Tutorial 5 - STAT 485/685

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Today's Plan

1. Recap of Tutorial 4
   - Chapter 3 Exercises
   - Validity of `lm()` Output
2. Models for Stationary Time Series
   - General Linear Processes
   - Moving Average Processes
   - Autoregressie Processes
   - Mixed Autoregressive and Moving Average Processes
   - Invertibility
3. Examples
   - Question 4.11
   - Question 4.21

For the time series $\{Y_t : t \in \mathcal{I}\}$, write each term as

$$Y_t = \mu_t + X_t,$$

where $E(Y_t) = \mu_t \Rightarrow E(X_t) = 0$.

Goal: Model and estimate μ_t.

- With the `wages` dataset, we fit the following regression models:
  - $\mu_t = \beta_0 + \beta_1 t$
  - $\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$

  Here, $t \in \mathcal{I} = \{1981.5, 1981.583, \cdots, 1986.917\}$.
  - ...residuals were not a random scatter around zero.
  - ...residuals were not a random scatter around zero.
    ...failed the runs test.
    ...the ACF showed that the residuals do not behave like white noise.

- With the `beersales` dataset, we fit the following regression models:
  - $\mu_t = \beta_1 I_{January} + \beta_2 I_{February} + \cdots + \beta_{12} I_{December}$
  - $\mu_t = \beta_1 I_{January} + \beta_2 I_{February} + \cdots + \beta_{12} I_{December} + \alpha_1 t + \alpha_2 t^2$

  Here, $t \in \mathcal{I} = \{1975.0, 1975.083, \cdots, 1990.917\}$.

  Results are similar to the `wages` dataset...

Recap of Tutorial 4

Validity of `lm()` Output

We simulate a random walk $\{Y_t : t \in \mathcal{I}\}$, with $\mathcal{I} = \{1, \cdots, n\}$. Recall that

$$Y_t = Y_{t-1} + e_t,$$

with $Y_0 = 0$, and $\{e_t : t \in \mathcal{I}\}$ is a (normal) white noise process. We fit the model

$$\begin{aligned} Y_t &= \mu_t + e_t \\ &= \beta_0 + \beta_1 t + e_t \end{aligned}$$

$\Rightarrow \beta_0 = \beta_1 = 0.$

If we use `lm()` in R, then

(1) the reported estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are *valid* ✓

- The estimates are *consistent* and centred around 0.

(2) the reported standard errors of $\hat{\beta}_0$ and $\hat{\beta}_1$ are *invalid* X

- The standard error estimates *underestimated* the simulated standard error.

Definition: $\{Y_t : t \in \mathcal{I}\}$ is a *general linear process* if it can be written as a (weighted) linear combination of present and past white noise terms:

$$\begin{aligned} Y_t &= e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \cdots \\ &= \sum_{j=0}^{\infty} \psi_j e_{t-j}, \end{aligned}$$

where $\psi_0 \equiv 1$.

- Also referred to as ψ-weight representation of $\{Y_t : t \in \mathcal{I}\}$.

In order for the infinite sum above to be *convergent*, we assume that

$$\sum_{j=1}^{\infty} \psi_j^2 < \infty.$$

As before, let's obtain $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$.

$$E(Y_t) = E\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}\right) = \sum_{j=0}^{\infty} \psi_j \underbrace{E(e_{t-j})}_{0} = 0.$$

$$Var(Y_t) = Var\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}\right) = \sum_{j=0}^{\infty} \psi_j^2 \underbrace{Var(e_{t-j})}_{\sigma_e^2} = \sigma_e^2 \underbrace{\sum_{j=0}^{\infty} \psi_j^2}_{<\infty} < \infty.$$

For $k \geq 0$:

$$\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}, \sum_{\ell=0}^{\infty} \psi_\ell e_{(t-k)-\ell}\right) \\
&= \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \psi_j \psi_\ell Cov(e_{t-j}, e_{t-(k+\ell)}) \\
&= \sum_{\ell=0}^{\infty} \psi_{k+\ell} \psi_\ell \underbrace{Cov(e_{t-(k+\ell)}, e_{t-(k+\ell)})}_{\sigma_e^2} + \sum_{\substack{j=0 \\ j \neq k+\ell}}^{\infty} \sum_{\ell=0}^{\infty} \psi_j \psi_\ell \underbrace{Cov(e_{t-j}, e_{t-(k+\ell)})}_{0} \\
&= \sigma_e^2 \sum_{\ell=0}^{\infty} \psi_{k+\ell} \psi_\ell.
\end{aligned}$$

