Models for Stationary Time Series:

AR Processes

Week VI: Video 16

STAT 485/685, Fall 2020, SFU

Sonja Isberg

Video 16 Learning Objectives

By the end of this video, we should be able to:

- Define an autoregressive process of order p, i.e. AR(p)
- Give the mean function, and some properties of the autocovariance and autocorrelation functions, for AR(1) and AR(2) processes
- Recognize some key properties of the general AR(p) process, including how values of ρ_k can be numerically obtained

Autoregressive Processes

Definition: An **autoregressive process of order** p (i.e. $\mathbf{AR}(p)$) is defined as:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

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What's In a Name? Y_t is represented as a "regression" on past values of Y.

i.e., Y_t is a linear combination of the p most recent past values of itself, plus an *innovation term* e_t.

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What's In a Name? Y_t is represented as a "regression" on past values of Y.

i.e., Y_t is a linear combination of the p most recent past values of itself, plus an *innovation term* e_t.

e_t incorporates anything new in the series that is not explained by past values.

So, we can assume: e_t is independent of any Y that occurred before time t (i.e., Y_{t-1}, Y_{t-2}, ...).

- But, e_t is *not* independent of Y_t, or any future Y's (i.e., Y_{t+1}, Y_{t+2}, ...)

Important assumption: Y_t has zero mean.

The First-Order AR Process

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$$Y_t = \phi Y_{t-1} + e_t$$

- Mean function: $E(Y_t) = 0$ (assumption)
- Variance function:

$$Var(Y_t) = Var(\phi Y_{t-1} + e_t)$$

$$Var(Y_t) = \phi^2 Var(Y_{t-1}) + Var(e_t) \qquad \text{(since } e_t \perp\!\!\!\perp Y_{t-1}\text{)}$$

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_e^2 \qquad \text{(assuming the process is stationary)}$$

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi^2}$$

Note: The variance must be positive, so we need: $|\phi| < 1$!

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

- Autocovariance function: (for $k \geq 1$)

$$Y_{t-k} Y_t = \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t \qquad \text{(multiply by } Y_{t-k}\text{)}$$

$$E(Y_{t-k} Y_t) = \phi E(Y_{t-k} Y_{t-1}) + E(Y_{t-k} e_t)$$

$$Cov(Y_{t-k}, Y_t) = \phi Cov(Y_{t-k}, Y_{t-1}) + Cov(Y_{t-k}, e_t) \qquad \text{(def'n of cov.; 0 means)}$$

$$\gamma_k = \phi \gamma_{k-1} \qquad \text{(stationarity; } e_t \perp\!\!\!\perp Y_{t-k}\text{)}$$

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

- Autocovariance function: (for $k \geq 1$)

$$\begin{aligned}
Y_{t-k} Y_t &= \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t && \text{(multiply by } Y_{t-k}\text{)} \\
E(Y_{t-k} Y_t) &= \phi E(Y_{t-k} Y_{t-1}) + E(Y_{t-k} e_t) \\
Cov(Y_{t-k}, Y_t) &= \phi Cov(Y_{t-k}, Y_{t-1}) + Cov(Y_{t-k}, e_t) && \text{(def'n of cov.; 0 means)} \\
\gamma_k &= \phi \gamma_{k-1} && \text{(stationarity; } e_t \perp\!\!\!\perp Y_{t-k}\text{)}
\end{aligned}$$

Examples:

$$\gamma_1 = \phi \gamma_0 = \phi \frac{\sigma_e^2}{1 - \phi^2} \qquad \gamma_2 = \phi \gamma_1 = \phi^2 \frac{\sigma_e^2}{1 - \phi^2}$$

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

- Autocovariance function: (for $k \geq 1$)

$$\begin{aligned}
Y_{t-k} Y_t &= \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t && \text{(multiply by } Y_{t-k}\text{)} \\
E(Y_{t-k} Y_t) &= \phi E(Y_{t-k} Y_{t-1}) + E(Y_{t-k} e_t) && \\
Cov(Y_{t-k}, Y_t) &= \phi Cov(Y_{t-k}, Y_{t-1}) + Cov(Y_{t-k}, e_t) && \text{(def'n of cov.; 0 means)} \\
\gamma_k &= \phi \gamma_{k-1} && \text{(stationarity; } e_t \perp\!\!\!\perp Y_{t-k}\text{)}
\end{aligned}$$

