Models for Non-Stationary Time Series:

ARIMA Processes - Part II

Week VII: Video 20

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Video 20 Learning Objectives

By the end of this video, we should be able to:

- Re-write an ARIMA process Y_t in terms of its differenced (stationary) process W_t
- Given the name of a simple ARIMA process, obtain its difference equation form for Y_t, using what we know about W_t
- Visualize a few examples of simple ARIMA processes

The ARIMA(p,d,q) Process

Definition: A process $\{Y_t\}$ is said to be an **integrated autoregressive moving average process of orders p and q and degree d** (i.e. **ARIMA(p,d,q)**) if:

The d^{th} difference $W_t = \nabla^d Y_t$ is a stationary ARMA(p,q) process.

In other words:

$$\begin{aligned} W_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} \\ + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \end{aligned}$$

for some $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$.

In most cases, $d = 1$ or $d = 2$ will suffice.

If there are no AR terms in $\{W_t\}$: $\{Y_t\}$ is ARIMA(0,d,q) = **IMA(d,q)**.

If there are no MA terms in $\{W_t\}$: $\{Y_t\}$ is ARIMA(p,d,0) = **ARI(p,d)**.

Re-Writing Y_t in Terms of W_t

Since $\{Y_t\}$ is not stationary, the autocovariance/autocorrelation functions we've derived for ARMA processes do not apply.

In order to derive these properties, it may be useful to re-write Y_t in terms of W_t, whose properties are known.

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For an ARIMA(p,1,q) process:

$$Y_t - Y_{t-1} = W_t$$

Note: Since $\{Y_t\}$ is not stationary, it can't be assumed to go infinitely into the past, so we'll assume it starts at some time $-m$.

Trick: We'll sum both sides from time $-m$ to time t.

Re-Writing Y_t in Terms of W_t (cont'd)

Then (for an ARIMA(p,1,q)):

$$\begin{aligned}
Y_t - Y_{t-1} &= W_t \\
\sum_{j=-m}^{t} (Y_j - Y_{j-1}) &= \sum_{j=-m}^{t} W_j \\
\sum_{j=-m}^{t} Y_j - \sum_{j=-m}^{t} Y_{j-1} &= \sum_{j=-m}^{t} W_j \\
\sum_{j=-m}^{t} Y_j - \sum_{j=-m-1}^{t-1} Y_j &= \sum_{j=-m}^{t} W_j \\
Y_t - Y_{-m-1} &= \sum_{j=-m}^{t} W_j \\
Y_t &= \sum_{j=-m}^{t} W_j
\end{aligned}$$

This way of expressing Y_t is useful for deriving its properties.

Re-Writing Y_t in Terms of W_t (cont'd)

Similarly, for an ARIMA(p,2,q):

$$Y_t - 2Y_{t-1} + Y_{t-2} = W_t$$

$$\sum_{j=-m}^{t} \sum_{i=-m}^{j} (Y_i - 2Y_{i-1} + Y_{i-2}) = \sum_{j=-m}^{t} \sum_{i=-m}^{j} W_i$$

$$\vdots$$

$$Y_t = \sum_{j=-m}^{t} \sum_{i=-m}^{j} W_i$$

$$Y_t = \sum_{j=0}^{t+m} (j+1) W_{t-j}$$

We'll see a few special cases of the usefulness of these expressions.

IMA(1,1)

Recall: IMA(1,1) = ARIMA(0,1,1).

In other words, $\{W_t\} = \{\nabla Y_t\}$ is an MA(1) model.

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Obtain the difference equation form of Y_t (in order to better understand Y_t):

$$W_t = e_t - \theta e_{t-1}$$

$$Y_t - Y_{t-1} = e_t - \theta e_{t-1}$$

$$Y_t = Y_{t-1} + e_t - \theta e_{t-1}$$

(Note that, as expected, this looks like an "ARMA(1,1)" model, but it is not stationary: $\phi = 1$.)

This is a very intuitive model, useful for many applications[1]. It is just the random walk model, with an extra lagged error term.

[1] Franses, P. H. (2020). IMA(1,1) as a new benchmark for forecast evaluation. *Applied Economics Letters, 27*(17), 1419-1423.

IMA(1,1) (cont'd)

We can use $Y_t = \sum_{j=-m}^{t} W_j$, and plug in $W_j = e_j - \theta e_{j-1}$ to obtain the full expression for Y_t in terms of the white noise terms (see pg. 93).

