Tutorial 6 - STAT 485/685

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Today's Plan

1. Recap of Tutorial 5
   - General Linear Process
   - Moving Average Process
   - Autoregressive Process
   - The Mixed Autoregressive Moving Average Model
2. Models for Nonstationary Time Series
   - Stationarity Through Differencing
   - ARIMA Models
   - Constant Terms in ARIMA Models
   - Other Transformations
3. Examples
   - Question 5.1
   - Question 5.6
   - Question 5.14

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Recap of Tutorial 5

General Linear Processes

Definition: $\{Y_t : t \in \mathcal{I}\}$ is a *general linear process* if it can be written as a (weighted) linear combination of present and past white noise terms:

$$\begin{aligned} Y_t &= e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \cdots \\ &= \sum_{j=0}^{\infty} \psi_j e_{t-j}, \end{aligned}$$

where $\psi_0 \equiv 1$.

- Also referred to as ψ-weight representation of $\{Y_t : t \in \mathcal{I}\}$.

In order for the infinite sum above to be *convergent*, we assume that

$$\sum_{j=1}^{\infty} \psi_j^2 < \infty.$$

What is $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$?

Recap of Tutorial 5

General Linear Processes

$$E(Y_t) = E\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}\right) = 0.$$

$$Var(Y_t) = \sigma_e^2 \underbrace{\sum_{j=0}^{\infty} \psi_j^2}_{<\infty} < \infty.$$

For $k \geq 0$:

$$Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell.$$

$$Corr(Y_t, Y_{t-k}) = \frac{\sum_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell}{\sum_{j=0}^{\infty} \psi_j^2}.$$

Remark: $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ do not depend on time t

$\Rightarrow \{Y_t : t \in \mathcal{I}\}$ is stationary.

Recap of Tutorial 5

Moving Average Processes

Definition: $\{Y_t : t \in \mathcal{I}\}$ is a moving average of order q if q of the ψ_j's of the general linear process are non-zero

$$\begin{aligned}
Y_t &= \sum_{j=0}^{\infty} \psi_j e_{t-j} \\
&= \sum_{j=0}^{q} \psi_j e_{t-j}, \quad \text{with } \psi_0 = 1, \\
&= e_t + \psi_1 e_{t-1} + \cdots + \psi_q e_{t-q}.
\end{aligned}$$

If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $MA(q)$ process.

People often express $MA(q)$ processes by slightly changing the notation $\Rightarrow$ letting $\theta_j = -\psi_j$:

$$\begin{aligned}
Y_t &= -\sum_{j=0}^{q} \theta_j e_{t-j}, \quad \text{with } \theta_0 = -1, \\
&= e_t - \theta_1 e_{t-1} - \cdots - \theta_1 e_{t-q}.
\end{aligned}$$

What is $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$?

$$E(Y_t) = E\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}\right) = 0.$$

$$Var(Y_t) = \sigma_e^2 \sum_{j=0}^{q} \theta_j^2.$$

For $k \geq 0$:

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \sum_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell & \text{if } k = 0, 1, \cdots, q \\ 0 & \text{if } k > q \end{cases}.$$

$$Corr(Y_t, Y_{t-k}) = \begin{cases} \dfrac{\sum_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell}{\sum_{j=0}^{q} \theta_j^2} & \text{if } k = 0, 1, \cdots, q \\ 0 & \text{if } k > q \end{cases}.$$

Remark 1: Note that the ACF *cuts off* after lag q

Remark 2: $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ do not depend on time t

$\Rightarrow \{Y_t : t \in \mathcal{I}\}$ is stationary.

Recap of Tutorial 5

Autoregressive Processes

- **Definition**: $\{Y_t : t \in \mathcal{I}\}$ is an autoregressive process of order p if the time series $\{Y_t : t \in \mathcal{I}\}$ satisfies the following equation

$$\begin{aligned} Y_t &= \sum_{j=1}^{p} \phi_j Y_{t-j} + e_t \\ &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t. \end{aligned}$$

If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $AR(p)$ process.