Examples:

$$\gamma_1 = \phi \gamma_0 = \phi \frac{\sigma_e^2}{1 - \phi^2} \qquad \gamma_2 = \phi \gamma_1 = \phi^2 \frac{\sigma_e^2}{1 - \phi^2}$$

In general, for $k \geq 1$:

$$\gamma_k = \phi^k \frac{\sigma_e^2}{1 - \phi^2}$$

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

The First-Order AR Process (cont'd)

$$Y_t = \phi Y_{t-1} + e_t$$

- Autocorrelation function: (for $k \geq 1$)

$$\begin{aligned}
\rho_k &= Corr(Y_t, Y_{t-k}) \\
&= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)Var(Y_{t-k})}} \\
&= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \qquad \text{(stationarity)} \\
&= \frac{\gamma_k}{\gamma_0} \\
&= \frac{\phi^k \sigma_e^2/(1-\phi^2)}{\sigma_e^2/(1-\phi^2)} \\
&= \phi^k
\end{aligned}$$

Note: Since $|\phi| < 1$, the correlation is exponentially decreasing in k.

The First-Order AR Process: Properties

$$Y_t = \phi Y_{t-1} + e_t$$

For now, we assume the process is stationary, which requires $|\phi| < 1$.

Properties of the AR(1) process:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \phi^k \frac{\sigma_e^2}{1-\phi^2}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \phi^k \quad \text{for } k \geq 1 \text{ (1 for } k = 0)$$

Notice: The correlation is exponentially decreasing in k, but is never zero!

The First-Order AR Process: ρ_k as a function of k

We've seen that, for the AR(1) process:

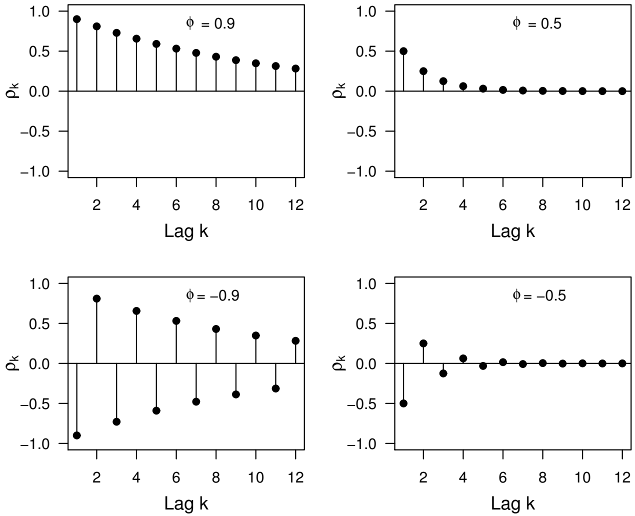
$$\rho_k = \phi^k \text{ for } k \geq 1$$

We know:

- The correlation exponentially decreases in k.
- If $\phi > 0$: all correlations are positive.
- If $\phi < 0$: correlations alternate between positive and negative.
- The exponential decay is slower for ϕ near ± 1, and faster for ϕ near 0.

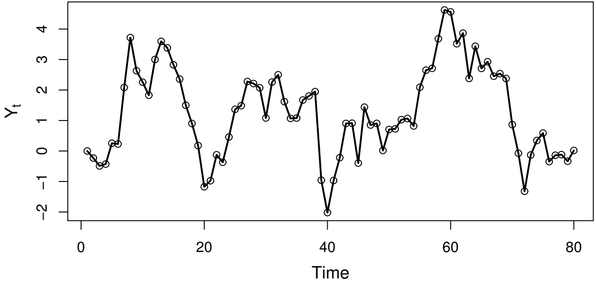
The First-Order AR Process: ρ_k as a function of k