This expression can then be used to derive the autocovariance function and autocorrelation function of Y_t.

Some useful results:

$$E(Y_t) = 0$$

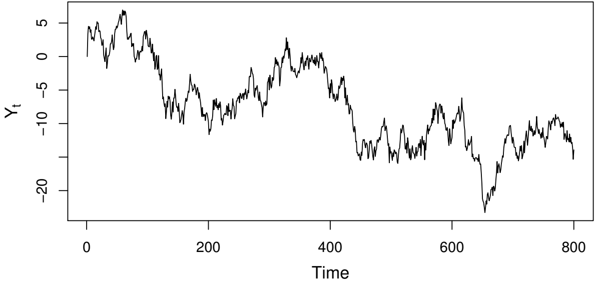
$$Var(Y_t) = [1 + \theta^2 + (1-\theta)^2(t+m)]\sigma_e^2$$

$$Corr(Y_t, Y_{t-k}) \approx \sqrt{\frac{t+m-k}{t+m}} \approx 1 \text{ for large } m \text{ and moderate } k$$

So, $Var(Y_t)$ increases with time, and the correlation between any Y_t and Y_{t-k} is strongly positive for later times.

IMA(1,1): Example

Example: IMA(1,1) process with $\theta = 0.2$:

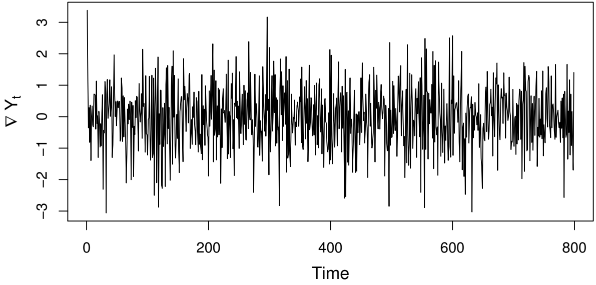



```
> e.vec <- rnorm(n=800, mean=0, sd=1)
> y.vec <- rep(NA, times=800)
> y.vec[1] <- 0
> for (t in 2:800)
  {
     y.vec[t] <- y.vec[t-1] + e.vec[t] - 0.2*e.vec[t-1]
  }
> plot(c(1:800), y.vec, type='l', xlab='Time', ylab=expression(Y[t]))
```

IMA(1,1): Example (cont'd)

Example: IMA(1,1) process with $\theta = 0.2$

Differenced series:

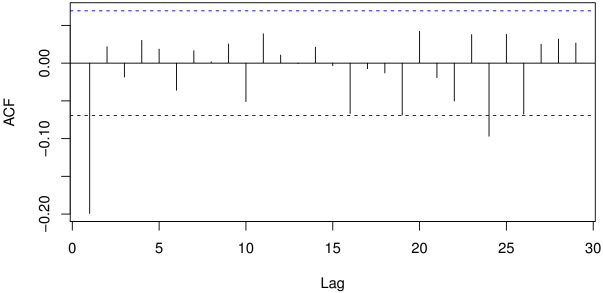



```
> w.vec <- diff(y.vec)
> plot(c(1:79), w.vec, type='l', xlab='Time',
      ylab=expression(paste(nabla,'Y'[t]))
```

IMA(1,1): Example (cont'd)

Example: IMA(1,1) process with $\theta = 0.2$

Sample ACF of the differenced series:




```
> acf(w.vec)
```

Does this support the statement that $\{W_t\} = \{\nabla Y_t\}$ is an MA(1) process?

IMA(2,2)

Recall: IMA(2,2) = ARIMA(0,2,2).

In other words, $\{W_t\} = \{\nabla^2 Y_t\}$ is an MA(2) model.

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Obtain the difference equation form of Y_t (in order to better understand Y_t):

$$W_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

$$Y_t - 2Y_{t-1} + Y_{t-2} = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

$$Y_t = 2Y_{t-1} - Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

(Note that, as expected, this looks like an "ARMA(2,2)" model, but it is not stationary: $\phi_1 + \phi_2 = 1$.)

IMA(2,2) (cont'd)

We can use $Y_t = \sum_{j=0}^{t+m}(j+1)W_{t-j}$, and plug in $W_j = e_j - \theta_1 e_{j-1} - \theta_2 e_{j-2}$ to obtain the full expression for Y_t in terms of the white noise terms (see pg. 94).

