Models for Stationary Time Series:

MA Processes

Week V: Video 15

STAT 485/685, Fall 2020, SFU

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Video 15 Learning Objectives

By the end of this video, we should be able to:

- Define a moving average process of order q, i.e. MA(q)
- Derive the mean function, autocovariance function and autocorrelation function for MA(1) and MA(2) processes
- Recognize some key properties of the general MA(q) process, including the behaviour of its autocorrelation function

Moving Average Processes

Definition: A **moving average process of order** q (i.e. $\mathbf{MA}(q)$) is defined as:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3} - \cdots - \theta_q e_{t-q}$$

(Notice that it *must* be finite, unlike the general linear process!)

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What's In a Name? It is a (weighted) average of white noise process terms, where the weights get shifted one spot over each time:

$$Y_{t+1} = e_{t+1} - \theta_1 e_t - \theta_2 e_{t-1} - \theta_3 e_{t-2} - \cdots - \theta_q e_{t-q+1}$$

$$Y_{t+2} = e_{t+2} - \theta_1 e_{t+1} - \theta_2 e_t - \theta_3 e_{t-1} - \cdots - \theta_q e_{t-q+2}$$

$$Y_{t+3} = e_{t+3} - \theta_1 e_{t+2} - \theta_2 e_{t+1} - \theta_3 e_t - \cdots - \theta_q e_{t-q+3}$$

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The First-Order MA Process

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$$\begin{aligned} &Cov(Y_t, Y_{t-k}) \\ &= Cov(e_t - \theta e_{t-1}, e_{t-k} - \theta e_{t-k-1}) \\ &= Cov(e_t, e_{t-k}) - \theta Cov(e_t, e_{t-k-1}) - \theta Cov(e_{t-1}, e_{t-k}) + \theta^2 Cov(e_{t-1}, e_{t-k-1}) \end{aligned}$$

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For $k = 0$: $\sigma_e^2 + \theta^2\sigma_e^2 = \sigma_e^2(1 + \theta^2)$

For $k = \pm 1$: $-\sigma_e^2\theta$

Otherwise: 0

The First-Order MA Process: Properties

$$Y_t = e_t - \theta e_{t-1}$$

Properties of the MA(1) process:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2\left(1+\theta^2\right) & \text{for } k = 0 \\ \sigma_e^2\left(-\theta\right) & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

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$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ -\frac{\theta}{1+\theta^2} & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

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Notice: The correlation cuts off after lag 1! This will be very important later.

The First-Order MA Process: About ρ_1

We've seen that, for the MA(1) process:

$$\rho_1 = -\frac{\theta}{1+\theta^2}$$

$$\rho_k = 0 \text{ for } k > 1$$

The correlations in the process are fully determined by the lag-1 autocorrelation.

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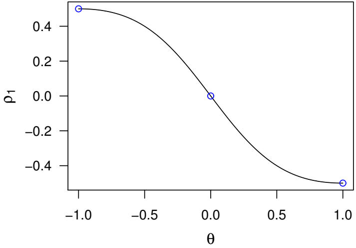
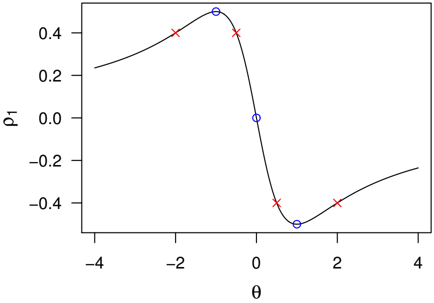


So, ρ_1 can be positive or negative, depending on the value of θ. The closer θ is to ± 1, the closer ρ_1 is to ∓ 0.5.

The First-Order MA Process: About ρ_1 (cont'd)

$$\rho_1 = -\frac{\theta}{1+\theta^2}$$

What if we consider a wider range of possible θ-values?



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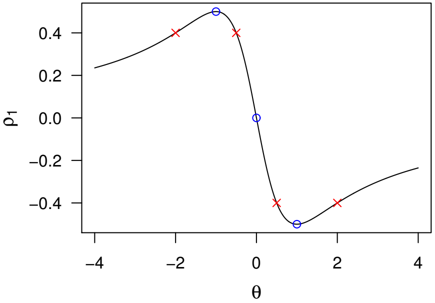
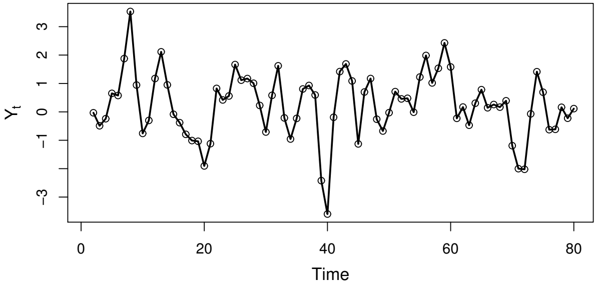


Notice: ρ_1 is the *same* for any θ vs. $1/\theta$.

