Models for Stationary Time Series: ARMA Processes

Week VI: Video 18

STAT 485/685, Fall 2020, SFU

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Video 18 Learning Objectives

By the end of this video, we should be able to:

- Define an autoregressive moving average process of orders p and q, i.e. ARMA(p,q)
- Give the mean function, autocovariance function and autocorrelation function for the ARMA(1,1) process
- Recognize that ARMA(p,q) is a very general class of model, which can be used to describe many different types of time series behaviours

Autoregressive Moving Average Processes

Review: Moving average process of order q (i.e. **MA(q)**):

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3} - \cdots - \theta_q e_{t-q}$$

Autoregressive process of order p (i.e. **AR(p)**):

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

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Autoregressive process of order p (i.e. **AR(p)**):

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Definition: An **autoregressive moving average process of orders p and q** (i.e. **ARMA(p,q)**) is defined as:

$$\begin{aligned} Y_t = {} & \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ & - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \end{aligned}$$

(Also sometimes referred to as a "mixed autoregressive moving average process".)

Autoregressive Moving Average Processes (cont'd)

ARMA(p,q) process:

$$\begin{aligned} Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3} - \cdots - \theta_q e_{t-q} \end{aligned}$$

This is a convenient model, because it incorporates:

- Linear combination of the past values of the variable (i.e., the "AR" part)
- Linear combination of white noise terms occurring at various times in the past (i.e., the "MA" part)

Autoregressive Moving Average Processes (cont'd)

ARMA(p,q) process:

$$\begin{aligned} Y_t = &\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ &- \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3} - \cdots - \theta_q e_{t-q} \end{aligned}$$

This is a convenient model, because it incorporates:

- Linear combination of the past values of the variable (i.e., the "AR" part)
- Linear combination of white noise terms occurring at various times in the past (i.e., the "MA" part)

The e_t's are assumed to be zero-mean white noise terms, with variance σ_e^2. They are independent of all past Y's.

AR and MA processes can be thought of as "special cases" of an ARMA process.

The ARMA(1,1) Process

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- Mean function: $E(Y_t) = 0$ (assumption)
- Autocovariance function:

$$Y_{t-k} Y_t = \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t - \theta Y_{t-k} e_{t-1}$$

$$E(Y_{t-k} Y_t) = \phi E(Y_{t-k} Y_{t-1}) + E(Y_{t-k} e_t) - \theta E(Y_{t-k} e_{t-1})$$

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Note:

- $E(Y_{t-k} Y_t) = Cov(Y_{t-k}, Y_t) + E(Y_{t-k})E(Y_t) = Cov(Y_{t-k}, Y_t) = \gamma_k$
- $E(Y_{t-k} Y_{t-1}) = Cov(Y_{t-k}, Y_{t-1}) = \gamma_{k-1}$

So:

$$\gamma_k = \phi \gamma_{k-1} + E(Y_{t-k} e_t) - \theta E(Y_{t-k} e_{t-1})$$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

$$\gamma_k = \phi \gamma_{k-1} + E(Y_{t-k} e_t) - \theta E(Y_{t-k} e_{t-1})$$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

$$\gamma_k = \phi\,\gamma_{k-1} + E(Y_{t-k}e_t) - \theta E(Y_{t-k}e_{t-1})$$

Now:

- What about $E(Y_{t-k}e_t)$?

  If $k = 0$: $E(Y_t e_t) = E[(\phi Y_{t-1} + e_t - \theta e_{t-1})e_t] = E(e_t^2) = \sigma_e^2$
  (since $e_t \perp\!\!\!\perp Y_{t-1}$, and $e_t \perp\!\!\!\perp e_{t-1}$)

  If $k > 0$: $E(Y_{t-k}e_t) = 0$ (since e_t is independent of any past Y's)

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

$$\gamma_k = \phi\,\gamma_{k-1} + E(Y_{t-k}e_t) - \theta E(Y_{t-k}e_{t-1})$$

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- What about $E(Y_{t-k}e_{t-1})$?

