09 - Discrete Differential Geometry of Surfaces

(II - Functions on surfaces)

Acknowledgements: Daniele Panozzo

In this lecture

- Functions defined on discrete surfaces
  - continuum and discrete
  - gradient
  - the Laplace-Beltrami operator
  - relations with curvature

Functions on surfaces

Functions defined on a surface

- Consider a (smooth) surface S and a smooth (C^2) scalar field

$$f : S \longrightarrow \mathbb{R}$$

- The concepts of partial derivatives, gradient, Laplacian, etc. can be generalized from traditional calculus of bivariate functions

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

to functions defined on S

Gradient

- Let $f(u,v)$ be a bivariate smooth (C^2) scalar field
- The *gradient* of f is defined as: $\nabla f = \begin{pmatrix} f_u \\ f_v \end{pmatrix}$
- The gradient $\nabla f(p)$ of a function f at a point p is a vector on the plane pointing to the direction of maximal ascent of f and having a size proportional to the rate of ascent
- $f : \mathcal{M} \longrightarrow \mathbb{R}$ defined on a manifold, the gradient $\nabla f_{\mathcal{M}}$ at a point p is a vector in the tangent plane at p pointing to the direction of maximal ascent of f and having a size proportional to the rate of ascent

Laplace Operator

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \Delta f : \mathbb{R}^3 \to \mathbb{R}$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \dots$$

function in Euclidean space

divergence operator

Cartesian coordinates

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Intuitive Explanation

The Laplacian $\Delta f(p)$ of a function f at a point p is the rate at which the average value of f over spheres centered at p deviates from $f(p)$ as the radius of the sphere grows, up to a constant depending on the dimension.

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

$$f : \mathcal{M} \to \mathbb{R} \qquad \Delta f : \mathcal{M} \to \mathbb{R}$$

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

function on surface M

divergence operator

Mean curvature from Laplace-Beltrami Operator

- For coordinate functions:

$$f_x(\mathbf{p}) = x \quad f_y(\mathbf{p}) = y \quad f_z(\mathbf{p}) = z$$

$$\mathbf{p} = (x, y, z)$$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{M}}\mathbf{p} = \mathrm{div}_{\mathcal{M}}\nabla_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n} \ \in \mathbb{R}^3$$

coordinate function on surface M

divergence operator

unit surface normal

Harmonic functions and surfaces

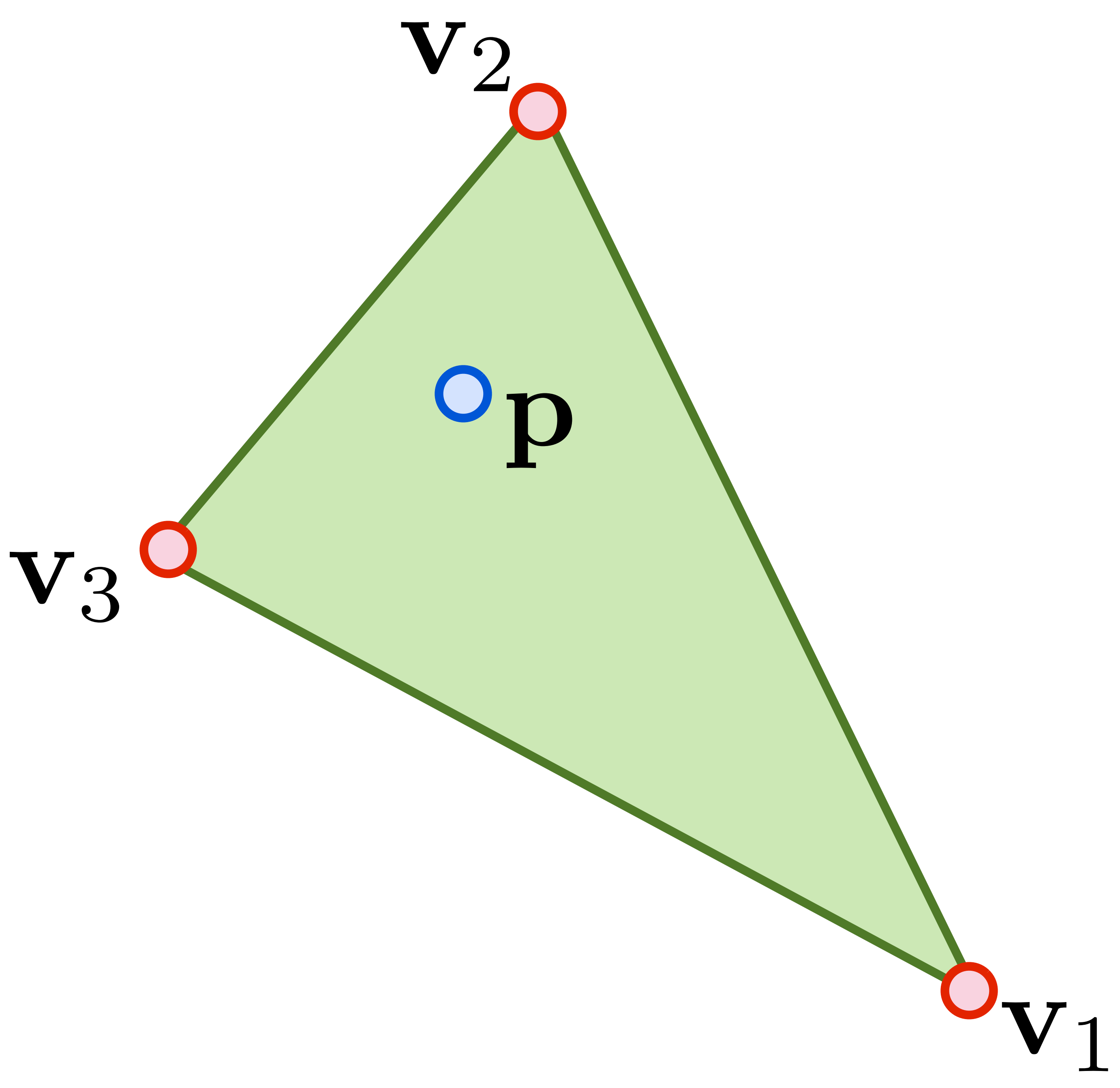
- A function is said to be ***harmonic*** if its Laplacian is null
- A ***minimal surface*** has null mean curvature (i.e., its coordinate functions are harmonic)
  - Harmonic functions and minimal surfaces are especially ***smooth***
- The Laplace-Beltrami operator and the Laplace PDE

$$\Delta_{\mathcal{M}} f = 0$$

will be very useful in the sequel to obtain smooth surfaces.

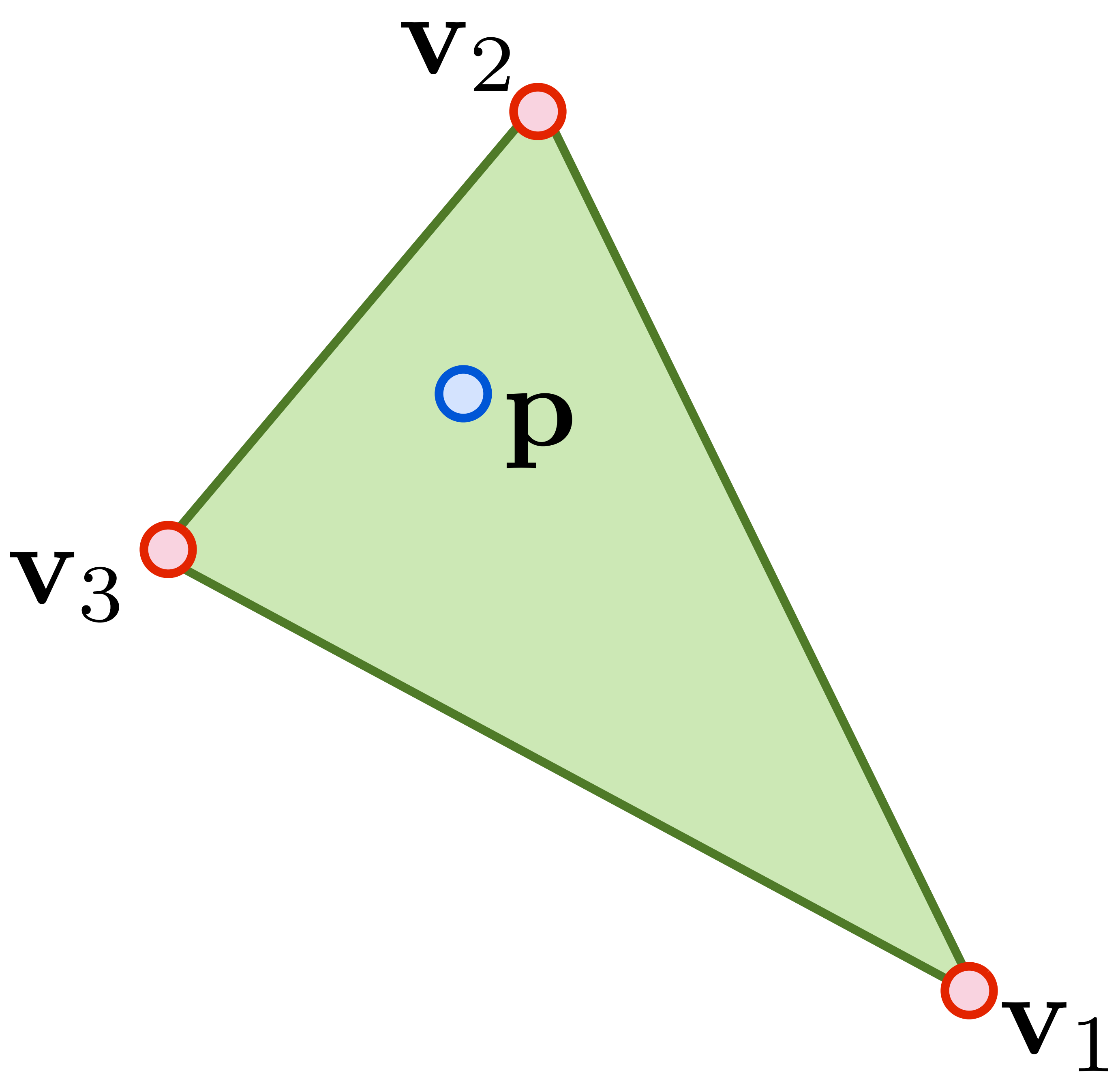
Discrete setting

Barycentric Coordinates



$$\mathbf{p} = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + w_3\mathbf{v}_3$$

Barycentric Coordinates



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Partition of unity: $w_1 + w_2 + w_3 = 1$

$$\mathbf{p} = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + (1 - w_1 - w_2)\mathbf{v}_3$$

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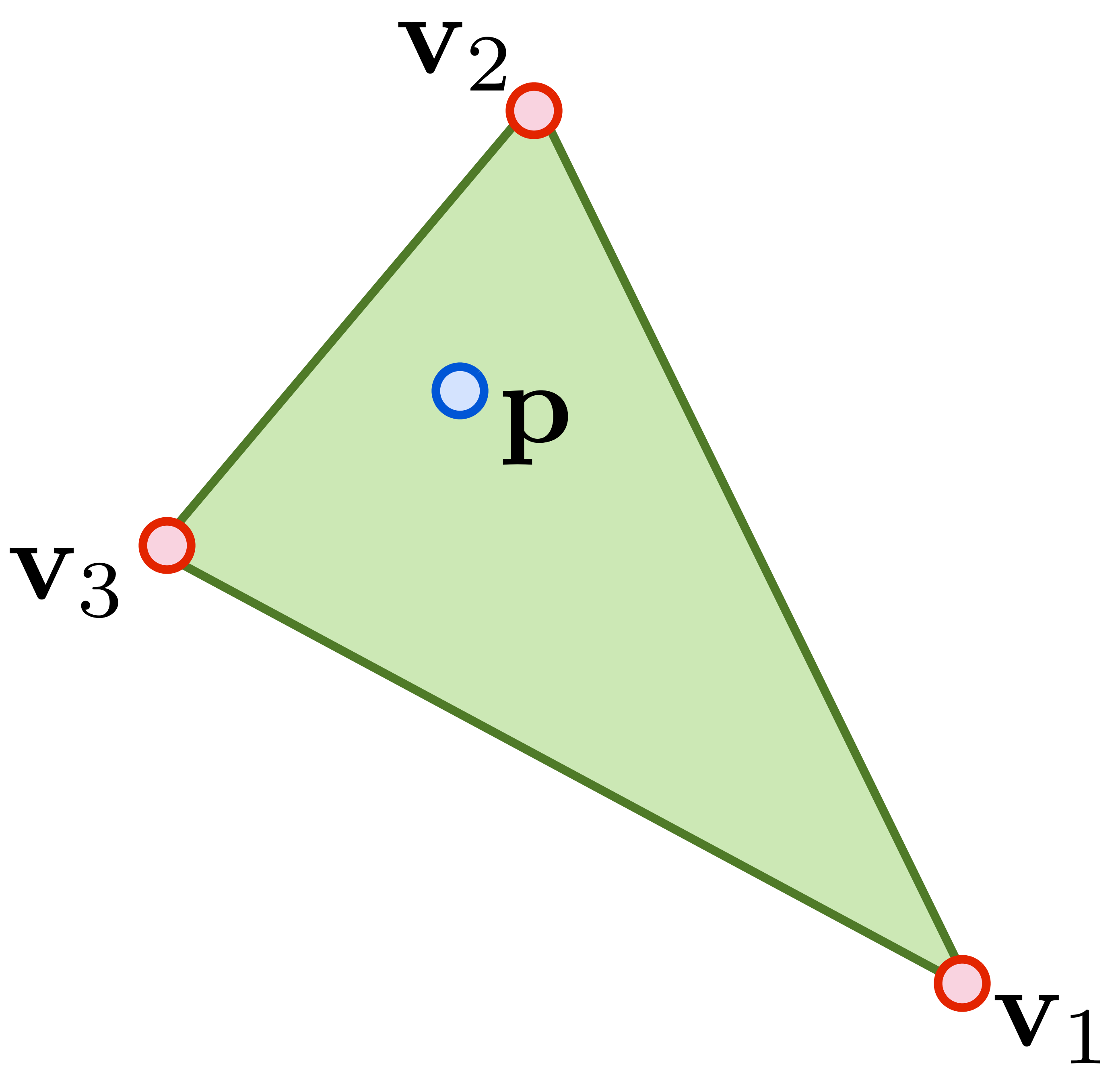

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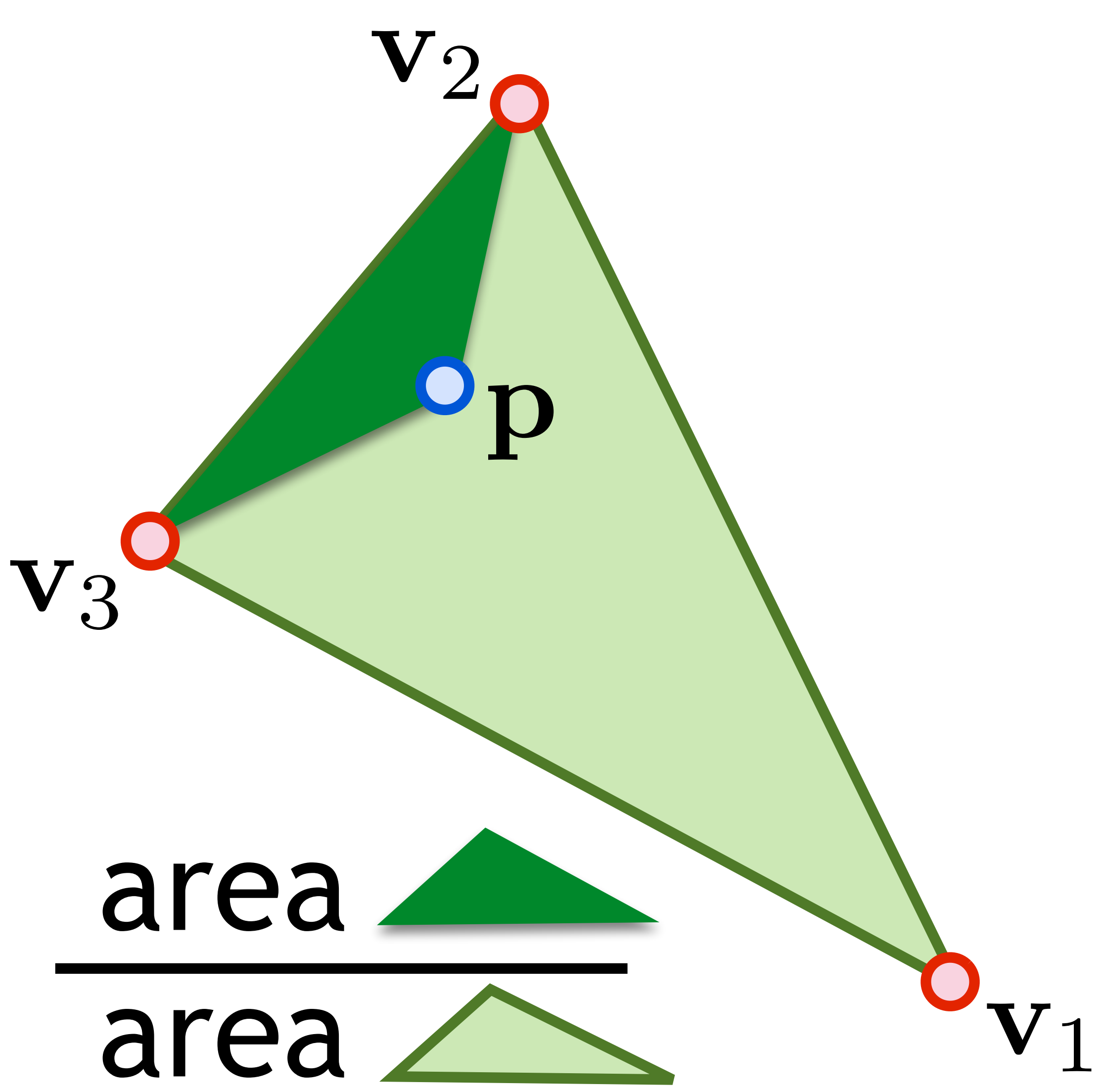

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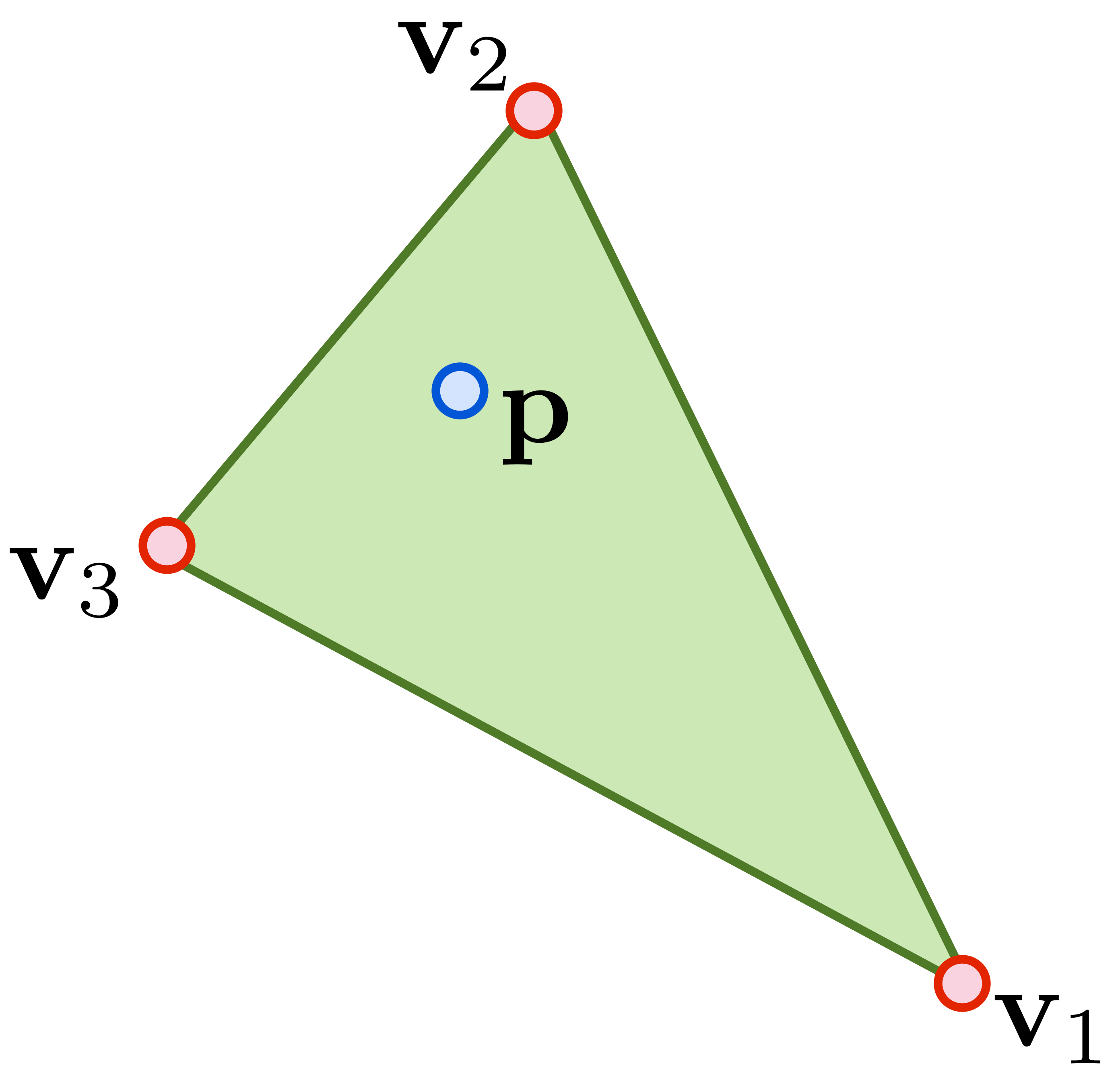

