07 - Elementary Differential Geometry

2 - Surfaces

Acknowledgements: Daniele Panozzo, Kai Hormann

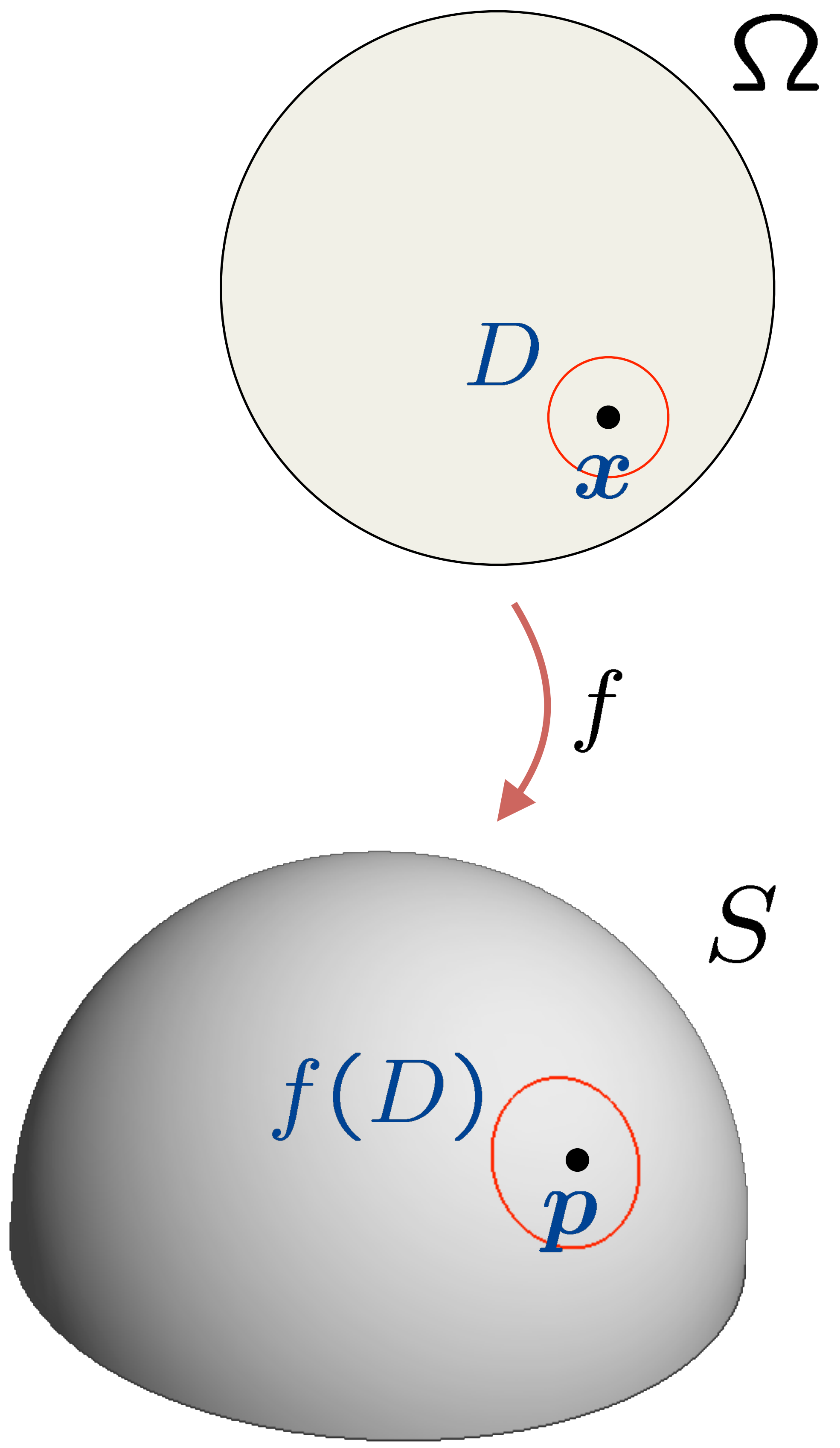
In this lecture

- Math of surfaces
  - in the continuum only

Parametric representation of surfaces