  If $k=0$: $E(Y_t e_{t-1}) = E[(\phi Y_{t-1} + e_t - \theta e_{t-1})e_{t-1}] = \phi\sigma_e^2 - \theta\sigma_e^2$

  If $k=1$: $E(Y_{t-1}e_{t-1}) = \cdots = \sigma_e^2$

  If $k>1$: $E(Y_{t-k}e_{t-1}) = 0$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

$$\gamma_k = \phi\gamma_{k-1} + E(Y_{t-k}e_t) - \theta E(Y_{t-k}e_{t-1})$$

where

$$E(Y_{t-k}e_t) = \begin{cases} \sigma_e^2 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

$$E(Y_{t-k}e_{t-1}) = \begin{cases} (\phi - \theta)\sigma_e^2 & \text{for } k = 0 \\ \sigma_e^2 & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

The ARMA(1,1) Process (cont'd)

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where

$$E(Y_{t-k}e_t) = \begin{cases} \sigma_e^2 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

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So:

$$\begin{cases} \gamma_0 = \phi\gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 = \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k = \phi\gamma_{k-1} \quad \text{for all } k \geq 2 \end{cases}$$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

$$\gamma_k = \begin{cases} \frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_e^2 & \text{for } k=0 \\ \phi\,\gamma_0 - \theta\sigma_e^2 & \text{for } k=1 \\ \phi\,\gamma_{k-1} & \text{for all } k \geq 2 \end{cases}$$

The ARMA(1,1) Process (cont'd)

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- Autocorrelation function:

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = \frac{\gamma_1}{\gamma_0} = \phi - \frac{\theta\sigma_e^2}{\gamma_0} = \cdots = \frac{(1-\phi\theta)(\phi-\theta)}{1-2\phi\theta+\theta^2} \\ \rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\phi\,\gamma_{k-1}}{\gamma_0} = \phi\rho_{k-1} \quad \text{for all } k \geq 2 \end{cases}$$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

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So:

$$\rho_k = \begin{cases} 1 & \text{for } k = 0 \\ \phi^{k-1}\frac{(1-\phi\theta)(\phi-\theta)}{1-2\phi\theta+\theta^2} & \text{for all } k \geq 1 \end{cases}$$

The ARMA(1,1) Process (cont'd)

- Autocovariance function:

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So:

$$\rho_k = \begin{cases} 1 & \text{for } k=0 \\ \phi^{k-1}\frac{(1-\phi\theta)(\phi-\theta)}{1-2\phi\theta+\theta^2} & \text{for all } k \geq 1 \end{cases}$$

Notice: The correlation is exponentially decreasing in k!

The ARMA(p,q) Process: Properties

- Autocorrelation function satisfies:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}$$

The shape of ρ_k can look very different, depending on the values of the ϕ's and θ's.

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  The shape of ρ_k can look very different, depending on the values of the ϕ's and θ's.

- It can be shown that the ARMA(p,q) is stationary *if and only if* each of the p roots of the AR characteristic equation is > 1 in absolute value:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

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$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

- The (stationary) ARMA(1,1) process can be re-written as:

$$Y_t = e_t + (\phi - \theta)[e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \phi^3 e_{t-4} + \dots]$$

  So, it is a general linear process with weights $\psi_j = (\phi - \theta)\phi^{j-1}$.

  The general (stationary) ARMA(p,q) process can also be re-written as a general linear process (see pg. 79).

Final Comments

That's all for now!

In this video, we've learned about the autoregressive moving average process of orders p and q, i.e. ARMA(p,q).

We derived some properties for the special case of ARMA(1,1), and we learned about some properties of ARMA(p,q) as well.

We are now equipped with a very general class of time series models, which are useful for representing many, many different types of behaviour!

Next Week in STAT 485/685: Models for *non-stationary* time series.

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.