Rather than writing an $AR(p)$ process in terms of Y_t, we can rearrange for e_t:

$$e_t = \phi(B) Y_t,$$

where

$$\begin{aligned} \phi(B) &= 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p, \\ B^k Y_t &= Y_{t-k}. \end{aligned}$$

$\Rightarrow$ B is the *backshift operator*
$\Rightarrow$ $\phi(B)$ is the *autoregressive process characteristic polynomial*.

- **Result**: An $AR(p)$ process is stationary if the roots of $\phi(B)$ lie outside the "unit circle" in $\mathbb{R}^p$-space.

Recap of Tutorial 5

Autoregressive Processes

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

$$e_t = \underbrace{\left[1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right]}_{\phi(B)} Y_t$$

$\Rightarrow$ Setting $\phi(B) = 0 \cdots$
$\Rightarrow \cdots$ Turns out we need

$$\phi_1 + \phi_2 + \cdots + \phi_p < 1,$$
$$|\phi_p| < 1$$

Examples:

$p = 1$: Stationarity condition: $|\phi_1| < 1$
$p = 2$: Stationarity conditions:

$$\phi_1 + \phi_2 < 1 \qquad \phi_2 - \phi_1 < 1 \qquad |\phi_2| < 1$$

To obtain γ_k and ρ_k, solve the Yule-Walker equations

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \cdots + \phi_p .
\end{aligned}$$

Definition: $\{Y_t : t \in \mathcal{I}\}$ is a mixed autoregressive moving average process of orders p and q, respectively, if the time series $\{Y_t : t \in \mathcal{I}\}$ satisfies the following equation

$$\begin{aligned} Y_t &= [\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t] + [-\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}] \\ &= \sum_{j=1}^{p} \phi_j Y_{t-j} - \sum_{j=0}^{q} \theta_j e_{t-j}, \end{aligned}$$

with $\theta_0 \equiv -1$. If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $ARMA(p, q)$ process.

The $ARMA(p, q)$ characteristic polynomial is

$$\begin{aligned} &\theta(B)e_t = \phi(B)Y_t, \\ &\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q \quad (MA(q) \text{ characteristic polynomial}) \\ &\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \quad (AR(p) \text{ characteristic polynomial}) \end{aligned}$$

It is assumed that $\theta(B)$ and $\phi(B)$ have no common factors.

- **Result**: An $ARMA(p, q)$ process is stationary if the roots of $\phi(B)$ lie outside the “unit circle” in $\mathbb{R}^p$-space.

$\Rightarrow$ Solve the Yule-Walker equations to derive γ_k and ρ_k
(see Appendix C, page 85).

Models for Nonstationary Time Series

Recall: For a time series $\{Y_t : t \in \mathcal{I}\}$, we can write each term as

$$Y_t = \mu_t + X_t,$$

where $E(Y_t) = \mu_t \Rightarrow E(X_t) = 0$.

$\Rightarrow$ If $\mu_t \neq \mu$ for all t (i.e. a constant), the time series cannot be stationary.

- We considered models for nonstationary time series in Chapter 3
  - Linear Trend: $\mu_t = \beta_0 + \beta_1 t$
  - Seasonal Means: $\mu_t = \mu_{t+k}$, with $k = 12$.
  - Cosine Trend: $\mu_t = \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t)$, with $f = \frac{1}{12}$.

Question: If $\{Y_t : t \in \mathcal{I}\}$ is not stationary, can we find a stationary time series $\{W_t : t \in \mathcal{I}\}$, such that W_t is derived from $\{Y_t : t \in \mathcal{I}\}$?

- **Approach 1**: Define $W_t = \nabla^d Y_t = \nabla(\nabla^{d-1} Y_t)$, where $d = 1$ or $d = 2$.
- **Approach 2**: Define $W_t = f(Y_t)$, for some function $f(.)$.