$$\rho_k = \phi^k \text{ for } k \geq 1$$



The First-Order AR Process: Example 1

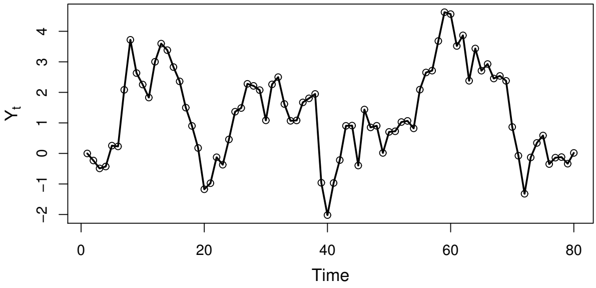
Example: AR(1) process with $\phi = 0.9$:




```
> e.vec <- rnorm(n=80, mean=0, sd=1)
> y.vec <- rep(NA, times=80)
> y.vec[1] <- 0
> for (t in 2:80)
  {
     y.vec[t] <- 0.9*y.vec[t-1] + e.vec[t]
  }
> plot(c(1:80), y.vec, type='o', xlab='Time', ylab=expression(Y[t]))
```

The First-Order AR Process: Example 1 (cont'd)

Example: AR(1) process with $\phi = 0.9$:



Due to the positive autocorrelations, observations "hang together" a lot.

The First-Order AR Process: Example 1 (cont'd)

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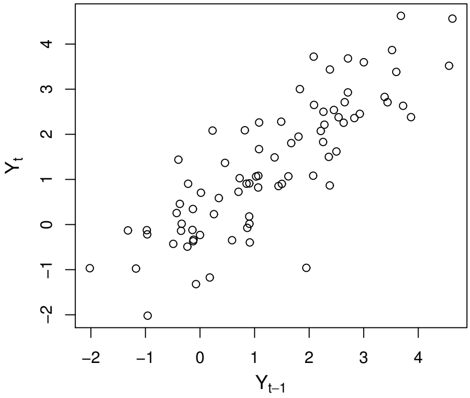

What do we see in the plot of Y_t vs. Y_{t-1}? (Recall: $\rho_1 = \phi$)

The First-Order AR Process: Example 1 (cont'd)

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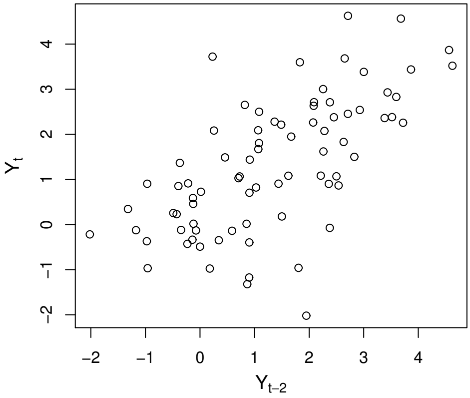
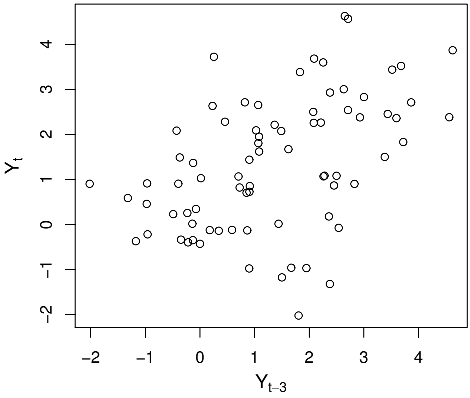


What do we see in the plot of Y_t vs. Y_{t-2}? (Recall: $\rho_2 = \phi^2$)

The First-Order AR Process: Example 1 (cont'd)

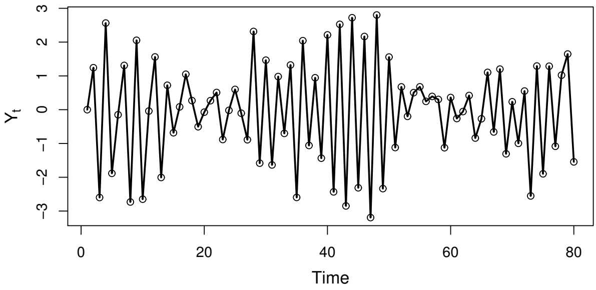
Example: AR(1) process with $\phi = 0.9$:



