Model Diagnostics

Week XII: Video 32

STAT 485/685, Fall 2020, SFU

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Our Roadmap

1. **Key Ideas:** Fundamental concepts (Ch. 1-2), Estimating trends (Ch. 3), Models for stationary time series (Ch. 4), Models for non-stationary time series (Ch. 5)
2. **Building a Model:**
   - **Model specification (Ch. 6):** How do we choose between the different models that we know?
   - **Parameter estimation (Ch. 7):** Now that we've chosen a model, there will be parameters whose values are unknown. How do we estimate these parameter values?
   - **Model diagnostics (Ch. 8):** How good is our chosen model? Should we be using a different model?
3. **Forecasting (Ch. 9)**
4. Other topics, as time permits.

Model Diagnostics: Introduction

There are two goals to the **model diagnostics** step:

- Assessing the goodness-of-fit of a model
- If the fit is poor, suggesting appropriate modifications

We have already seen many of these concepts in Chapter 3. Now we apply them to fitted time series models!

Video 32 Learning Objectives

By the end of this video, we should be able to:

- Assess the goodness-of-fit of a fitted time series model, using several different types of residual plots
- Use conclusions made from the residual plots to inform the next steps in model specification

Residuals of an ARMA Model

We first encountered **residuals** in Video 10 (Ch. 3), where they represented the stochastic component remaining in the data after the deterministic trend had been fitted.

Residuals for an ARMA model follow a slightly different (but related) definition:

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Important Result:

We saw that an invertible MA(q) process can be written as an AR(∞). Therefore, any ARMA(p,q) model (with invertible MA terms) can be written as

$$Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t$$

for some coefficients $\pi_1, \pi_2, \pi_3, \ldots$

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In Ch. 9, we will see that the best forecast of Y_t based on Y_{t-1}, Y_{t-2}, ... is

$$\hat{Y}_t = \hat{\pi}_1 Y_{t-1} + \hat{\pi}_2 Y_{t-2} + \hat{\pi}_3 Y_{t-3} + \ldots$$

Residuals of an ARMA Model (cont'd)

$$\hat{Y}_t = \hat{\pi}_1 Y_{t-1} + \hat{\pi}_2 Y_{t-2} + \hat{\pi}_3 Y_{t-3} + \ldots$$

For example, for an AR(1):

$$\begin{aligned} \text{Model:} \quad & Y_t = \phi Y_{t-1} + e_t \\ \implies \text{Fitted Values:} \quad & \hat{Y}_t = \hat{\phi} Y_{t-1} \\ \text{Residuals:} \quad & \hat{e}_t = Y_t - \hat{Y}_t \end{aligned}$$

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For an MA(1):

$$\begin{aligned} \text{Model:} \quad & Y_t = e_t - \theta e_{t-1} = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t \\ \implies \text{Fitted Values:} \quad & \hat{Y}_t = \hat{\pi}_1 Y_{t-1} + \hat{\pi}_2 Y_{t-2} + \hat{\pi}_3 Y_{t-3} + \dots \\ \text{Residuals:} \quad & \hat{e}_t = Y_t - \hat{Y}_t \end{aligned}$$

The calculation of the $\hat{\pi}_j$'s will be covered in Ch. 9.

Residuals of an ARMA Model (cont'd)

If the model is correctly specified, and the parameter estimates are close to the true values, then we would expect the residuals to be "similar" to white noise:

- Independent and identically distributed
- Normally distributed
- Zero mean
- Common variance

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To test this, we will analyze the following:

1. Plot of the residuals over time
2. Q-Q plot of the residuals
3. Sample ACF plot of the residuals

Note: We usually plot **standardized residuals** or **studentized residuals**. These values standardize the residual in some way, to make it easier to interpret.

Residuals vs. Time: Example 1

We begin by plotting the residuals over time. If the model is a good fit, and the parameters are accurately estimated, we would expect the residuals to look like a white noise process: a random scatter around 0.

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Example:

The `hare` dataset in the `TSA` package gives the annual number of hares.

In Ch. 6 (pg. 136), it was found that a square-root transformation would be most appropriate for this dataset.

On pg. 137, we found that an AR(2) model might be appropriate for the square root series. What if we try fitting an AR(1) model instead?

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```
data(hare)
hare.ar1.model <- arima(sqrt(hare), order=c(1,0,0))
hare.ar1.model
```

$\hat{\phi}_1 = 0.73$	$\hat{\mu} = 5.81$	$\hat{\theta}_0 = \hat{\mu}(1 - \hat{\phi}_1) = 1.58$

$$\implies \sqrt{Y_t} = 1.58 + 0.73\sqrt{Y_{t-1}} + e_t$$

Residuals vs. Time: Example 1 (cont'd)

```
plot(rstandard(hare.ar1.model), type='o', ylab='Standardized Residuals')
```

These observations "hang together" too much, i.e. there is some evidence of a positive correlation between the residuals.

There may also be some unequal variances.

Residuals vs. Time: Example 2

Let's simulate data from a true underlying AR(2) model. Then, we will "forget" where it came from and try to fit an AR(2) model to it:

