Trends:

Cyclical/Seasonal Trends

Week III: Video 9

STAT 485/685, Fall 2020, SFU

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Review: Trends in Time Series

Suppose our process of interest, $\{Y_t\}$, has some mean function μ_t (which may or may not be a function of t).

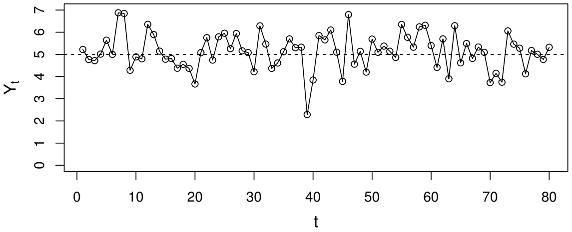
We can separate out the mean from the rest of the process by writing:

$$Y_t = \mu_t + X_t$$

where $\{X_t\}$ is the "de-trended" version of the process, i.e. $E(X_t) = 0$.

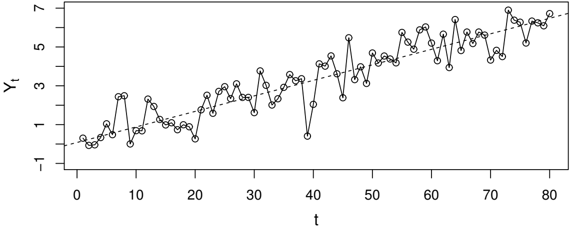
Some examples of how μ_t may look:

- Constant trend: $\mu_t = \mu$ for all t

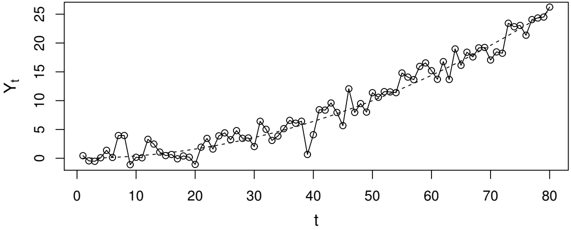


Review: Trends in Time Series (cont'd)

- Linear trend: $\mu_t = \beta_0 + \beta_1 t$

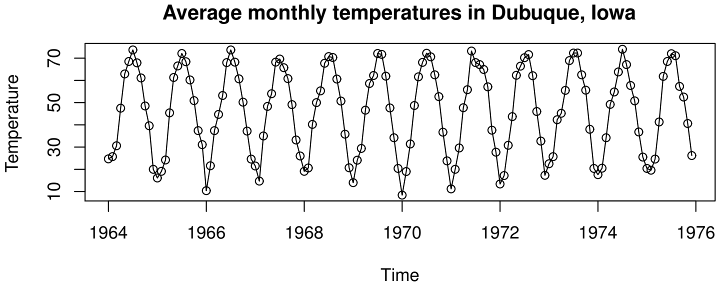


- Quadratic trend: $\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$



Review: Trends in Time Series (cont'd)

- Cyclical/seasonal trend: e.g., $\mu_t = \mu_{t-12}$ for all t



Video 9 Learning Objectives

By the end of this video, we should be able to:

- Recognize a seasonal trend from a given time series plot
- Obtain estimates of the mean at different times, using a *seasonal means model*
- Obtain estimates of the mean at different times, using a *cosine trend model*
- Identify some advantages and disadvantages of each of the two models

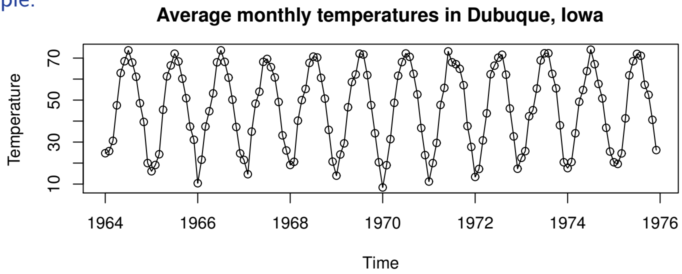
Cyclical or Seasonal Trend

Suppose:

$$Y_t = \mu_t + X_t,$$

where μ_t is some seasonal trend.