What do we see in the plot of Y_t vs. Y_{t-3}? (Recall: $\rho_3 = \phi^3$)

The First-Order AR Process: Example 2

Example: AR(1) process with $\phi = -0.9$:

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> for (t in 2:80)
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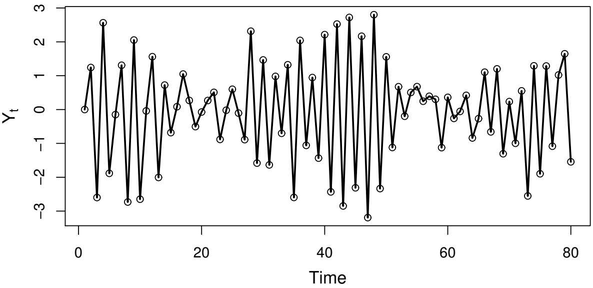
  }
> plot(c(1:80), y.vec, type='o', xlab='Time', ylab=expression(Y[t]))
```

The First-Order AR Process: Example 2 (cont'd)

Example: AR(1) process with $\phi = -0.9$:



Due to the negative lag-1 autocorrelation, the plot is very "jagged" over time.

The First-Order AR Process: Example 2 (cont'd)

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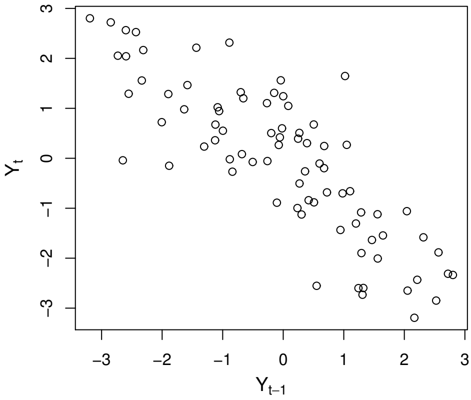

What do we see in the plot of Y_t vs. Y_{t-1}? (Recall: $\rho_1 = \phi$)

The First-Order AR Process: Example 2 (cont'd)

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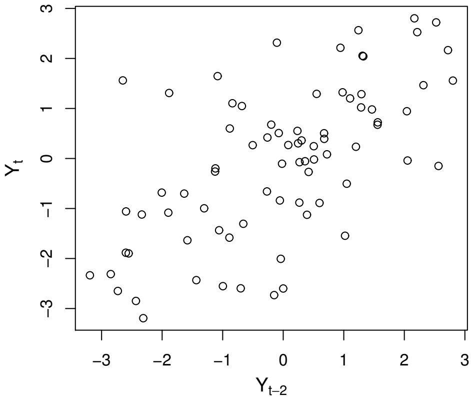

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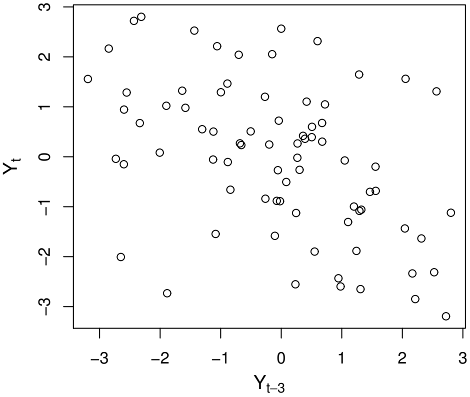

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The Second-Order AR Process

Definition: The second-order autoregressive process, i.e. **AR(2)**, is:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

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- Autocovariance function: (for $k \geq 1$)

$$Y_{t-k} Y_t = \phi_1 Y_{t-k} Y_{t-1} + \phi_2 Y_{t-k} Y_{t-2} + Y_{t-k} e_t$$

$$E(Y_{t-k} Y_t) = \phi_1 E(Y_{t-k} Y_{t-1}) + \phi_2 E(Y_{t-k} Y_{t-2}) + E(Y_{t-k} e_t)$$

$$Cov(Y_{t-k}, Y_t) = \phi_1 Cov(Y_{t-k}, Y_{t-1}) + \phi_2 Cov(Y_{t-k}, Y_{t-2}) + Cov(Y_{t-k}, e_t)$$

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \tag{1}$$

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$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \tag{1}$$

- Autocorrelation function: (for $k \geq 1$)

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \tag{2}$$

Equations (1) & (2) are called the **Yule-Walker equations**.

The Second-Order AR Process (cont'd)

Yule-Walker equation for the autocorrelation function: (for $k \geq 1$)

$$\rho_k = \phi_1 \, \rho_{k-1} + \phi_2 \rho_{k-2}$$

For $k = 1$:

$$\rho_1 = \phi_1 \, \rho_0 + \phi_2 \rho_{-1}$$

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

The Second-Order AR Process (cont'd)

Yule-Walker equation for the autocorrelation function: (for $k \geq 1$)

$$\rho_k = \phi_1 \, \rho_{k-1} + \phi_2 \rho_{k-2}$$

For $k = 1$:

$$\rho_1 = \phi_1 \, \rho_0 + \phi_2 \rho_{-1}$$

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

For $k = 2$:

$$\rho_2 = \phi_1 \, \rho_1 + \phi_2 \rho_0$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_2 = \frac{\phi_2(1 - \phi_2) + \phi_1^2}{1 - \phi_2}$$

For $k > 2$: ρ_k can be obtained using the Yule-Walker equation.

The Second-Order AR Process (cont'd)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