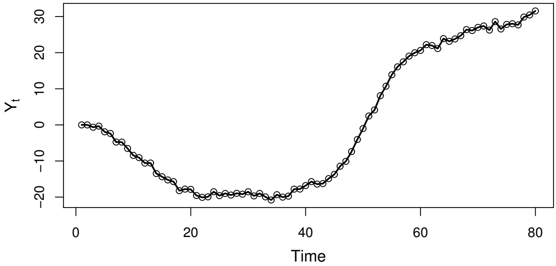
This expression can then be used to derive the autocovariance function and autocorrelation function of Y_t.

Some useful results:

- $E(Y_t) = 0$
- $Var(Y_t)$ increases rapidly with time
- Just like for IMA(1,1), the correlation between any Y_t and Y_{t-k} is strongly positive for all moderate k

IMA(2,2): Example

Example: IMA(2,2) process with $\theta_1 = 1$ & $\theta_2 = -0.6$:




```
> e.vec <- rnorm(n=80, mean=0, sd=1)
> y.vec <- rep(NA, times=80)
> y.vec[1] <- 0; y.vec[2] <- 0
> for (t in 3:80)
  {
     y.vec[t] <- 2*y.vec[t-1] - y.vec[t-2] + e.vec[t] - 1*e.vec[t-1] +
                 0.6*e.vec[t-2]
  }
> plot(c(1:80), y.vec, type='o', xlab='Time', ylab=expression(Y[t]))
```

ARI(1,1)

Recall: ARI(1,1) = ARIMA(1,1,0).

In other words, $\{W_t\} = \{\nabla Y_t\}$ is an AR(1) model.

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Recall: ARI(1,1) = ARIMA(1,1,0).

In other words, $\{W_t\} = \{\nabla Y_t\}$ is an AR(1) model.

Obtain the difference equation form of Y_t (in order to better understand Y_t):

$$W_t = \phi W_{t-1} + e_t$$

$$Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + e_t$$

$$Y_t = (1+\phi)Y_{t-1} - \phi Y_{t-2} + e_t$$

(Note that, as expected, this looks like an "AR(2) = ARMA(2,0)" model, but it is not stationary: $\phi_1' + \phi_2' = (1+\phi) - \phi = 1$.)

ARI(1,1) (cont'd)

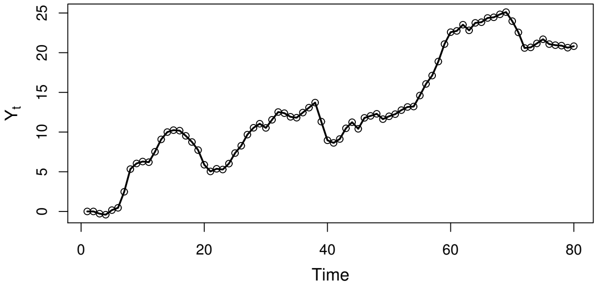
We can use $Y_t = \sum_{j=-m}^{t} W_j$, and plug in $W_j = \phi W_{j-1} + e_j$ to obtain the full expression for Y_t in terms of the white noise terms (see pg. 94).

This expression can then be used to derive the autocovariance function and autocorrelation function of Y_t.

However, the results here are a bit less intuitive.

ARI(1,1): Example

Example: ARI(1,1) process with $\phi = 0.5$:




```
> e.vec <- rnorm(n=80, mean=0, sd=1)
> y.vec <- rep(NA, times=80)
> y.vec[1] <- 0; y.vec[2] <- 0
> for (t in 3:80)
  {
     y.vec[t] <- (1+0.5)*y.vec[t-1] - 0.5*y.vec[t-2] + e.vec[t]
  }
> plot(c(1:80), y.vec, type='o', xlab='Time', ylab=expression(Y[t]))
```

Final Comments

That's all for now!

In this video, we've seen how an ARIMA process can be re-written in terms of its (stationary) differenced series, and how this expression can be useful for deriving autocovariance, autocorrelation, etc.

We've also seen a few examples of some special cases of ARIMA processes, and practiced obtaining the difference equation form for each.

Coming Up Next: Other transformations for obtaining a stationary process from a non-stationary one.

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Franses, P. H. (2020). IMA(1,1) as a new benchmark for forecast evaluation. *Applied Economics Letters, 27*(17), 1419-1423.

[3] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.