Example:



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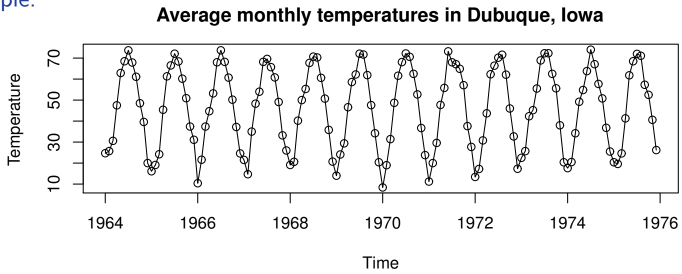
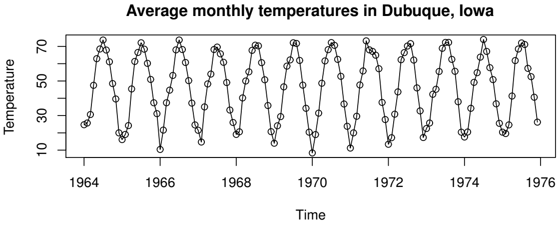


We will consider **two** possible ways of modelling μ_t:

- Seasonal means model
- Cosine trend model

Seasonal Means Model



We know that our data is seasonal by month, so we propose the model:

$$\mu_t = \begin{cases} \beta_1 & \text{for } t = 1, 13, 25, \ldots \\ \beta_2 & \text{for } t = 2, 14, 26, \ldots \\ \vdots & \\ \beta_{12} & \text{for } t = 12, 24, 36, \ldots \end{cases}$$

So, the parameter β_1 is the mean for January, β_2 is the mean for February, etc.

Seasonal Means Model: Estimating Parameters

$$\mu_t = \begin{cases} \beta_1 & \text{for } t = 1, 13, 25, \ldots \\ \beta_2 & \text{for } t = 2, 14, 26, \ldots \\ \vdots & \\ \beta_{12} & \text{for } t = 12, 24, 36, \ldots \end{cases}$$

Each parameter β_j is the mean for the j^{th} month (for $j = 1, 2, \ldots, 12$).

Each β_j can be estimated by taking the sample mean of the observations for that month:

$$\hat{\beta}_j = \frac{1}{N} \sum_{i=0}^{N-1} Y_{j+12i},$$

where N is the number of (complete) years of data.

Then, the estimate of the mean at month j is: $\hat{\mu}_j = \hat{\beta}_j$ (for $j = 1, 2, \ldots, 12$).

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Note: This $\hat{\beta}_j$ is similar to our estimate of the constant mean, $\hat{\mu} = \bar{Y}$, except we are only using every 12^{th} time point.

These computations will mostly be done by statistical software.

Seasonal Means Model: Properties of $\hat{\beta}_j$

Each β_j is estimated by taking the sample mean of the observations for that month.

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This is similar to our estimate of the constant mean, $\hat{\mu} = \bar{Y}$, except we are only using every 12^{th} time point.

If we assume that $\{X_t\}$ is a stationary process with autocorrelation function ρ_k:

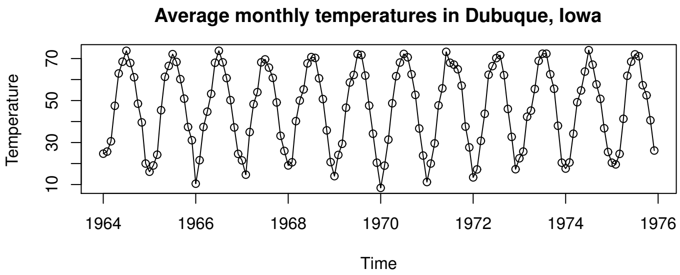
$$Var(\hat{\beta}_j) = \frac{\gamma_0}{N}\left[1 + 2\sum_{k=1}^{N-1}\left(1 - \frac{k}{N}\right)\rho_{12k}\right],$$

where N is the number of years of data.

Some special cases:

- If $\{X_t\}$ is a white noise process: $Var(\hat{\beta}_j) = \gamma_0/N$.
- Even if some of the ρ's are non-zero, as long as $\rho_{12k} = 0$ (i.e., observations between the same month are independent): $Var(\hat{\beta}_j) = \gamma_0/N$.

R Example



R Code:

```
data(tempdub)
month. <- season(tempdub)
model.seasonal <- lm(tempdub~month.-1)
summary(model.seasonal)
```

R Example(cont'd)

