2. (Exercise 2.4 in Cryer & Chan). Let $\{e_t\}$ be a zero-mean white noise process with a variance denoted by σ_e^2. Suppose that the observed process is $Y_t = e_t + \theta e_{t-1}$, where θ is some constant value.

(a) Find the autocovariance function for $\{Y_t\}$, in terms of θ. (Hint. Remember to consider all possible combinations of s and t: when they are equal, when they are 1 unit apart, etc).

$Y_t = e_t + \theta e_{t-1}$

① if $t=s$: $Cov(Y_t, Y_s) = Var(Y_t) = Var(e_t + \theta e_{t-1})$

$= Var(e_t) + \theta^2 Var(e_{t-1}) = \sigma_e^2 + \theta^2\sigma_e^2 = \sigma_e^2(1+\theta^2)$

② if $|t-s|=1$

$Cov(Y_t, Y_s) = Cov(Y_t, Y_{t-1}) = Cov(\underbrace{e_t + \theta e_{t-1}}_{Y_t}, \underbrace{e_{t-1} + \theta e_{t-2}}_{Y_{t-1}})$

$= \underbrace{\cancel{Cov(e_t, e_{t-1})}}_{0} + \theta\underbrace{\cancel{(e_t, e_{t-2})}}_{0} + \theta Cov(e_{t-1}, e_{t-1}) + \theta^2 \underbrace{\cancel{Cov(e_{t-1}, e_{t-2})}}_{0}$

$= \theta Var(e_{t-1}) = \theta\sigma_e^2$.

③ if $|t-s|>1$

$Cov(Y_t, Y_s) = Cov(Y_t, Y_{t-2}) = Cov(e_t + \theta e_{t-1}, e_{t-2} + \theta e_{t-3})$

$= \underbrace{\cancel{Cov(e_t, e_{t-2})}}_{0} + \theta \underbrace{\cancel{Cov(e_t, e_{t-3})}}_{0} + \theta \underbrace{\cancel{Cov(e_{t-1}, e_{t-2})}}_{0} + \theta^2 \underbrace{\cancel{Cov(e_{t-1}, e_{t-3})}}_{0}$

$\vdots$

0

$$\gamma_{t,s} = \begin{cases} \sigma_e^2(1+\theta^2) & \text{if } t=s \\ \theta\sigma_e^2 & \text{if } |t-s|=1 \\ 0 & \text{if } |t-s|>1 \end{cases}$$

(b) Find the autocorrelation function for $\{Y_t\}$, in terms of θ.

$$Corr(Y_t, Y_s) = \frac{Cov(Y_t, Y_s)}{\sqrt{Var(Y_t)\,Var(Y_s)}}, \qquad Var(Y_t) =$$

$$\rho_{t,s} = \begin{cases} 1 & \text{if } t=s \quad (Corr(Y_t, Y_t) = 1.) \\ \frac{\theta}{1+\theta^2} & \text{if } |t-s|=1 \quad (Cov(Y_t, Y_s) = \theta\sigma_e^2) \\ 0 & \text{if } |t-s|>1 \quad (Cov(Y_t, Y_s) = 0.) \end{cases}$$

$$\frac{\theta\cancel{\sigma_e^2}}{\sqrt{\cancel{\sigma_e^2}(1+\theta^2)\cdot\cancel{\sigma_e^2}(1+\theta^2)}} = \frac{\theta}{1+\theta^2}$$

(c) Suppose that θ may be equal to either 3 or 1/3. Evaluate the autocorrelation

if $\theta=3$.
$$\rho_{t,s} = \begin{cases} 1 & \text{if } t=s \\ \frac{3}{1+9} = \frac{3}{10} & \text{if } |t-s|=1 \\ 0 & \text{if } |t-s|<1. \end{cases}$$

if $\theta=\frac{1}{3}$
$$\rho_{t,s} = \begin{cases} 1 & \text{if } t=s \\ \frac{\frac{1}{3}}{1+\frac{1}{9}} = \frac{3}{10} & \text{if } |t-s|=1 \\ 0 & \text{if } |t-s|=1 \end{cases}$$

(d) Suppose you observe a time series $\{Y_1, Y_2, \cdots, Y_n\}$ and use it obtain an estimate of the autocorrelation function (using methods we will learn later in the course). Would you be able to use this estimate of $\rho_{t,s}$ to determine whether θ is equal to 3 or 1/3? Why or why not?

No, because in both case, when $|t-s|=1$, they have same value.