For $k \geq 0$:

$$\begin{aligned}
Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-k})}} \\
&= \frac{\sigma_e^2 \sum\limits_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell}{\sqrt{\sigma_e^2 \sum\limits_{j=0}^{\infty} \psi_j^2} \times \sqrt{\sigma_e^2 \sum\limits_{j=0}^{\infty} \psi_j^2}} \\
&= \frac{\sum\limits_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell}{\sum\limits_{j=0}^{\infty} \psi_j^2}.
\end{aligned}$$

Remark: $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ do not depend on time t

$\Rightarrow \{Y_t : t \in \mathcal{I}\}$ is stationary.

- **Example 1** (see page 55 of the textbook): Suppose $\psi_j = \phi^j$, where $-1 < \phi < 1$. Then

$$Y_t = \sum_{j=0}^{\infty} \phi^j e_{t-j}.$$

What is $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$?

- **Example 1** (see page 55 of the textbook): Suppose $\psi_j = \phi^j$, where $-1 < \phi < 1$. Then

$$Y_t = \sum_{j=0}^{\infty} \phi^j e_{t-j}.$$

What is $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$?

$$E(Y_t) = 0.$$

$$Var(Y_t) = \sigma_e^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma_e^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma_e^2}{1-\phi^2}.$$

$$Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell = \sigma_e^2 \sum_{\ell=0}^{\infty} \phi^{k+\ell}\phi^\ell = \sigma_e^2 \phi^k \sum_{\ell=0}^{\infty} \phi^{2\ell} = \frac{\sigma_e^2 \phi^k}{1-\phi^2}.$$

$$Corr(Y_t, Y_{t-k}) = \frac{\sum_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell}{\sum_{j=0}^{\infty} \psi_j^2} = \frac{\sum_{\ell=0}^{\infty} \phi^{k+\ell}\phi^\ell}{\sum_{j=0}^{\infty} \phi^{2j}} = \frac{\phi^k \sum_{\ell=0}^{\infty} \phi^{2\ell}}{\sum_{j=0}^{\infty} \phi^{2j}} = \underbrace{\phi^k}_{\in(-1,1)}.$$

Note: For a geometric series with $-1 < r < 1$,

$$\sum_{\ell=0}^{\infty} ar^\ell = \frac{a}{1-r}. \quad (r = \phi^2, a = \sigma_e^2 \text{ or } a = \sigma_e^2\phi^k)$$

- **Example 2**: Suppose

$$\psi_j = \begin{cases} 1 & \text{if } j = 0, 1, \cdots, t-1 \\ 0 & \text{if } j \geq t \end{cases}.$$

Then

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \sum_{j=0}^{t-1} \psi_j e_{t-j} + \sum_{j=t}^{\infty} \psi_j e_{t-j} = \sum_{j=0}^{t-1} e_{t-j} = \sum_{u=1}^{t} e_u.$$

We see that $\{Y_t : t \in \mathcal{I}\}$ corresponds to the *random walk*!

Note: The random walk doesn't need to be written as an infinite sum...

- ...can express Y_t as a finite sum of white noise terms.

Are there other processes like this?

- **Definition**: $\{Y_t : t \in \mathcal{I}\}$ is a moving average of order q if q of the ψ_j's of the general linear process are non-zero

$$
\begin{aligned}
Y_t &= \sum_{j=0}^{\infty} \psi_j e_{t-j} \\
&= \sum_{j=0}^{q} \psi_j e_{t-j}, \quad \text{with } \psi_0 = 1, \\
&= e_t + \psi_1 e_{t-1} + \cdots + \psi_q e_{t-q}.
\end{aligned}
$$

If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $MA(q)$ process.

People often express $MA(q)$ processes by slightly changing the notation $\Rightarrow$ letting $\theta_j = -\psi_j$:

$$
\begin{aligned}
Y_t &= -\sum_{j=0}^{q} \theta_j e_{t-j}, \quad \text{with } \theta_0 = -1, \\
&= e_t - \theta_1 e_{t-1} - \cdots - \theta_1 e_{t-q}.
\end{aligned}
$$

What is $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k \geq 0$?