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Recall Exercise 2.9:

- Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t : t \in \mathcal{I}\}$ is a zero-mean stationary series with autocovariance function γ_k, and β_0 and β_1 are non-zero constants.

  Show that $\{Y_t : t \in \mathcal{I}\}$ is not stationary but $\{W_t : t \in \mathcal{I}\}$ is stationary, where $W_t = \nabla Y_t = Y_t - Y_{t-1}$.

Recall Exercise 2.9:

- Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t : t \in \mathcal{I}\}$ is a zero-mean stationary series with autocovariance function γ_k, and β_0 and β_1 are non-zero constants.

  Show that $\{Y_t : t \in \mathcal{I}\}$ is not stationary but $\{W_t : t \in \mathcal{I}\}$ is stationary, where $W_t = \nabla Y_t = Y_t - Y_{t-1}$.

We see that

$$E(Y_t) = E(\beta_0 + \beta_1 t + X_t) = \beta_0 + \beta_1 t + \underbrace{E(X_t)}_{0} = \beta_0 + \beta_1 t.$$

Therefore, $\{Y_t : t \in \mathcal{I}\}$ is not stationary since $E(Y_t)$ is not constant over time.

However,

$$W_t = \nabla Y_t = \underbrace{Y_t}_{\beta_0+\beta_1 t+X_t} - \underbrace{Y_{t-1}}_{\beta_0+\beta_1(t-1)+X_{t-1}} = \beta_1 + X_t - X_{t-1}$$

$$\Rightarrow E(W_t) = E(\beta_1 + X_t - X_{t-1}) = \beta_1 + \underbrace{E(X_t)}_{0} + \underbrace{E(X_{t-1})}_{0} = \beta_1$$

$$\begin{aligned}\Rightarrow Cov(W_t, W_{t-k}) &= Cov(\beta_1 + X_t - X_{t-1}, \beta_1 + X_{t-k} - X_{t-k-1}) \\ &= \underbrace{Cov(X_t, X_{t-k})}_{\gamma_k} - \underbrace{Cov(X_t, X_{t-k-1})}_{\gamma_{k+1}} - \underbrace{Cov(X_{t-1}, X_{t-k})}_{\gamma_{k-1}} + \underbrace{Cox(X_{t-1}, X_{t-k-1})}_{\gamma_k} \\ &= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}.\end{aligned}$$

Since $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ do not depend on time, $\{W_t : t \in \mathcal{I}\}$ is stationary.

Recall: the Random Walk $\{Y_t : t \in \mathcal{I}\}$, where

$$Y_t = Y_{t-1} + e_t.$$

We already showed that $\{Y_t : t \in \mathcal{I}\}$ is not stationary.

Note that this is an $AR(1)$ process $Y_t = \phi_1 Y_{t-1} + e_t$ with $\phi_1 = 1$.

- If $\{Y_t : t \in \mathcal{I}\}$ is stationary, we need $|\phi_1| < 1$.

If we let $W_t = \nabla Y_t = Y_t - Y_{t-1}$, we see from the random walk that $W_t = e_t$.

Therefore, $\{W_t : t \in \mathcal{I}\}$ is stationary.

Definition: $\{Y_t : t \in \mathcal{I}\}$ is an integrated autoregressive moving average model if the dth difference $W_t = \nabla^d Y_t$ is a stationary $ARMA(p, q)$. That is, we can construct $\{W_t : t \in \mathcal{I}\}$, where

$$\begin{aligned} W_t &= [\phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} + e_t] + [-\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}] \\ &= \sum_{j=1}^{p} \phi_j W_{t-j} - \sum_{j=0}^{q} \theta_j e_{t-j}. \end{aligned}$$

If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $ARIMA(p, d, q)$ process.

Note: For practical purposes, we only allow for $d \in \{0, 1, 2\}$.

We can then apply the models from Chapter 4 with $\{W_t : t \in \mathcal{I}\}$.

