2D Fourier Transform for Images

Digital Signal and Image Processing

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Outline

2D Fourier Transform

Fourier Filtering

2D Fourier Transform

The 2D Fourier Transform can be easily derived from the 1D.

- **Fourier Transform:**

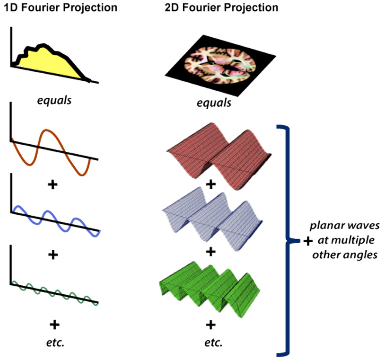
$$F(\omega_1, \omega_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-i(\omega_1 x + \omega_2 y)} dx dy$$

- **Inverse of the Fourier Transform:**

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2$$

ω_1 and ω_2 are spatial frequencies.

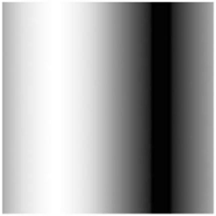
Sinusoidal waves



- Similarly to the 1D case, the FT is based on a decomposition into sinusoidal functions that form an orthonormal basis

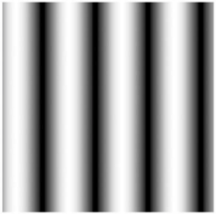
$$e^{-i(\omega_1 x+\omega_2 y)} = \cos(\omega_1 x + \omega_2 y) + i \sin(\omega_1 x + \omega_2 y)$$

- the terms are sinusoids on the x, y plane

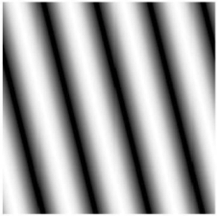


$\omega_1 = 1, \omega_2 = 0$

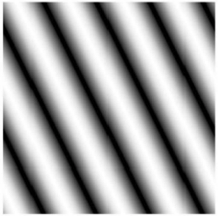
$\omega_1 = 2, \omega_2 = 0$



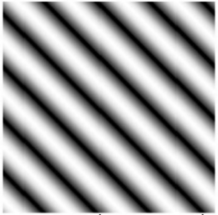
$\omega_1 = 4, \omega_2 = 0$



$\omega_1 = 4, \omega_2 = 1$

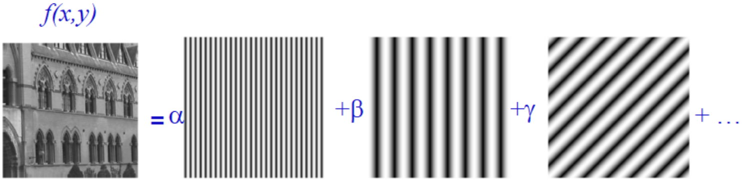


$\omega_1 = 4, \omega_2 = 2$



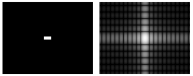
$\omega_1 = 4, \omega_2 = 4$

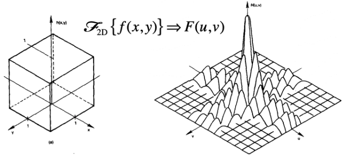
Putting things together...



Example 1 - FT pairs

rectangle centred at origin
with sides of length X and Y





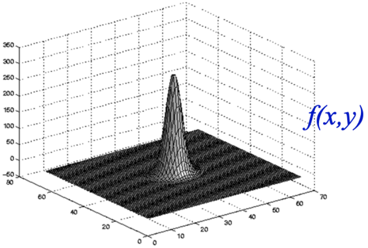
$$
\begin{aligned}
F(\omega_1, \omega_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) e^{-i(\omega_1 x + \omega_2 y)} dxdy \\
&= \int_{-X/2}^{X/2} e^{-i\omega_1 x} dx \int_{-Y/2}^{Y/2} e^{-i\omega_2 y} dy = \\
&= \left[\frac{e^{-i\omega_1 x}}{-i\omega_1} \right]_{-X/2}^{X/2} \left[\frac{e^{-i\omega_2 y}}{-i\omega_2} \right]_{-Y/2}^{Y/2} = \\
&= -\frac{1}{2i\omega_1} (e^{-i\omega_1 X} - e^{i\omega_1 X})(-\frac{1}{2i\omega_2})(e^{-i\omega_2 Y} - e^{i\omega_2 Y}) = \\
&= \frac{1}{2} XY \frac{\sin(X\omega_1)}{X\omega_1} \frac{\sin(Y\omega_2)}{Y\omega_2} = \frac{1}{2} XY sinc(X\omega_1) sinc(Y\omega_2)
\end{aligned}
$$

