1 - G_1^2)G_2^{k+1}}{(G_1 - G_2)(1 + G_1 G_2)}, \text{ where}$$

$$G_1 = 1/(\text{first root}) \;\&\; G_2 = 1/(\text{second root})$$

$\Rightarrow$ Exponentially decaying in k.

Autocorrelation Function of the AR(2) Process (cont'd)

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$$G_1 = 1/(\text{first root}) \;\&\; G_2 = 1/(\text{second root})$$

$\Rightarrow$ Exponentially decaying in k.

b) If the roots of the characteristic polynomial are real and equal to each other (i.e., $\phi_1^2 + 4\phi_2 = 0$):

$$\rho_k = \left(1 + \frac{1 + \phi_2}{1 - \phi_2} k\right) \left(\frac{\phi_1}{2}\right)^k$$

$\Rightarrow$ Exponentially decaying in k.

Autocorrelation Function of the AR(2) Process (cont'd)

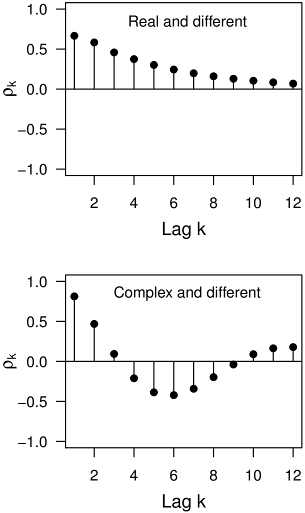
c) If the roots of the characteristic polynomial are complex and different from each other (i.e., $\phi_1^2 + 4\phi_2 < 0$):

$$\rho_k = R^k \frac{sin(\Theta k + \Phi)}{sin(\Phi)}, \text{ where}$$

$$R = \sqrt{-\phi_2} \;\&\; \Theta = acos\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right) \;\&\; \Phi = atan\left(\frac{1-\phi_2}{1+\phi_2}\right)$$

$\Rightarrow$ Exponentially decaying in k, as a *damped sine wave*.

Autocorrelation Function of the AR(2) Process (cont'd)



Autocorrelation Function of the AR(p) Process

Recall: AR(p) process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Autocorrelation function for an AR(p) process:

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \phi_3 \rho_{k-3} + \cdots + \phi_p \rho_{k-p} & \text{for all } k \geq 1 \end{cases}$$

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$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Autocorrelation function for an AR(p) process:

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What if we want a more explicit formula for ρ_k?

ρ_k will be a linear combination of:

- Exponentially decaying terms (corresponding to the real roots of the characteristic equation), and
- Damped sine wave terms (corresponding to the complex roots of the characteristic equation)

MA and AR Processes as General Linear Processes

Recall: General linear process:

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \dots$$

MA and AR Processes as General Linear Processes

Recall: General linear process:

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \dots$$

MA(q) process:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

This is just a general linear processes that truncates after e_{t-q}, with $\psi_j = -\theta_j$!

MA and AR Processes as General Linear Processes (cont'd)

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \dots$$

AR(1) process:

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t = e_t + \phi e_{t-1} + \phi^2 Y_{t-2} \\ &= e_t + \phi e_{t-1} + \phi^2(\phi Y_{t-3} + e_{t-2}) = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 Y_{t-3} \end{aligned}$$

MA and AR Processes as General Linear Processes (cont'd)

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \dots$$

AR(1) process:

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In general: $\quad Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k}$

Letting $k \to \infty$: $\quad Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots$

MA and AR Processes as General Linear Processes (cont'd)

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \dots$$

AR(1) process:

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In general: $\quad Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k}$

Letting $k \to \infty$: $\quad Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots$

This is just a general linear processes, with $\psi_j = \phi^j$!

We've seen this in Video 14, and derived its mean, autocovariance and autocorrelation functions! Check that these match what we derived for AR(1) in Video 16.

The MA(1) Process

$$Y_t = e_t - \theta e_{t-1}$$

Recall: Properties of the MA(1) process:

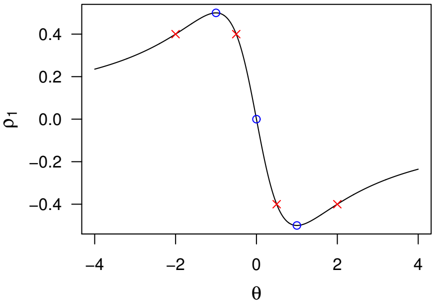
$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2\left(1+\theta^2\right) & \text{for } k=0 \\ \sigma_e^2\left(-\theta\right) & \text{for } k=1 \\ 0 & \text{for } k>1 \end{cases}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k=0 \\ -\frac{\theta}{1+\theta^2} & \text{for } k=1 \\ 0 & \text{for } k>1 \end{cases}$$

The MA(1) Process: About ρ_1

$$\rho_1 = -\frac{\theta}{1+\theta^2}$$



Notice: ρ_1 is the *same* for any θ vs. $1/\theta$.

This raises the question of **invertibility**: When given a value (or estimate) of ρ_1, we can't determine the true value of θ, unless we place some restrictions on θ.

Invertibility of the MA(q) Process

Definition: The **MA characteristic polynomial** for the MA(q) process is:

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$$

The corresponding **MA characteristic equation** is:

$$1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q = 0$$

Invertibility of the MA(q) Process

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The corresponding **MA characteristic equation** is:

$$1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q = 0$$

The MA characteristic equation has q roots.

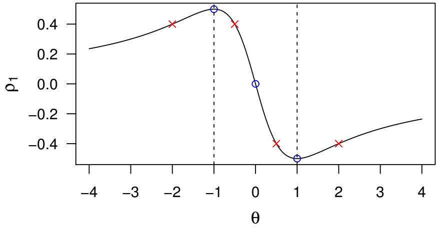
It can be shown that the MA(q) process is invertible *if and only if* each of the q roots of the MA characteristic equation is > 1 (in absolute value).

Example: For MA(1): $1 - \theta x = 0 \quad \Rightarrow \quad x = 1/\theta,$ so we require $|\theta| < 1$.

Invertibility of the MA(q) Process: Meaning

What does it mean for an MA(q) process to be *invertible*?

- For a given autocorrelation function ρ_k: There is only one set of θ-parameters that can yield this function while also satisfying the invertibility conditions.



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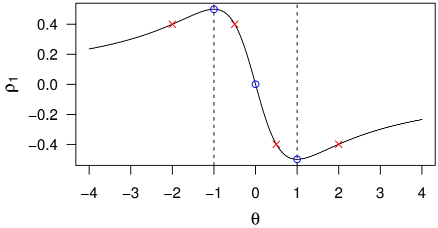

- If an MA(q) process is invertible, there exist some coefficients π_j such that:

$$Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t$$

i.e., it can "inverted" into an AR(∞) process.

Final Comments

That's all for now!

In this video, we've learned about the stationarity conditions for an AR process, and the invertibility conditions for an MA process.

We've also learned how to express the AR(1) process as a general linear process, and we've seen some more properties of the autocorrelation function for an AR process.

Coming Up Next: ARMA models!

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.