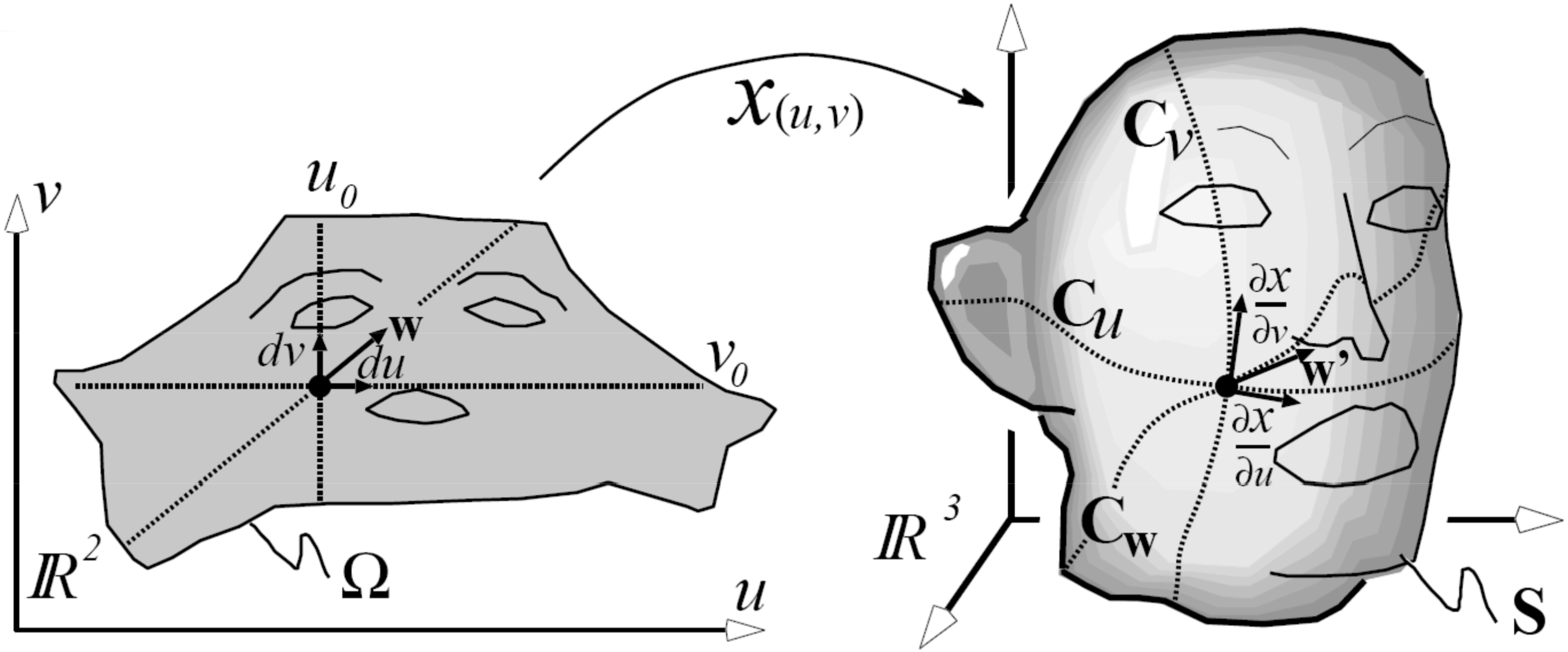
- surface $S \subset \mathbb{R}^3$
- parameter domain $\Omega \subset \mathbb{R}^2$
- mapping $f : \Omega \rightarrow S$

  f is a bi-variate vector function mapping points of the 2D plane to points in 3D space