By expressing the $MA(q)$ process in terms of the general linear process (from earlier)

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = -\sum_{j=0}^{q} \theta_j e_{t-j},$$

where

$$\psi_\ell = \begin{cases} -\theta_\ell & \text{if } \ell \leq q \\ 0 & \text{if } \ell > q \end{cases}.$$

$\Rightarrow$ We can use results from general linear processes to help us!

$$E(Y_t) = E\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}\right) = E\left(-\sum_{j=0}^{q} \theta_j e_{t-j}\right) = -\sum_{j=0}^{q} \theta_j \underbrace{E(e_{t-j})}_{0} = 0.$$

$$\begin{aligned} Var(Y_t) &= \sigma_e^2 \sum_{j=0}^{\infty} \psi_j^2 \qquad \text{(from earlier)} \\ &= \sigma_e^2 \sum_{j=0}^{q} (-\theta_j)^2 \\ &= \sigma_e^2 \sum_{j=0}^{q} \theta_j^2. \end{aligned}$$

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= \sigma_e^2 \sum_{\ell=0}^{\infty} \psi_{k+\ell}\psi_\ell \qquad \text{(from earlier)} \\ &= \sigma_e^2 \sum_{\ell=0}^{q-k} \psi_{k+\ell}\psi_\ell \end{aligned}$$

We consider cases:

- **Case 1**: $q - k < 0 \Rightarrow k > q \Rightarrow Cov(Y_t, Y_{t-k}) = 0$
- **Case 2**: $q - k \geq 0 \Rightarrow k \leq q$

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= \sigma_e^2 \sum_{\ell=0}^{q-k} (-\theta_{k+\ell})(-\theta_\ell) \\ &= \sigma_e^2 \sum_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell \end{aligned}$$

Therefore,

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2 \sum_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell & \text{if } k = 0, 1, \cdots, q \\ 0 & \text{if } k > q \end{cases}.$$

$$Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-k})}}.$$

We consider cases:

- **Case 1**: $q - k < 0 \Rightarrow k > q \Rightarrow Corr(Y_t, Y_{t-k}) = 0$
- **Case 2**: $q - k \geq 0 \Rightarrow k \leq q$

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{\sigma_e^2 \sum\limits_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell}{\sqrt{\sigma_e^2 \sum\limits_{j=0}^{q} \theta_j^2} \times \sqrt{\sigma_e^2 \sum\limits_{j=0}^{q} \theta_j^2}} \\ &= \frac{\sum\limits_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell}{\sum\limits_{j=0}^{q} \theta_j^2}. \end{aligned}$$

Therefore,

$$Corr(Y_t, Y_{t-k}) = \begin{cases} \dfrac{\sum\limits_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell}{\sum\limits_{j=0}^{q} \theta_j^2} & \text{if } k = 0, 1, \cdots, q \\ 0 & \text{if } k > q \end{cases}.$$

Remark 1: Note that the ACF *cuts off* after lag q:

$$Corr(Y_t, Y_{t-k}) = \begin{cases} \dfrac{\sum\limits_{\ell=0}^{q-k} \theta_{k+\ell}\theta_\ell}{\sum\limits_{j=0}^{q} \theta_j^2} & \text{if } k = 0, 1, \cdots, q \\ 0 & \text{if } k > q \end{cases}.$$

Remark 2: $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ do not depend on time t

$\Rightarrow \{Y_t : t \in \mathcal{I}\}$ is stationary.

- **Example 2 (Continued)**: Recall the random walk process $\{Y_t : t \in \mathcal{I}\}$:

$$\begin{aligned} Y_t &= \sum_{j=0}^{t-1} e_{t-j} \\ &= \sum_{j=0}^{t-1} \psi_j e_{t-j}, \end{aligned}$$

where $\psi_j = 1$ for all $j \in \{0, 1, \cdots, t-1\}$.

$\Rightarrow$ This is "essentially" an $MA(t-1)$ process

- ...but be careful: q should not depend on t!
  (Since the time series won't be stationary if it does!)