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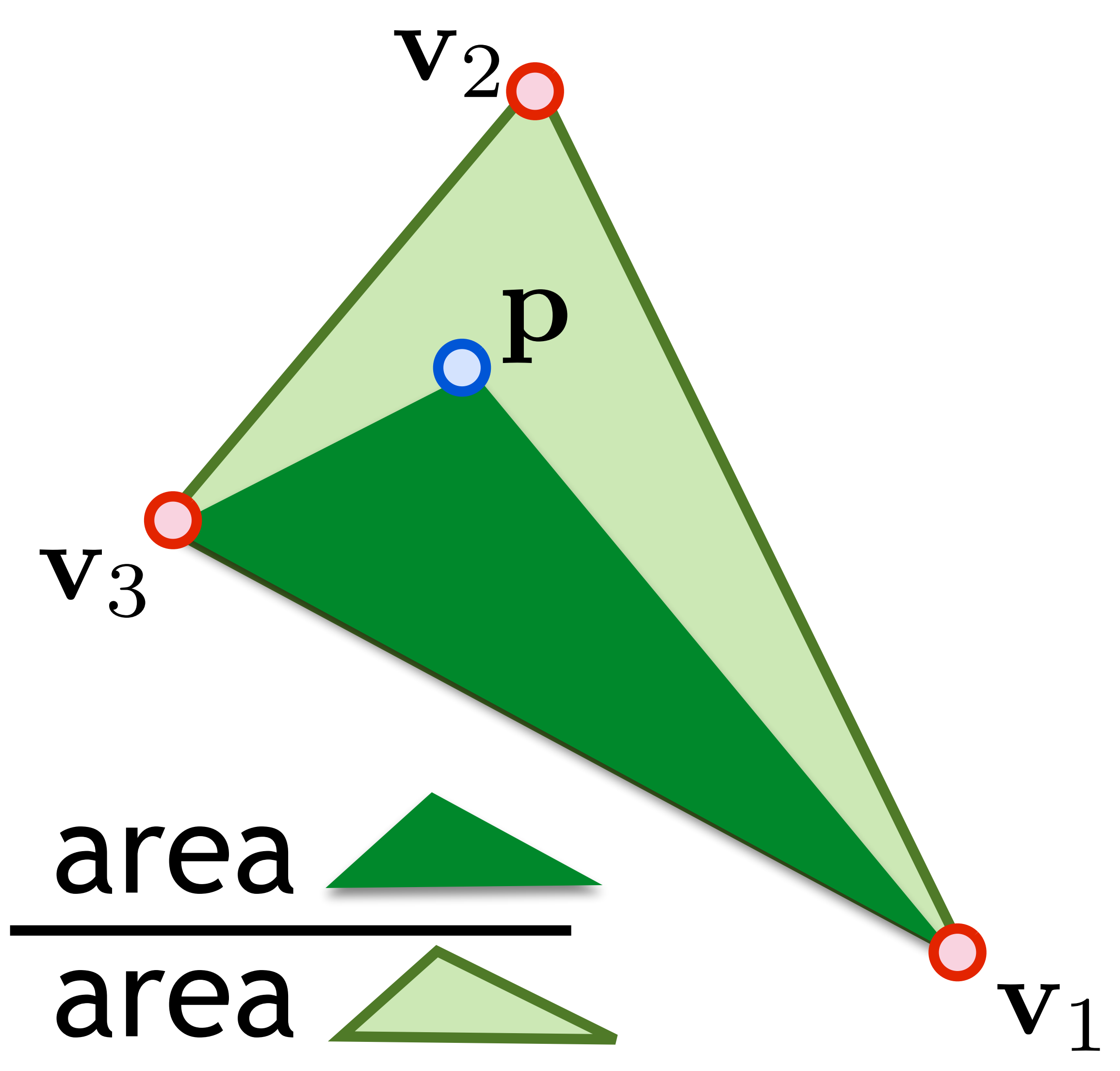

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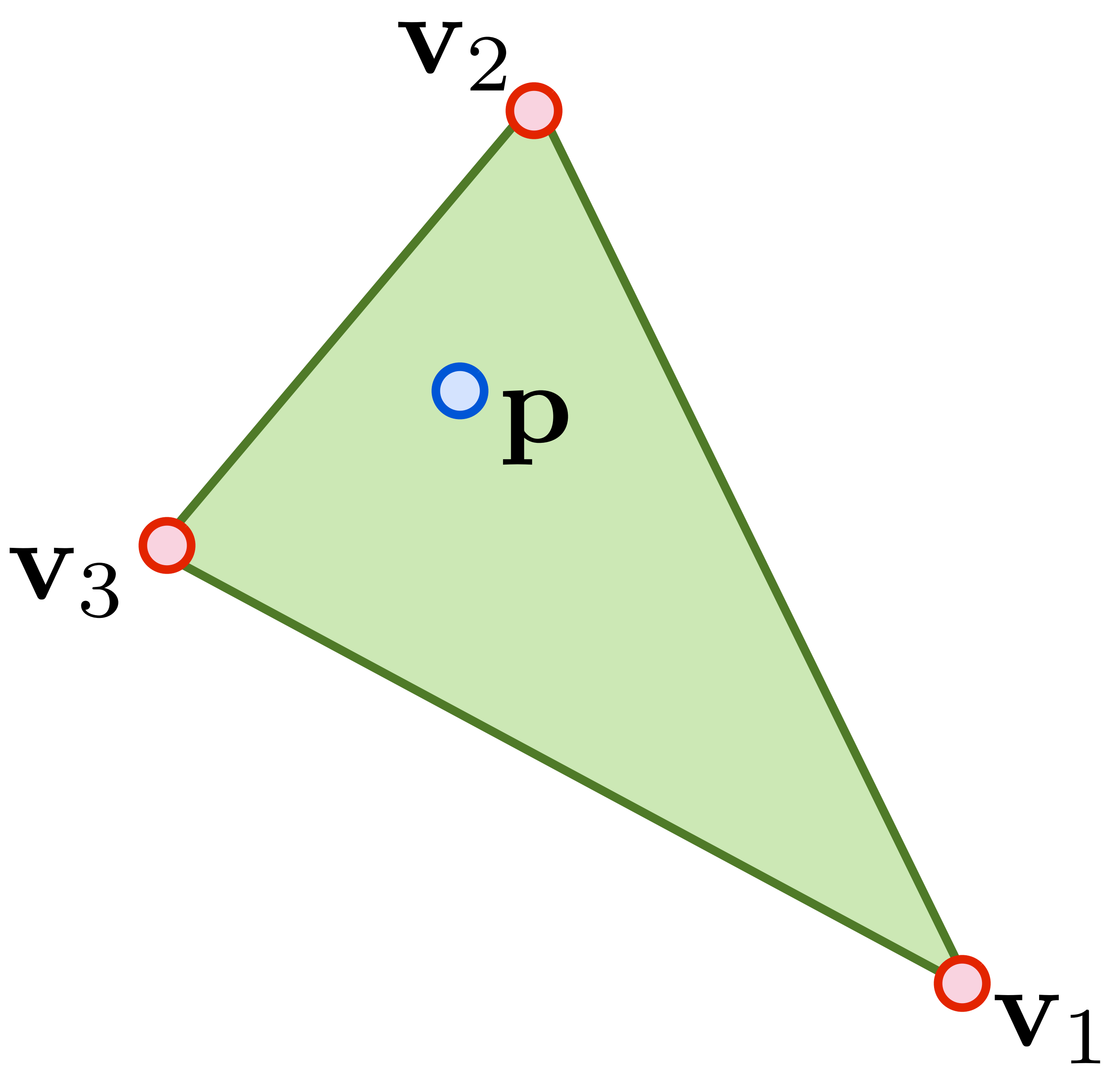

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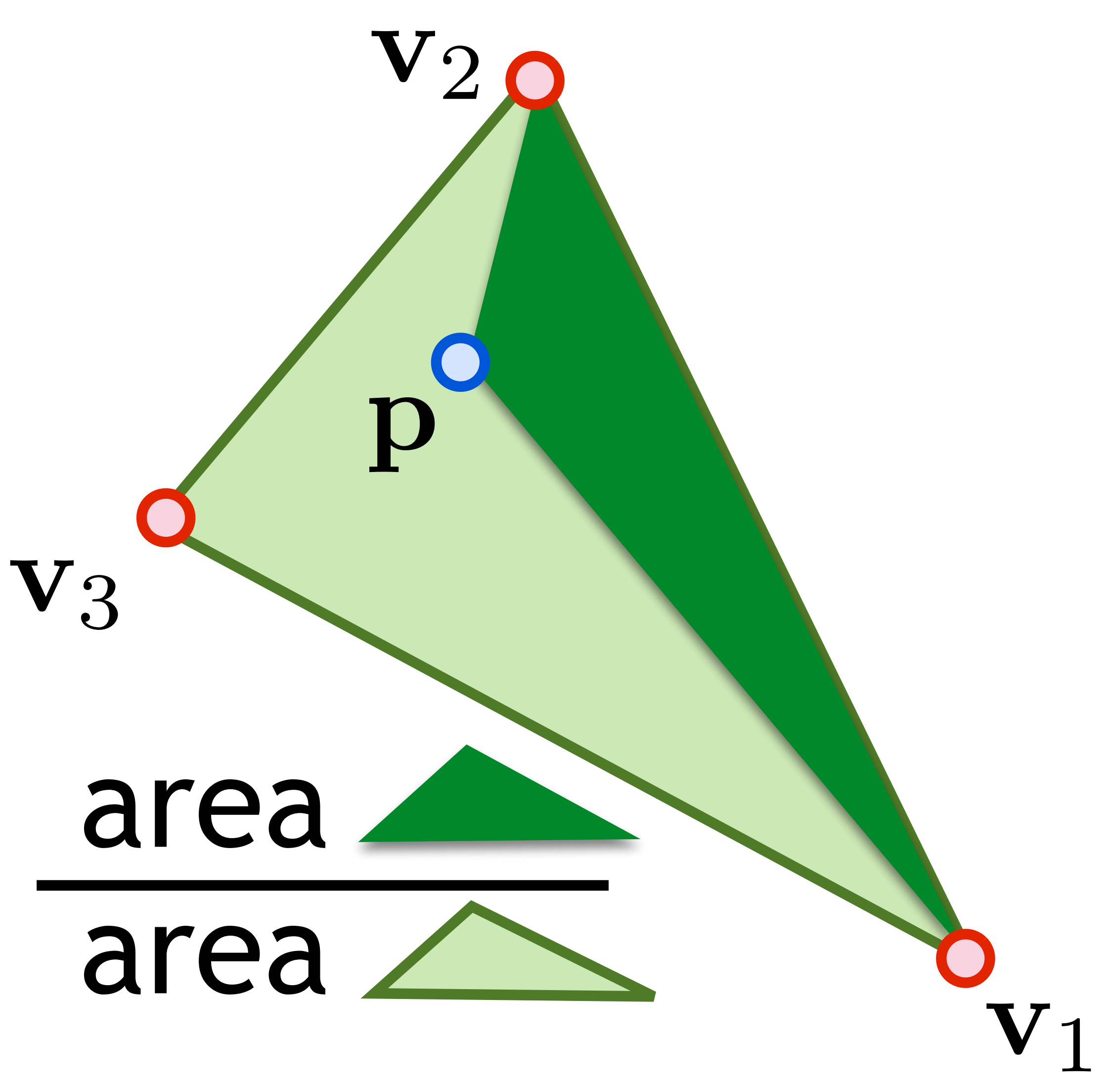
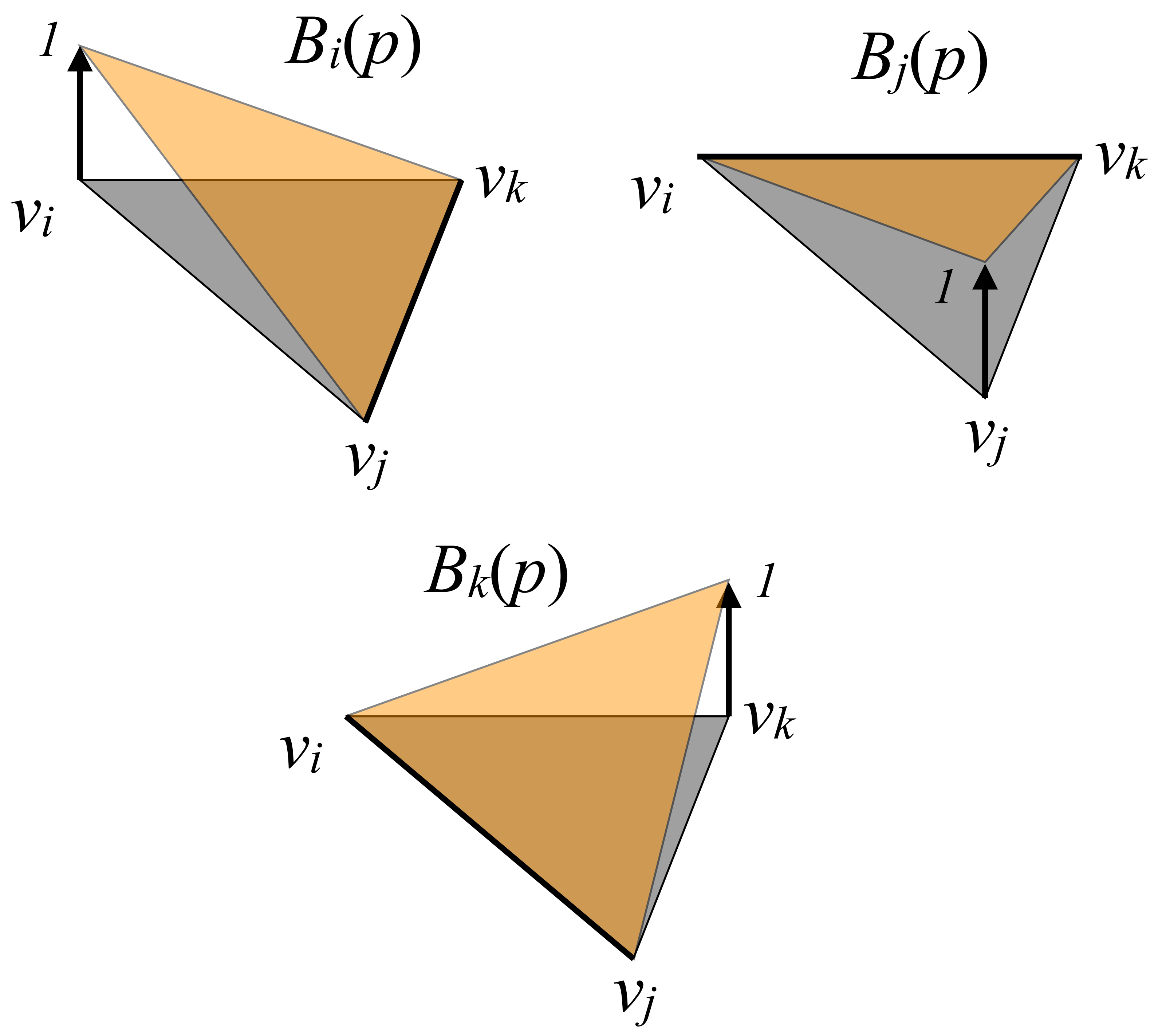


Piecewise linear functions on meshes

Hat functions and PL interpolation

$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$

$$B_i(\mathbf{p}) + B_j(\mathbf{p}) + B_k(\mathbf{p}) = 1$$



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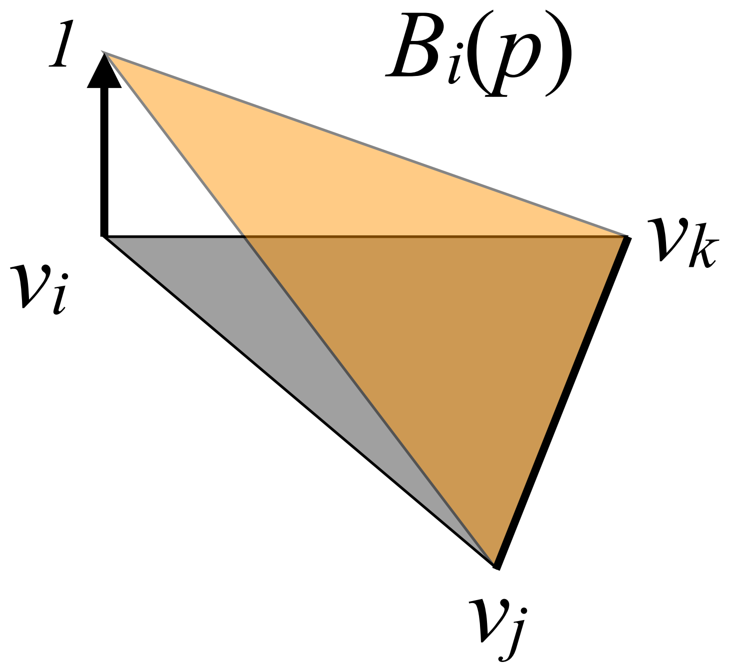
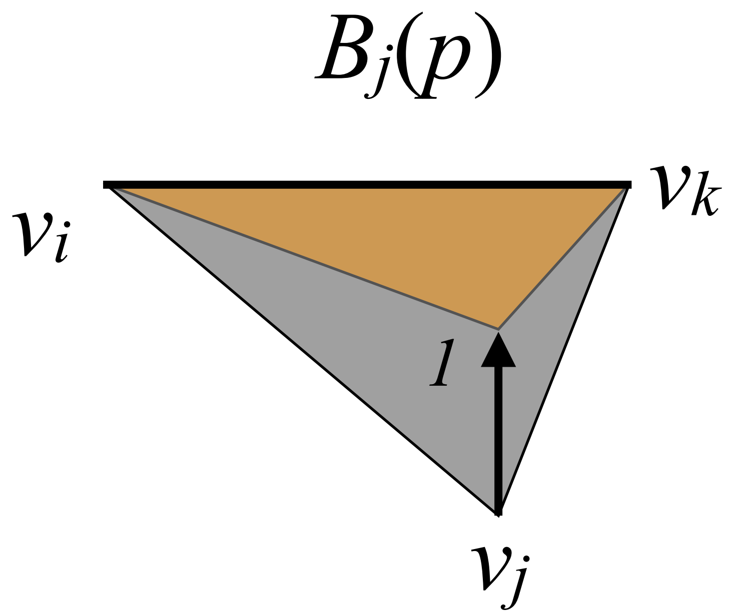
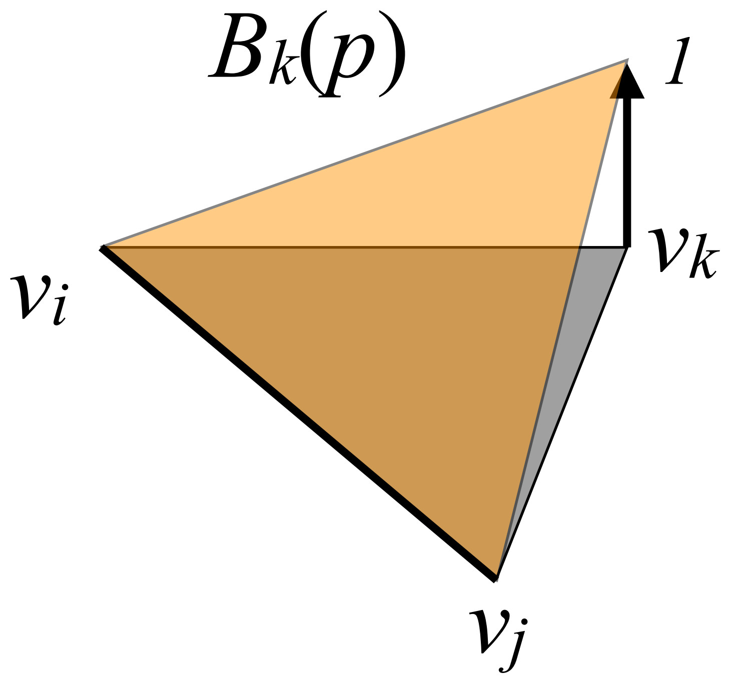




Gradients

$$\nabla f(\mathbf{p}) = \nabla B_i(\mathbf{p}) f_i + \nabla B_j(\mathbf{p}) f_j + \nabla B_k(\mathbf{p}) f_k$$

$$\nabla B_i(\mathbf{p}) + \nabla B_j(\mathbf{p}) + \nabla B_k(\mathbf{p}) = 0$$



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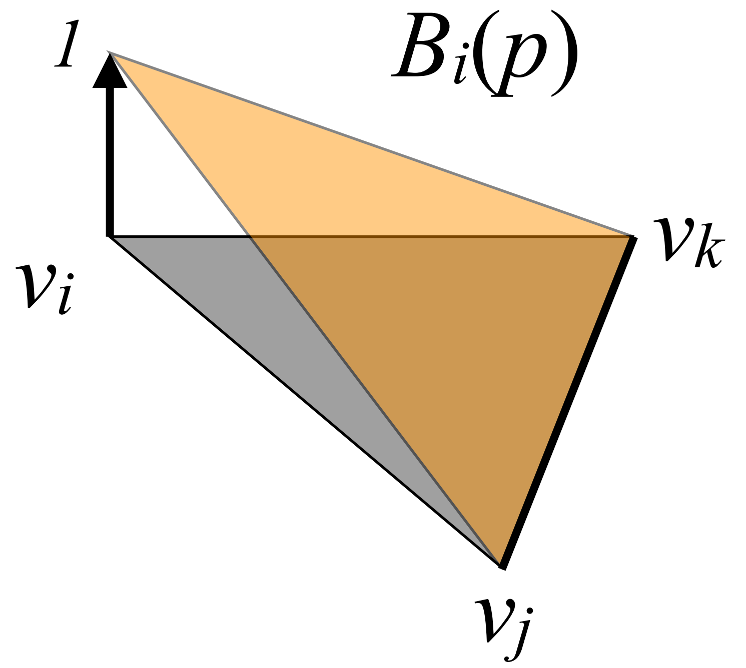

