Trends:

Additional Topics

Week IV: Video 11

STAT 485/685, Fall 2020, SFU

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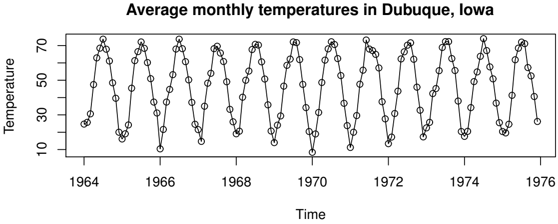
Video 11 Learning Objectives

In this video, we'll cover a few additional topics from Ch. 3 which have come up in discussions and assignments:

- Coding of t in seasonal data: What if t is years, not months?
- Writing the seasonal means model as a regression, using indicator variables
- What to look for in a residuals vs. time plot

Coding of t in Seasonal Data



We know two different models for estimating a seasonal trend:

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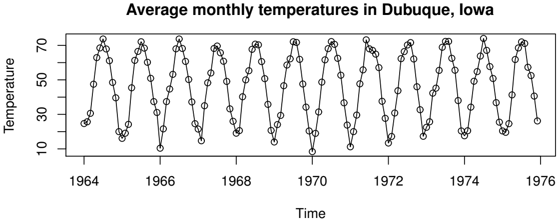

We know two different models for estimating a seasonal trend:

Seasonal means model:

$$\mu_t = \begin{cases} \beta_1 & \text{for all Januarys} \\ \beta_2 & \text{for all Februarys} \\ \vdots & \\ \beta_{12} & \text{for all Decembers} \end{cases}$$

Cosine trend model:

$$\mu_t = \beta_0 + \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t)$$

where f is the frequency of the cosine curve.

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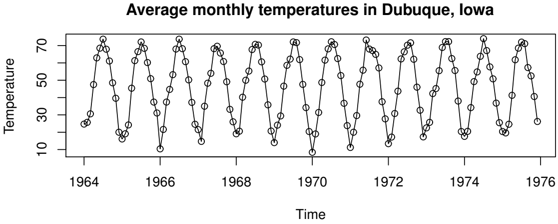

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Cosine trend model:

$$\mu_t = \beta_0 + \beta_1 \, cos(2\pi ft) + \beta_2 \, sin(2\pi ft)$$

where f is the frequency of the cosine curve.

When using either of these models, we have to very carefully look at how t is defined in the dataset!

Coding of t in Seasonal Data: Cosine Trend Model

$$\mu_t = \beta_0 + \beta_1 \, cos(2\pi ft) + \beta_2 \, sin(2\pi ft)$$

where f is the frequency of the cosine curve.

Case A: Each time unit is one month:

e.g., January is $t = 1, 13, 25, \ldots$

February is $t = 2, 14, 26, \ldots$

$\vdots$

December is $t = 12, 24, 36, \ldots$

The pattern repeats itself every 12 time points. So: $f = 1/12$.

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The pattern repeats itself every 12 time points. So: $f = 1/12$.

Case B: Each time unit is one year:

e.g., January is $t = 1986.000, 1987.000, 1988.000, \ldots$

February is $t = 1986.083, 1987.083, 1988.083, \ldots$

$\vdots$

December is $t = 1986.917, 1987.917, 1988.917, \ldots$

The pattern repeats itself every 1 time point. So: $f = 1$.

Coding of t in Seasonal Data: Seasonal Means Model

$$\mu_t = \begin{cases} \beta_1 & \text{for all Januarys} \\ \beta_2 & \text{for all Februarys} \\ \vdots & \\ \beta_{12} & \text{for all Decembers} \end{cases}$$

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Writing the Seasonal Means Model as a Linear Model

Seasonal means model:

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Alternative formulation:

$$\mu_t = \beta_1 I_{January} + \beta_2 I_{February} + \cdots + \beta_{12} I_{December}$$

where I's are *indicator variables* for each month:

$$e.g., \quad I_{January} = \begin{cases} 1 & \text{if } t \text{ is a January} \\ 0 & \text{if } t \text{ is not a January} \end{cases}$$

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So, for all Januarys:

$$\mu_t = \beta_1(1) + \beta_2(0) + \cdots + \beta_{12}(0) = \beta_1$$

Seasonal Means + Linear Time Trend Model

If we wish to add a linear time trend to the seasonal means model:

$$\mu_t = \beta_1 I_{January} + \beta_2 I_{February} + \cdots + \beta_{12} I_{December} + \alpha\, t$$