```
ar2.data <- arima.sim(model=list(ar=c(0.3,-0.2)), n=100)
ar2.model <- arima(ar2.data, order=c(2,0,0))
```

$\hat{\phi}_1 = 0.30$	$\hat{\phi}_2 = -0.33$	$\hat{\mu} = -0.02$ (with a relatively large s.e.)

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plot(rstandard(ar2.model), type='o', ylab='Standardized Residuals')
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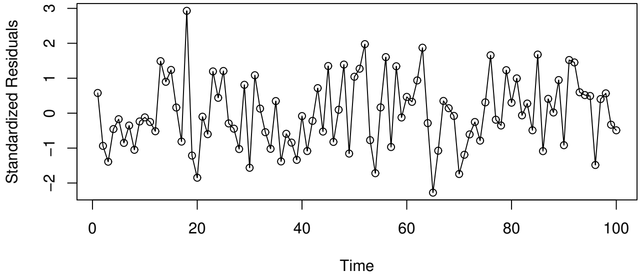

This looks a lot more like a white noise process.

Q-Q Plot of Residuals: Example 1

Q-Q plot of residuals will tell us whether or not the residuals appear to be normal.

Recall: A Q-Q plot displays the quantiles of residuals vs. the theoretical quantiles of a normal distribution. If the residuals are approximately normal, we should see a straight line.

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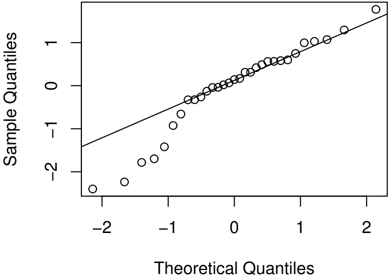
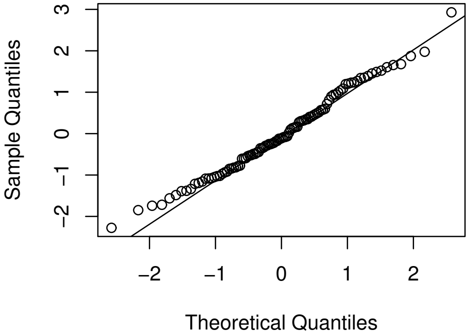
```
hare.ma3.model <- arima(sqrt(hare), order=c(0,0,3))
qqnorm(rstandard(hare.ma3.model))
qqline(rstandard(hare.ma3.model))
```



Q-Q Plot of Residuals: Example 2

```
qqnorm(rstandard(ar2.model))
qqline(rstandard(ar2.model))
```



The residuals from this model appear much more normally distributed.

Sample ACF Plot of Residuals: Example 1

Sample ACF plot of residuals will tell us if there is any dependence in $\{\hat{e}_t\}$. If they are truly independent variables, we should see no significant correlations.

Note: If they are truly independent, the residuals are not expected to be *exactly* white noise process terms. Therefore, their variances will be slightly different than the $1/n$ values assumed in the sample ACF plots. However, the values are close enough that we can still use the dashed lines, especially for larger k. *(See pg. 180-182 for more information.)*

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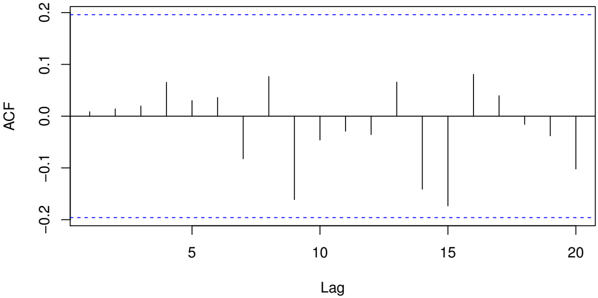
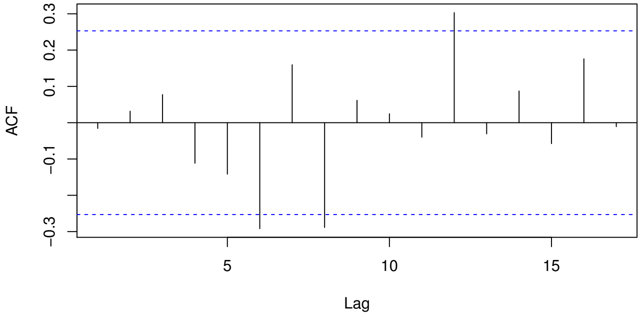
```
acf(rstandard(ar2.model))
```



Sample ACF Plot of Residuals: Example 2

Suppose we were to fit an AR(2) model to the `rwalk` dataset:

```
data(rwalk)
rwalk.ar2.model <- arima(rwalk, order=c(2,0,0))
acf(rstandard(rwalk.ar2.model))
```



There appear to potentially be some positive correlations in these residuals (as we would expect!).

Back to Model Specification?

If we find that our fitted model is not performing well, we need to consider a different model.

We should go back to the model specification step, and choose another model that appears to fit the data well.

General principles:

- Modify the model one term at a time (i.e., increasing the AR or MA order by 1 – but not both at once!)
- Use the model diagnostics plots to inform your next choice of model (e.g., a significant lag-2 correlation after fitting an MA(1) model may suggest you need an MA(2) model instead)

Overfitting

Sometimes, even if your chosen model is adequate, it may be good practice to try fitting a slightly more complicated one. The textbook refers to this as **overfitting**.

For example, even if your AR(1) model performs well, you may wish to try an AR(2) model as well and compare them.

The same "general principles" apply (i.e., only add one parameter at a time, and use the model diagnostics to inform your choice of new model).

Then, compare the two models. Consider:

- Is the newly added parameter significantly different from zero?
- Does this new model do well in model diagnostics?
- Does the new model have a lower AIC value than the previous model?

Final Comments

That's all for now!

In this video, we've learned about a few different methods for assessing a fitted time series model's goodness of fit. We've seen several different residual plots, and learned how to interpret them to assess model assumptions.

We also learned how this information can help guide us when going back to the model specification step.

Coming Up Next: Forecasting!

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.