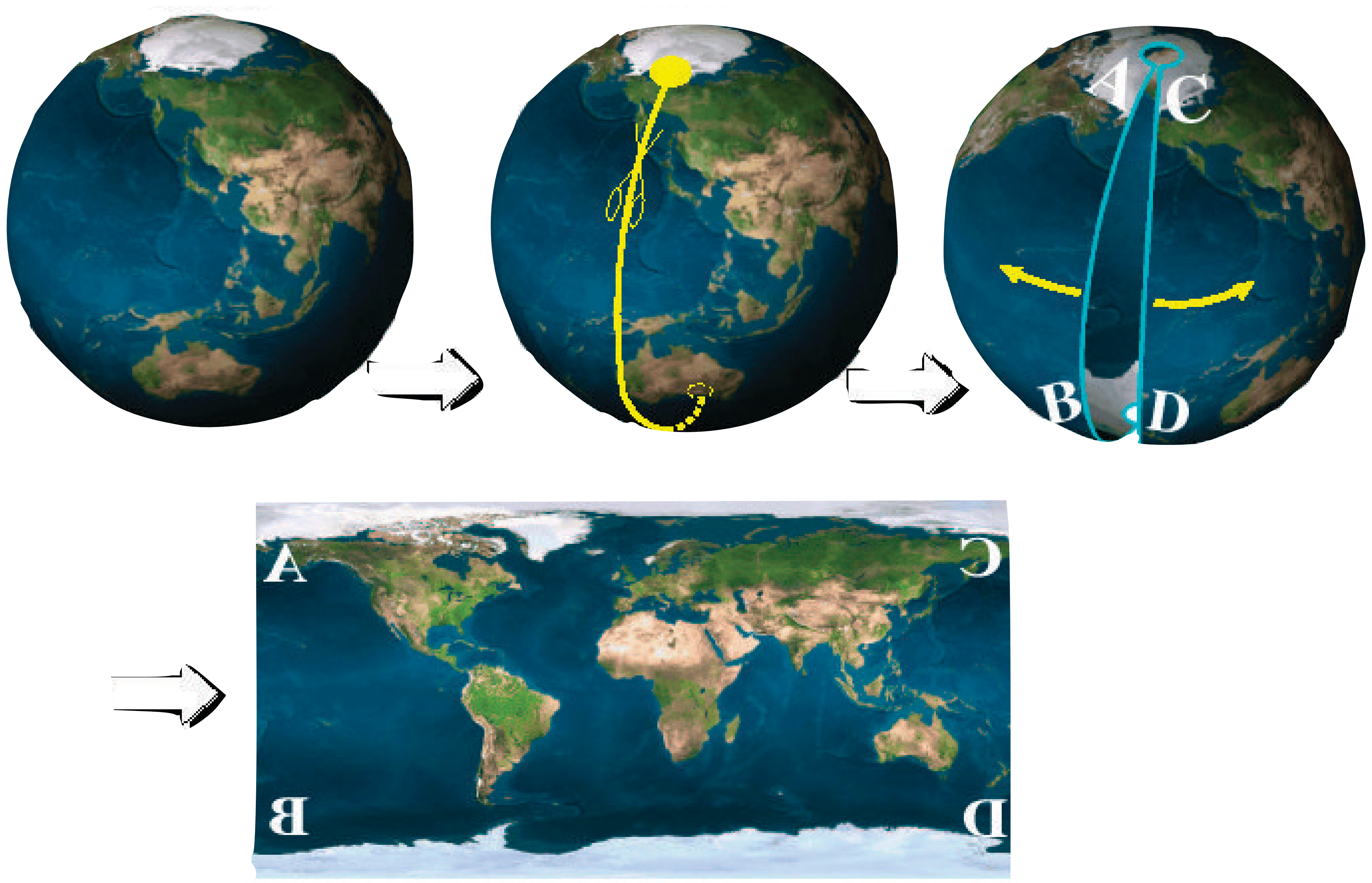
Parametric representation of surfaces

- We can define an injective parametrization iff surface S is homeomorphic to a (punctured) disk



Parametric representation of surfaces

- For a surface S that is *not* homeomorphic to a punctured disk, parametrization needs to *cut* S at the image of borders of Ω