FT pair example 2

Gaussian centred on origin

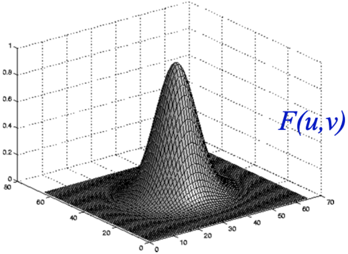
$$f(r) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

where $r^2 = x^2 + y^2$.



$$F(u,v) = F(\rho) = e^{-2\pi^2\rho^2\sigma^2}$$

where $\rho^2 = u^2 + v^2$.

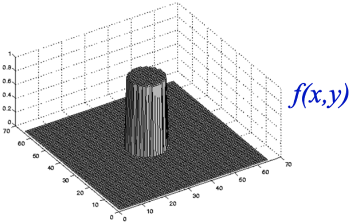


- FT of a Gaussian is a Gaussian
- Note inverse scale relation

FT pair example 3

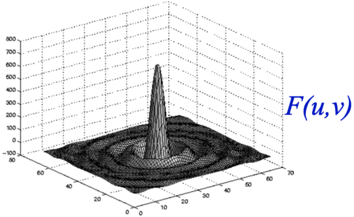
Circular disk unit height and radius *a* centred on origin

$$f(x,y) = \begin{cases} 1, & |r| < a, \\ 0, & |r| \geq a. \end{cases}$$



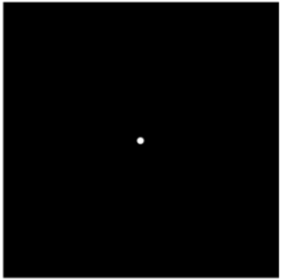
$$F(u,v) = F(\rho) = aJ_1(\pi a\rho)/\rho$$

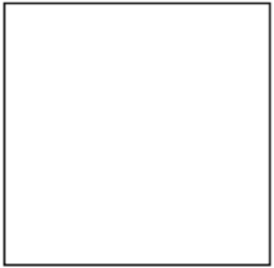
where $J_1(x)$ is a Bessel function.



- rotational symmetry
- a '2D' version of a sinc

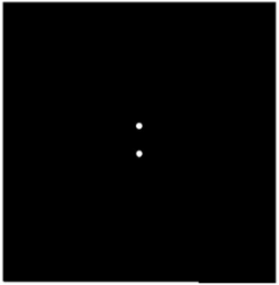
FT pairs





$$
\begin{aligned}
f(x,y) &= \delta(x,y) = \delta(x)\delta(y) \\
F(\omega_1,\omega_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x,y) e^{-i(\omega_1 x + \omega_2 y)} dxdy = 1
\end{aligned}
$$

FT pairs



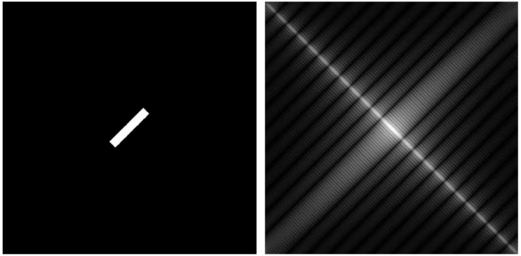
$$
\begin{aligned}
f(x,y) &= \frac{1}{2}(\delta(x,y-a)+\delta(x,y+a)) \\
F(\omega_1,\omega_2) &= \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x,y-a)+\delta(x,y+a)e^{-i(\omega_1 x+\omega_2 y)}dxdy = \\
&= \frac{1}{2}(e^{-ia\omega_2}+e^{ia\omega_2}) = \cos(a\omega_2)
\end{aligned}
$$

Complex Conjugate symmetry of a real function

- An image is usually a 2D real-valued signal.
- If $f(x, y)$ is real, $f(x, y) = f^*(x, y)$, then $F^*(\omega_1, \omega_2) = F(-\omega_1, -\omega_2)$;
- It follows that $|F(\omega_1, \omega_2)| = |F(-\omega_1, -\omega_2)|$, which says that the spectrum of the Fourier transform is symmetric.
- In other words, there exist negative frequencies which are mirror images of the corresponding positive frequencies .