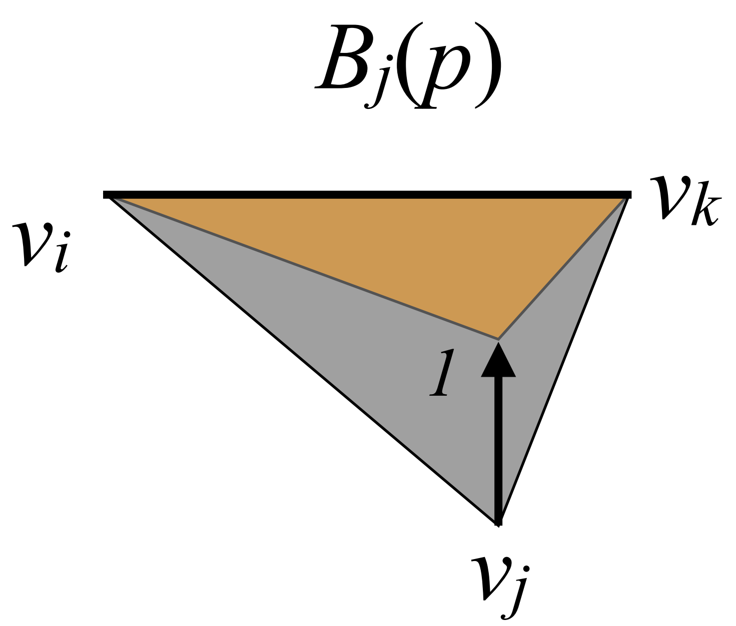

Gradients

$$\nabla f(\mathbf{p}) = \nabla B_i(\mathbf{p})f_i + \nabla B_j(\mathbf{p})f_j + \nabla B_k(\mathbf{p})f_k$$

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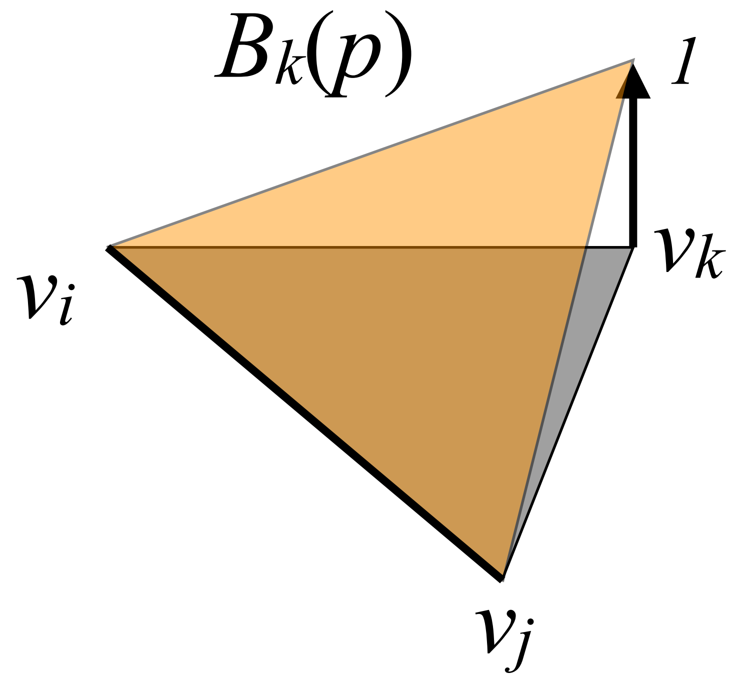
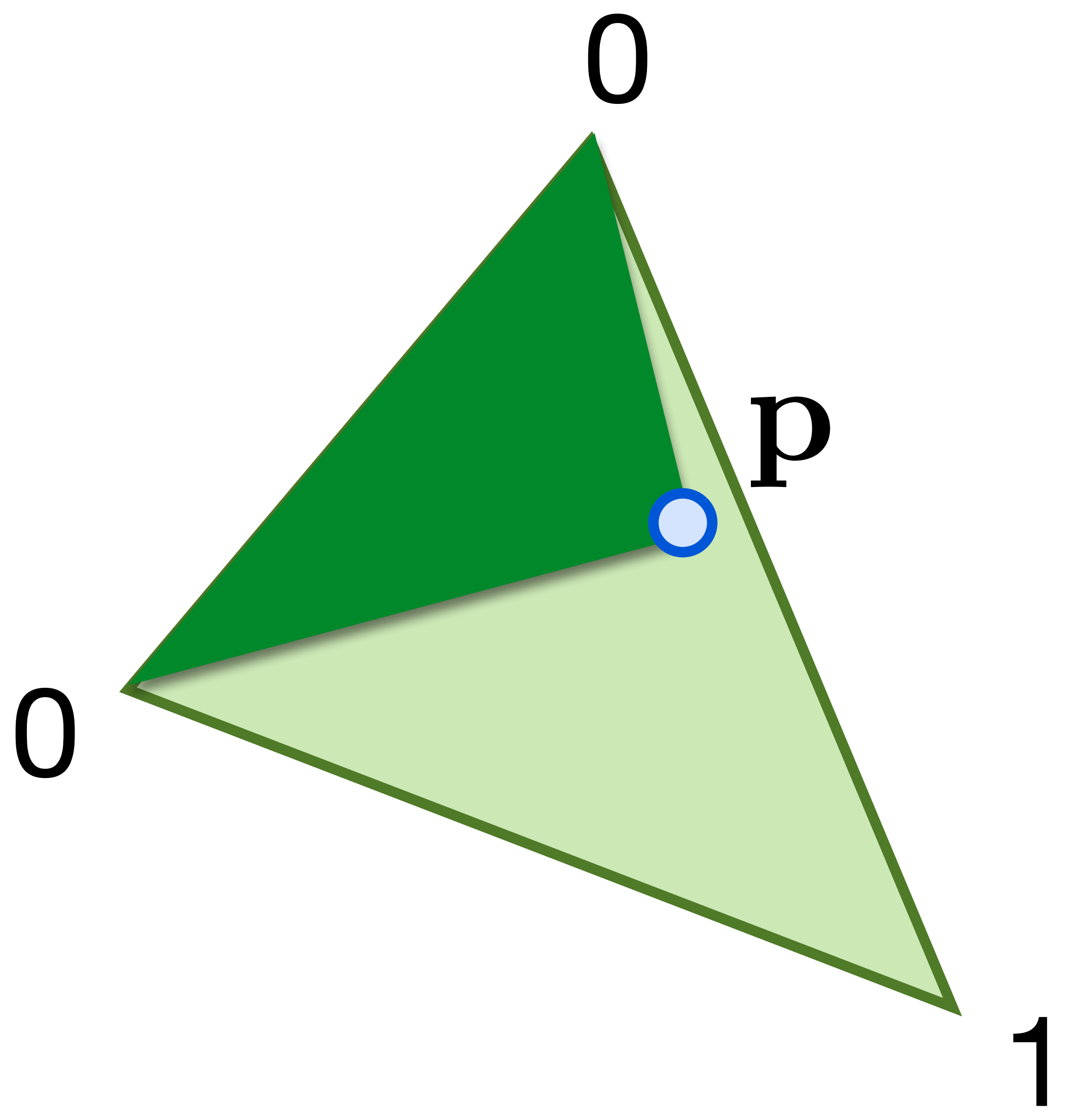


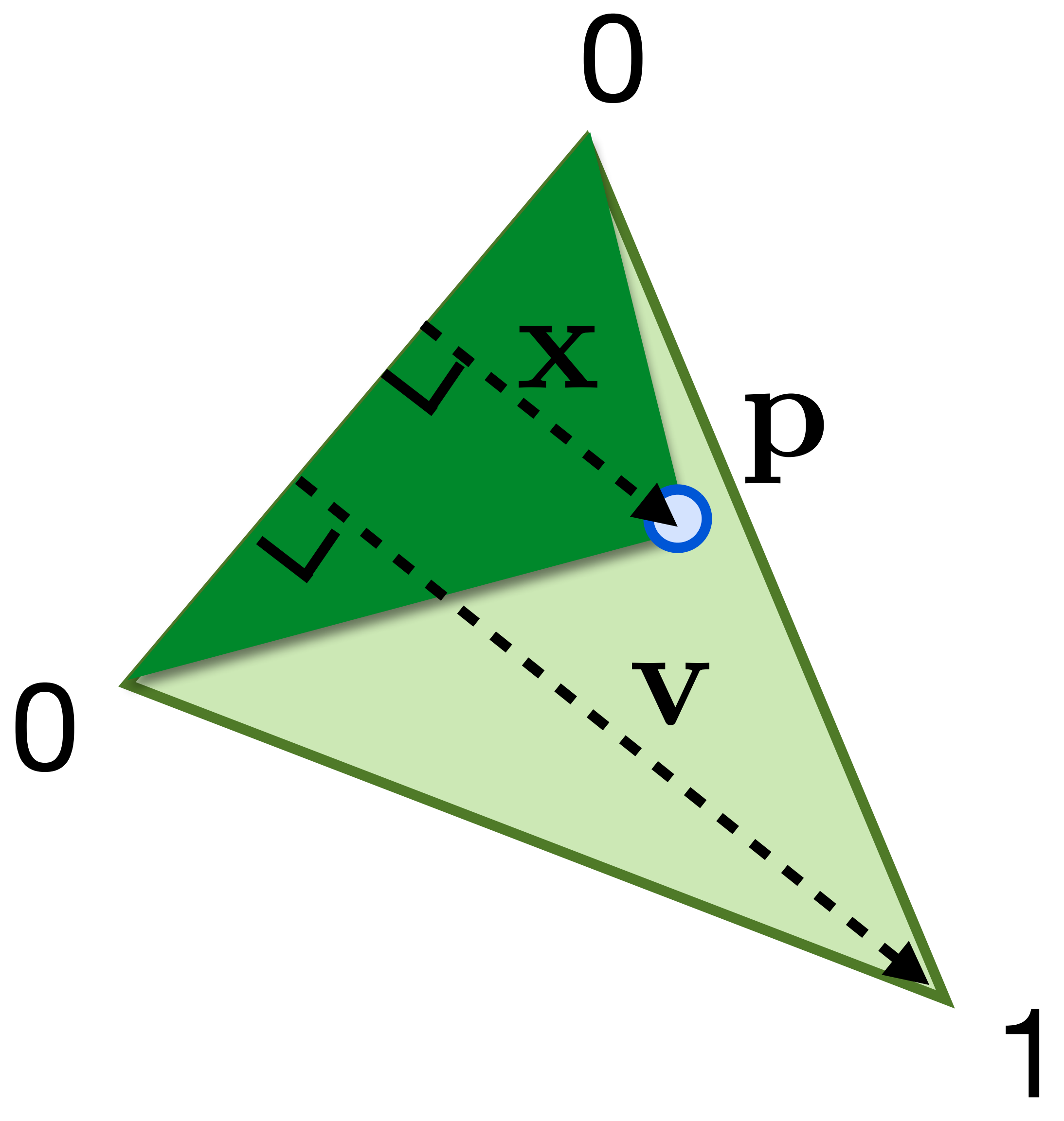
$$\nabla f(\mathbf{p}) = (f_j - f_i)\nabla B_j(\mathbf{p}) + (f_k - f_i)\nabla B_k(\mathbf{p})$$

The hat function



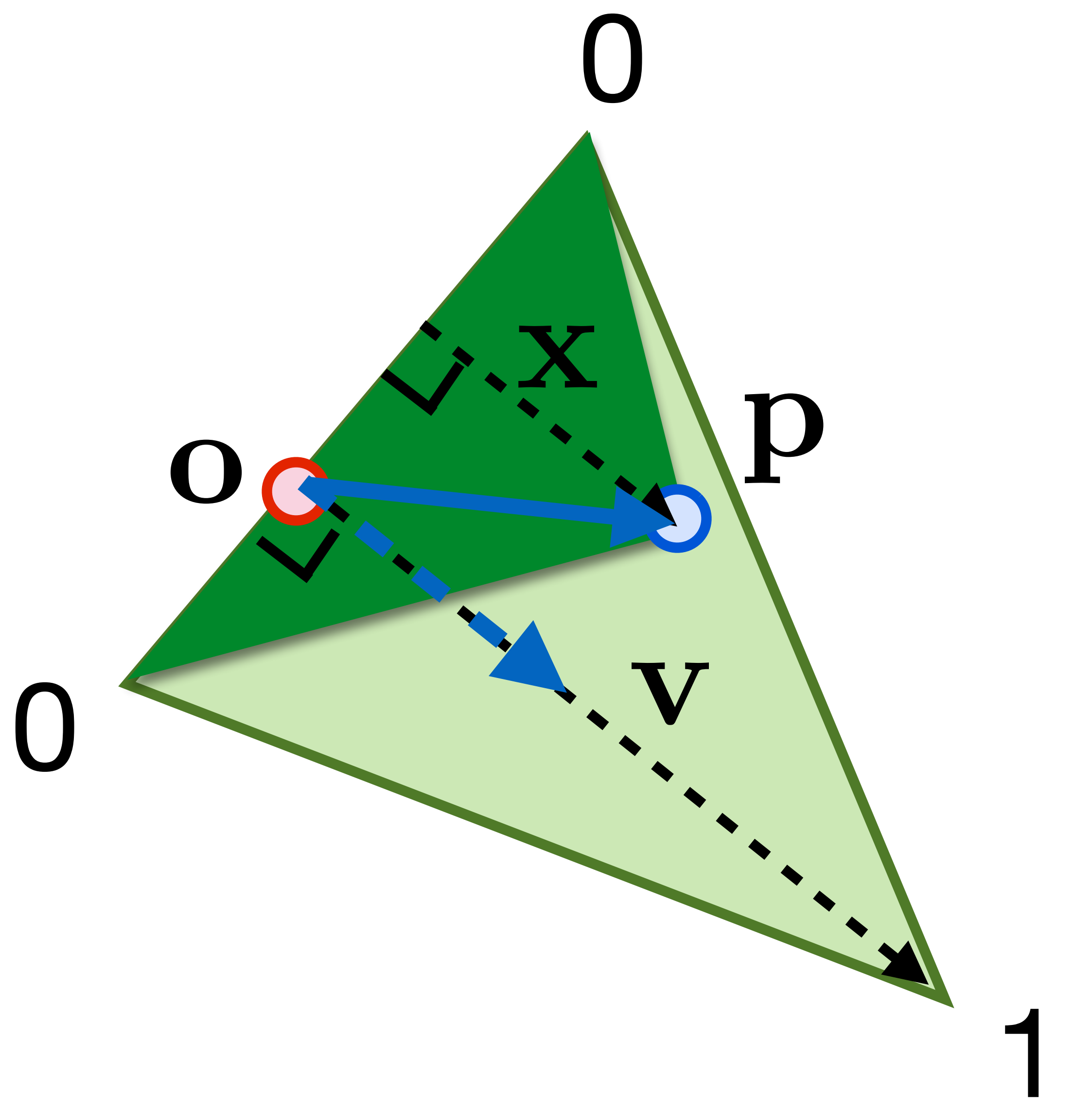
$$B(\mathbf{p}) = \frac{\text{area}}{\text{area}}$$

The hat function



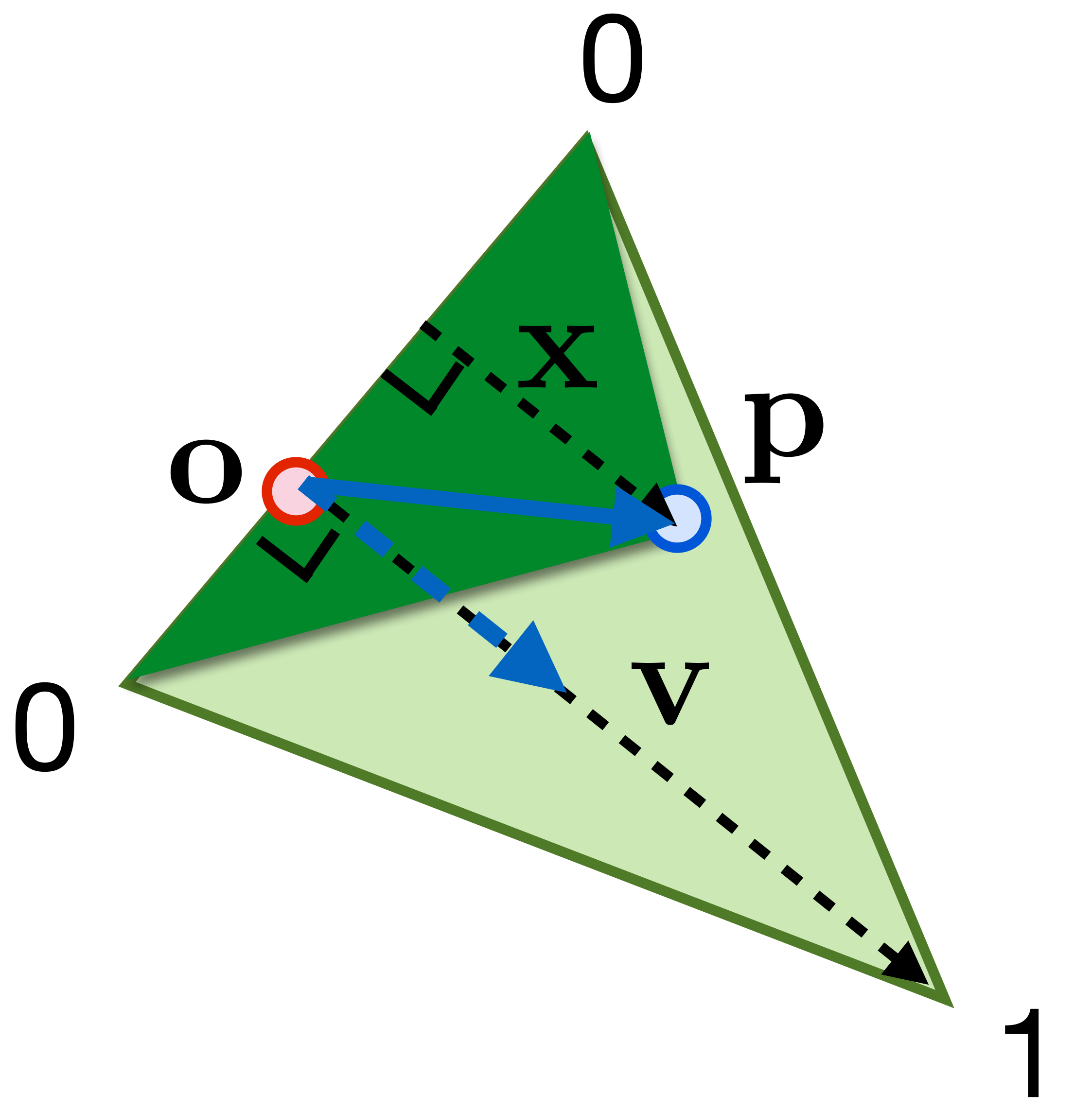
$$B(\mathbf{p}) = \frac{\text{area}}{\text{area}} = \frac{\|\mathbf{x}\|}{\|\mathbf{v}\|}$$

The hat function



$$B(\mathbf{p}) = \frac{\text{area}}{\text{area}} = \frac{\|\mathbf{x}\|}{\|\mathbf{v}\|} = \frac{(\mathbf{p}-\mathbf{o})\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}}{\|\mathbf{v}\|}$$

The hat function



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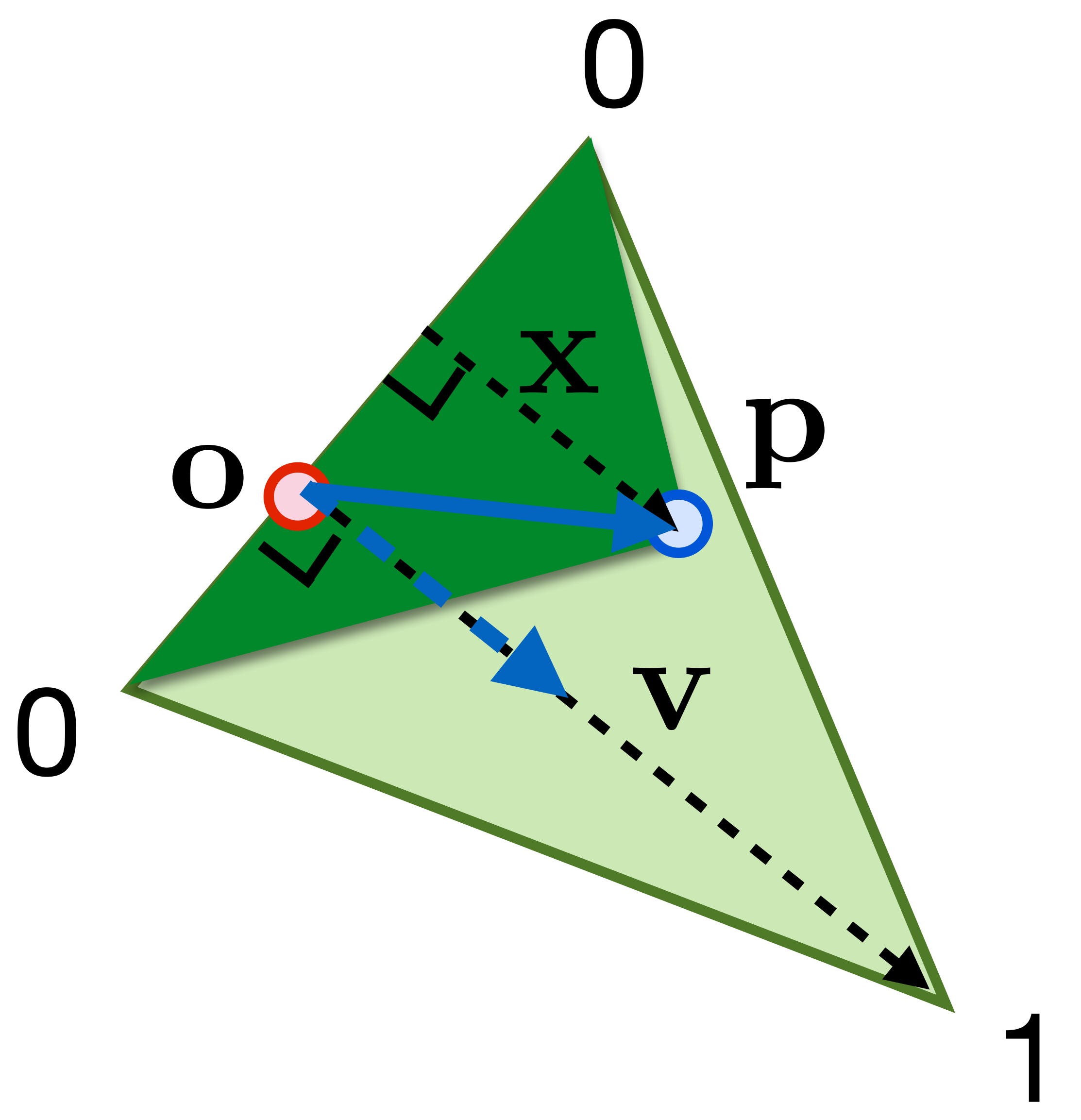
Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