Models for Stationary Time Series

$MA(1)$ Process

- **Special Case 1**: What if $q = 1$?

$$Y_t = -\sum_{j=0}^{1} \theta_j e_{t-j} = e_t - \theta_1 e_{t-1}.$$

We can then derive $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k > 0$:

$$E(Y_t) = 0$$

$$Var(Y_t) = \sigma_e^2 \sum_{j=0}^{1} \theta_j^2 = \sigma_e^2 (1 + \theta_1^2).$$

$$Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{\ell=0}^{1-k} \theta_{k+\ell} \theta_\ell$$

- for $k = 0 : Cov(Y_t, Y_t) = Var(Y_t)$.
- for $k = 1 : Cov(Y_t, Y_{t-1}) = \sigma_e^2 \sum_{\ell=0}^{0} \theta_{1+\ell} \theta_\ell = \sigma_e^2 (\theta_1)(-1) = -\theta_1 \sigma_e^2$.
- for $k > 1 : Cov(Y_t, Y_{t-k}) = 0$.

Therefore,

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2(1+\theta_1^2) & \text{if } k = 0 \\ -\theta_1\sigma_e^2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} .$$

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-k})}} \\ &= \frac{Cov(Y_t, Y_{t-k})}{\sigma_e^2(1+\theta_1^2)} \\ &= \begin{cases} 1 & \text{if } k = 0 \\ \frac{-\theta_1}{(1+\theta_1^2)} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Note that $Corr(Y_t, Y_{t-k})$ *cuts off* after lag 1.

- **Special Case 2**: What if $q = 2$?

$$Y_t = -\sum_{j=0}^{2} \theta_j e_{t-j} = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

We can then derive $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$, for $k > 0$:

$$E(Y_t) = 0$$

$$Var(Y_t) = \sigma_e^2 \sum_{j=0}^{2} \theta_j^2 = \sigma_e^2 (1 + \theta_1^2 + \theta_2^2).$$

$$Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{\ell=0}^{2-k} \theta_{k+\ell}\theta_\ell$$

- for $k = 0 : Cov(Y_t, Y_t) = Var(Y_t)$.
- for $k = 1 : Cov(Y_t, Y_{t-1}) = \sigma_e^2 \sum_{\ell=0}^{1} \theta_{1+\ell}\theta_\ell = \sigma_e^2(-\theta_1 + \theta_2\theta_1)$.
- for $k = 2 : Cov(Y_t, Y_{t-2}) = \sigma_e^2 \sum_{\ell=0}^{0} \theta_{2+\ell}\theta_\ell = -\theta_2\sigma_e^2$.
- for $k > 2 : Cov(Y_t, Y_{t-k}) = 0$.

Therefore,

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2(1+\theta_1^2+\theta_2^2) & \text{if } k=0 \\ \sigma_e^2(-\theta_1+\theta_2\theta_1) & \text{if } k=1 \\ -\theta_2\sigma_e^2 & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases}.$$

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-k})}} \\ &= \frac{Cov(Y_t, Y_{t-k})}{\sigma_e^2(1+\theta_1^2+\theta_2^2)} \\ &= \begin{cases} 1 & \text{if } k=0 \\ \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k=1 \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Note that $Corr(Y_t, Y_{t-k})$ *cuts off* after lag 2.

- **Definition**: $\{Y_t : t \in \mathcal{I}\}$ is an autoregressive process of order p if the time series $\{Y_t : t \in \mathcal{I}\}$ satisfies the following equation

$$
\begin{aligned}
Y_t &= \sum_{j=1}^{p} \phi_j Y_{t-j} + e_t \\
&= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t .
\end{aligned}
$$

If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $AR(p)$ process.

Rather than writing an $AR(p)$ process in terms of Y_t, we can rearrange for e_t:

$$
\begin{aligned}
e_t &= Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} \\
&= (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) Y_t \\
&= \phi(B) Y_t ,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(B) &= 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p , \\
B^k Y_t &= Y_{t-k} .
\end{aligned}
$$

$\Rightarrow$ B is the *backshift operator*
$\Rightarrow$ $\phi(B)$ is the *autoregressive process characteristic polynomial.*

- **Result**: An $AR(p)$ process is stationary if the roots of $\phi(B)$ lie outside the "unit circle" in $\mathbb{R}^p$-space.