Properties: Rotation in space

Let as assume g to be an in-plane rotation of f by an angle θ, that is $(x', y') = R_\theta(x, y)$
Then, $g(x, y) = f(x \cos\ theta + y \sin\theta, -x \sin\ theta + y \cos\theta)$
By the linearity property of the FT
$G(\omega_1, \omega_2) = F(\omega_1 \cos\ theta - \omega_2 \sin\theta, \omega_1 \sin\ theta + \omega_2 \cos\theta)$
The FT rotates by the same angle of a rotated image

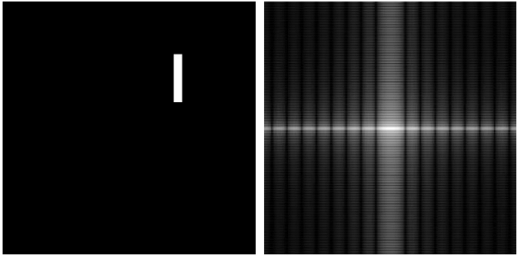


Properties: translation in space

(FT properties), Given $g(x - x_0, y - y_0)$, its FT $G(\omega_1, \omega_2)$ is related to $F(\omega_1, \omega_2)$ by a phase factor:

$$G(\omega_1, \omega_2) = e^{-i(x_0\omega_1 + y_0\omega_2)}F(\omega_1, \omega_2)$$

As for the spectrum: $|G(w)| = |e^{-i(x_0\omega_1 + y_0\omega_2)}||\,F(\omega_1, \omega_2)| = |\,F(\omega_1, \omega_2)|$
The spectrum $|F(\omega_1, \omega_2)|$ is insensitive to image translation



Properties : shift in frequency

(Notice: duality)
given $G(\omega_1 + o_1, \omega_2 + o_2)$, then its IFT $g(x, y)$ is related to $f(x, y)$ by a phase factor

$$g(x, y) = e^{i(o_1 x + o_2 y)} f(x, y)$$

Outline

2D Fourier Transform

Fourier Filtering

Fourier Filtering

To filter an image in the frequency domain:

- Compute the DFT of the image $F(u, v)$
- Select and appropriate filter H (or filter transfer function)
- Multiply $F(u, v)$ by a filter function $H(u, v)$:

$$G(u, v) = H(u, v)F(u, v)$$

- Compute the inverse DFT of $G(u, v)$

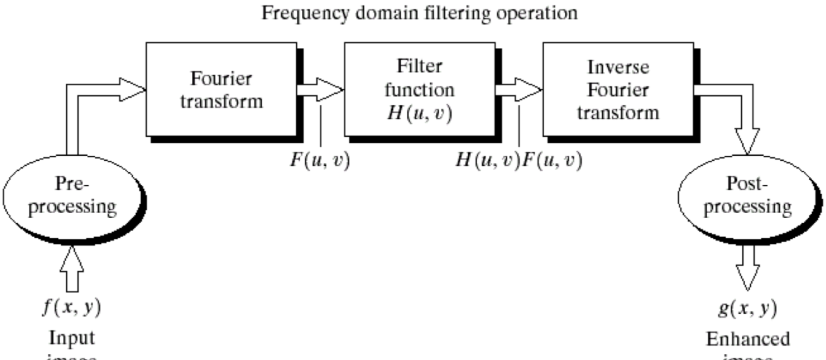


Image filtering and convolution theorem

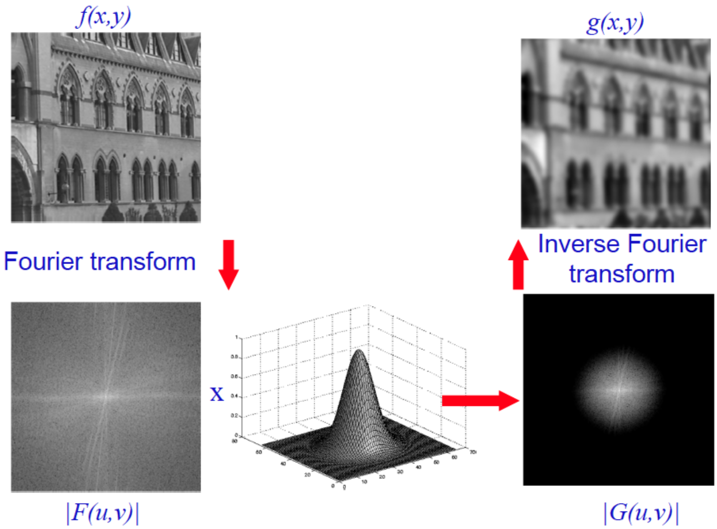
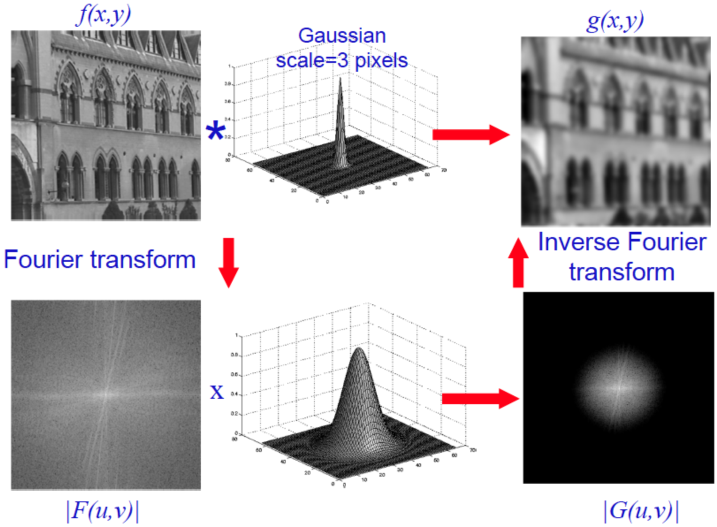
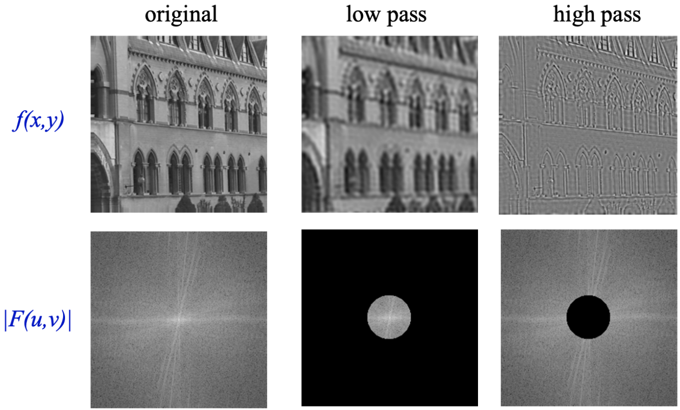


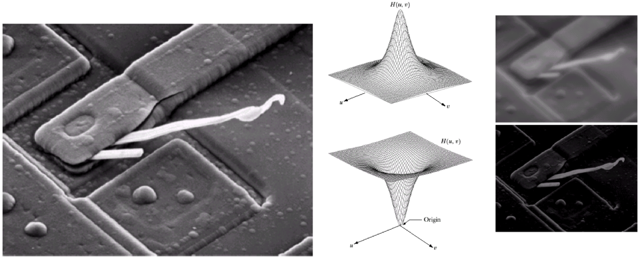
Image filtering and convolution theorem



Filters in space and frequencies



Low-pass and high-pass filtering



From: Digital Image Processing 3rd ed. Gonzalez & Woods

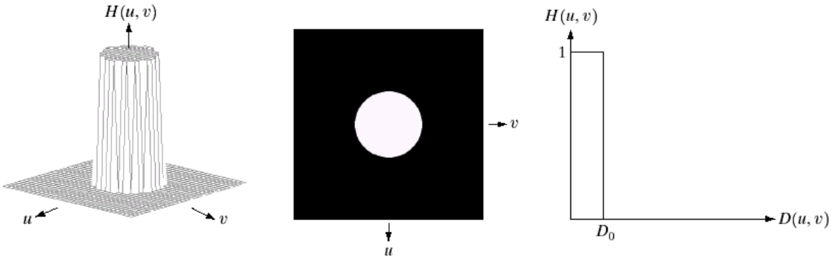
Image smoothing

Smoothing is achieved in the frequency domain by dropping out the high frequency components. This effect is obtained by applying a low-pass filter.
We can take into account different low-pass filters, for instance:

- Ideal low-pass
- Butterworth
- Gaussian

Ideal low-pass filter

Simply cuts off all high frequency components that are a specified distance D_0 from the origin of the transform.