This raises the question of **invertibility**: When given a value (or estimate) of ρ_1, we can't determine the true value of θ, unless we place some restrictions on θ.

The First-Order MA Process: Example 1

Example: MA(1) process with $\theta = -0.9$: (i.e., $\rho_1 = 0.4972$)




```
> e.vec <- rnorm(n=80, mean=0, sd=1)
> y.vec <- rep(NA, times=80)
> for (t in 2:80)
  {
     y.vec[t] <- e.vec[t] + 0.9*e.vec[t-1]
  }
> plot(c(1:80), y.vec, type='o', xlab='Time', ylab=expression(Y[t]))
```

The First-Order MA Process: Example 1 (cont'd)

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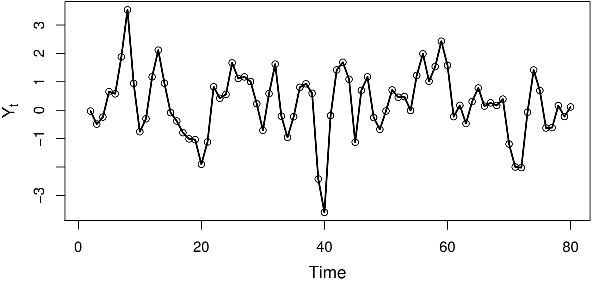

Due to the positive lag-1 autocorrelation, observations "hang together" quite a bit.

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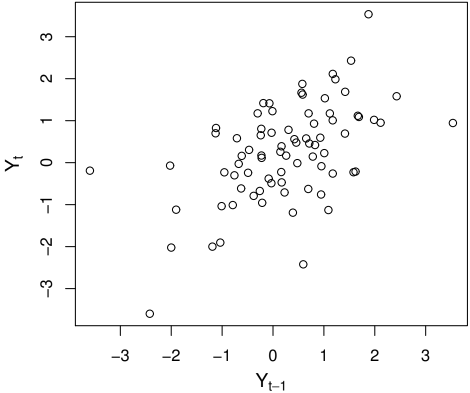

What do we see in the plot of Y_t vs. Y_{t-1}?

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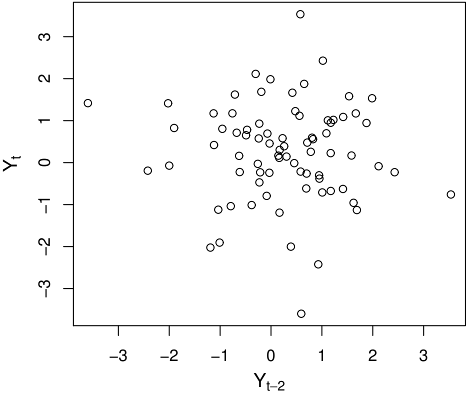
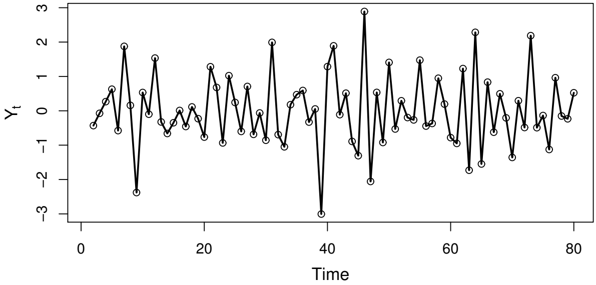


What do we see in the plot of Y_t vs. Y_{t-2}?

The First-Order MA Process: Example 2

Example: MA(1) process with $\theta = +0.9$: (i.e., $\rho_1 = -0.4972$)

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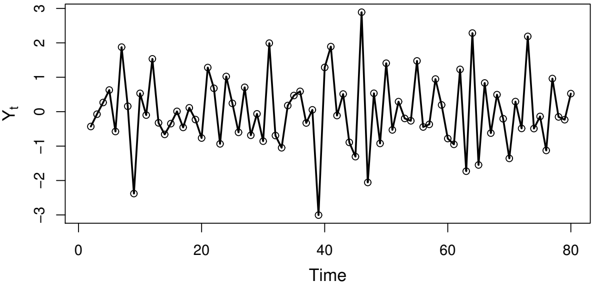

Due to the negative lag-1 autocorrelation, the plot is quite "jagged" over time.

