Trends:

Linear Trends and Regression Methods

Week III: Video 8

STAT 485/685, Fall 2020, SFU

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Review: Trends in Time Series

Suppose our process of interest, $\{Y_t\}$, has some mean function μ_t (which may or may not be a function of t).

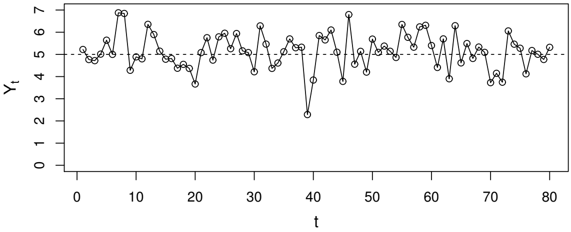
We can separate out the mean from the rest of the process by writing:

$$Y_t = \mu_t + X_t$$

where $\{X_t\}$ is the "de-trended" version of the process, i.e. $E(X_t) = 0$.

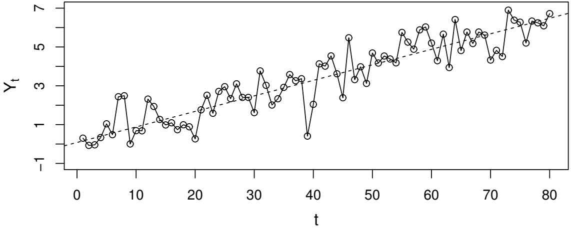
Some examples of how μ_t may look:

- Constant trend: $\mu_t = \mu$ for all t

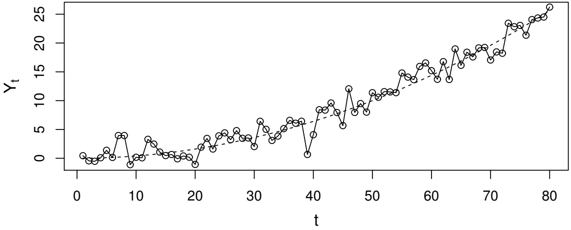


Review: Trends in Time Series (cont'd)

- Linear trend: $\mu_t = \beta_0 + \beta_1 t$

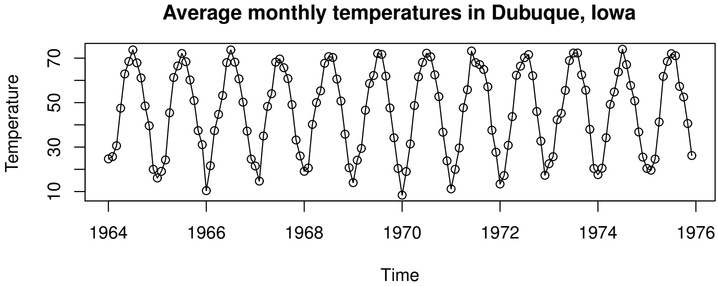


- Quadratic trend: $\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$



Review: Trends in Time Series (cont'd)

- Cyclical/seasonal trend: e.g., $\mu_t = \mu_{t-12}$ for all t



Video 8 Learning Objectives

By the end of this video, we should be able to:

- Recognize a linear trend from a given time series plot
- Obtain estimates of the mean at different times, using a linear trend model
- Identify why R time series output for linear trend models must be interpreted carefully

Linear Trend

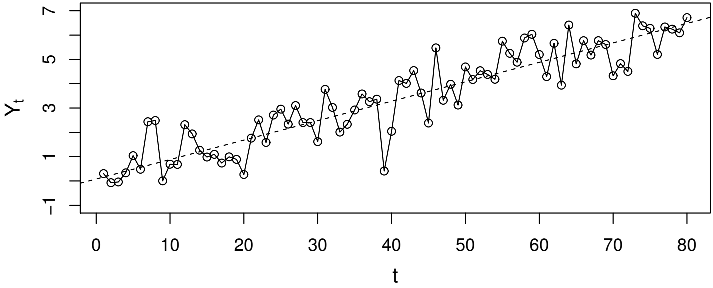
Suppose:

$$Y_t = \mu_t + X_t, \quad \text{where} \quad \mu_t = \beta_0 + \beta_1 t$$

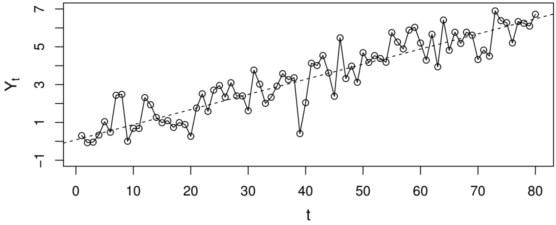
In other words, the mean μ_t is a linear function of time.

The unknown parameters β_0 and β_1 are the intercept and slope of the line.

Example:



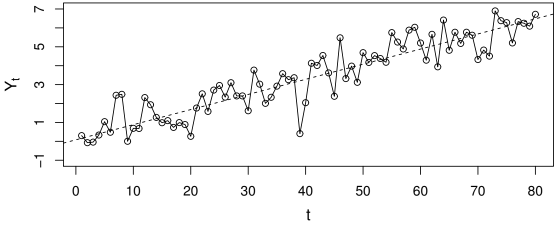
Estimating the Linear Trend



In order to understand the linear trend μ_t, we need to estimate the two unknown parameters: β_0 & β_1.