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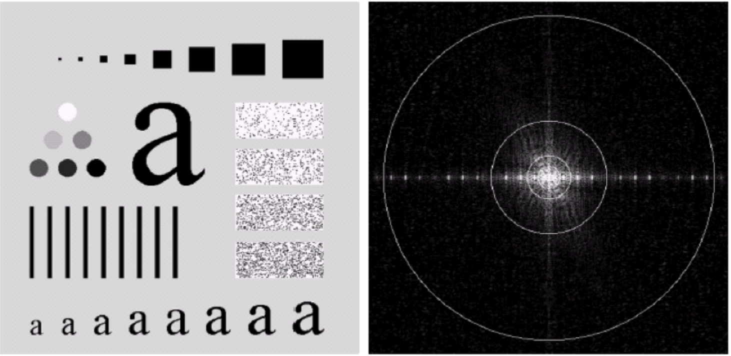
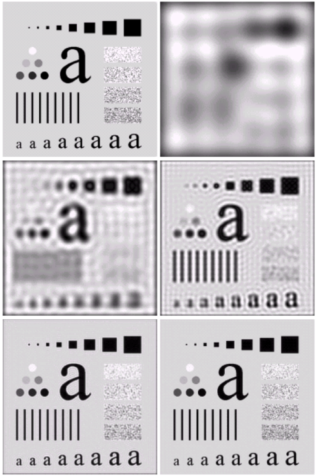
Ideal low-pass filter



Above we show an image, its Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it.

From: Digital Image Processing 3rd ed. Gonzalez & Woods

Ideal low-pass filter (notice the ringing effects)



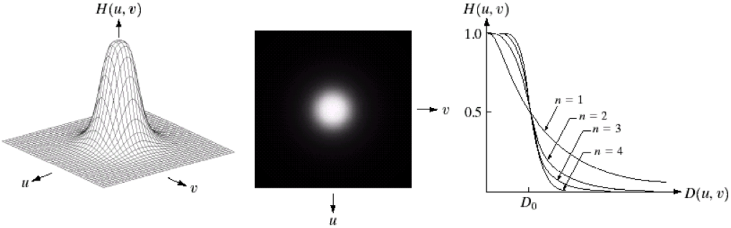
The original image and the five different filtered images with $D_0 = 5, 15, 30, 80, 230$.

From: Digital Image Processing 3rd ed. Gonzalez & Woods

Butterworth low-pass filter

The transfer function of a Butterworth lowpass filter of order n with cutoff frequency at distance D_0 from the origin is defined as:

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$



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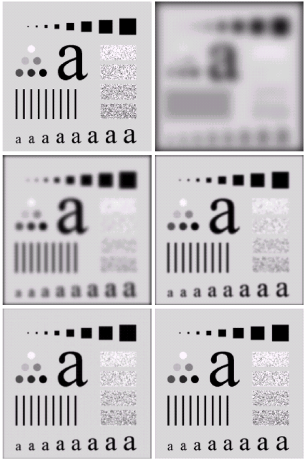
Low-pass filtering



The original image and the five different images filtered by a Butterworth filter of order 2 and $D_0 = 5, 15, 30, 80, 230$.