- **Special Case 1**: What if $p = 1$?

$$Y_t = \phi_1 Y_{t-1} + e_t.$$

The characteristic polynomial is

$$\begin{aligned} e_t &= \phi(B) Y_t \\ \phi(B) &= 1 - \phi_1 B \end{aligned}$$

$\Rightarrow$ Setting $\phi(B) = 0$,
$\Rightarrow$ $B = \frac{1}{\phi_1}$ is the root of the characteristic polynomial.
$\Rightarrow$ This falls outside of the unit circle if and only if $\frac{1}{|\phi_1|} > 1 \Rightarrow |\phi_1| < 1$.

General Linear Process Representation: Note that if we rearrange for Y_t:

$$\begin{aligned} Y_t &= \frac{e_t}{1 - \phi_1 B} \\ &= e_t(1 + \phi_1 B + \phi_1^2 B^2 + \cdots) \\ &= \sum_{j=0}^{\infty} \psi_j e_{t-j}, \end{aligned}$$

where $\psi_0 \equiv 1$, and $\psi_j = \phi_1^j$ for $j \geq 1$.

$\Rightarrow$ See **Example 1** for $E(Y_t)$, $Var(Y_t)$, $Cov(Y_t, Y_{t-k})$, and $Corr(Y_t, Y_{t-k})$ for $k \geq 0$.

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- **Special Case 2**: What if $p = 2$?

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

The characteristic polynomial is

$$e_t = \phi(B) Y_t$$
$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

$\Rightarrow$ Setting $\phi(B) = 0 \cdots$
$\Rightarrow \cdots$ roots are $B = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$ are the (complex) roots of the characteristic polynomial.
$\Rightarrow \cdots$ Turns out we need

$$\phi_1 + \phi_2 < 1$$
$$\phi_2 - \phi_1 < 1$$
$$|\phi_2| < 1$$

$\Rightarrow$ see Appendix B, page 84.

Can proceed to write Y_t as a general linear process...easier approach is to write down the **Yule-Walker equations**:

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \quad \text{for } k = 1, 2, 3, \cdots$$
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad \text{for } k = 1, 2, 3, \cdots$$

Let's derive these!

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

Start by deriving $E(Y_t) = \mu_t \equiv \mu$:

$$\begin{aligned}
E(Y_t) &= E(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t) \\
&= \phi_1 E(Y_{t-1}) + \phi_2 E(Y_{t-2}) + 0 \\
\therefore \mu &= \phi_1 \mu + \phi_2 \mu \qquad \text{(Assuming!)} \\
\Rightarrow \mu(1 - \phi_1 - \phi_2) &= 0.
\end{aligned}$$

Note that one of the stationarity conditions is $\phi_1 + \phi_2 < 1 \Rightarrow 1 - \phi_1 - \phi_2 > 0.$
$\Rightarrow \mu = 0.$

Then find $Var(Y_t)$:

$$\begin{aligned}
Var(Y_t) &= Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t) \\
&= Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2}) + Var(e_t), \quad \text{since } e_t \perp\!\!\!\perp Y_{t-1}, Y_{t-2}. \\
&= [\phi_1^2 \underbrace{Var(Y_{t-1})}_{\gamma_0} + \phi_2^2 \underbrace{Var(Y_{t-2})}_{\gamma_0} + 2\phi_1\phi_2 \underbrace{Cov(Y_{t-1}, Y_{t-2})}_{\gamma_1}] + \sigma_e^2 \\
\therefore \gamma_0 &= \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + 2\phi_1\phi_2 \underbrace{\gamma_1}_{(*)} + \sigma_e^2
\end{aligned}$$

$(*)$: What is γ_1?

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

For some fixed $k \geq 0$, the Yule-Walker equations are obtained as

$$Y_t Y_{t-k} = \phi_1 Y_{t-1} Y_{t-k} + \phi_2 Y_{t-2} Y_{t-k} + Y_{t-k} e_t$$
$$\underbrace{E(Y_t Y_{t-k})}_{Cov(Y_t, Y_{t-k})} = \phi_1 \underbrace{E(Y_{t-1} Y_{t-k})}_{Cov(Y_{t-1}, Y_{t-k})} + \phi_2 \underbrace{E(Y_{t-2} Y_{t-k})}_{Cov(Y_{t-2}, Y_{t-k})} + \underbrace{E(Y_{t-k} e_t)}_{Cov(Y_{t-k}, e_t)=0}$$

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}.$$

$\Rightarrow$ Setting $k = 1$:

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$$
$$\Rightarrow \gamma_1 = \frac{\phi_1 \gamma_0}{1 - \phi_2},$$

since $\gamma_{-1} = \gamma_1$.