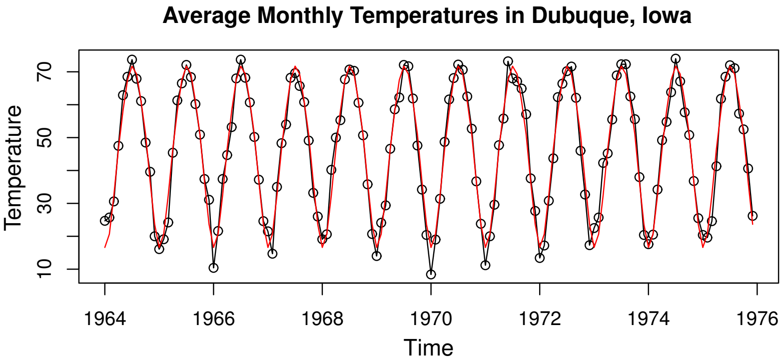
Output:

```
                Estimate Std. Error t value Pr(>|t|)    
month.January     16.608      0.987   16.83   <2e-16 ***
month.February    20.650      0.987   20.92   <2e-16 ***
month.March       32.475      0.987   32.90   <2e-16 ***
month.April       46.525      0.987   47.14   <2e-16 ***
month.May         58.092      0.987   58.86   <2e-16 ***
month.June        67.500      0.987   68.39   <2e-16 ***
month.July        71.717      0.987   72.66   <2e-16 ***
month.August      69.333      0.987   70.25   <2e-16 ***
month.September   61.025      0.987   61.83   <2e-16 ***
month.October     50.975      0.987   51.65   <2e-16 ***
month.November    36.650      0.987   37.13   <2e-16 ***
month.December    23.642      0.987   23.95   <2e-16 ***
```

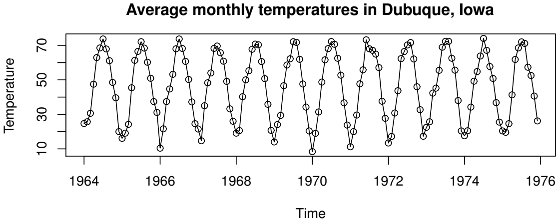
R Example (cont'd)

Plotting:

```
plot(tempdub, type='o', xlab='Time', ylab='Temperature',
     main='Average Monthly Temperatures in Dubuque, Iowa')
fitted.vals <- ts(fitted(model.seasonal), freq=12, start=c(1964,1))
lines(fitted.vals, col='red')
```



Cosine Trend Model



Some drawbacks of the seasonal means model:

- Requires the estimation of 12 parameters.
- Parameters are considered to be independent, and therefore do not take the shape of the trend into account.

A **cosine trend model** can incorporate the smooth change expected from one time point to the next.

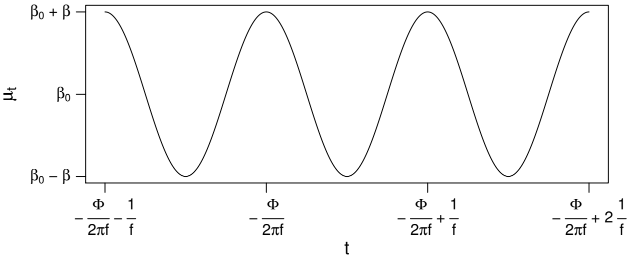
Cosine Trend Model (cont'd)

Suppose:

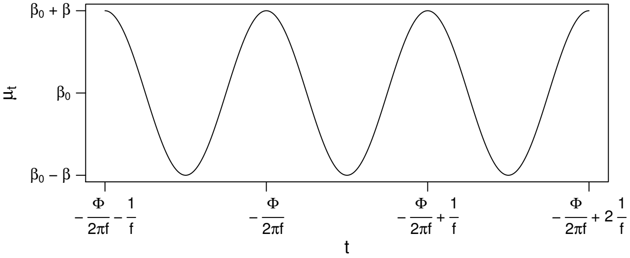
$$\mu_t = \beta\, cos\left[2\pi f\left(t + \frac{\Phi}{2\pi f}\right)\right] + \beta_0.$$

This model contains the parameters:

- β (> 0): *amplitude* of the curve
- f: *frequency* of the curve
- Φ: *phase* of the curve
- β_0: vertical shift of the curve



Cosine Trend Model (cont'd)



What we know:

- The cycle repeats every $1/f$ time units. So, for monthly seasonal data we set $f = 1/12$.
- Oscillates between a maximum of $\beta_0 + \beta$ and a minimum of $\beta_0 - \beta$.
- Φ serves to set the time at which the curve starts.

Cosine Trend Model: Estimating Parameters

The model can be re-written as:

$$\mu_t = \beta_0 + \beta_1\, cos(2\pi ft) + \beta_2\, sin(2\pi ft)$$

We have only three unknown parameters: β_0, β_1 and β_2.

The parameters can be estimated using a linear regression, where instead of t our predictors are $cos(2\pi ft)$ and $sin(2\pi ft)$. These computations will mostly be done by statistical software.

Then, the estimate of the mean at time t is:

$\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1\, cos(2\pi ft) + \hat{\beta}_2\, sin(2\pi ft)$.

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Then, the estimate of the mean at time t is:
$\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 \, cos(2\pi ft) + \hat{\beta}_2 \, sin(2\pi ft)$.