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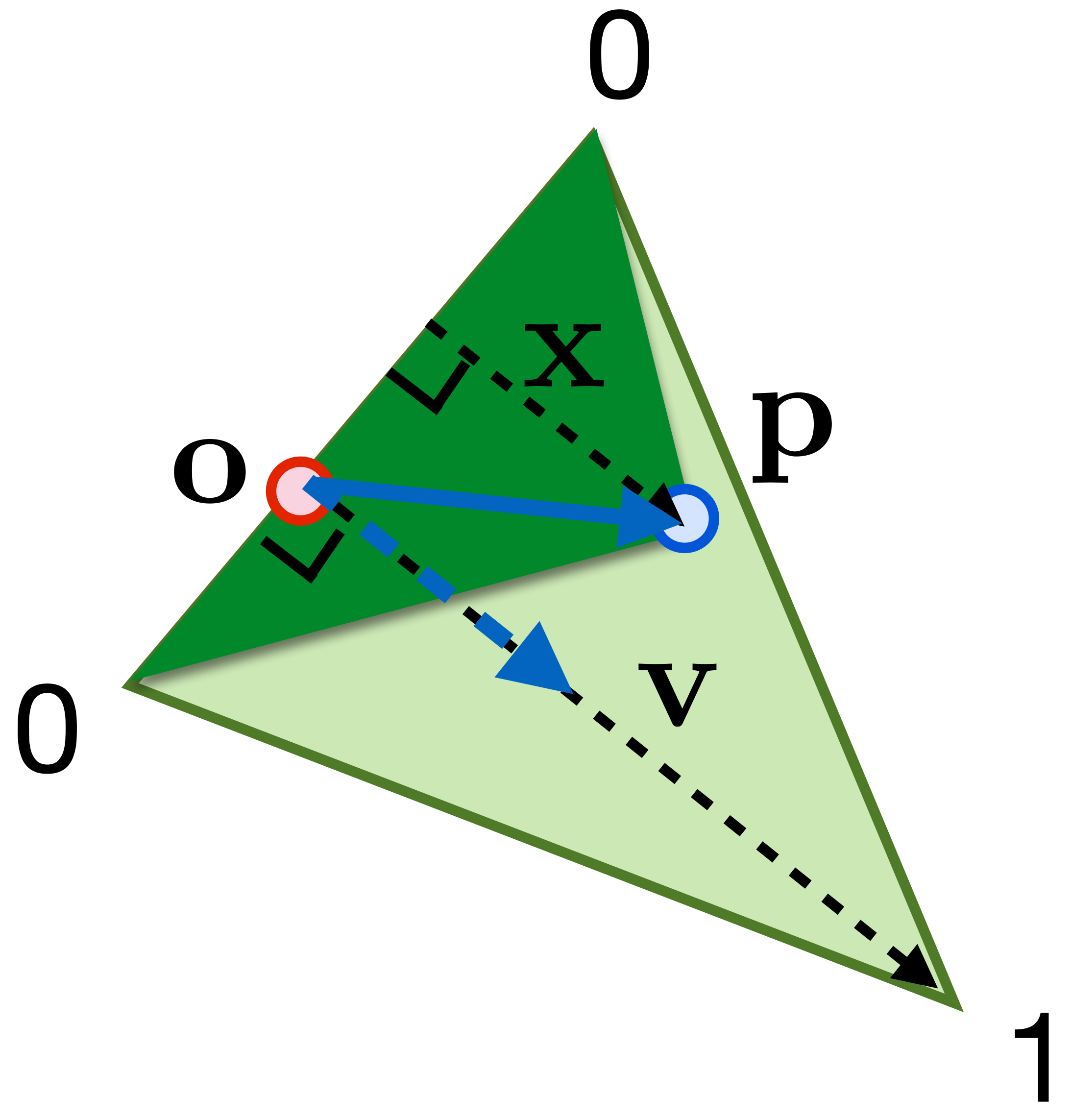

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{v}\|}$$

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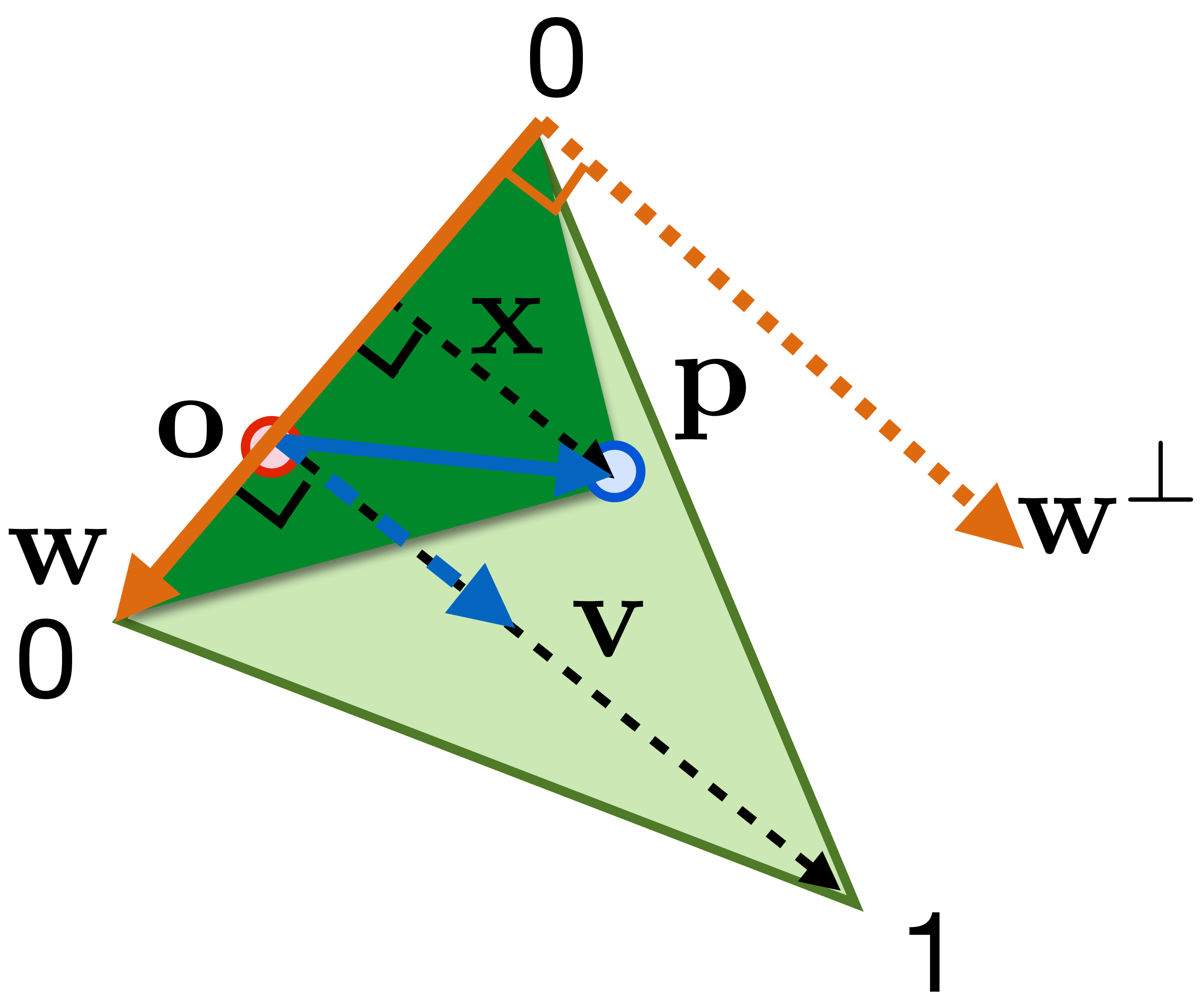

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^{\perp}}{\|\mathbf{w}^{\perp}\|}$$

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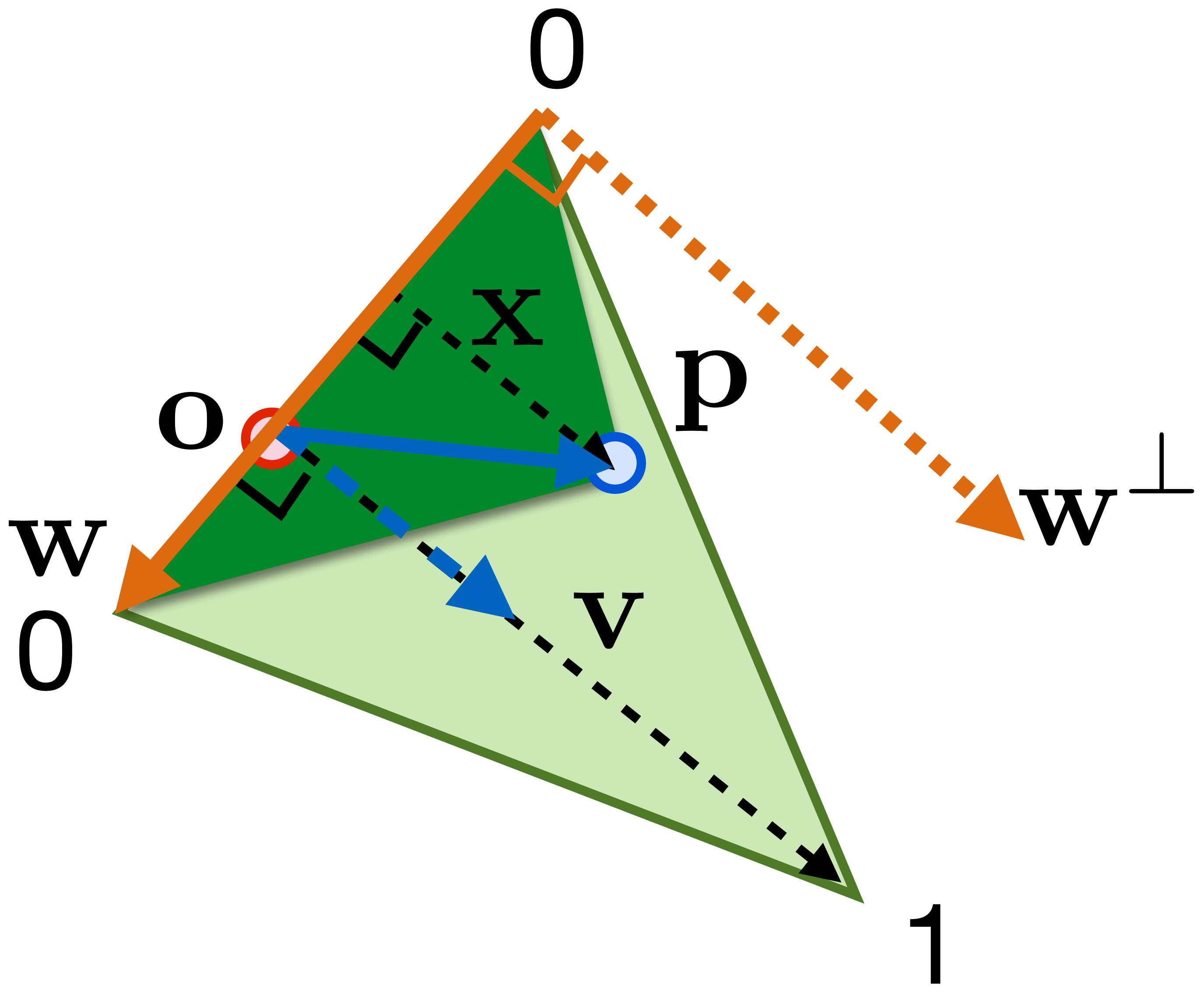

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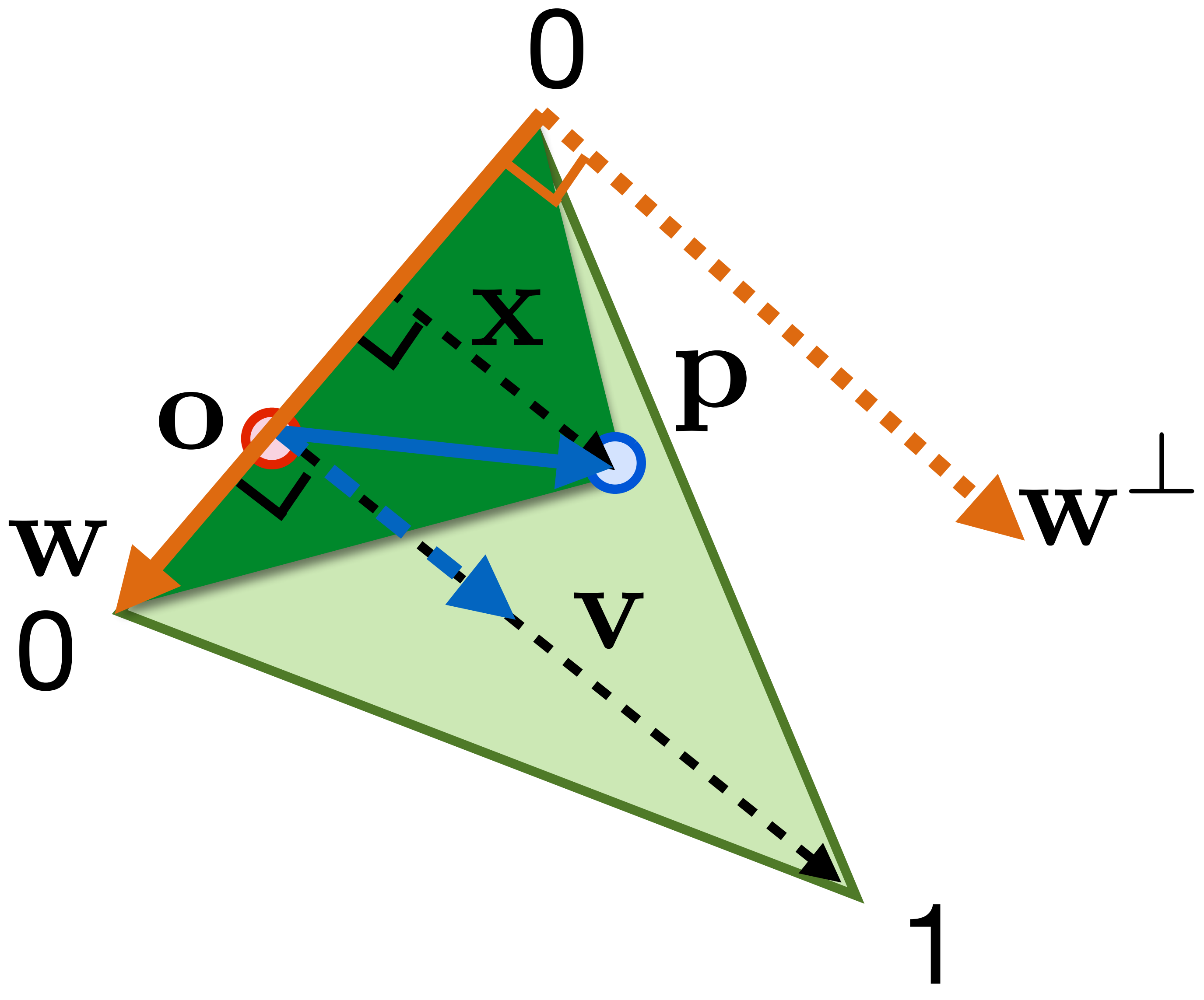

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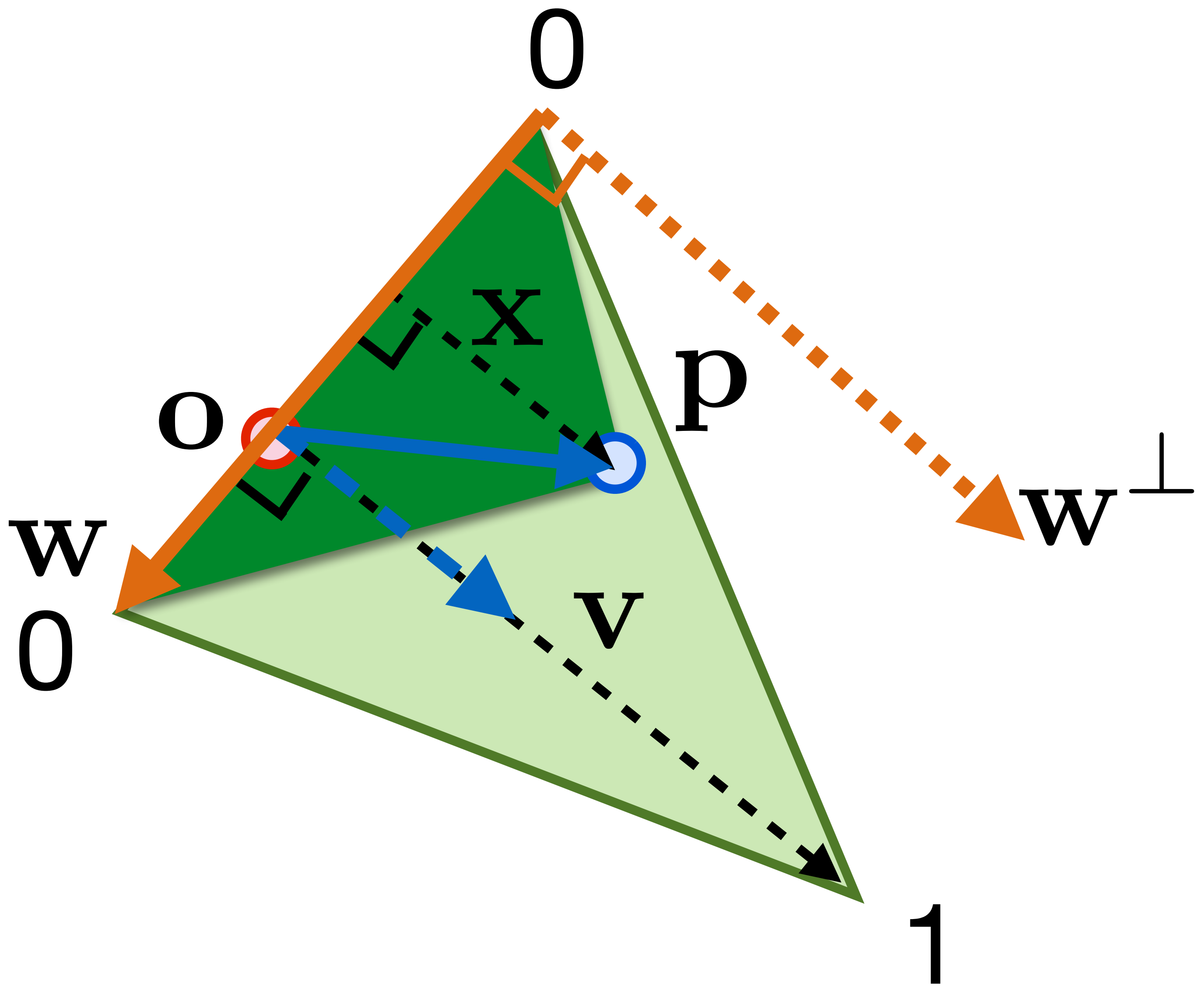

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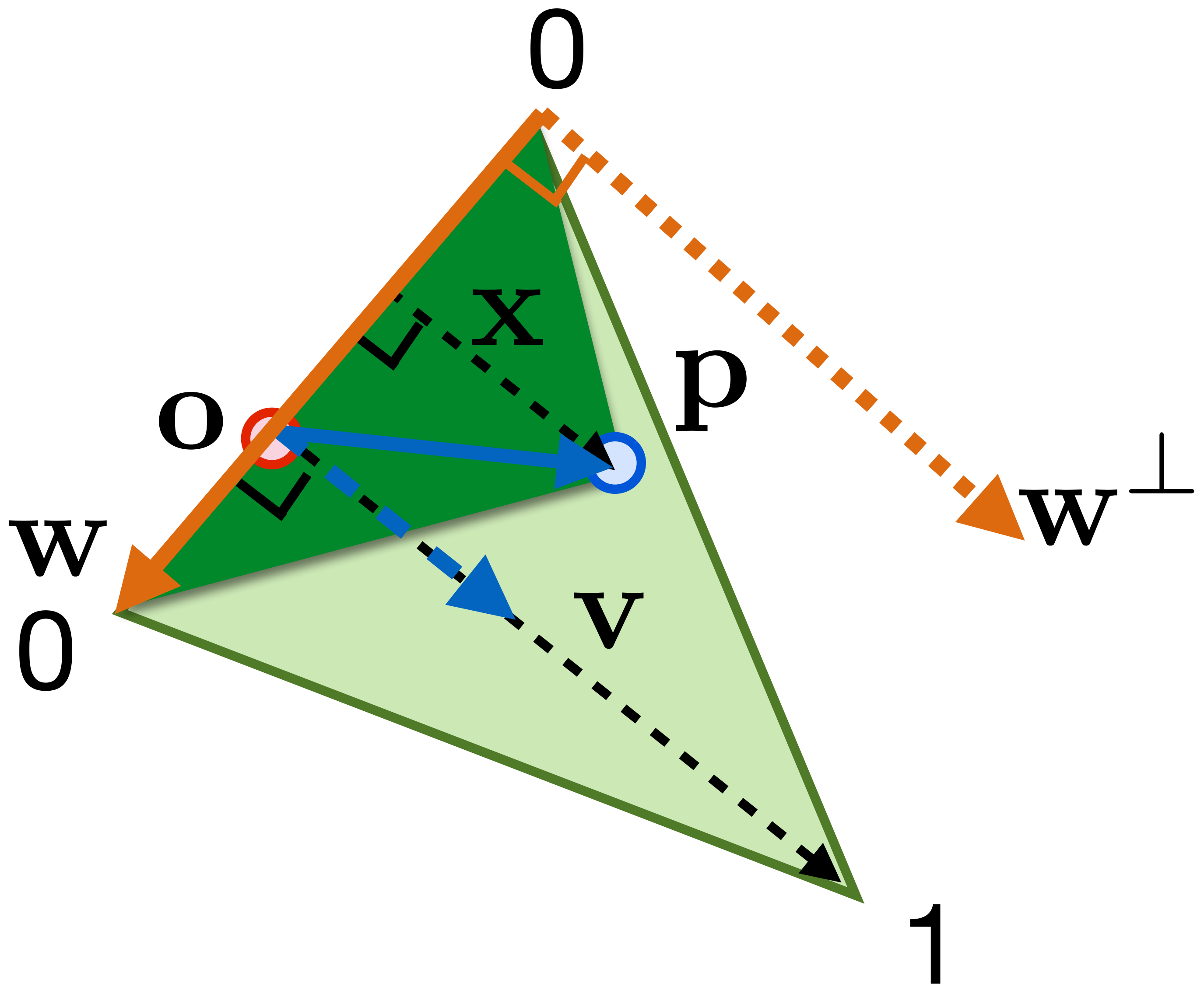