By inserting γ_1 in the equation in the previous slide,

$$\gamma_0 = \frac{(1 - \phi_2)\sigma_e^2}{(1 - \phi_2)(1 - \phi_1^2 - \phi_2^2) - 2\phi_2 \phi_1^2}.$$

$\Rightarrow$ Use the Yule-Walker equations to obtain $\gamma_1, \gamma_2, \cdots$.

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

As for the autocorrelation function:

$$\begin{aligned}\rho_k &= \frac{\gamma_k}{\gamma_0} \\ &= \frac{\phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}}{\gamma_0} \\ &= \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}\end{aligned}$$

$\Rightarrow$ Setting $k = 1$:

$$\begin{aligned}\rho_1 &= \phi_1 \underbrace{\rho_0}_{1} + \phi_2 \rho_1 \\ \Rightarrow \rho_1 &= \frac{\phi_1}{1 - \phi_2},\end{aligned}$$

since $\rho_{-1} = \rho_1$.

$\Rightarrow$ Use the Yule-Walker equations to obtain $\rho_1, \rho_2, \cdots$.

- **Remark**: We (initially) wrote $E(Y_t) = \mu$ and $Cov(Y_t, Y_{t-k}) = \gamma_k$. That is, we *require* $\{Y_t : t \in \mathcal{I}\}$ to be a stationary process.

- For an $AR(p)$ process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

The characteristic polynomial is

$$\begin{aligned} e_t &= \phi(B) Y_t \\ \phi(B) &= 1 - \phi_1 B - \phi_2 B^2 - \phi_p B^p. \end{aligned}$$

$\Rightarrow$ Setting $\phi(B) = 0 \cdots$
$\Rightarrow \cdots$ Turns out we need

$$\begin{aligned} \phi_1 + \phi_2 + \cdots + \phi_p &< 1, \\ |\phi_p| &< 1 \end{aligned}$$

To obtain γ_k and ρ_k, solve the Yule-Walker equations

$$\begin{aligned} \rho_1 &= \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1} \\ \rho_2 &= \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2} \\ &\vdots \\ \rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \cdots + \phi_p. \end{aligned}$$

- **Definition**: $\{Y_t : t \in \mathcal{I}\}$ is a mixed autoregressive moving average process of orders p and q, respectively, if the time series $\{Y_t : t \in \mathcal{I}\}$ satisfies the following equation

$$
\begin{aligned}
Y_t &= [\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t] + [-\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}] \\
&= \sum_{j=1}^{p} \phi_j Y_{t-j} - \sum_{j=0}^{q} \theta_j e_{t-j},
\end{aligned}
$$

with $\theta_0 \equiv -1$. If so, we say that $\{Y_t : t \in \mathcal{I}\}$ is an $ARMA(p, q)$ process.

The $ARMA(p, q)$ characteristic polynomial is

$$
\begin{aligned}
&\theta(B) e_t = \phi(B) Y_t, \\
&\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q \quad (MA(q) \text{ characteristic polynomial}) \\
&\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \quad (AR(p) \text{ characteristic polynomial})
\end{aligned}
$$

It is assumed that $\theta(B)$ and $\phi(B)$ have no common factors.

- **Result**: An $ARMA(p, q)$ process is stationary if the roots of $\phi(B)$ lie outside the “unit circle” in $\mathbb{R}^p$-space.

$\Rightarrow$ Proceed to show that $E(Y_t) = 0$, and solve the Yule-Walker equations to derive γ_k and ρ_k (see Appendix C, page 85).

With a general linear process:

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j},$$

this allowed us to write an autoregressive process as a moving average process.

Can we “invert” this relationship, and write a moving average process as an autoregressive process

$$e_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}, \tag{1}$$

where $\pi_0 \equiv 1$ and for some π_j for $j \geq 1$?

- **Result**: An $MA(q)$ process can be written as (1) if the roots of $\theta(B)$ lie outside the “unit circle” in $\mathbb{R}^q$-space.
- **Result**: An $ARMA(p, q)$ process can be written as (1) if the roots of $\theta(B)$ lie outside the “unit circle” in $\mathbb{R}^q$-space.

Note: This is referred to as π-weight representation of $\{e_t : t \in \mathcal{I}\}$.