The First-Order MA Process: Example 2 (cont'd)

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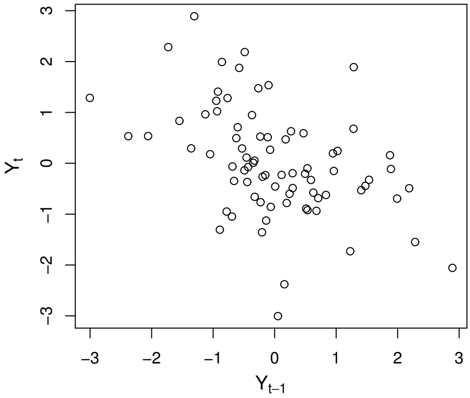

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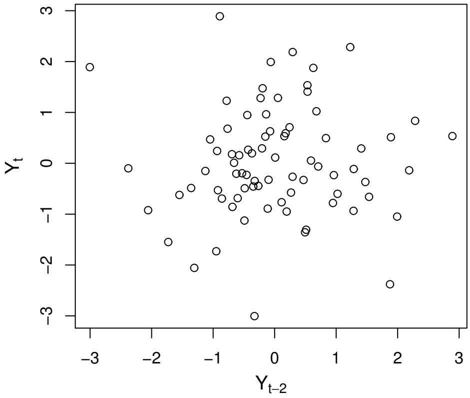

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- Autocovariance function:

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&= Cov(e_t, e_{t-k}) - \theta_1 Cov(e_t, e_{t-k-1}) - \theta_2 Cov(e_t, e_{t-k-2}) \\
&\quad - \theta_1 Cov(e_{t-1}, e_{t-k}) + \theta_1^2 Cov(e_{t-1}, e_{t-k-1}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-k-2}) \\
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The Second-Order MA Process

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&= Cov(e_t, e_{t-k}) - \theta_1 Cov(e_t, e_{t-k-1}) - \theta_2 Cov(e_t, e_{t-k-2}) \\
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\end{aligned}
$$

For $k = 0$: $\sigma_e^2 + \theta_1^2\sigma_e^2 + \theta_2^2\sigma_e^2 = \sigma_e^2\,(1 + \theta_1^2 + \theta_2^2)$

For $k = \pm 1$: $-\theta_1\sigma_e^2 + \theta_1\theta_2\sigma_e^2 = \sigma_e^2\,(-\theta_1 + \theta_1\theta_2)$

For $k = \pm 2$: $\sigma_e^2\,(-\theta_2)$

Otherwise: 0

The Second-Order MA Process: Properties

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

Properties of the MA(2) process:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \left(1 + \theta_1^2 + \theta_2^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_1 + \theta_1\theta_2\right) & \text{for } k = 1 \\ \sigma_e^2 \left(-\theta_2\right) & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}$$

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$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \left(-\theta_1+\theta_1\theta_2\right) / \left(1+\theta_1^2+\theta_2^2\right) & \text{for } k = 1 \\ -\theta_2 / \left(1+\theta_1^2+\theta_2^2\right) & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}$$

Notice: The correlation cuts off after lag 2!

The Second-Order MA Process: Example

Example: MA(2) process:

$$Y_t = e_t - 0.9e_{t-1} + 0.7e_{t-2}$$

i.e., $\theta_1 = 0.9$ & $\theta_2 = -0.7$.

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Then:

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-0.9 + (0.9)(-0.7)}{1 + (0.9)^2 + (-0.7)^2} = -0.67$$

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{0.7}{1 + (0.9)^2 + (-0.7)^2} = 0.30$$

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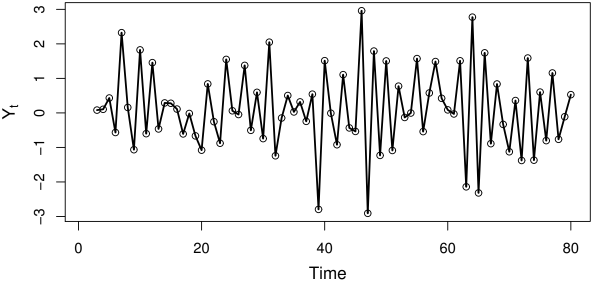

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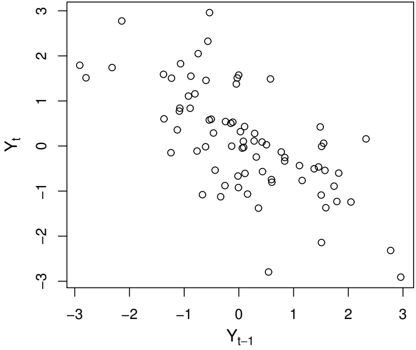

What do we see in the plot of Y_t vs. Y_{t-1}?