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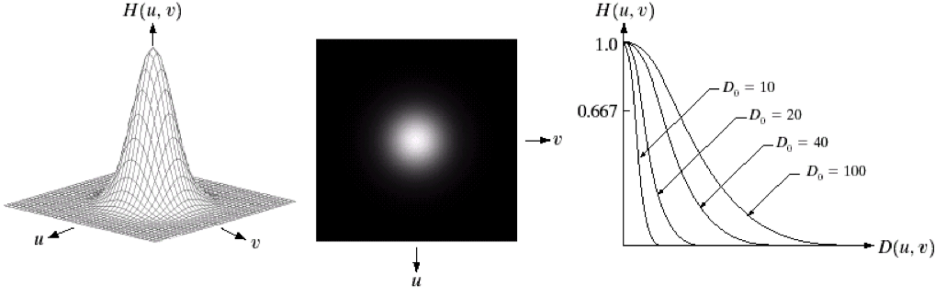
Gaussian low-pass filter

The transfer function of a Gaussian lowpass filter with cutoff frequency at distance D_0 from the origin is defined as:

$$H(u,v) = e^{-\frac{D^2(u,v)}{2D_0^2}}$$



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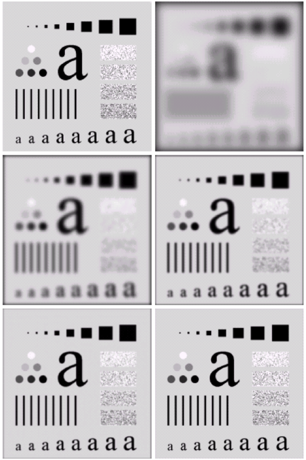
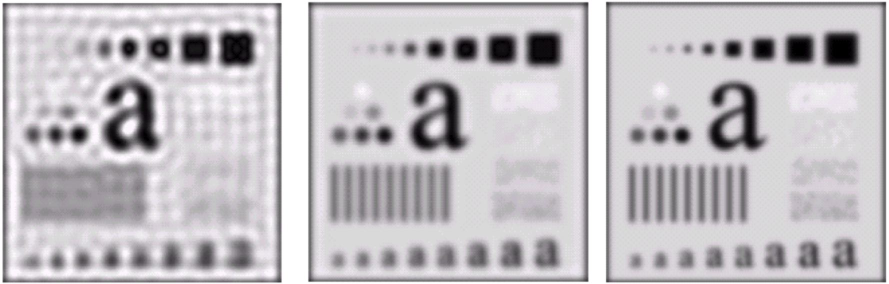
Low-pass filtering



The original image and the five different images filtered by a Gauss filter with $D_0 = 5, 15, 30, 80, 230$.

Low-pass filtering comparison



The test image filtered by: (left) Ideal low-pass filter $D_0 = 15$, (center) butterworth low-pass filter $D_0 = 15$, (right) Gaussian low-pass filter $D_0 = 15$

Low-pass filtering application

A low pass Gaussian filter is used to connect broken text or to remove blemishes in a photograph

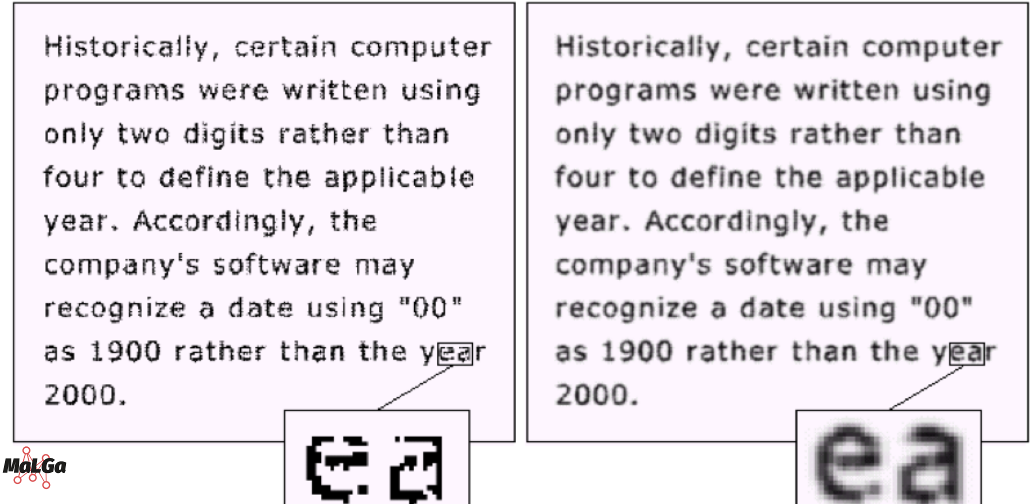
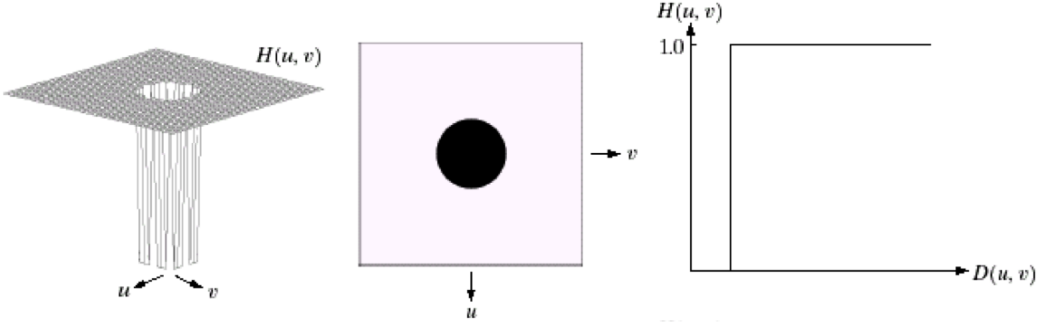