Tangent plane

- **Jacobian** of f

$$J_f = [f_u, f_v] \in \mathbb{R}^{3\times 2}$$

- **tangent plane** at $\boldsymbol{p}$

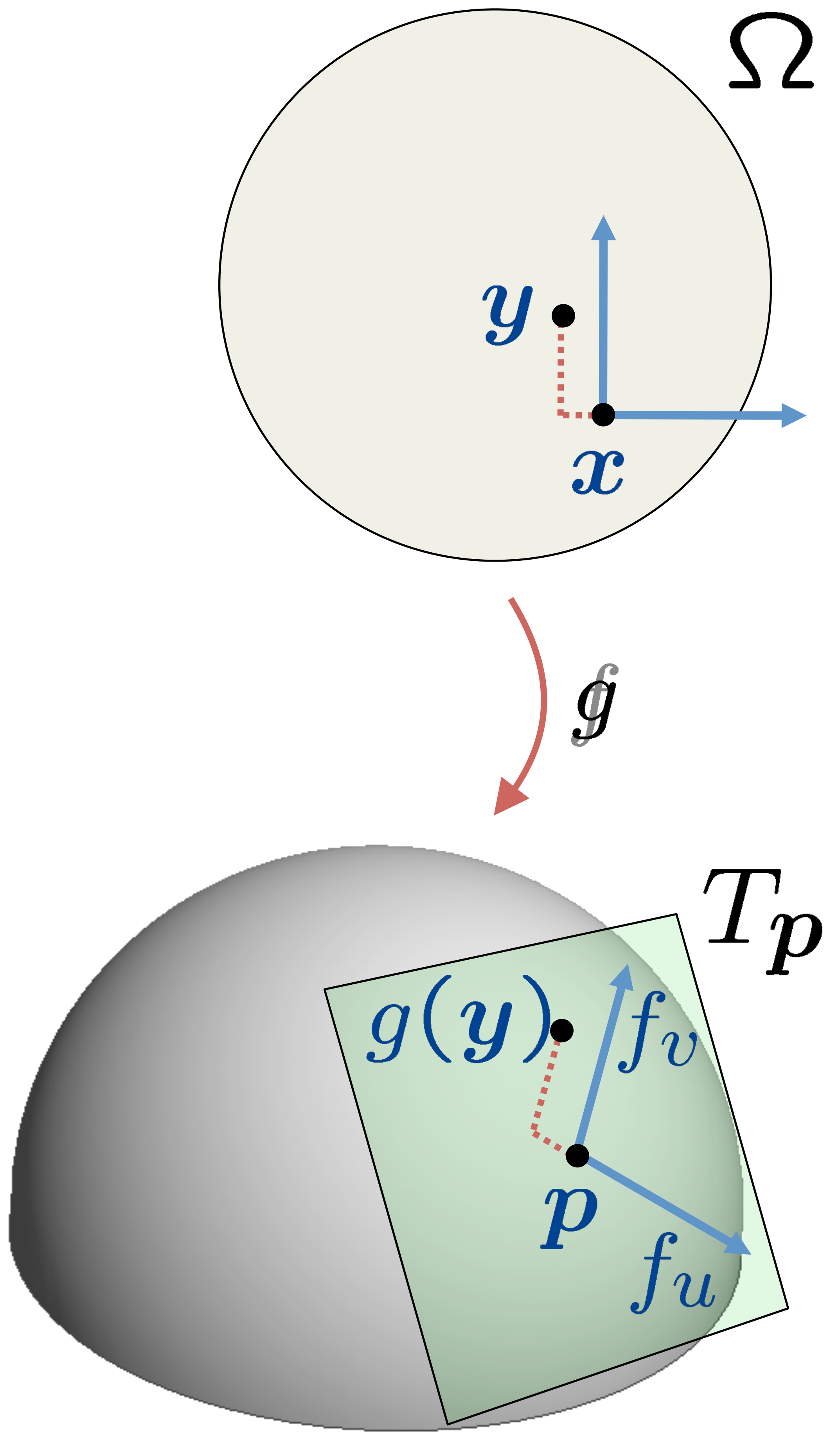
$$T_{\boldsymbol{p}} = \{\boldsymbol{p} + \alpha f_u + \beta f_v : \alpha, \beta \in \mathbb{R}\}$$

- **Taylor expansion** of f

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + J_f(\boldsymbol{y} - \boldsymbol{x}) + \dots$$

- **first order approximation** of f

$$g(\boldsymbol{y}) = \boldsymbol{p} + J_f(\boldsymbol{y} - \boldsymbol{x}) \in T_{\boldsymbol{p}}$$