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$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

$$\boxed{A = \frac{\|\mathbf{v}\|\|\mathbf{w}\|}{2}}$$

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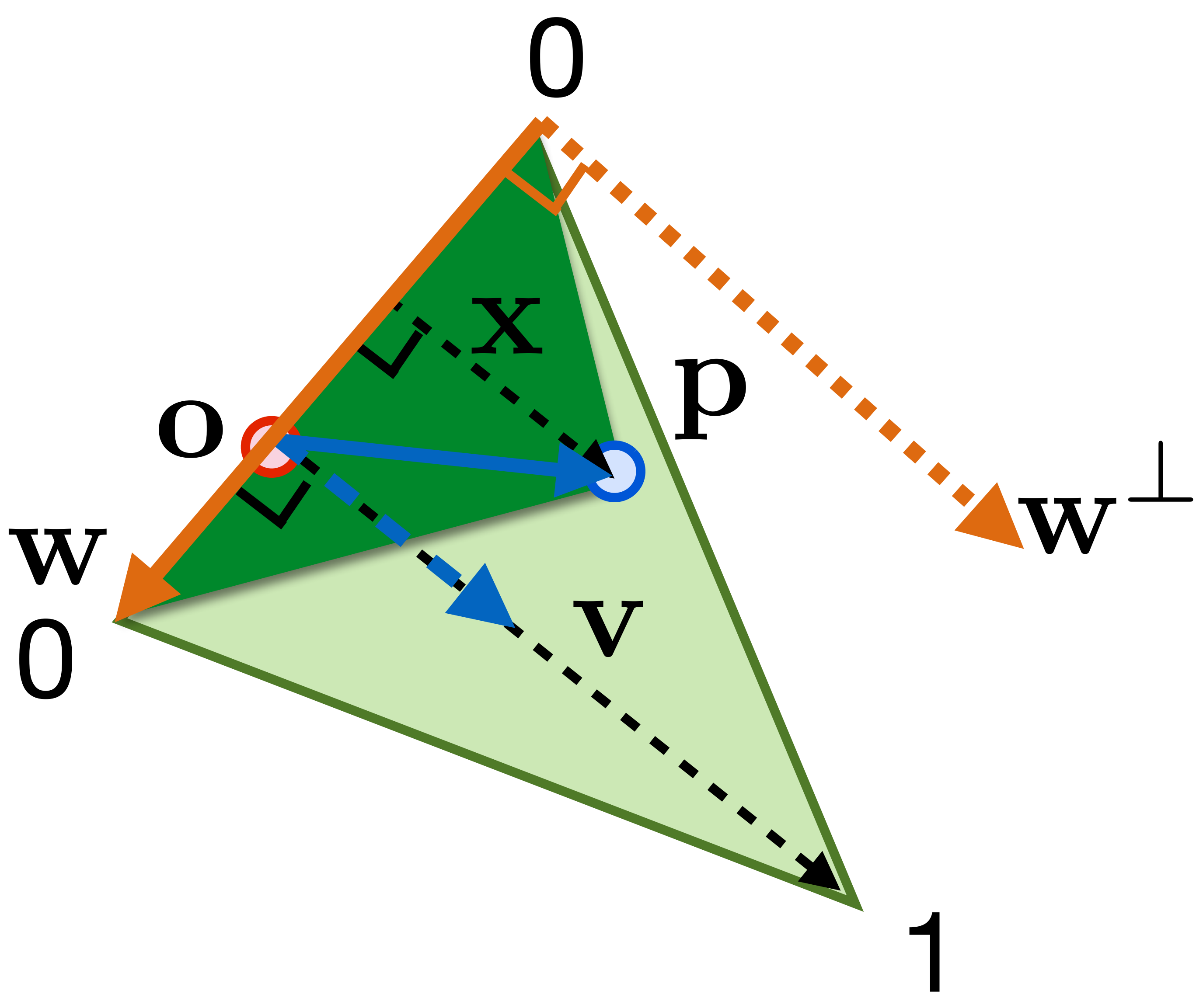

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$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{2A}$$

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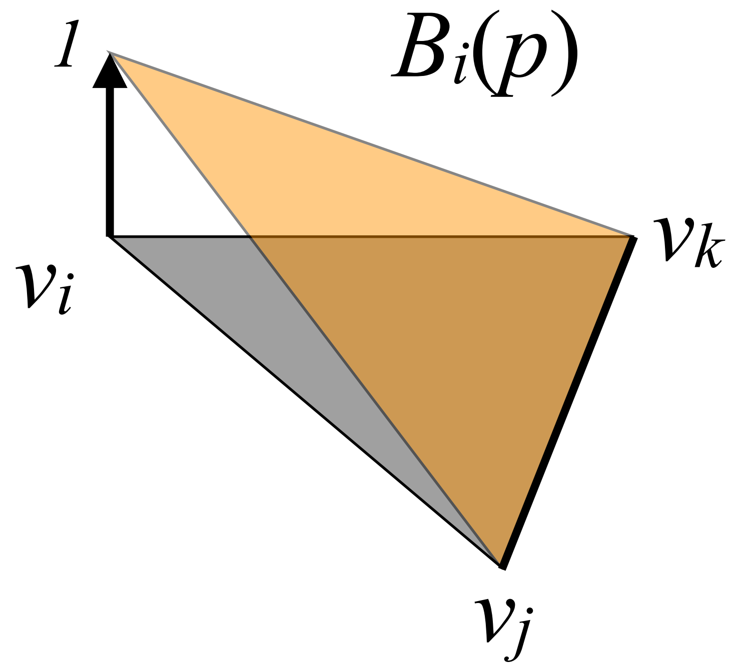

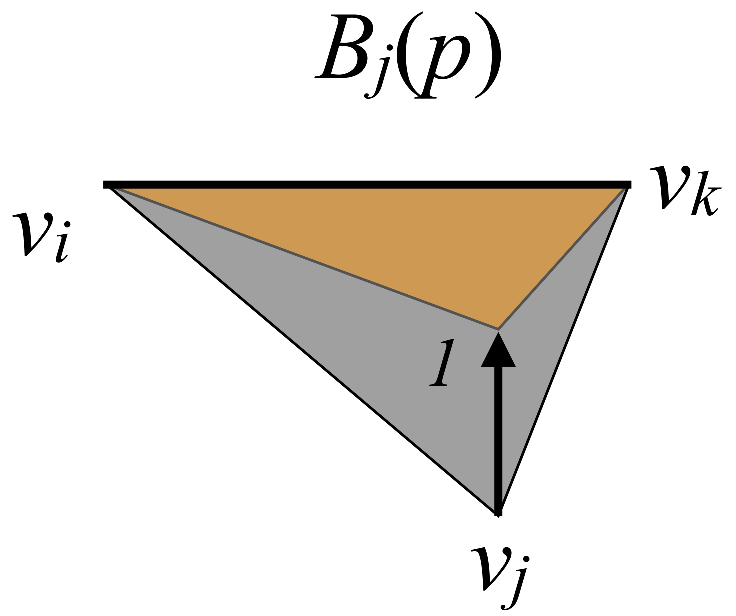

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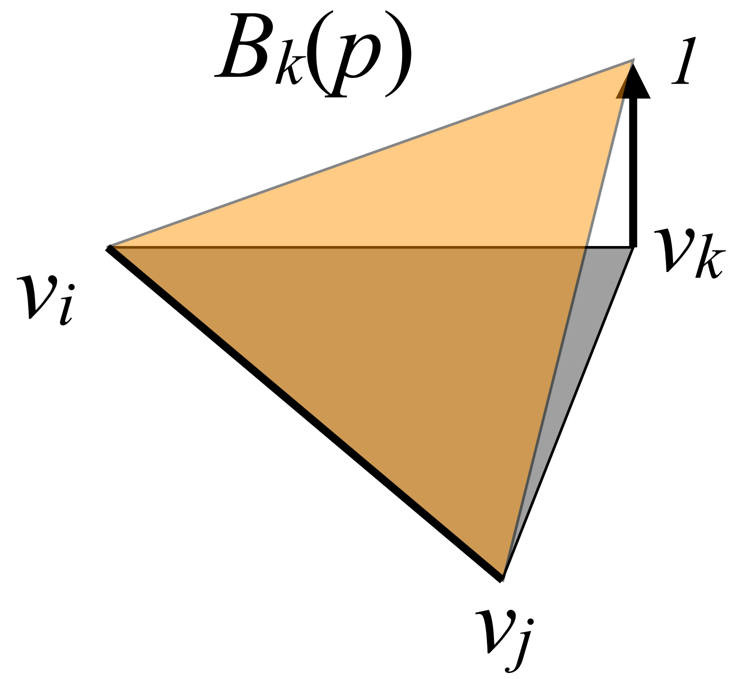

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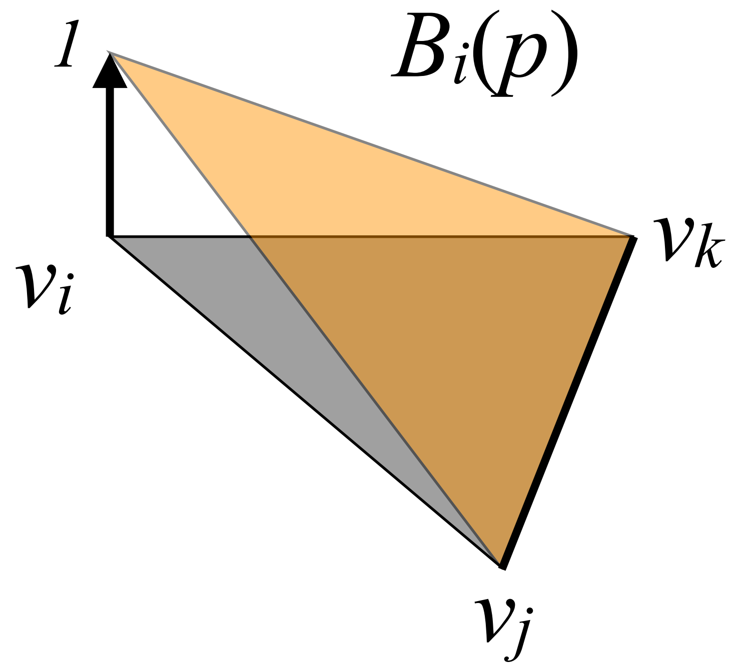

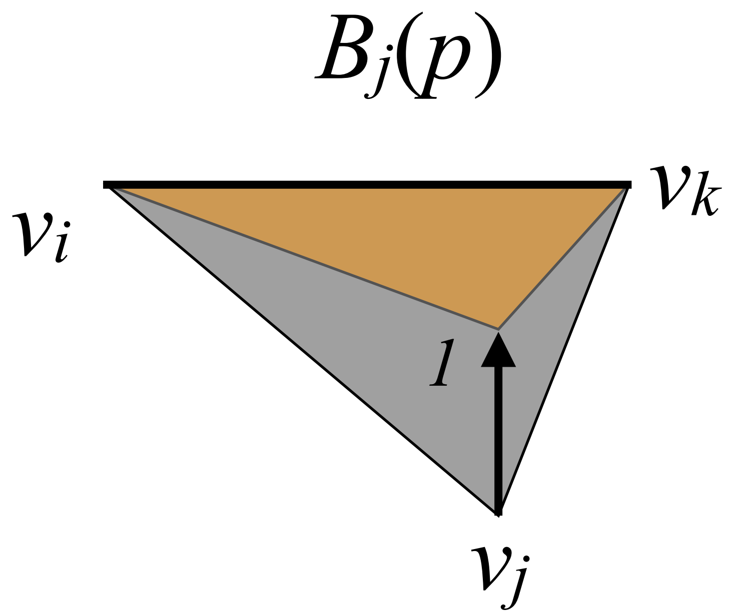

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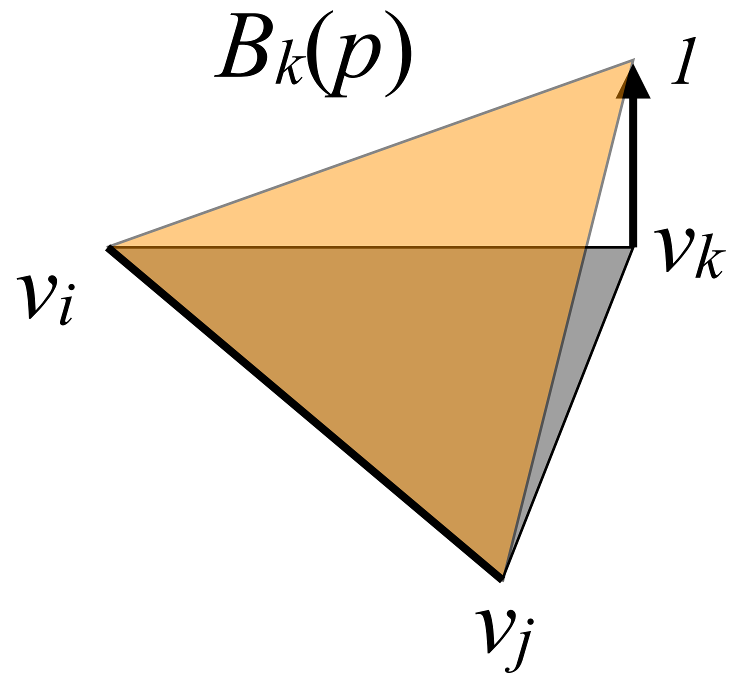


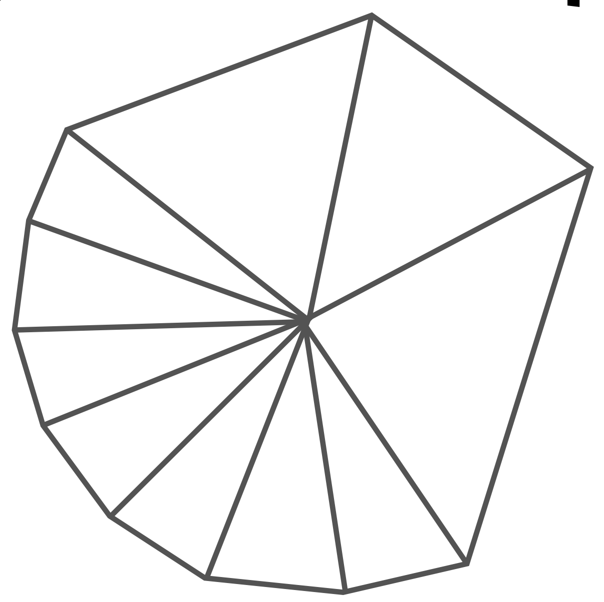
$$\nabla f(\mathbf{p}) = (f_j - f_i)\frac{(\mathbf{v_i} - \mathbf{v_k})^{\perp}}{2A} + (f_k - f_i)\frac{(\mathbf{v_j} - \mathbf{v_i})^{\perp}}{2A}$$

Discrete Laplace-Beltrami - uniform

- Directly measure the difference between function at a vertex with respect to the average of its neighbors

$$\Delta f(v_i) = \frac{1}{\mathcal{N}_1(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} (f_j - f_i) = \left(\frac{1}{\mathcal{N}_1(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} f_j \right) - f_i$$

- Simple and efficient - mesh geometry not taken into account
  - assumption: all edges have unit length, all triangles are equilateral
- not accurate for non-uniform meshing

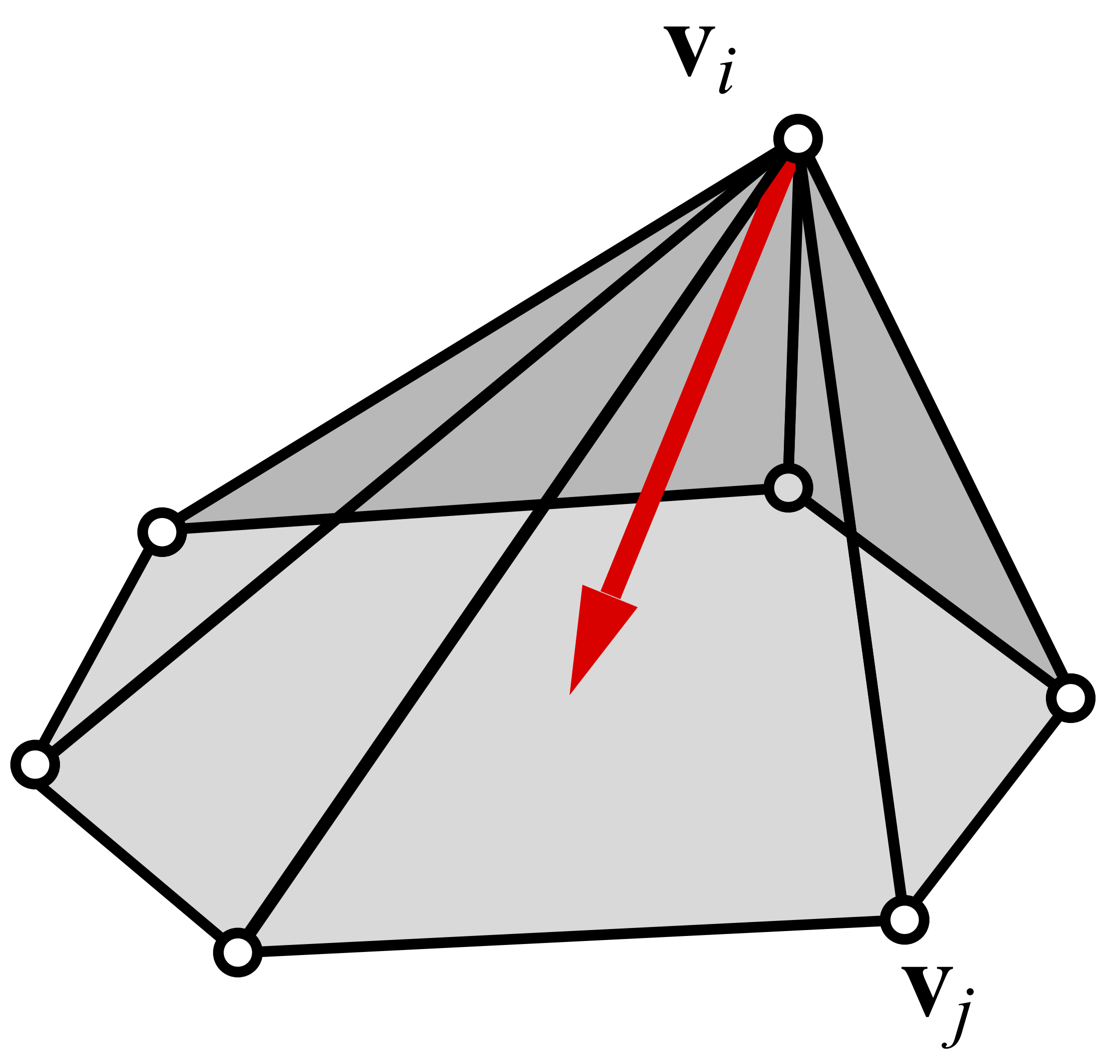


Discrete Laplace-Beltrami - uniform

- For the position function:

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

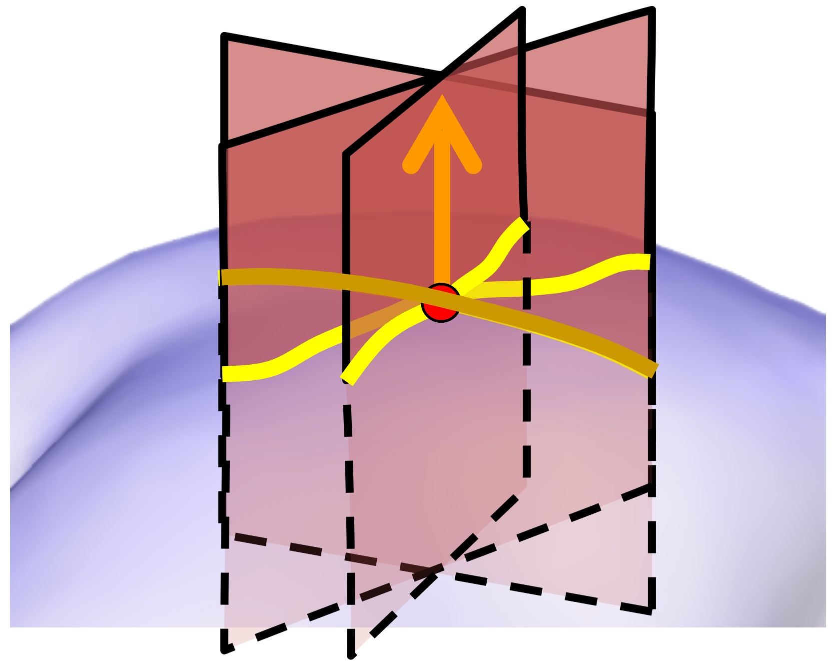
$$L_u(\mathbf{v}_i) = \frac{1}{|\mathcal{N}(i)|}\sum_{j\in\mathcal{N}(i)}(\mathbf{v}_j - \mathbf{v}_i) = \left(\frac{1}{|\mathcal{N}(i)|}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j\right) - \mathbf{v}_i$$