$\Rightarrow$ Use the fact that $W_t = \nabla^d Y_t$ to then apply the model to $\{Y_t : t \in \mathcal{I}\}$.

Special Cases:

- $ARIMA(0, d, q) \Rightarrow IMA(d, q)$
- $ARIMA(p, d, 0) \Rightarrow ARI(p, d)$
- $ARIMA(p, 0, q) \Rightarrow ARMA(p, q)$

Recall: If $\{W_t : t \in \mathcal{I}\}$ is an $ARMA(p,q)$ process, then $E(W_t) = \mu = 0$.

- What if we instead consider the time series $\{W_t^* : t \in \mathcal{I}\}$, where
$$W_t^* = W_t + c,$$
for some $c \neq 0$.

  $\Rightarrow E(W_t^*) = c$

  $\Rightarrow Cov(W_t^*, W_{t-k}^*) = Cov(W_t, W_{t-k})$.

$\Rightarrow$ Can we model $\{W_t^* : t \in \mathcal{I}\}$?

$$W_t = \sum_{j=1}^{p} \phi_j W_{t-j} - \sum_{j=0}^{q} \theta_j e_{t-j}$$

$$(W_t^* - c) = \sum_{j=1}^{p} \phi_j (W_{t-j}^* - c) - \sum_{j=0}^{q} \theta_j e_{t-j}$$

$$W_t^* = \underbrace{\left[c - \sum_{j=1}^{p} c\phi_j\right]}_{\theta_0} + \sum_{j=1}^{p} \phi_j W_{t-j}^* - \sum_{j=0}^{q} \theta_j e_{t-j}$$

$$= \theta_0 + \sum_{j=1}^{p} \phi_j W_{t-j}^* - \sum_{j=0}^{q} \theta_j e_{t-j}.$$

This looks like an $ARMA(p,q)$ process, except θ_0 is an intercept term!

$$W_t^* = \theta_0 + \sum_{j=1}^{p} \phi_j W_{t-j}^* - \sum_{j=0}^{q} \theta_j e_{t-j}$$

Therefore, we see that

$$\theta_0 = c - \sum_{j=1}^{p} c\phi_j$$

$$c = \frac{\theta_0}{1 - \sum_{j=1}^{p} \phi_j}$$

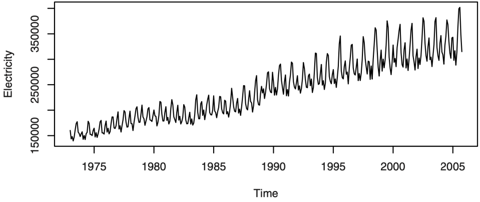
$\Rightarrow$ if we include an intercept term in an $ARMA(p, q)$ model, we can model stationary processes with non-zero means.

Special case: $\theta_0 = 0 \Rightarrow c = 0$, which is what we considered in Chapter 4.

- **Approach 1**: Define $W_t = \nabla^d Y_t = \nabla(\nabla^{d-1} Y_t)$, where $d = 1$ or $d = 2$. ✓
- **Approach 2**: Define $W_t = f(Y_t)$, for some function $f(.)$.

What if $E(Y_t)$ and $Var(Y_t)$ are related?

$$E(Y_t) = \mu_t$$
$$Var(Y_t) = \mu_t^2 \sigma^2$$



Solution: Consider the transformation function $f(Y_t) = \log Y_t$

Consider a (first-order) Taylor series expansion of $\log(Y_t)$ evaluated about μ_t:

$$\log Y_t \approx \log \mu_t + \frac{Y_t - \mu_t}{\mu_t}$$

$$\therefore E(\log Y_t) \approx E\left(\log \mu_t + \frac{Y_t - \mu_t}{\mu_t}\right) = \log \mu_t + \frac{E(Y_t) - \mu_t}{\mu_t} = \log \mu_t$$

$$\therefore Var(\log Y_t) \approx Var\left(\log \mu_t + \frac{Y_t - \mu_t}{\mu_t}\right) = \frac{1}{\mu_t^2} Var(Y_t) = \frac{1}{\mu_t^2}\mu_t^2\sigma^2 = \sigma^2.$$

That is, the variance of $\log Y_t$ no longer depends on μ_t,

$\Rightarrow$ the variance is *stabilized*.

