Tutorial 2 - STAT 485/685

Trevor Thomson

Department of Statistics & Actuarial Science
Simon Fraser University, BC, Canada

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Today's Plan

1. Recap of Tutorial 1
   - Example 1: Random Walk
   - Example 2: Moving Average
2. Stationarity
3. Examples
   - Example 1: Question 2.14
   - Example 2: Question 2.19
   - Example 3: Question 2.21

Recap of Tutorial 1

- **Stochastic Process**: A collection of random variables indexed by some set $\mathcal{I}$:

$$\{Y_t : t \in \mathcal{I}\}$$

E.g.

- $\mathcal{I} = \mathbb{N} = \{1, 2, 3, \cdots\}$
- $\mathcal{I} = \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- $\mathcal{I} = \mathbb{R}$ (set of real numbers)
- $\mathcal{I} = \mathbb{R}^+$ (set of positive real numbers)

If $\mathcal{I}$ is coloured in red, $\{Y_t : t \in \mathcal{I}\}$ is a discrete-time stochastic process.

If $\mathcal{I}$ is coloured in blue, $\{Y_t : t \in \mathcal{I}\}$ is a continuous-time stochastic process.

Recap of Tutorial 1

- **Stochastic Process**: A collection of random variables indexed by some set $\mathcal{I}$:

$$\{Y_t : t \in \mathcal{I}\}$$

- **Mean function**: $\mu_t = E(Y_t)$, for $t \in \mathcal{I}$

- **Autocovariance function**: $\gamma_{t,s} = Cov(Y_t, Y_s)$, for $t, s \in \mathcal{I}$
  Note that $\gamma_{t,t} = Var(Y_t)$.

- **Autocorrelation function**: $\rho_{t,s} = Corr(Y_t, Y_s) = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}} \times \sqrt{\gamma_{s,s}}}$, for $t, s \in \mathcal{I}$.

SFU

Let $e_1, e_2, \cdots$ be a sequence of independent, identically distributed (iid) random variables, where $E(e_t) = 0$, $Var(e_t) = \sigma_e^2$.

The sequence $\{e_t : t \in \mathbb{N}\}$ is a white noise process

Let $Y_t = Y_{t-1} + e_t$, where $Y_0 = 0$. Note that $\{Y_t : t \in \mathbb{N}\}$ is a stochastic process.

$$
\begin{aligned}
Y_1 &= e_1 \\
Y_2 &= Y_1 + e_2 = e_1 + e_2 \\
Y_3 &= Y_2 + e_3 = e_1 + e_2 + e_3 \\
&\vdots \\
Y_t &= \sum_{u=1}^{t} e_u
\end{aligned}
$$

Question: What is μ_t? What is $\gamma_{t,s}$? What is $\rho_{t,s}$?

μ_t:

$$\mu_t = E(Y_t) = E\left(\sum_{u=1}^{t} e_u\right) = \sum_{u=1}^{t} E(e_u) = \sum_{u=1}^{t} 0 = 0 \quad \checkmark$$

$\gamma_{t,s}$:

$$\gamma_{t,t} = Var(Y_t) = Var\left(\sum_{u=1}^{t} e_u\right) \underbrace{=}_{\text{why?}} \sum_{u=1}^{t} Var(e_u) = \sum_{u=1}^{t} \sigma_e^2 = t\sigma_e^2 \quad \checkmark$$

$$\gamma_{t,s} = Cov(Y_t, Y_s) = Cov\left(\sum_{u=1}^{t} e_u \,,\, \sum_{v=1}^{s} e_v\right) = \min\{t,s\}\sigma_e^2 \quad ?$$

$\rho_{t,s}$:

$$\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}} \times \sqrt{\gamma_{s,s}}} = \frac{\min\{t,s\}\sigma_e^2}{\sqrt{t\sigma_e^2} \times \sqrt{s\sigma_e^2}} = \begin{cases} \sqrt{\frac{t}{s}} & \text{if } 1 \le t \le s \\ \sqrt{\frac{s}{t}} & \text{if } 1 \le s \le t \end{cases} \quad \checkmark$$

Recap of Tutorial 1

Example 1: Random Walk

$$\gamma_{t,s} = Cov(Y_t, Y_s) = Cov\left(\sum_{u=1}^{t} e_u \; , \; \sum_{v=1}^{s} e_v\right)$$