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Then:

$$\mu_t = \begin{cases} \beta_1 + \alpha\, t & \text{for all Januarys} \\ \beta_2 + \alpha\, t & \text{for all Februarys} \\ \vdots & \\ \beta_{12} + \alpha\, t & \text{for all Decembers} \end{cases}$$

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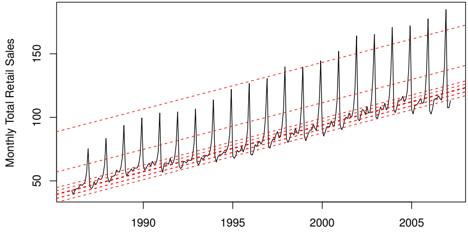
Separate lines of μ_t vs. t for each month, where the lines are parallel:



Reading Residuals vs. Time Plots

Plots of residuals vs. time tell us:

1. What the process $\{X_t\}$ might look like.
2. Whether the trend we've fitted is adequate.

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1. What the process $\{X_t\}$ might look like.
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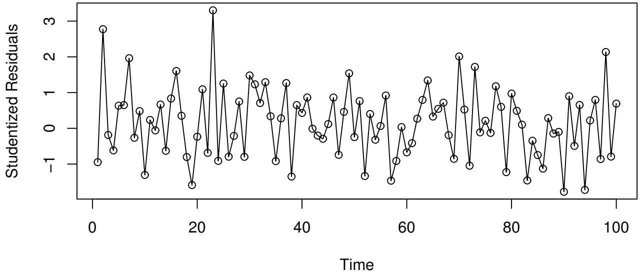
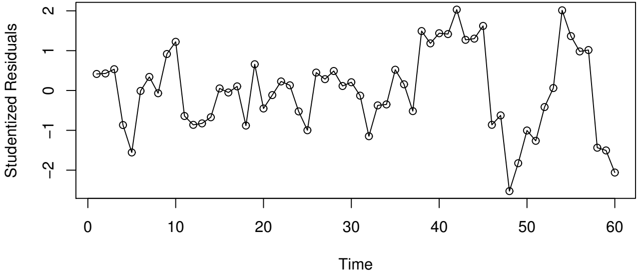
Ideally, the plot should look like a random scatter about 0:



There should be no discernible patterns/shapes in the plot.

Reading Residuals vs. Time Plots (cont'd)

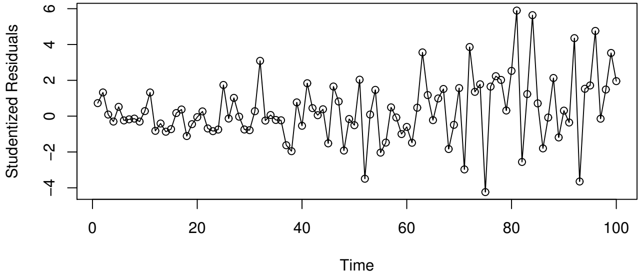
Case 1: Residuals "hang together" too much:



This suggests that the observations are not independent.

Reading Residuals vs. Time Plots (cont'd)

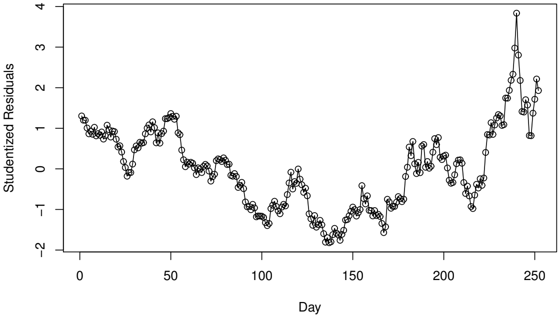
Case 2: Residuals have an unequal variance over time:



This suggests that $\{X_t\}$ has a non-stationary variance.

Reading Residuals vs. Time Plots (cont'd)

Case 3: Residuals form a "U"-shape, or an upside-down "U"-shape:



This may suggest that $\{X_t\}$ really behaves in this way (and is therefore not white noise).

Or: Perhaps we have fitted a linear trend when a quadratic one would have been more appropriate.

Final Comments

That's all for now!

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- Writing the seasonal means model as a regression, using indicator variables
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Coming Up Next: Info about Midterm 1!

Thank you!

References:

[1] Cryer, J. D., & Chan, K. S. (2008). *Time series analysis: with applications in R.* Springer Science and Business Media.

[2] Chan, K. S., & Ripley, B. (2020). TSA: Time Series Analysis. R package version 1.2.1.