If $\mu_t \neq \mu$, we can consider $W_t = \nabla \log Y_t = \log Y_t - \log Y_{t-1}$ and assess if $\{W_t : t \in \mathcal{I}\}$ is stationary.

Note: The Taylor series of a function $f(x)$ that is infinitely differentiable about a number a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Here, $f(x) = \log x$, $f'(x) = \frac{1}{x}$, $a = \mu_t$, and we only considered $n = 0$ and $n = 1$.

If we want to transform our data, which function should we use?

Examples:

- $f(x) = \log x$
- $f(x) = \sqrt{x}$
- $f(x) = \frac{1}{x}$
- $\ldots$

Box-Cox Power Transformations: For a given value of λ and for $Y_t > 0$ for all $t \in \mathcal{I}$, a *power transformation* with parameter λ is defined by

$$g(x) = \begin{cases} \frac{x^{\lambda}-1}{\lambda} & \text{if } \lambda \neq 0 \\ \log x & \text{if } \lambda = 0 \end{cases}.$$

We see that if

- $\lambda = 0 \Rightarrow$ logarithm transformation
- $\lambda = \frac{1}{2} \Rightarrow$ square-root transformation
- $\lambda = -1 \Rightarrow$ inverse transformation
- $\lambda = 1 \Rightarrow$ no transformation

Models for Nonstationary Time Series

Other Transformations

Remark: If some values in $\{Y_t : t \in \mathcal{I}\}$ are not positive, apply the power transformation to $\{Y_t^* : t \in \mathcal{I}\}$, where $Y_t^* = Y_t + m$, where m is some constant such that $Y_t^* > 0$ for all t.

In R: `BoxCox.ar`

- Computes a log-likelihood function for a grid of λ-values based on a normal likelihood function.
- Generates a 95% confidence interval for λ, where the centre is $\hat{\lambda}$.
- Use the 95% confidence interval to guide us in selecting a proper λ.

Let's look at the `electricity` dataset in R!

- See the R file `Tutorial6.R`

Question 5.1: Identify the following as specific $ARIMA(p, d, q)$ models:

(a) $Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$

(b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t$

(c) $Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$

Examples

Question 5.1

Question 5.1: Identify the following as specific $ARIMA(p, d, q)$ models:

(a) $Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$

(b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t$

(c) $Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$

(a) This appears to be an $ARIMA(2, 0, 1)$ process with $\phi_1 = 1$, $\phi_2 = -0.25$, and $\theta_1 = 0.1$; but we need to first check the stationarity conditions.

Recall that an $ARMA(p, q)$ process is stationary if the autoregressive characteristic polynomial roots lie outside of the $\mathbb{R}^p$ “unit circle”,

The stationarity conditions are

$$\phi_1 + \phi_2 = -0.75 < 1 \quad \checkmark$$
$$\phi_2 - \phi_1 = -1.25 < 1 \quad \checkmark$$
$$|\phi_2| = 0.25 < 1 \quad \checkmark$$

Therefore, $\{Y_t : t \in \mathcal{I}\}$ is an $ARIMA(2, 0, 1)$ process with $\phi_1 = 1$, $\phi_2 = -0.25$, and $\theta_1 = 0.1$.

(b) This appears to be an $AR(2)$ process with $\phi_1 = 2$, and $\phi_2 = -1$. However, the stationarity conditions are not satisfied (actually, all three stationarity conditions fail!)

Consider the difference $W_t = \nabla Y_t = Y_t - Y_{t-1} = W_{t-1} + e_t$. This appears to be an $AR(1)$ model with $\phi_1 = 1$, but since $|\phi_1| \not< 1$, the stationarity conditions are not satisfied.