Case 1: $\min\{s,t\} = s$

$$\begin{aligned}
\gamma_{t,s} &= Cov\left(\sum_{u=1}^{s} e_u + \sum_{u=s+1}^{t} e_u \; , \; \sum_{v=1}^{s} e_v\right) \\
&= Cov\left(\sum_{u=1}^{s} e_u \; , \; \sum_{v=1}^{s} e_v\right) + \underbrace{Cov\left(\sum_{u=s+1}^{t} e_u \; , \; \sum_{v=1}^{s} e_v\right)}_{0} \\
&= \sum_{u=1}^{s} \underbrace{Cov\left(e_u, e_u\right)}_{Var(e_u)} + \sum_{u=1}^{s} \sum_{\substack{v=1 \\ u \neq v}}^{s} Cov\left(e_u, e_v\right) \\
&= \sum_{u=1}^{s} \sigma_e^2 + 0 \\
&= s\sigma_e^2.
\end{aligned}$$

Recap of Tutorial 1

Example 1: Random Walk

$$\gamma_{t,s} = Cov(Y_t, Y_s) = Cov\left(\sum_{u=1}^{t} e_u \ , \ \sum_{v=1}^{s} e_v\right)$$

Case 2: $\min\{s,t\} = t$

Exercise 1: Show that $\gamma_{t,s} = t\sigma_e^2$.

Let $\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots$ be a sequence of independent, identically distributed (iid) random variables, where $E(e_t) = 0$, $Var(e_t) = \sigma_e^2$.

The sequence $\{e_t : t \in \mathbb{Z}\}$ is a white noise process

Let $Y_t = \frac{e_t + e_{t-1}}{2}$. Note that $\{Y_t : t \in \mathbb{Z}\}$ is a stochastic process.

Question: What is μ_t? What is $\gamma_{t,s}$? What is $\rho_{t,s}$?

Let $\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots$ be a sequence of independent, identically distributed (iid) random variables, where $E(e_t) = 0$, $Var(e_t) = \sigma_e^2$.

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Let $Y_t = \frac{e_t + e_{t-1}}{2}$. Note that $\{Y_t : t \in \mathbb{Z}\}$ is a stochastic process.

Question: What is μ_t? What is $\gamma_{t,s}$? What is $\rho_{t,s}$?

Solution: See Tutorial 1.

Stationarity

Purpose: We need to make simplifying (yet reasonable) assumptions about the structure of $\{Y_t : t \in \mathcal{I}\}$

- Probability laws that govern the behaviour of the stochastic process do not change over time.
- **Later**: Chapter...
  - ...4 studies models for stationary time series.
  - ...5 studies models for non-stationary time series.

Stationarity

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1. **Strictly Stationary**: Joint distribution of $Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}$ is the same as the joint distribution of $Y_{t_1-k}, Y_{t_2-k}, \cdots, Y_{t_n-k}$, for any value of k, and all choices of time points $t_1, t_2, \cdots, t_n$, and a specified n.
   - Very strong (i.e. impractical) assumption!

2. **Weakly Stationary** (AKA Second-Order Stationary): $\{Y_t : t \in \mathcal{I}\}$ is weakly stationary if it satisfies conditions (A) and (B) below:
   (A) $E(Y_t) = \mu$ - mean function is constant over time
   (B) $\gamma_{t,t-k} = \gamma_{0,k}$ - variance is constant over time, and $Cov(Y_t, Y_{t-k})$ depends only on k.

Stationarity

- **Note**: If we say $\{Y_t : t \in \mathcal{I}\}$ is *stationary* $\Rightarrow$ $\{Y_t : t \in \mathcal{I}\}$ is *weakly stationary*.

Stationarity

- **Note**: If we say $\{Y_t : t \in \mathcal{I}\}$ is *stationary* $\Rightarrow$ $\{Y_t : t \in \mathcal{I}\}$ is *weakly stationary*.

- **Exercise 2**: Is the random walk from Example 1, $\{Y_t : t \in \mathbb{N}\}$ with $Y_t = \sum_{u=1}^{t} e_u$, stationary?

- **Exercise 3**: Is the moving average from Example 2, $\{Y_t : t \in \mathbb{Z}\}$ with $Y_t = \frac{e_t + e_{t-1}}{2}$, stationary?

Example 1: Question 2.14

Evaluate the mean and covariance function for each of the following processes. In each case, determine whether or not the process is stationary.