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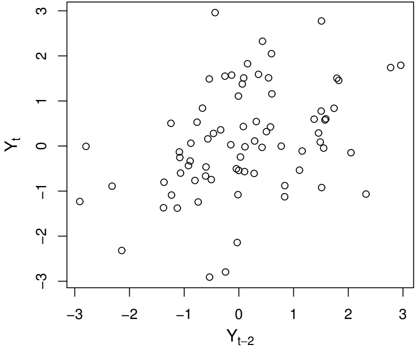

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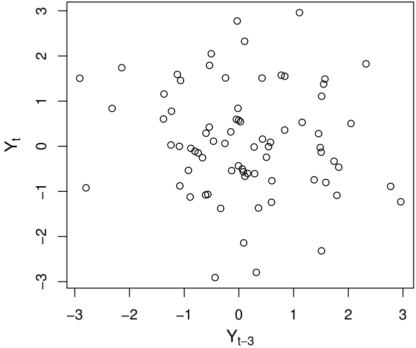

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The General MA(q) Process

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Properties:

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$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \left(1 + \sum_{j=1}^{q} \theta_j^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_k + \sum_{j=k+1}^{q} \theta_{j-k}\theta_j\right) & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

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Try $q = 1$:

$$\gamma_k = \begin{cases} \sigma_e^2 \left(1 + \theta^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta\right) & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

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$$\gamma_k = \begin{cases} \sigma_e^2 (1 + \theta^2) & \text{for } k = 0 \\ \sigma_e^2 (-\theta) & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

Try $q = 2$:

$$\gamma_k = \begin{cases} \sigma_e^2 (1 + \theta_1^2 + \theta_2^2) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_1 + \sum_{j=2}^{2} \theta_{j-1}\theta_j\right) = \sigma_e^2 (-\theta_1 + \theta_1\theta_2) & \text{for } k = 1 \\ \sigma_e^2 (-\theta_2) & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}$$

The General MA(q) Process

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \left(1 + \sum_{j=1}^{q} \theta_j^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_k + \sum_{j=k+1}^{q} \theta_{j-k}\theta_j\right) & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

Try $q = 3$:

$$\gamma_k = \begin{cases} \sigma_e^2 \left(1 + \theta_1^2 + \theta_2^2 + \theta_3^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_1 + \sum_{j=2}^{3} \theta_{j-1}\theta_j\right) = \sigma_e^2 \left(-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3\right) & \text{for } k = 1 \\ \sigma_e^2 \left(-\theta_2 + \sum_{j=3}^{3} \theta_{j-2}\theta_j\right) = \sigma_e^2 \left(-\theta_2 + \theta_1\theta_3\right) & \text{for } k = 2 \\ \sigma_e^2 \left(-\theta_3\right) & \text{for } k = 3 \\ 0 & \text{for } k > 3 \end{cases}$$

The General MA(q) Process (cont'd)

Definition: The moving average process of order q, i.e. MA(q), is:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3} - \cdots - \theta_q e_{t-q}$$

Properties:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \left(1 + \sum_{j=1}^{q} \theta_j^2\right) & \text{for } k = 0 \\ \sigma_e^2 \left(-\theta_k + \sum_{j=k+1}^{q} \theta_{j-k}\theta_j\right) & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \left(-\theta_k + \sum_{j=k+1}^{q} \theta_{j-k}\theta_j\right) / \left(1 + \sum_{j=1}^{q} \theta_j^2\right) & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

Notice: The correlation cuts off after lag q!

Final Comments

That's all for now!

In this video, we've learned about the moving average process of order q, i.e. MA(q).

We derived the properties for the special cases MA(1) and MA(2), and we learned how the autocorrelation function for an MA(q) process cuts off after lag q.

Next Week in STAT 485/685: Autoregressive (AR) processes, ARMA processes, and more.

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.