For the $ARMA(1,2)$ model $Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$, show that

(a) $\rho_k = 0.8\rho_{k-1}$ for $k > 2$.

(b) $\rho_2 = 0.8\rho_1 + 0.6\frac{\sigma_e^2}{\gamma_0}$.

Examples

Question 4.11

For the $ARMA(1,2)$ model $Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$, show that

(a) $\rho_k = 0.8\rho_{k-1}$ for $k > 2$.

(b) $\rho_2 = 0.8\rho_1 + 0.6\frac{\sigma_e^2}{\gamma_0}$.

First, we show that $E(Y_t) = 0$:

$$\begin{aligned}
E(Y_t) &= E(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}) \\
&= 0.8E(Y_{t-1}) \\
\mu &= 0.8\mu \\
\therefore \mu &= 0.
\end{aligned}$$

Now obtain the Yule-Walker equations:

$$\begin{aligned}
Y_t &= 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2} \\
Y_tY_{t-k} &= 0.8Y_{t-1}Y_{t-k} + Y_{t-k}e_t + 0.7Y_{t-k}e_{t-1} + 0.6Y_{t-k}e_{t-2} \\
\underbrace{E(Y_tY_{t-k})}_{Cov(Y_t,Y_{t-k})} &= 0.8\underbrace{E(Y_{t-1}Y_{t-k})}_{Cov(Y_{t-1},Y_{t-k})} + \underbrace{E(Y_{t-k}e_t)}_{Cov(Y_{t-k},e_t)} + 0.7\underbrace{E(Y_{t-k}e_{t-1})}_{Cov(Y_{t-k},e_{t-1})} + 0.6\underbrace{E(Y_{t-k}e_{t-2})}_{Cov(Y_{t-k},e_{t-2})} \\
\gamma_k &= 0.8\gamma_{k-1} + Cov(Y_{t-k}, e_t) + 0.7Cov(Y_{t-k}, e_{t-1}) + 0.6Cov(Y_{t-k}, e_{t-2}).
\end{aligned}$$

Examples

Question 4.11

$$\gamma_k = 0.8\gamma_{k-1} + \underbrace{Cov(Y_{t-k}, e_t)}_{\text{(see below)}} + 0.7\underbrace{Cov(Y_{t-k}, e_{t-1})}_{\text{(see below)}} + 0.6\underbrace{Cov(Y_{t-k}, e_{t-2})}_{\text{(see below)}}$$

$$\begin{aligned} Cov(Y_{t-k}, e_{t-j}) &= Cov(0.8Y_{t-k-1} + e_{t-k} + 0.7e_{t-k-1} + 0.6e_{t-k-2}, e_{t-j}) \\ &= 0.8Cov(Y_{t-k-1}, e_{t-j}) + Cov(e_{t-k}, e_{t-j}) + 0.7Cov(e_{t-k-1}, e_{t-j}) \\ &\quad + 0.6Cov(e_{t-k-2}, e_{t-j}) \end{aligned}$$

$j = 0$:
- $Cov(Y_{t-k-1}, e_t) = 0$ for all $k \geq 0$, since $Y_{t-k-1} \perp\!\!\!\perp e_t$.
- $Cov(e_{t-k}, e_t)$: non-zero if $k = 0$.
- $Cov(e_{t-k-1}, e_t)$: non-zero if $k = -1$.
- $Cov(e_{t-k-2}, e_t)$: non-zero if $k = -2$.

$j = 1$:
- $Cov(Y_{t-k-1}, e_{t-1}) = 0$ for all $k \geq 0$, since $Y_{t-k-1} \perp\!\!\!\perp e_{t-1}$.
- $Cov(e_{t-k}, e_{t-1})$: non-zero if $k = 1$.
- $Cov(e_{t-k-1}, e_{t-1})$: non-zero if $k = 0$.
- $Cov(e_{t-k-2}, e_{t-1})$: non-zero if $k = -1$.

$j = 2$:
- $Cov(Y_{t-k-1}, e_{t-2}) = 0$ for all $k \geq 0$, since $Y_{t-k-1} \perp\!\!\!\perp e_{t-2}$.
- $Cov(e_{t-k}, e_{t-2})$: non-zero if $k = 2$.
- $Cov(e_{t-k-1}, e_{t-2})$: non-zero if $k = 1$.
- $Cov(e_{t-k-2}, e_{t-2})$: non-zero if $k = 0$.