Image enhancement

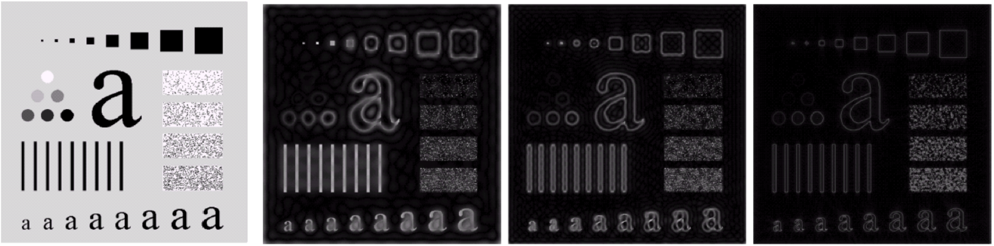
- Fine details in images are associated with high frequency components
- High pass filters only pass the high frequencies, dropping the low ones
- High pass filters are the reverse of low pass filters, so:

$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

Ideal High-pass filter



Ideal High-pass filter

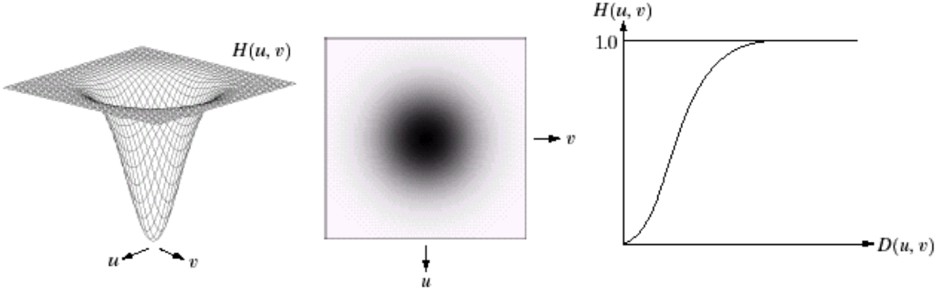


The original image and three different images filtered by an ideal high-pass filter with $D_0 = 15, 30, 80$.

Gaussian high-pass filter

The Gaussian high-pass filter with cutoff frequency at distance D_0 from the origin is defined as:

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$



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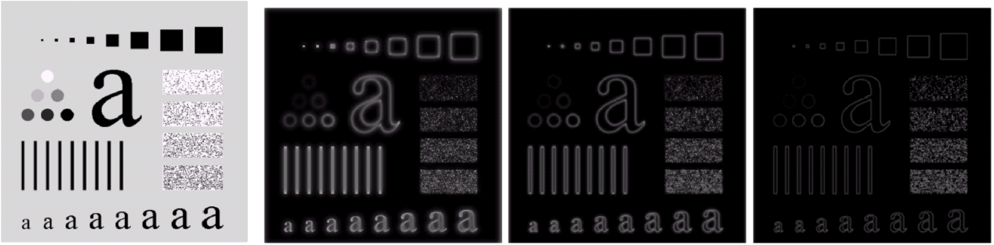
Gaussian high-pass filtering



The original image and three different images filtered by a Gauss filter with $D_0 = 15, 30, 80$.