Discrete Laplace-Beltrami - uniform

- Intuition for uniform discretization

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$



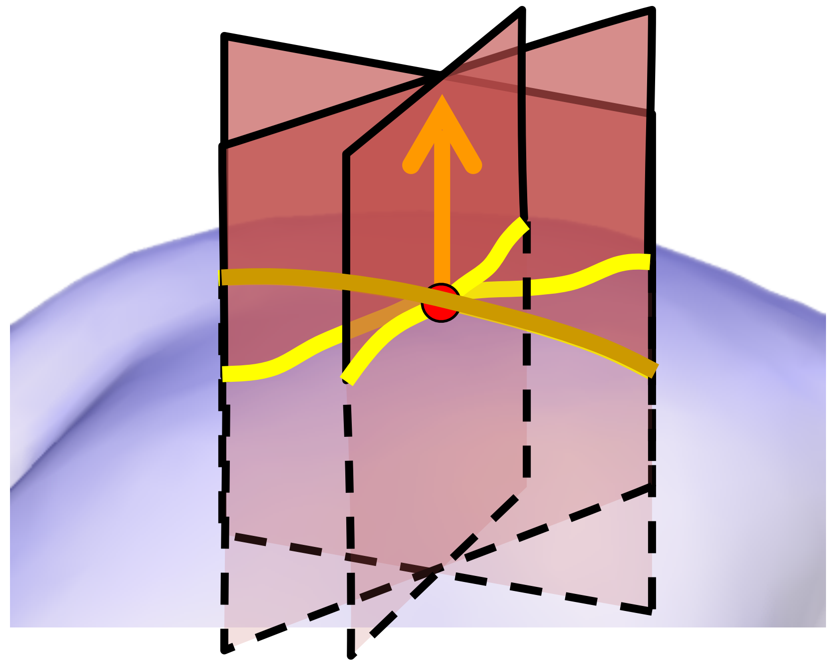
$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)d\varphi \qquad \boldsymbol{\kappa}\mathbf{n} = \gamma''$$

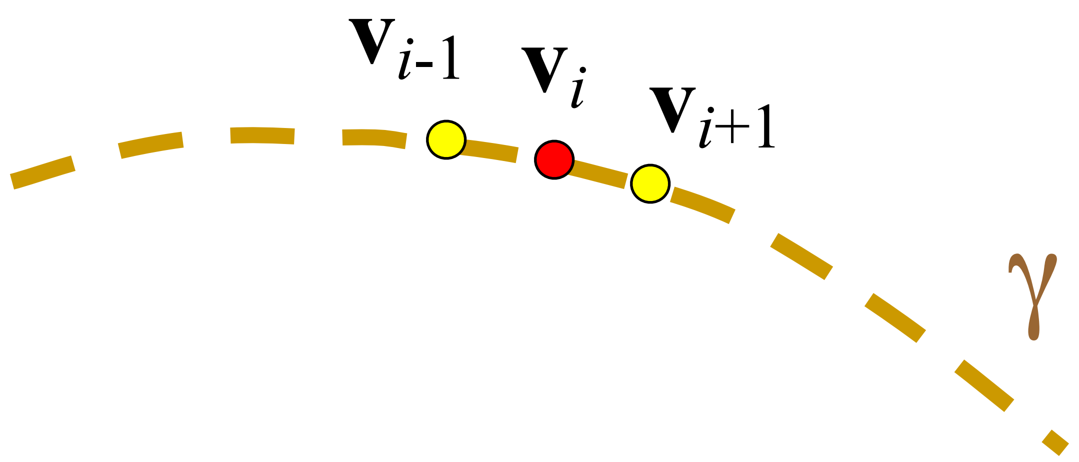
$$-2H\mathbf{n} = -2\left(\frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\, d\varphi\right)\mathbf{n} = -\frac{1}{\pi}\int_0^{2\pi} \kappa(\varphi)\mathbf{n}\, d\varphi = -\frac{1}{\pi}\int_0^{2\pi} \gamma''\, d\varphi$$

Discrete Laplace-Beltrami - uniform

- Intuition for uniform discretization

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$





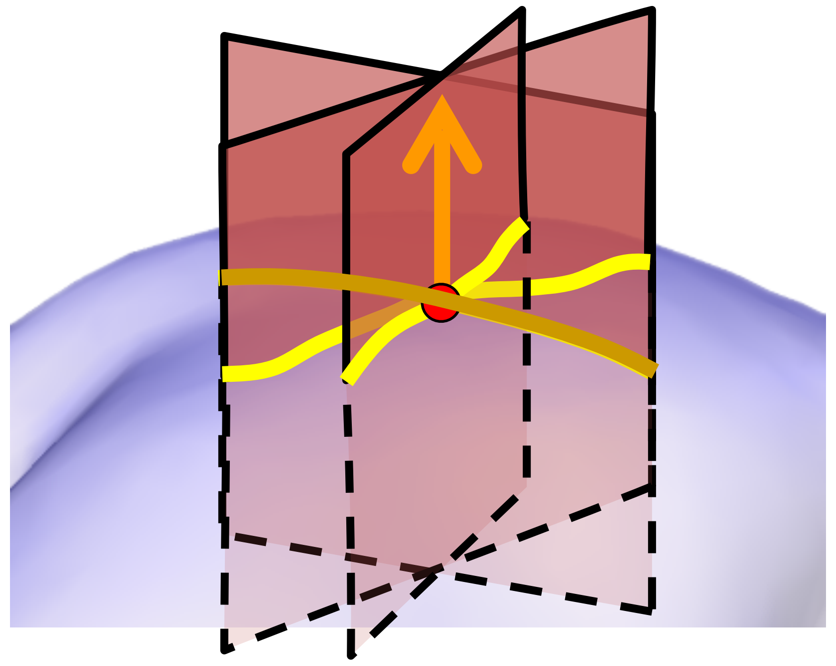
$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)d\varphi \qquad\qquad \boldsymbol{\kappa}\mathbf{n} = \gamma''$$

$$\gamma'' \approx \frac{1}{h}\left(\frac{\mathbf{v}_{i+1}-\mathbf{v}_i}{h} - \frac{\mathbf{v}_i-\mathbf{v}_{i-1}}{h}\right) = -\frac{2}{h^2}\left(\frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_{i+1}) - \mathbf{v}_i\right)$$

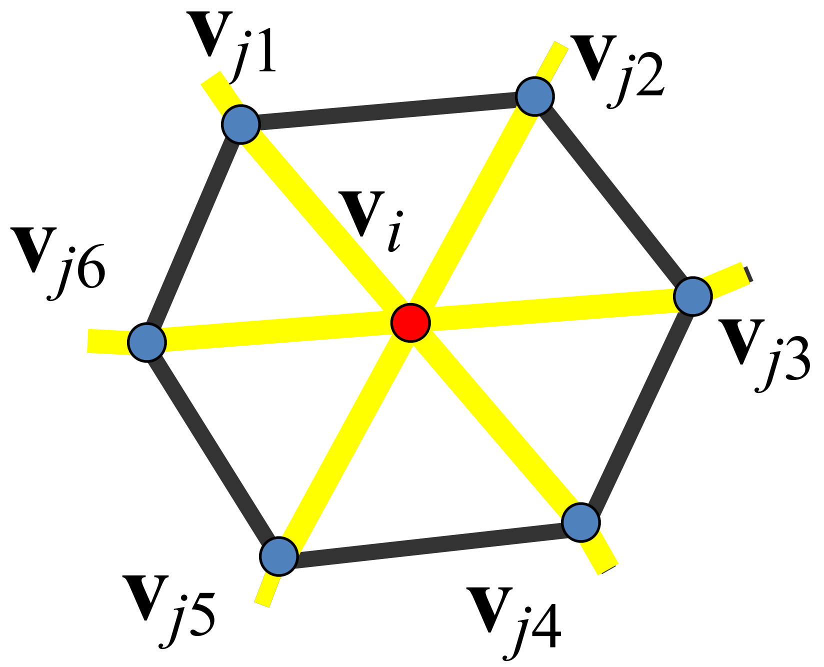
Discrete Laplace-Beltrami - uniform

- Intuition for uniform discretization

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$



$$\gamma'' \approx \frac{1}{h}\left(\frac{\mathbf{v}_{i+1}-\mathbf{v}_i}{h} - \frac{\mathbf{v}_i-\mathbf{v}_{i-1}}{h}\right) = -\frac{2}{h^2}\left(\frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_{i+1}) - \mathbf{v}_i\right)$$



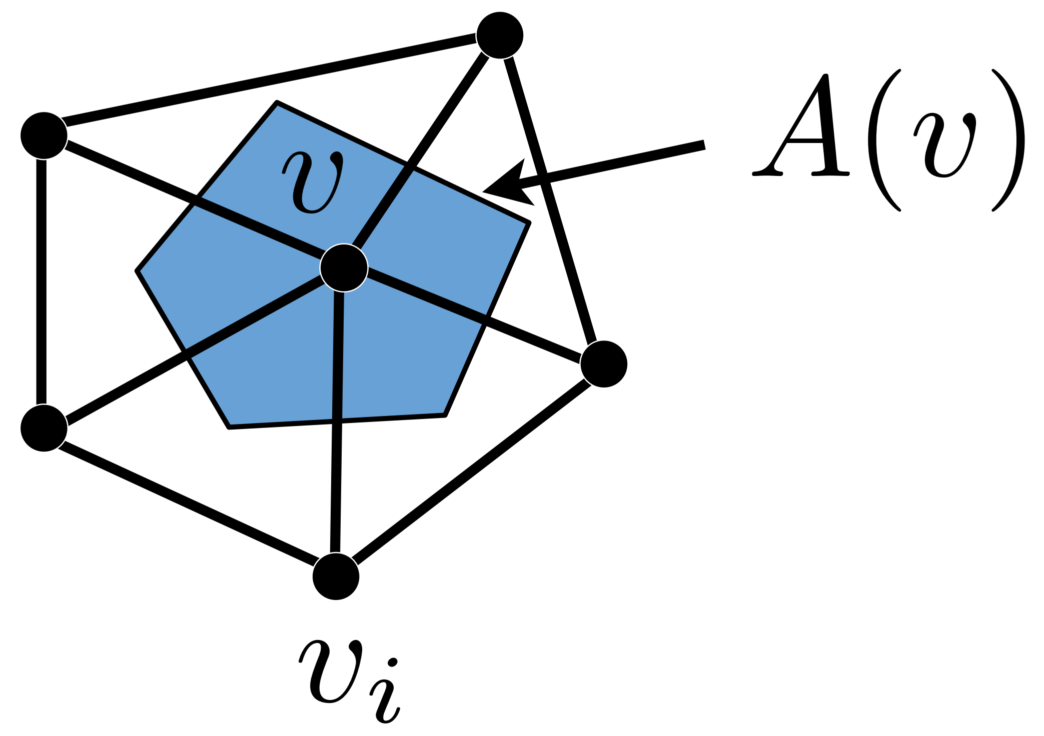
$$\Delta_{\mathcal{M}}\mathbf{p} = -\frac{1}{\pi}\int_0^{2\pi}\gamma'' d\varphi$$

$$\begin{aligned}
&\frac{1}{2}(\mathbf{v}_{j_1}+\mathbf{v}_{j_4}) - \mathbf{v}_i \;\; + \\
&\frac{1}{2}(\mathbf{v}_{j_2}+\mathbf{v}_{j_5}) - \mathbf{v}_i \;\; + \\
&\frac{1}{2}(\mathbf{v}_{j_3}+\mathbf{v}_{j_6}) - \mathbf{v}_i = \frac{1}{2}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j - 3\mathbf{v}_i = 3\left(\underbrace{\frac{1}{6}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j - \mathbf{v}_i}_{\boxed{L_u(\mathbf{v}_i)}}\right)
\end{aligned}$$

Discrete Laplace-Beltrami - cotangent

- Cotangent formula:
  - we integrate the value of the Laplacian over the averaging region $A(v)$ about a vertex v

$$\Delta f(v) = \frac{1}{A(v)} \iint_{A(v)} \Delta f(\mathbf{u}) dA$$



Discrete Laplace-Beltrami - cotangent

- Cotangent formula:
  - makes use of the divergence theorem:

$$\iint_A \nabla \cdot \mathbf{F}(\mathbf{u}) dA = \oint_{\partial A} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

where $\mathrm{n}(\mathrm{u})$ is the outward pointing unit normal at each point u of the boundary of region A

Discrete Laplace-Beltrami - cotangent

- Cotangent formula:

$$\int_{A(v)} \Delta f(\mathbf{u}) dA = \int_{A(v)} \mathrm{div} \nabla f(\mathbf{u}) dA = \int_{\partial A(v)} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

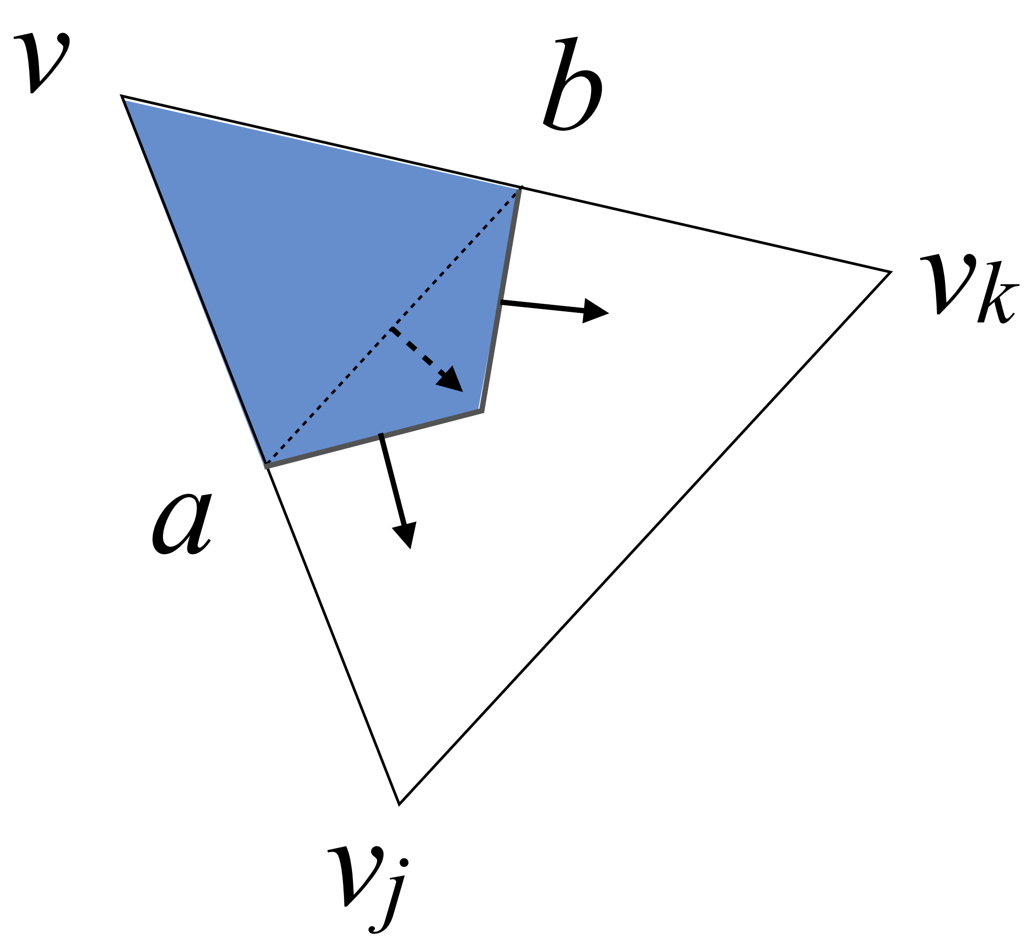
  - we split the integral separately for each triangle of $A(v)$:

$$\int_{\partial A(v)} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds = \sum_{t \in star(v)} \int_{\partial A(v) \cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

Discrete Laplace-Beltrami - cotangent

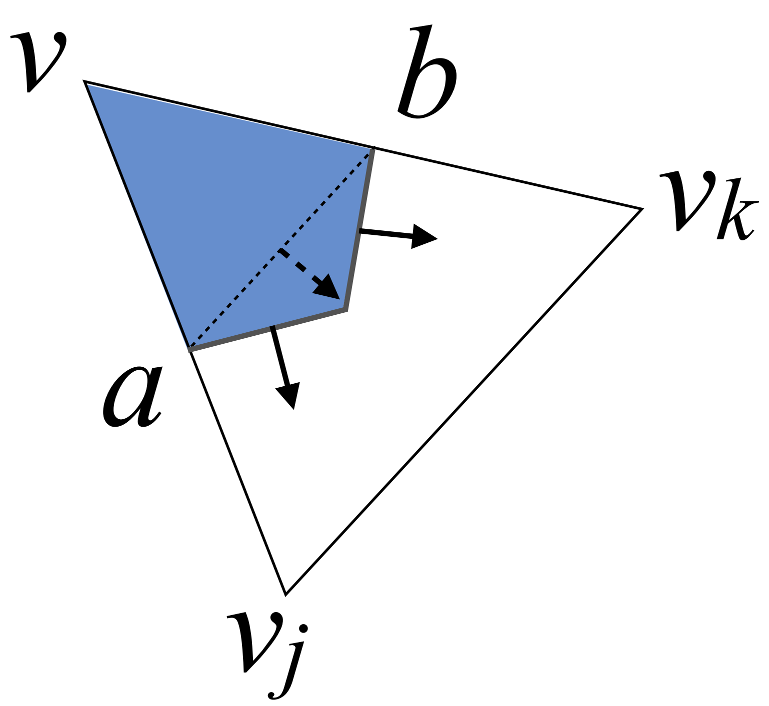
- Cotangent formula: gradient is constant in each triangle

$$\int_{\partial A(v)\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds = \nabla f(\mathbf{u}) \cdot (a-b)^{\perp} = \frac{1}{2} \nabla f(\mathbf{u}) \cdot (v_j - v_k)^{\perp}$$



Discrete Laplace-Beltrami - cotangent

- Cotangent formula: plug in gradient formula



$$\int_{\partial A(v)\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds =$$

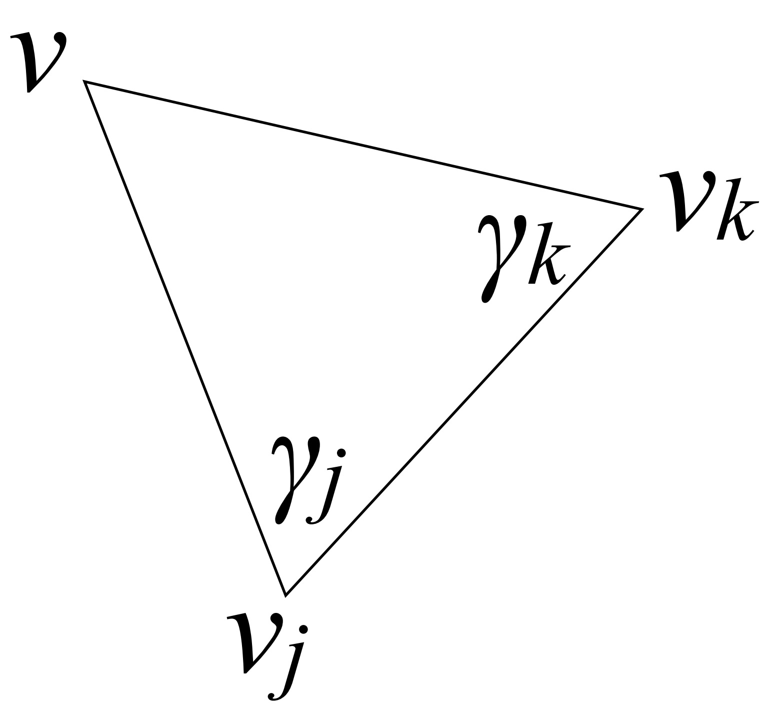
$$= (f(v_j) - f(v)) \frac{(v - v_k)^{\perp} \cdot (v_j - v_k)^{\perp}}{4A_t} +$$

$$+ (f(v_k) - f(v)) \frac{(v_j - v)^{\perp} \cdot (v_j - v_k)^{\perp}}{4A_t}$$

Discrete Laplace-Beltrami - cotangent

- Cotangent formula: area of triangle

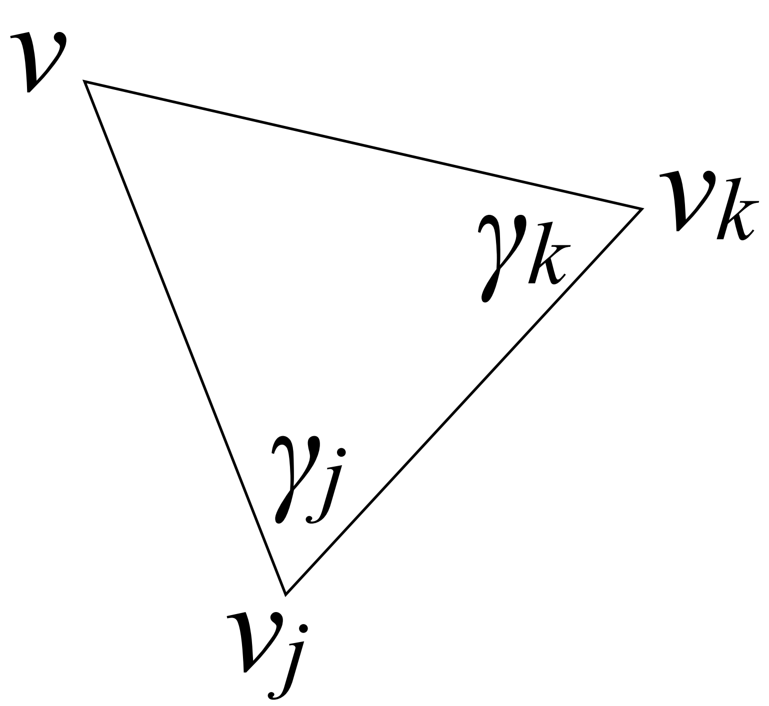
$$A_t = \frac{1}{2} \sin \gamma_j ||v_j - v|| ||v_j - v_k|| =$$
$$= \frac{1}{2} \sin \gamma_k ||v - v_k|| ||v_j - v_k||$$



$$\cos \gamma_j = \frac{(v_j - v) \cdot (v_j - v_k)}{||v_j - v|| ||v_j - v_k||} \qquad \cos \gamma_k = \frac{(v - v_k) \cdot (v_j - v_k)}{||v - v_k|| ||v_j - v_k||}$$