- Variance function:

$$Var(Y_t) = Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t)$$

$$Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + Var(e_t)$$

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2$$

The Second-Order AR Process (cont'd)

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$$Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + Var(e_t)$$

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2$$

Also use: $\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$ (from Yule-Walker equation).

Therefore:

$$\gamma_0 = \left(\frac{1-\phi_2}{1+\phi_2}\right) \frac{\sigma_e^2}{(1-\phi_2)^2 - \phi_1^2}$$

The Second-Order AR Process: Properties

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Properties of the AR(2) process:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \left(\frac{1-\phi_2}{1+\phi_2}\right) \frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2} & \text{for } k = 0 \\ \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} & \text{for all } k \geq 1 \end{cases}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1/(1-\phi_2) & \text{for } k = 1 \\ [\phi_2(1-\phi_2) + \phi_1^2]/(1-\phi_2) & \text{for } k = 2 \\ \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} & \text{for all } k \geq 1 \end{cases}$$

The General AR(p) Process

Definition: The autoregressive process of order p, i.e. **AR(p)**, is:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Properties:

$$\mu_t = E(Y_t) = 0 \quad \text{for all } t$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \begin{cases} \gamma_0 & \text{for } k = 0 \\ \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \phi_3 \gamma_{k-3} + \cdots + \phi_p \gamma_{k-p} & \text{for all } k \geq 1 \end{cases}$$

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \phi_3 \rho_{k-3} + \cdots + \phi_p \rho_{k-p} & \text{for all } k \geq 1 \end{cases}$$

$$\gamma_0 = Var(Y_t) = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}$$

The General AR(p) Process

$$\rho_k = Corr(Y_t, Y_{t-k}) = \begin{cases} 1 & \text{for } k = 0 \\ \phi_1\, \rho_{k-1} + \phi_2\, \rho_{k-2} + \phi_3\, \rho_{k-3} + \cdots + \phi_p \rho_{k-p} & \text{for all } k \geq 1 \end{cases}$$

Written as a set of equations:

$$\rho_1 = \phi_1 + \phi_2\, \rho_1 + \phi_3\, \rho_2 + \cdots + \phi_p\, \rho_{p-1}$$

$$\rho_2 = \phi_1\, \rho_1 + \phi_2 + \phi_3 \rho_1 + \cdots + \phi_p \rho_{p-2}$$

$$\rho_3 = \phi_1\, \rho_2 + \phi_2\, \rho_1 + \phi_3 + \cdots + \phi_p \rho_{p-3}$$

$$\vdots$$

Final Comments

That's all for now!

In this video, we've learned about the autoregressive process of order p, i.e. AR(p).

We derived the properties for the special cases AR(1) and AR(2), and we learned about some properties of AR(p) as well.

Coming Up Next: Some more important properties of MA and AR models.

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.