Models for Non-Stationary Time Series: Constant Terms in ARIMA Models

Week VII: Video 22

STAT 485/685, Fall 2020, SFU

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Introduction

In Videos 19-21, we've seen how differencing, log-transformations, differences of logs, and power transformations, can be used to obtain a stationary process from a non-stationary one.

Recall that, in addition to stationarity, we have another assumption on our ARMA processes: *zero means*.

In today's video, we will learn some techniques for handling constant, non-zero means in a stationary time series.

Video 22 Learning Objectives

By the end of this video, we should be able to:

- Re-write an ARMA(p,q) model to incorporate a non-zero constant mean term
- Identify how a non-zero mean in the differenced series $\{W_t\}$ will affect the original ARIMA series $\{Y_t\}$

Recall: The ARIMA(p,d,q) Process

Definition: A process $\{Y_t\}$ is said to be an **integrated autoregressive moving average process of orders** p **and** q **and degree** d (i.e. **ARIMA(**p**,**d**,**q**)**) if:

The d^{th} difference $W_t = \nabla^d Y_t$ is a stationary ARMA(p,q) process.

In other words:

$$W_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} \\ + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

for some $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$.

Note: Our standard assumption is that stationary models have a zero mean. But what if they have a non-zero (but still constant) mean?

Accommodating a Non-Zero Mean: Method 1

If the process $\{W_t\}$ has a constant, non-zero mean μ, this can be accommodated in two ways.

Method 1:

State that the process $\{W_t - \mu\}$ is then a stationary, zero-mean, ARMA(p,q) process.

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Method 1:

State that the process $\{W_t - \mu\}$ is then a stationary, zero-mean, ARMA(p,q) process.

In other words:

$$W_t - \mu = \phi_1(W_{t-1} - \mu) + \phi_2(W_{t-2} - \mu) + \cdots + \phi_p(W_{t-p} - \mu) \\ + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

Check: $E(W_t - \mu) = E(W_t) - \mu = \mu - \mu = 0$

Accommodating a Non-Zero Mean: Method 2

Method 2:

Starting with our equation from Method 1:

$$
\begin{aligned}
W_t - \mu = &\ \phi_1(W_{t-1} - \mu) + \phi_2(W_{t-2} - \mu) + \cdots + \phi_p(W_{t-p} - \mu) \\
&+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\
W_t = &\ \mu + \phi_1 W_{t-1} - \phi_1\mu + \phi_2 W_{t-2} - \phi_2\mu + \cdots + \phi_p W_{t-p} - \phi_p\mu \\
&+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\
W_t = &\ \mu(1 - \phi_1 - \phi_2 - \cdots - \phi_p) + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} \\
&+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\
W_t = &\ \theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} \\
&+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}
\end{aligned}
$$

where

$$\theta_0 = \mu(1 - \phi_1 - \phi_2 - \cdots - \phi_p)$$

Accommodating a Non-Zero Mean: Method 2 (cont'd)

$$W_t = \theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p}$$
$$+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

where

$$\theta_0 = \mu(1 - \phi_1 - \phi_2 - \cdots - \phi_p)$$

Therefore: $\{W_t\}$ is a stationary ARMA(p,q) process with "drift term" θ_0.

Method 1 and Method 2 are just alternative parameterizations, but they are mathematically equivalent. So we use whichever one is convenient.

How θ_0 Affects Y_t

Suppose $\{W_t\}$ has a non-zero mean of μ, i.e.

$$\begin{aligned} W_t = &\theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} \\ &+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \end{aligned}$$

How does this affect Y_t? The value θ_0 also gets added to the expression for Y_t.

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How does this affect Y_t? The value θ_0 also gets added to the expression for Y_t.

Example: Suppose $\{W_t\} = \{\nabla Y_t\}$ is an ARMA(1,1) process with mean μ. Then:

$$W_t = \theta_0 + \phi W_{t-1} + e_t - \theta e_{t-1}$$

where $\theta_0 = \mu(1 - \phi)$.

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where $\theta_0 = \mu(1 - \phi)$.

Then, the difference equation form for Y_t is:

$$Y_t - Y_{t-1} = \theta_0 + \phi(Y_{t-1} - Y_{t-2}) + e_t - \theta e_{t-1}$$
$$Y_t = \theta_0 + (1 + \phi)Y_{t-1} - \phi Y_{t-2} + e_t - \theta e_{t-1}$$

So, $\{Y_t\}$ is an ARIMA(1,1,1) process with "drift term" θ_0.

How θ_0 Affects the Mean of Y_t

How does θ_0 affect the mean of Y_t?

ARIMA(p,1,q):

Recall that, for an ARIMA(p,1,q) process Y_t, we can usually obtain properties by expressing Y_t in terms of W_t:

$$Y_t = \sum_{j=-m}^{t} W_j$$

where $-m$ is the time point at which Y_t is assumed to start.

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Then:

$$E(Y_t) = E\left[\sum_{j=-m}^{t} W_j\right] = \sum_{j=-m}^{t} E(W_j) = (t+m+1)\mu$$

So, the mean of $\{Y_t\}$ is a linear function of t.

How θ_0 Affects the Mean of Y_t (cont'd)

ARIMA(p,2,q):

Similarly, for an ARIMA(p,2,q) process Y_t:

$$Y_t = \sum_{j=-m}^{t} \sum_{i=-m}^{j} W_i$$

Then:

$$E(Y_t) = E\left[\sum_{j=-m}^{t} \sum_{i=-m}^{j} W_i\right] = \sum_{j=-m}^{t} \sum_{i=-m}^{j} E(W_j) = \ldots$$

The mean of $\{Y_t\}$ will be a quadratic function of t.

How θ_0 Affects the Mean of Y_t (cont'd)

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Then:

$$E(Y_t) = E\left[\sum_{j=-m}^{t} \sum_{i=-m}^{j} W_i\right] = \sum_{j=-m}^{t} \sum_{i=-m}^{j} E(W_j) = \dots$$

The mean of $\{Y_t\}$ will be a quadratic function of t.

ARIMA(p,d,q):

It can be shown that, for a general ARIMA(p,d,q) model where $E(\nabla^d Y_t) \neq 0$:

$$Y_t = \tilde{Y}_t + \mu_t$$

where $\tilde{Y}_t$ is a mean-zero ARIMA(p,d,q) process, and μ_t is some deterministic polynomial of degree d.

Final Comments

That's all for now!

In this video, we've seen how to incorporate constant non-zero means into an ARMA model.

We have also seen how these constant means will affect the original ARIMA model.

Next Week in STAT 485/685: We begin the Model Building Process, and we learn about its first step: *model specification*.

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.