Discrete Laplace-Beltrami - cotangent

- Cotangent formula: substitute

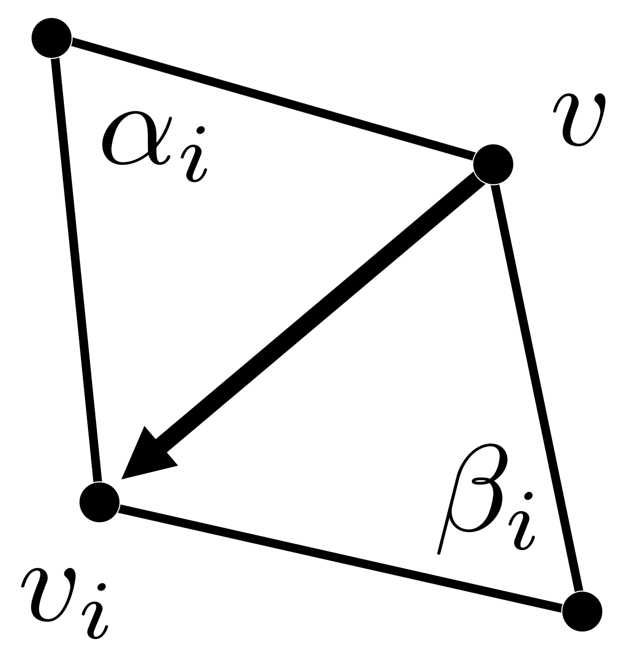


$$\int_{\partial A(v)\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds =$$

$$= \frac{1}{2}(\cot \gamma_k (f(v_j) - f(v)) + \cot \gamma_j (f(v_k) - f(v)))$$

Discrete Laplace-Beltrami - cotangent

- Cotangent formula: sum on all triangles



$$\int_{A(v)} \Delta f(\mathbf{u}) dA = \frac{1}{2} \sum_{v_i \in N_1(v)} (\cot \alpha_i + \cot \beta_i)(f(v_i) - f(v))$$

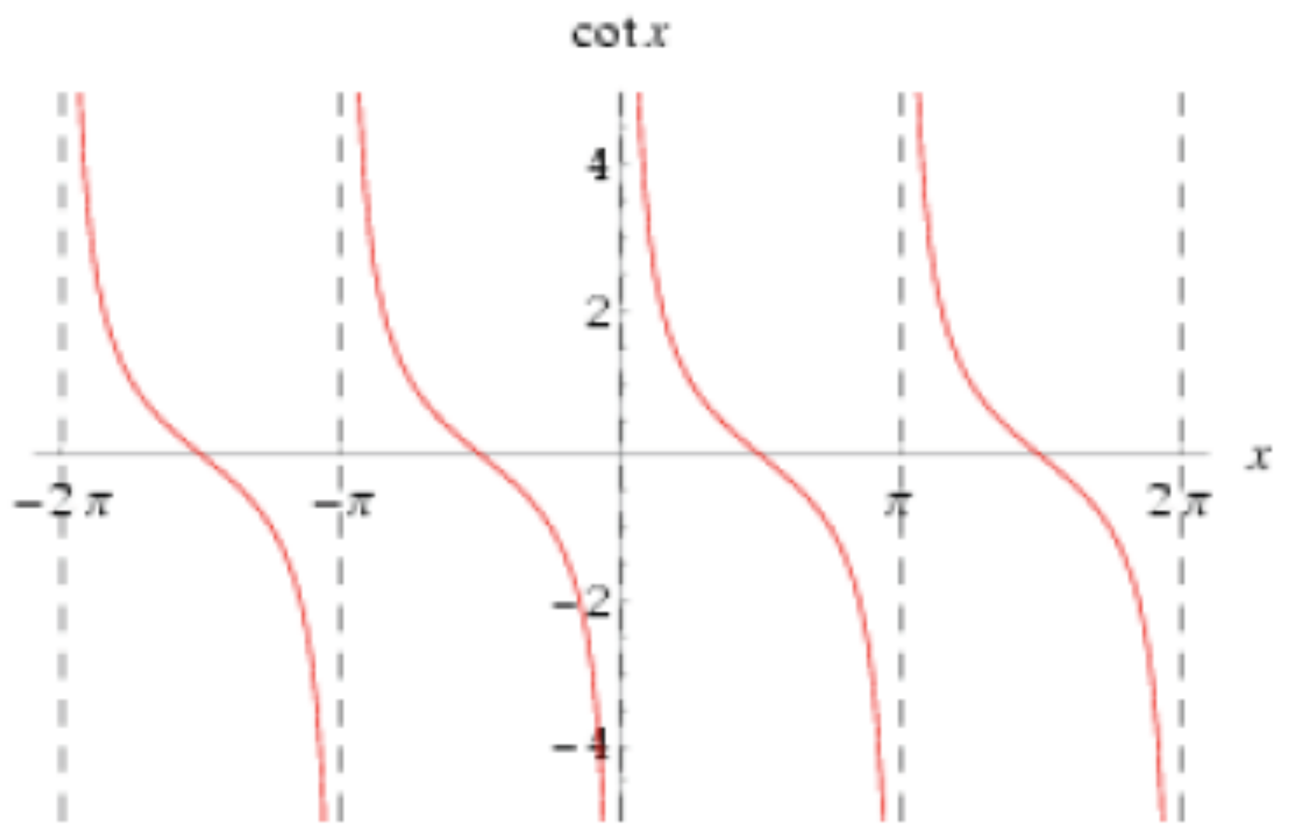
$$\Delta f(v) = \frac{1}{2A(v)} \sum_{v_i \in N_1(v)} (\cot \alpha_i + \cot \beta_i)(f(v_i) - f(v))$$

Discrete Laplace-Beltrami - cotangent

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$

- Accounts for mesh geometry
- Potentially negative/infinite weights



Discrete Laplace-Beltrami - cotangent

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$

- It can also be derived using linear Finite Elements
- Nice property: gives zero for planar 1-rings!

Discrete Laplace-Beltrami - general

- There exist also other methods that use different weights for the discrete integral

- General formula:

$$\Delta f(v_i) = \frac{1}{w_i} \sum_{v_j \in N_1(v_i)} w_{ij}(f(v_j) - f(v_i))$$

- w_i vertex weight

- w_{ij} edge weights

Laplacian matrix

- We define the following matrices:
  - Lumped mass matrix: $\mathbf{M}$ = diag($w_1, \ldots, w_n$) diagonal matrix of *vertex weights*
  - Stiffness matrix: $\mathbf{L}_w$ defined as
    $$l_{i,j} = \begin{cases} -\sum_{v_j \in N_1(v_i)} w_{ij} & \text{if } \ i = j \\ w_{ij} & \text{if } \ v_j \in N_1(v_i) \\ 0 & \text{otherwise} \end{cases}$$
    symmetric matrix of *edge weights*
  - $\mathbf{L} = \mathbf{M}^{-1}\mathbf{L}_w$ the *Laplacian matrix*

Laplacian matrix

- Then we have:

$$\begin{pmatrix} \Delta f(v_1) \\ \cdot \\ \cdot \\ \Delta f(v_n) \end{pmatrix} = \mathbf{L} \begin{pmatrix} f(v_1) \\ \cdot \\ \cdot \\ f(v_n) \end{pmatrix} = \mathbf{M}^{-1}\mathbf{L}_w \begin{pmatrix} f(v_1) \\ \cdot \\ \cdot \\ f(v_n) \end{pmatrix}$$

- Notice:
  - both $\mathbf{L}$ and $\mathbf{L}_w$ are sparse; $\mathbf{M}$ is diagonal
  - $\mathbf{L}_w$ is symmetric, $\mathbf{L}$ is not

Linear system involving the Laplacian

- Suppose we have a linear system (discrete Poisson equation):

$$\mathbf{L}\mathbf{x} = \mathbf{b}$$

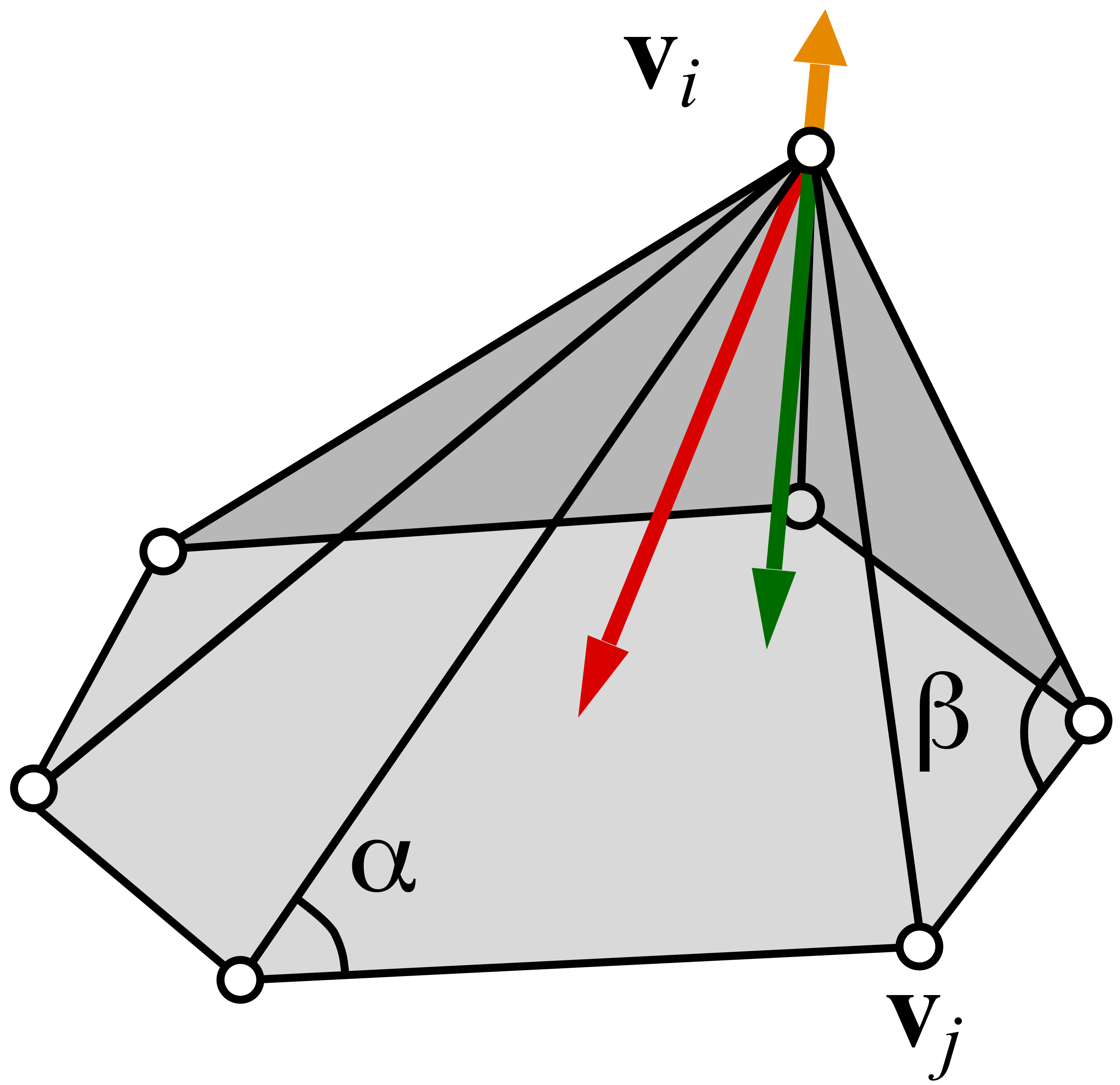
using the asymmetric cotangent Laplacian

- The right hand side vector $\mathbf{b}$ represents a scalar field, storing the desired Laplacian value at each vertex

- Solving symmetric sparse linear systems matrices is much more efficient, so we prefer to solve:

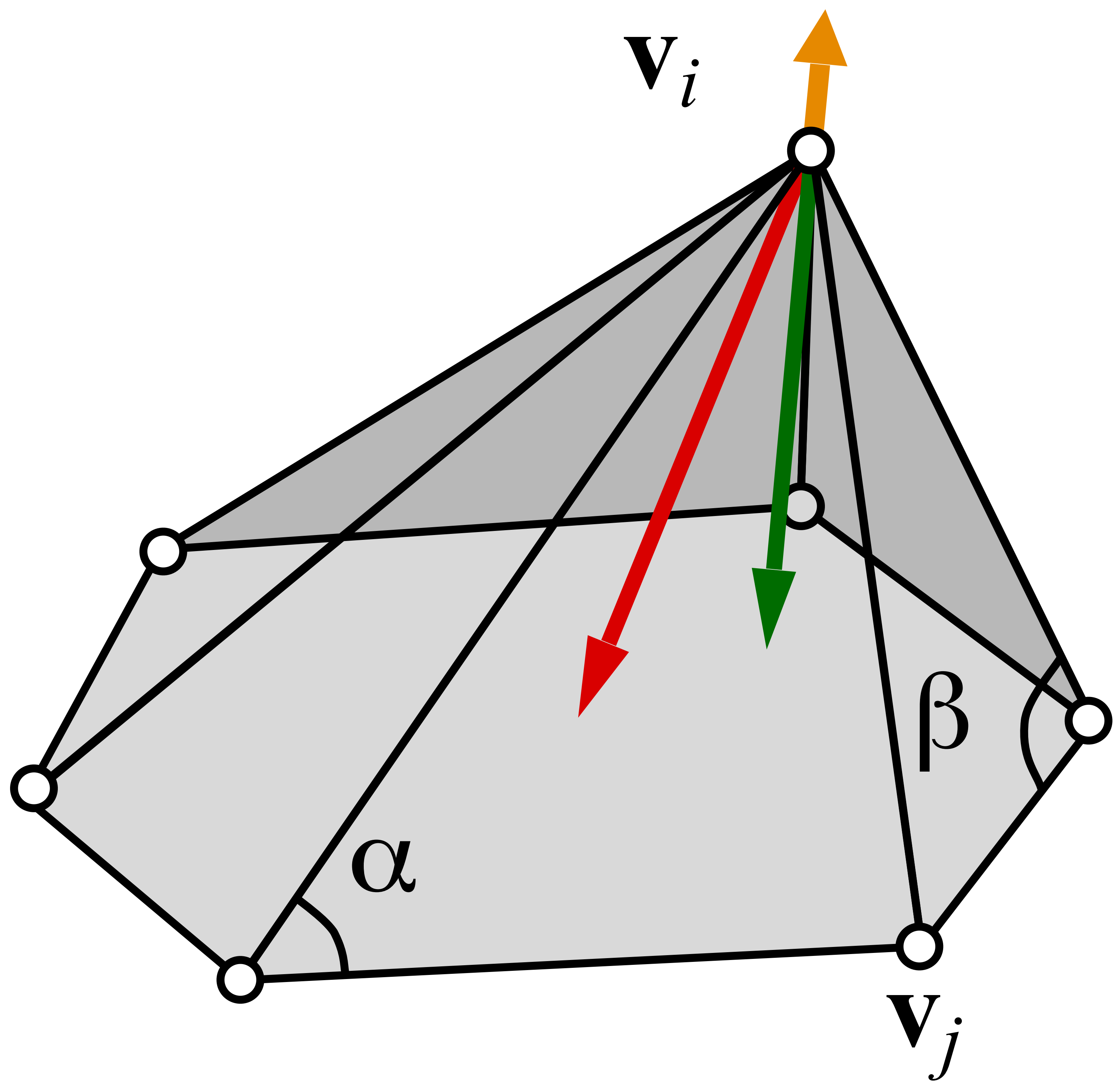
$$\mathbf{L}_w\mathbf{x} = \mathbf{M}\mathbf{b}$$

Discrete Laplace-Beltrami geometric meaning



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

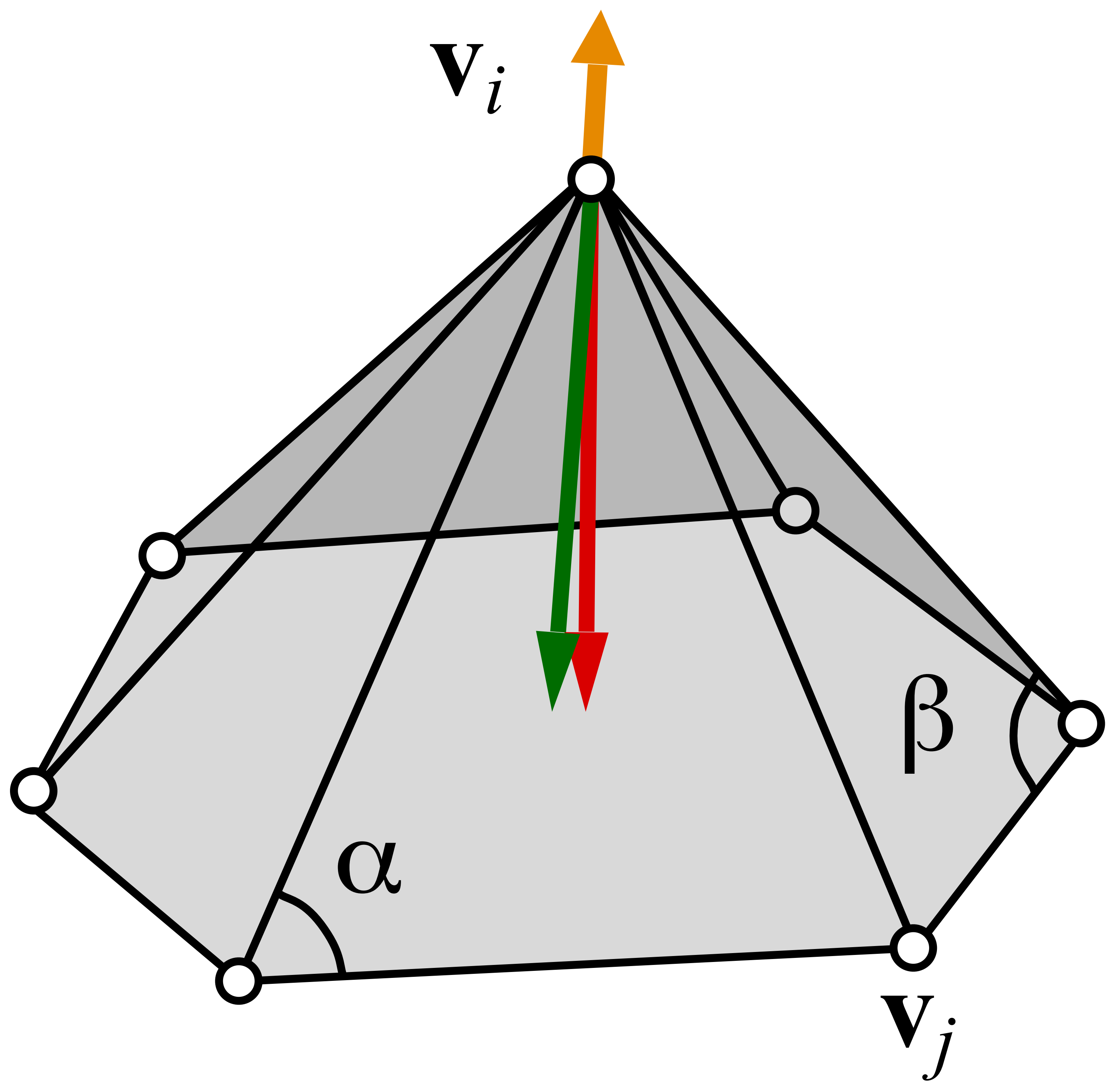
Discrete Laplace-Beltrami geometric meaning



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

Cotan Laplacian allows computing discrete normal

Discrete Laplace-Beltrami geometric meaning



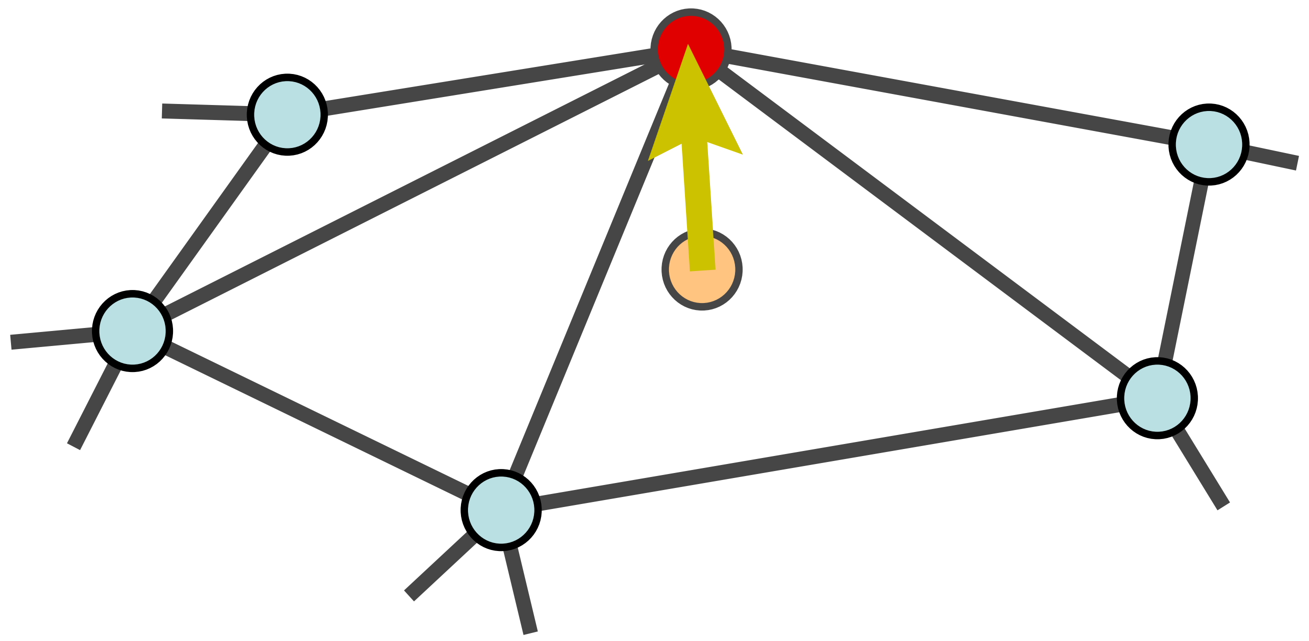
- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