Consider the difference $X_t = \nabla W_t = W_t - W_{t-1} = e_t$, which is white noise.

Therefore, $\{Y_t : t \in \mathcal{I}\}$ is an $ARIMA(0, 2, 0)$ process.

Question 5.1: Identify the following as specific $ARIMA(p, d, q)$ models:

(a) $Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$

(b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t$

(c) $Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$

(c) This appears to be an $ARIMA(2, 0, 2)$ process with $\phi_1 = 0.5$, $\phi_2 = -0.5$, $\theta_1 = 0.5$, and $\theta_2 = -0.25$.

The stationarity conditions are

$$\phi_1 + \phi_2 = 0 < 1 \quad \checkmark$$
$$\phi_2 - \phi_1 = -1 < 1 \quad \checkmark$$
$$|\phi_2| = 0.5 < 1 \quad \checkmark$$

Therefore, $\{Y_t : t \in \mathcal{I}\}$ is an $ARIMA(2, 0, 2)$ process with $\phi_1 = 0.5$, $\phi_2 = -0.5$, $\theta_1 = 0.5$, and $\theta_2 = -0.25$.

Examples

Question 5.6

Question 5.6: Consider a stationary process $\{Y_t : t \in \mathcal{I}\}$. Show that if $\rho_1 < \frac{1}{2}$, ∇Y_t has a larger variance than Y_t.

Examples

Question 5.6

Question 5.6: Consider a stationary process $\{Y_t : t \in \mathcal{I}\}$. Show that if $\rho_1 < \frac{1}{2}$, ∇Y_t has a larger variance than Y_t.

$$Var(\nabla Y_t) = Var(Y_t - Y_{t-1}) = \underbrace{Var(Y_t)}_{\gamma_0} + \underbrace{Var(Y_{t-1})}_{\gamma_0} - 2\underbrace{Cov(Y_t, Y_{t-1})}_{\gamma_1} = \gamma_0 + \underbrace{\gamma_0 - 2\gamma_1}_{(*)}.$$

If $Var(\nabla Y_t) > \gamma_0$, we need to show that $(*) > 0$, i.e. $\gamma_0 - 2\gamma_1 > 0$.

Since $\rho_1 = \frac{\gamma_1}{\gamma_0}$, then $\gamma_1 = \rho_1 \gamma_0$.

Therefore,

$$\begin{aligned}
\gamma_0 - 2\gamma_1 &> 0 \\
\Rightarrow \gamma_0 - 2(\rho_1 \gamma_0) &> 0 \\
\Rightarrow \gamma_0(1 - 2\rho_1) &> 0 \\
\Rightarrow 1 - 2\rho_1 &> 0 \quad \text{(since } \gamma_0 > 0\text{)} \\
\Rightarrow -2\rho_1 &> -1 \\
\Rightarrow \rho_1 &< \frac{1}{2}.
\end{aligned}$$

Takeaway Message: If the autocorrelation is weak, modelling the process $\{W_t : t \in \mathcal{I}\}$, where $W_t = \nabla Y_t$ will exhibit more variability than $\{Y_t : t \in \mathcal{I}\}$.

If the autocorrelation is weak (i.e. $\rho_1 < \frac{1}{2}$), is it worth it?

Examples

Question 5.14

Question 5.14: Consider the `larain` dataset. The quantile-quantile normal plot of these data convinced us that the data are not normally distributed.

(a) Use R to determine the “best” value of λ for a power transformation of the data.

(b) Display a quantile-quantile plot of the transformed data. Does the data appear to be normally distributed?

(c) Produce a time series plot of the transformed values.

(d) Use the transformed values to display a plot of Y_t vs. Y_{t-1}. Should we expect the transformation to change the dependence or lack of dependence in the series?

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See the R file `Tutorial6.R` for the solutions.