(a) $Y_t = \theta_0 + te_t$.

(b) $W_t = \nabla Y_t = Y_t - Y_{t-1}$, where $Y_t = \theta_0 + te_t$.

(c) $Y_t = e_t e_{t-1}$ (Assume that $\{e_t\}$ is normal white noise).

Example 1: Question 2.14

(a) $Y_t = \theta_0 + te_t$.

$$E(Y_t) = E(\theta_0 + te_t) = \theta_0 + t\,\underbrace{E(e_t)}_{0} = \theta_0.$$

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(\theta_0 + te_t,\ \theta_0 + (t-k)e_{t-k}) \\ &= Cov(te_t,\ (t-k)e_{t-k}) \\ &= t(t-k) \times Cov(e_t, e_{t-k}) \end{aligned}$$

$$\underbrace{Cov(e_t, e_{t-k})}_{(*)} = \begin{cases} \sigma_e^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$(*)$ Only non-zero term is when $t - k = t \Rightarrow k = 0$.

$$Cov(Y_t, Y_{t-k}) = \begin{cases} t^2\sigma_e^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $Cov(Y_t, Y_{t-k})$ is a function of t, $\{Y_t\}$ is not stationary.

Example 1: Question 2.14

(b) $W_t = \nabla Y_t = Y_t - Y_{t-1}$, where $Y_t = \theta_0 + te_t$.

$W_t = \theta_0 + te_t - [\theta_0 + (t-1)e_{t-1}] = te_t - (t-1)e_{t-1}$

$$E(W_t) = E(te_t - (t-1)e_{t-1}) = t\,\underbrace{E(e_t)}_{0} - (t-1)\,\underbrace{E(e_{t-1})}_{0} = 0.$$

$$\begin{aligned} Cov(W_t, W_{t-k}) &= Cov(te_t - (t-1)e_{t-1}\ ,\ (t-k)e_{t-k} - (t-k-1)e_{t-k-1}) \\ &= t(t-k)\underbrace{Cov(e_t, e_{t-k})}_{\text{non-zero if } k=0} - t(t-k-1)\underbrace{Cov(e_t, e_{t-k-1})}_{\text{non-zero if } k=-1} \\ &\quad - (t-1)(t-k)\underbrace{Cov(e_{t-1}, e_{t-k})}_{\text{non-zero if } k=1} + (t-1)(t-k-1)\underbrace{Cov(e_{t-1}, e_{t-k-1})}_{\text{non-zero if } k=0} \end{aligned}$$

$$\begin{aligned} k = -1: &\quad Cov(W_t, W_{t+1}) = 0 - t^2\sigma_e^2 - 0 + 0 = -t^2\sigma_e^2 \\ k = 0: &\quad Cov(W_t, W_{t-0}) = t^2\sigma_e^2 - 0 - 0 + (t-1)^2\sigma_e^2 = [t^2 + (t-1)^2]\sigma_e^2 \\ k = 1: &\quad Cov(W_t, W_{t-1}) = 0 - 0 - (t-1)^2\sigma_e^2 + 0 = -(t-1)^2\sigma_e^2 \\ k \notin \{-1, 0, 1\}: &\quad Cov(W_t, W_{t-k}) = 0 \end{aligned}$$

Example 1: Question 2.14

(b) $W_t = \nabla Y_t = Y_t - Y_{t-1}$, where $Y_t = \theta_0 + te_t$.

Therefore,

$$Cov(W_t, W_{t-k}) = \begin{cases} -t^2\sigma_e^2 & \text{if } k = -1 \\ [t^2 + (t-1)^2]\sigma_e^2 & \text{if } k = 0 \\ -(t-1)^2\sigma_e^2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $Cov(W_t, W_{t-k})$ is a function of t, $\{W_t\}$ is not stationary.