Examples

Question 4.11

(a) for $k > 2$, we have

$$\begin{aligned}\gamma_k &= 0.8\gamma_{k-1} + \underbrace{Cov(Y_{t-k}, e_t)}_{0} + 0.7\underbrace{Cov(Y_{t-k}, e_{t-1})}_{0} + 0.6\underbrace{Cov(Y_{t-k}, e_{t-2})}_{0} \\ &= 0.8\gamma_{k-1}.\end{aligned}$$

Therefore,

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{0.8\gamma_{k-1}}{\gamma_0} = 0.8\rho_{k-1}.$$

(b) for $k = 2$, we have

$$\begin{aligned}\gamma_2 &= 0.8\gamma_1 + \underbrace{Cov(Y_{t-2}, e_t)}_{0} + 0.7\underbrace{Cov(Y_{t-2}, e_{t-1})}_{0} + 0.6\underbrace{Cov(Y_{t-2}, e_{t-2})}_{\sigma_e^2} \\ &= 0.8\gamma_1 + 0.6\sigma_e^2.\end{aligned}$$

Therefore,

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.8\gamma_1 + 0.6\sigma_e^2}{\gamma_0} = 0.8\rho_1 + 0.6\frac{\sigma_e^2}{\gamma_0}.$$

Examples

Question 4.21

Consider the model $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$

(a) Find the autocovariance function for this process.

(b) Show that this model corresponds to an $ARMA(p, q)$ model. Determine p and q.

Consider the model $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$

(a) Find the autocovariance function for this process.

(b) Show that this model corresponds to an $ARMA(p, q)$ model. Determine p and q.

(a) For $k \geq 0$:

$$
\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov(e_{t-1} - e_{t-2} + 0.5e_{t-3}, e_{t-k-1} - e_{t-k-2} + 0.5e_{t-k-3}) \\
&= \underbrace{Cov(e_{t-1}, e_{t-k-1})}_{\text{non-zero if } k = 0} - \underbrace{Cov(e_{t-1}, e_{t-k-2})}_{\text{non-zero if } k = -1} + 0.5\underbrace{Cov(e_{t-1}, e_{t-k-3})}_{\text{non-zero if } k = -2} \\
&\quad - \underbrace{Cov(e_{t-2}, e_{t-k-1})}_{\text{non-zero if } k = 1} + \underbrace{Cov(e_{t-2}, e_{t-k-2})}_{\text{non-zero if } k = 0} - 0.5\underbrace{Cov(e_{t-2}, e_{t-k-3})}_{\text{non-zero if } k = -1} \\
&\quad + 0.5\underbrace{Cov(e_{t-3}, e_{t-k-1})}_{\text{non-zero if } k = 2} - 0.5\underbrace{Cov(e_{t-3}, e_{t-k-2})}_{\text{non-zero if } k = 1} + 0.25\underbrace{Cov(e_{t-3}, e_{t-k-3})}_{\text{non-zero if } k = 0} \\
&= \begin{cases} 2.25\sigma_e^2 & \text{if } k = 0 \\ -1.5\sigma_e^2 & \text{if } k = 1 \\ 0.5\sigma_e^2 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

$$Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$$

(b) Note that $E(Y_t) = 0$, and so $\{Y_t : t \in \mathcal{I}\}$ is stationary. We can then "shift" the process back one time unit, by introducing $\{a_t : t \in \mathcal{I}\}$, and letting

$$Y_t = a_t - a_{t-1} + 0.5a_{t-2},$$

where $a_t = e_{t-1}$. Note that $E(Y_t)$ and $Cov(Y_t, Y_{t-k})$ remain unchanged.

$\Rightarrow$ We see that $\{Y_t : t \in \mathcal{I}\}$ is an $ARMA(0, 2)$ process. That is, an $MA(2)$ process with $\theta_1 = 1$ and $\theta_2 = -0.5$.

Recall that

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2(1 + \theta_1^2 + \theta_2^2) & \text{if } k = 0 \\ \sigma_e^2(-\theta_1 + \theta_2\theta_1) & \text{if } k = 1 \\ -\theta_2\sigma_e^2 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}.$$

By setting $\theta_1 = 1$ and $\theta_2 = -0.5$, we get the same result as in part (a).