What method do we use to estimate the intercept and slope of a line? (How do we find the "line of best fit", given a particular dataset?)

Estimating the Linear Trend



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What method do we use to estimate the intercept and slope of a line? (How do we find the "line of best fit", given a particular dataset?)

Linear regression!

We find the "least-squares estimates" of β_0 and β_1. These are the values that minimize

$$\sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$$

Estimating the Linear Trend (cont'd)

$$\sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$$

By taking the partial derivatives of the above sum with respect to the two parameters, and setting them to zero, we obtain:

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(Y_t - \bar{Y})(t - \bar{t})}{\sum_{t=1}^{n}(t - \bar{t})^2}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{t}$$

where

$$\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t = \frac{n+1}{2}.$$

Then, the estimate of the trend at time t is: $\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 t$.

Estimating the Linear Trend (cont'd)

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where

$$\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t = \frac{n+1}{2}.$$

Then, the estimate of the trend at time t is: $\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 t$.

The $\hat{\beta}$-computations will mostly be done by statistical software. Some R examples will be in the Week 4 tutorial and practice session.

Properties of $\hat{\beta}_0$ and $\hat{\beta}_1$

Analytical expressions exist for $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_1)$, but they are quite complicated.

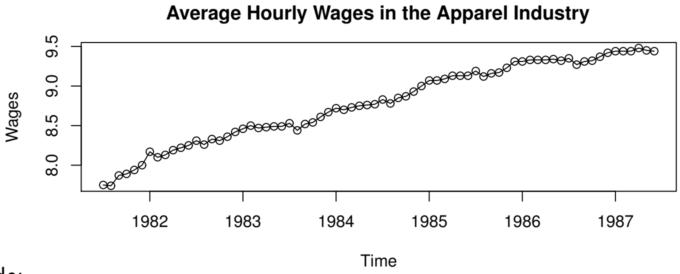
R output usually provides estimates of the standard deviations of the $\hat{\beta}$'s, but relies heavily on some assumptions on $\{X_t\}$ which are usually not true:

- Usually assumes that $\{X_t\}$ is a white noise process (i.e., iid random variables).
- Sometimes even assumes that the X_t's must be approximately Normal.

So the test statistics and p-values shown in R are usually *not reliable*.

Some R examples will be shown in the tutorial.

R Example



R Code:

```
data(wages)
model.lin <- lm(wages~time(wages))
summary(model.lin)
```

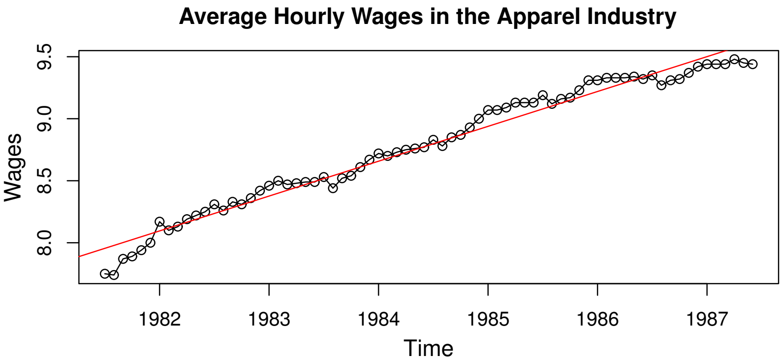
Output:

```
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -5.490e+02  1.115e+01  -49.24   <2e-16 ***
time(wages)  2.811e-01  5.618e-03   50.03   <2e-16 ***
```

R Example (cont'd)

Plotting:

```
plot(wages, type='o', xlab='Time', ylab='Wages',
     main='Average Hourly Wages in the Apparel Industry')
abline(model.lin, col='red')
```



Quadratic Trends, etc.

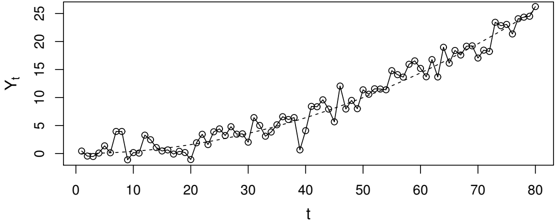
We might also have reason to assume that:

$$Y_t = \mu_t + X_t, \quad \text{where} \quad \mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

or even

$$Y_t = \mu_t + X_t, \quad \text{where} \quad \mu_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

Example:



For these types of models, we can still use linear regression to obtain the $\hat{\beta}$'s.

However, the choice of model for μ_t implies that this trend applies for all t, i.e. forever. So we should have a good reason for choosing such a model!

Final Comments

That's all for now!

In this video, we've learned how to estimate a linear trend using regression methods.

We've also learned about the possibility of incorporating higher-order terms in the model, and the potential risks of doing so.

Coming Up Next: Seasonal trends.

References

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.