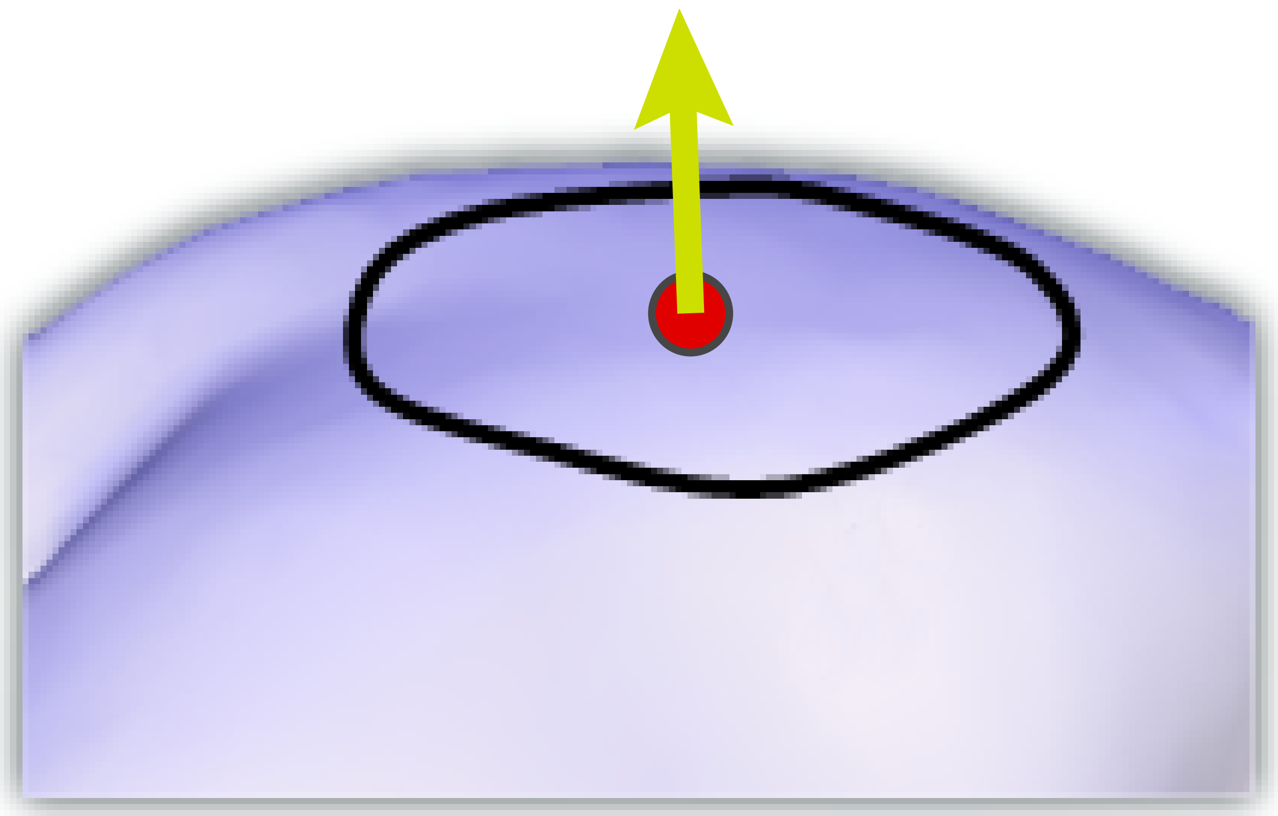
- For nearly equal edge lengths
  Uniform ≈ Cotangent

Differential coordinates

- The vector $\Delta X(v)$ defines the *differential coordinates* of v with respect to its neighbors, and it approximates the local shape characteristics of surface S at v

- "How much to move from the average of neighbors in normal direction to reach v"





Mean curvature from Laplacian

- The norm of $\Delta X(v)$ is twice the mean curvature of S at v

$$\Delta X = -2H \cdot \mathbf{n}$$

- This is true in the continuum, and approximated with discrete estimation on meshes:

$$|H(v_i)| = \|L(v_i)\|/2$$

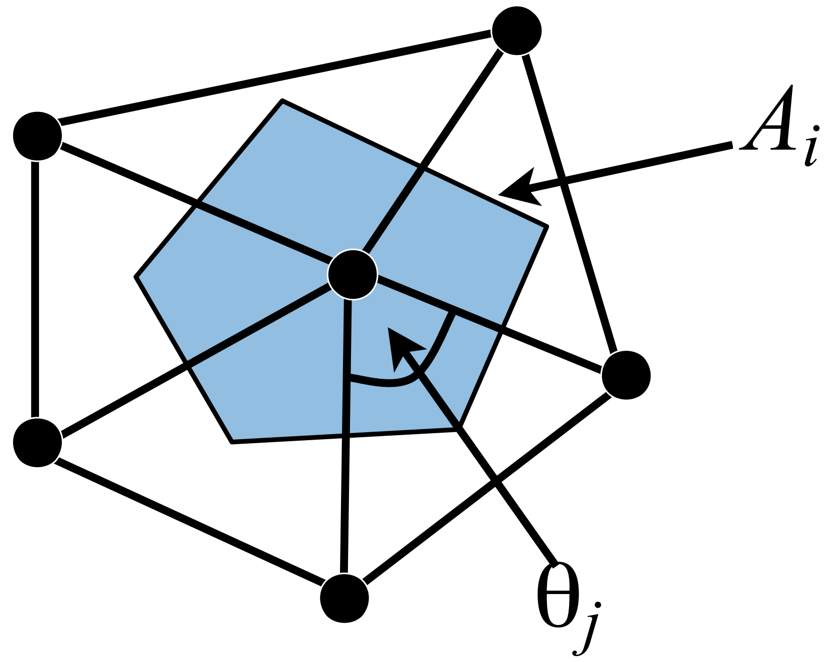
Recap: discrete curvatures

- Mean curvature (sign defined according to normal)

$$|H(\mathbf{v}_i)| = \|L_c(\mathbf{v}_i)\|/2$$

- Gaussian curvature

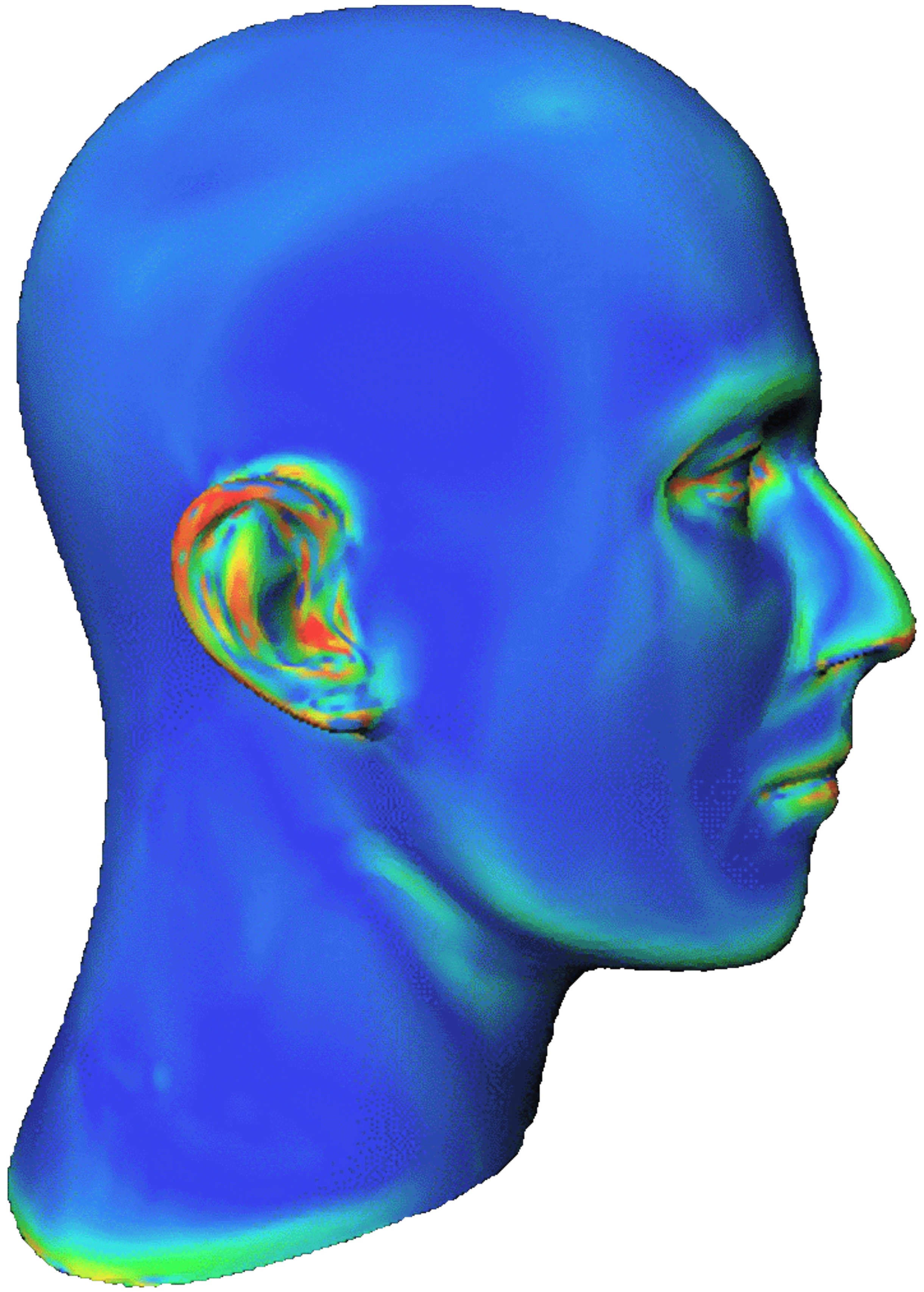
$$K(\mathbf{v}_i) = \frac{1}{A_i}(2\pi - \sum_j \theta_j)$$



- Principal curvatures

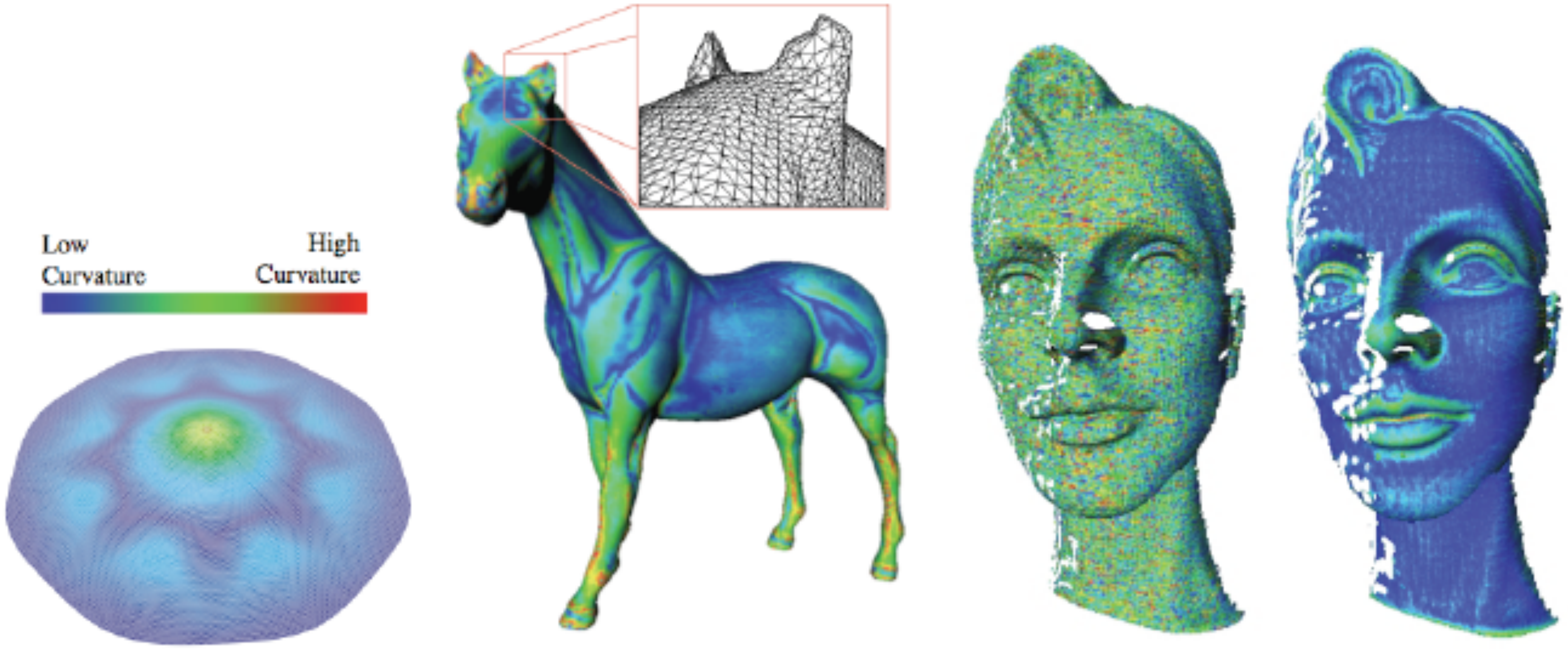
$$\kappa_1 = H - \sqrt{H^2 - K} \qquad \kappa_2 = H + \sqrt{H^2 - K}$$

Example: Discrete Mean Curvature



Links and Literature

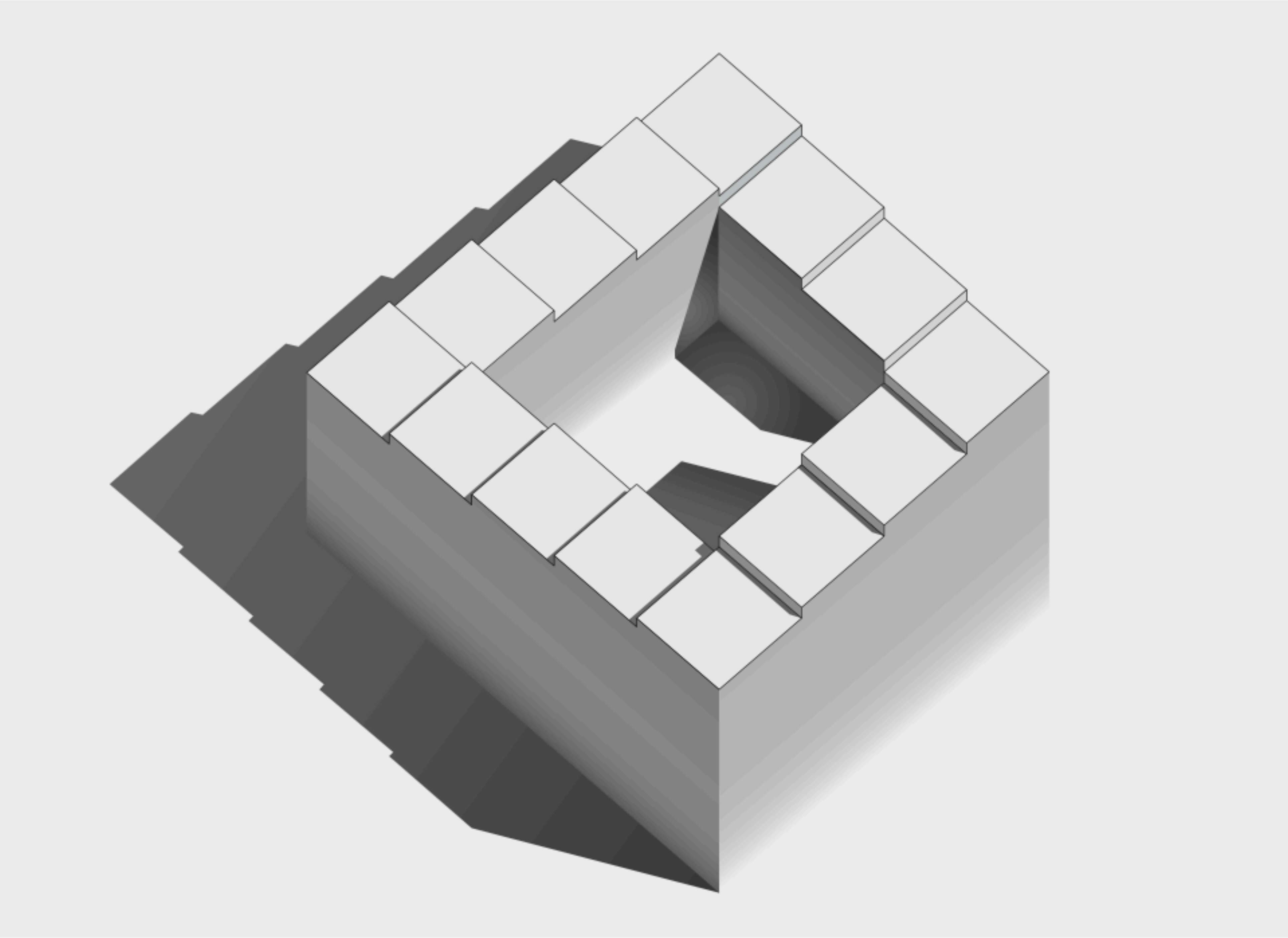
- M. Meyer, M. Desbrun, P. Schroeder, A. Barr
  Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, VisMath, 2002



Links and Literature

Discrete Differential Geometry: An Applied Introduction

Keenan Crane

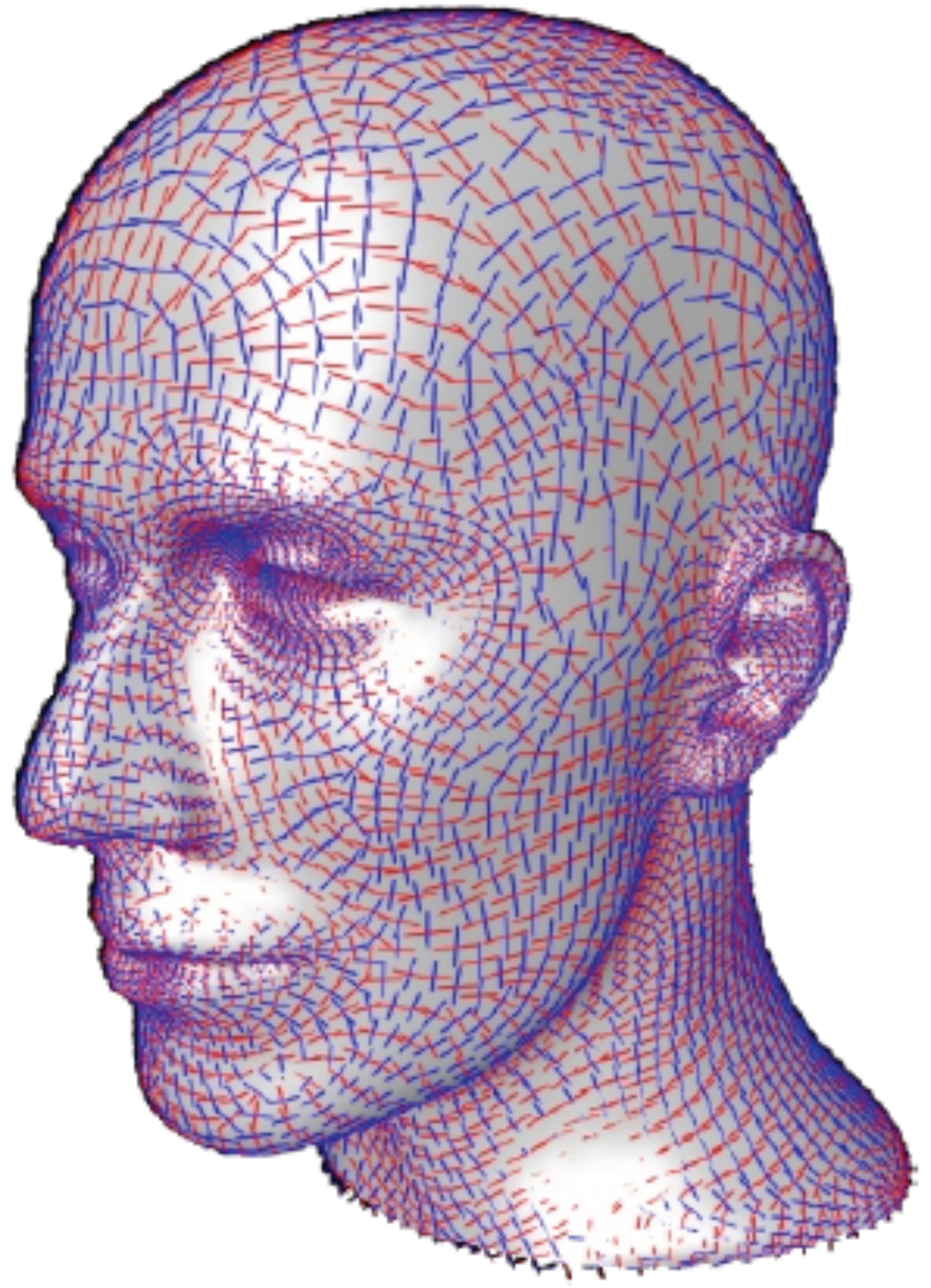


https://www.cs.cmu.edu/~kmcrane/Projects/DGPDEC/

Links and Literature

- igl implements many discrete differential operators
- See the tutorial!

  http://libigl.github.io/libigl/tutorial/tutorial.html



Principal Directions

Thank you