Example 1: Question 2.14

(c) $Y_t = e_t e_{t-1}$ (Assume that $\{e_t\}$ is normal white noise).

$$E(Y_t) = E(e_t e_{t-1}) = \underbrace{Cov(e_t, e_{t-1})}_{0} + \underbrace{E(e_t)}_{0}\,\underbrace{E(e_{t-1})}_{0} = 0.$$

$$Cov(Y_t, Y_{t-k}) = \underbrace{Cov(e_t e_{t-1}, e_{t-k} e_{t-k-1})}_{(**)}$$

$(**)$ Non-zero if $t = t-k$, & $t-1 = t-k-1 \Rightarrow k = 0$

$$\begin{aligned}
Cov(Y_t, Y_{t-0}) &= Cov(e_t e_{t-1}, e_t e_{t-1}) \\
&= E(e_t^2 e_{t-1}^2) - \underbrace{E(e_t e_{t-1})^2}_{E(Y_t)^2=0} \\
&= E(e_t^2 e_{t-1}^2) \\
&= \underbrace{Cov(e_t^2, e_{t-1}^2)}_{0} + \underbrace{E(e_t^2)}_{Var(e_t)}\ \underbrace{E(e_{t-1}^2)}_{Var(e_{t-1})} \\
&= \sigma_e^2 \sigma_e^2 \\
&= \sigma_e^4. \\
Cov(Y_t, Y_{t-k}) &= 0, \quad \text{if } k \neq 0.
\end{aligned}$$

Example 1: Question 2.14

(c) $Y_t = e_t e_{t-1}$ (Assume that $\{e_t\}$ is normal white noise).

Therefore

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^4 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ are not functions of t, $\{Y_t\}$ is stationary.

Example 2: Question 2.19

Let $Y_1 = \theta_0 + e_1$ and then for $t > 1$, define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here, θ_0 is a constant. The process $\{Y_t\}$ is called a random walk with drift.

(a) Show that $Y_t = t\theta_0 + \sum_{u=1}^{t} e_u$

(b) Find the mean function for Y_t.

(c) Find the autocovariance function for Y_t.

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Example 2: Question 2.19

(a)

Let's consider a few values of t:

$$t = 1 : Y_1 = \theta_0 + e_1 = 1 \times \theta_0 + \sum_{u=1}^{1} e_u$$

$$t = 2 : Y_2 = \theta_0 + Y_1 + e_2 = \theta_0 + (\theta_0 + e_1) + e_2 = 2\theta_0 + \sum_{u=1}^{2} e_u$$

$$t = 3 : Y_3 = \theta_0 + Y_2 + e_3 = \theta_0 + (2\theta_0 + e_1 + e_2) + e_3 = 3\theta_0 + \sum_{u=1}^{3} e_u$$

The pattern continues, so that $Y_t = t\theta_0 + \sum_{u=1}^{t} e_u$.

Example 2: Question 2.19

(a)

Let's consider a few values of t:

$$t = 1 : Y_1 = \theta_0 + e_1 = 1 \times \theta_0 + \sum_{u=1}^{1} e_u$$

$$t = 2 : Y_2 = \theta_0 + Y_1 + e_2 = \theta_0 + (\theta_0 + e_1) + e_2 = 2\theta_0 + \sum_{u=1}^{2} e_u$$

$$t = 3 : Y_3 = \theta_0 + Y_2 + e_3 = \theta_0 + (2\theta_0 + e_1 + e_2) + e_3 = 3\theta_0 + \sum_{u=1}^{3} e_u$$

The pattern continues, so that $Y_t = t\theta_0 + \sum_{u=1}^{t} e_u$.

- **Remark**: You can prove $Y_t = t\theta_0 + \sum_{u=1}^{t} e_u$ by *induction*.

Example 2: Question 2.19

(b)

$$E(Y_t) = E\left(t\theta_0 + \sum_{u=1}^{t} e_u\right) = t\theta_0 + \sum_{u=1}^{t} \underbrace{E(e_u)}_{0} = t\theta_0.$$

(c)

$$\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov\left(t\theta_0 + \sum_{u=1}^{t} e_u \,,\, (t-k)\theta_0 + \sum_{v=1}^{t-k} e_v\right) \\
&= Cov\left(\sum_{u=1}^{t-k} e_u + \sum_{u=t-k+1}^{t} e_u \,,\, \sum_{v=1}^{t-k} e_v\right) \\
&= Cov\left(\sum_{u=1}^{t-k} e_u \,,\, \sum_{v=1}^{t-k} e_v\right) + \underbrace{Cov\left(\sum_{u=t-k+1}^{t} e_u \,,\, \sum_{v=1}^{t-k} e_v\right)}_{0} \\
&= \sum_{u=1}^{t-k} \underbrace{Cov\left(e_u, e_u\right)}_{Var(e_u)} + \sum_{\substack{u=1 \\ u \neq v}}^{t-k} \sum_{v=1}^{t-k} Cov\left(e_u, e_v\right) \\
&= \sum_{u=1}^{t-k} \sigma_e^2 + 0 \\
&= (t-k)\sigma_e^2.
\end{aligned}$$

Example 3: Question 2.21

For a random walk with a random starting value, let $Y_t = Y_0 + \sum_{u=1}^{t} e_u$ for $t > 0$, where Y_0 has a distribution with mean μ_0 and variance σ_0^2. Suppose further that $Y_0, e_1, \cdots, e_t$ are independent.