Caution: We must be very careful when defining f for monthly seasonality. If our data records the time points as $t = 1$ for January, $t = 2$ for February, etc., then the monthly pattern repeats itself every 12 time points, and we should be using $f = 1/12$.
If, however, our data records the time points as each time point being one *year*, then the monthly pattern repeats itself every 1 time point, and we should be using $f = 1$.

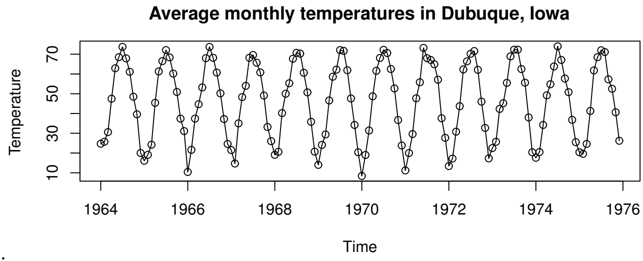
Cosine Trend Model: Properties of $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$

The estimates of β_0, β_1 and β_2 are obtained using linear regression, where instead of t our predictors are $cos(2\pi ft)$ and $sin(2\pi ft)$.

Analytical expressions exist for $Var(\hat{\beta}_0)$, $Var(\hat{\beta}_1)$ and $Var(\hat{\beta}_2)$, but they are quite complicated.

If $\{X_t\}$ is a white noise process: $Var(\hat{\beta}_1) = Var(\hat{\beta}_2) = 2\gamma_0/n$.

R Example



R Code:

```
data(tempdub)
har. <- harmonic(tempdub, 1)
model.seasonal2 <- lm(tempdub~har.)
summary(model.seasonal2)
```

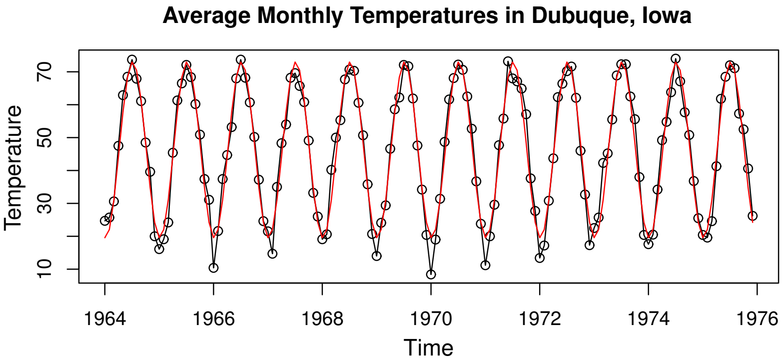
Output:

```
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       46.2660     0.3088 149.816  < 2e-16 ***
har.cos(2*pi*t) -26.7079     0.4367 -61.154  < 2e-16 ***
har.sin(2*pi*t)  -2.1697     0.4367  -4.968 1.93e-06 ***
```

R Example (cont'd)

Plotting:

```
plot(tempdub, type='o', xlab='Time', ylab='Temperature',
     main='Average Monthly Temperatures in Dubuque, Iowa')
fitted.vals2 <- ts(fitted(model.seasonal2), freq=12, start=c(1964,1))
lines(fitted.vals2, col='red')
```



Seasonal Means Model vs. Cosine Trend Model

Both the seasonal means model and cosine trend model are fairly easy to fit using R. They are both fitted using least-squares estimates.

The cosine trend model:

- Requires the estimation of fewer parameters.
- Takes the shape of the trend into account.
- However: Has a strong model assumption.

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Assume $\{X_t\}$ is a white noise process. Let's compare the estimates of the mean in January:

- Seasonal means model: $\hat{\mu}_1 = \hat{\beta}_1$. Then: $Var(\hat{\mu}_1) = Var(\hat{\beta}_1) = \gamma_0/N$.
- Cosine trend model: $\hat{\mu}_1 = \hat{\beta}_0 + \hat{\beta}_1 \cos\left(\frac{2\pi}{12}\right) + \hat{\beta}_2 \sin\left(\frac{2\pi}{12}\right)$. Then: $Var(\hat{\mu}_1) = \cdots = 3\gamma_0/n$.

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- Since generally $N \approx n/12$:

$$\frac{\text{Variance of } \hat{\mu}_1 \text{ for seasonal means model}}{\text{Variance of } \hat{\mu}_1 \text{ for cosine trend model}} = \frac{\gamma_0/(n/12)}{3\gamma_0/n} = 4$$

Final Comments

That's all for now!

In this video, we've learned about two different approaches to modelling a seasonal trend: the **seasonal means model** and the **cosine trend model.**

We've also learned about some of the properties of the estimates arising from each of these models.

Coming Up Next: Residual analysis, and determining whether a chosen model matches the data well.

References

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.