(a) Show that $E(Y_t) = \mu_0$ for all t.

(b) Show that $Var(Y_t) = t\sigma_e^2 + \sigma_0^2$.

(c) Show that $Cov(Y_t, Y_s) = \min\{t, s\}\sigma_e^2 + \sigma_0^2$.

(d) Show that $Corr(Y_t, Y_s) = \sqrt{\frac{t\sigma_e^2+\sigma_0^2}{s\sigma_e^2+\sigma_0^2}}$ for $0 \le t \le s$.

Example 3: Question 2.21

(a)

$$E(Y_t) = E\left(Y_0 + \sum_{u=1}^{t} e_u\right) = E(Y_0) + \sum_{u=1}^{t} \underbrace{E(e_u)}_{0} = \mu_0 + 0 = \mu_0.$$

(b)

$$\begin{aligned} Var(Y_t) &= Var\left(Y_0 + \sum_{u=1}^{t} e_u\right) \\ &= Var(Y_0) + \sum_{u=1}^{t} Var(e_u), \quad \text{since } Y_0, e_1, \cdots, e_t, \text{ are independent} \\ &= \sigma_0^2 + \sum_{u=1}^{t} \sigma_e^2 \\ &= \sigma_0^2 + t\sigma_e^2. \end{aligned}$$

(c)

$$
\begin{aligned}
Cov(Y_t, Y_s) &= Cov\left(Y_0 + \sum_{u=1}^{t} e_u \ ,\ Y_0 + \sum_{v=1}^{s} e_v\right) \\
&= \underbrace{Cov\,(Y_0, Y_0)}_{\sigma_0^2} + \underbrace{Cov\left(Y_0,\ \sum_{v=1}^{s} e_v\right)}_{0} + \underbrace{Cov\left(\sum_{u=1}^{t} e_u \ , Y_0\right)}_{0} + \underbrace{Cov\left(\sum_{u=1}^{t} e_u \ ,\ \sum_{v=1}^{s} e_v\right)}_{\min\{t,\, s\}\sigma_e^2, \text{ from Example 1}} \\
&= \sigma_0^2 + \min\{t, s\}\sigma_e^2
\end{aligned}
$$

(d)

Suppose that $0 \leq t \leq s \Rightarrow \min\{t, s\} = t$

- $Var(Y_t) = \sigma_0^2 + t\sigma_e^2$, from Part (b)
- $Cov(Y_t, Y_s) = \sigma_0^2 + t\sigma_e^2$, from Part (c)

Example 3: Question 2.21

(d)

Therefore,

$$
\begin{aligned}
Corr(Y_t, Y_s) &= \frac{Cov(Y_t, Y_s)}{\sqrt{Var(Y_t)} \times \sqrt{Var(Y_s)}} \\
&= \frac{Var(Y_t)}{\sqrt{Var(Y_t)} \times \sqrt{Var(Y_s)}} \\
&= \sqrt{\frac{Var(Y_t)}{Var(Y_s)}} \\
&= \sqrt{\frac{\sigma_0^2 + t\sigma_e^2}{\sigma_0^2 + s\sigma_e^2}}.
\end{aligned}
$$

Example 3: Question 2.21

(d)

Therefore,

$$\begin{aligned}
Corr(Y_t, Y_s) &= \frac{Cov(Y_t, Y_s)}{\sqrt{Var(Y_t)} \times \sqrt{Var(Y_s)}} \\
&= \frac{Var(Y_t)}{\sqrt{Var(Y_t)} \times \sqrt{Var(Y_s)}} \\
&= \sqrt{\frac{Var(Y_t)}{Var(Y_s)}} \\
&= \sqrt{\frac{\sigma_0^2 + t\sigma_e^2}{\sigma_0^2 + s\sigma_e^2}}.
\end{aligned}$$

- **Exercise 4**: When will the stochastic process $\{Y_t\}$ reduce to the stochastic process $\{Y_t\}$ from Example 1?