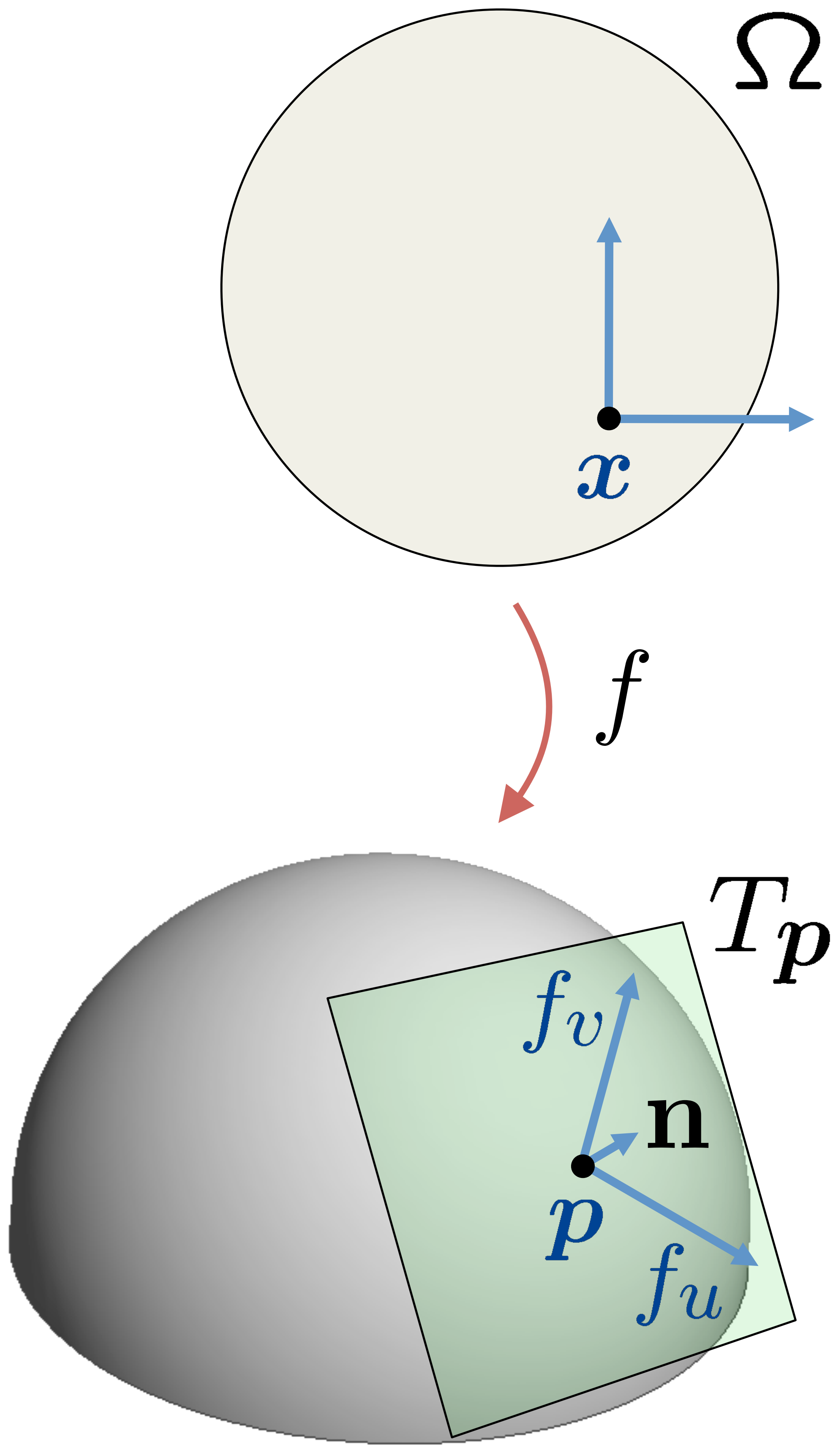


Surface normal

- **Surface normal** at a point p is the unit-length normal vector $\mathbf{n}$ of tangent plane T_p pointing outwards from S

- If $p = f(x)$ then

$$\mathbf{n} = \frac{f_u \times f_v}{||f_u \times f_v||}$$



Directional derivatives

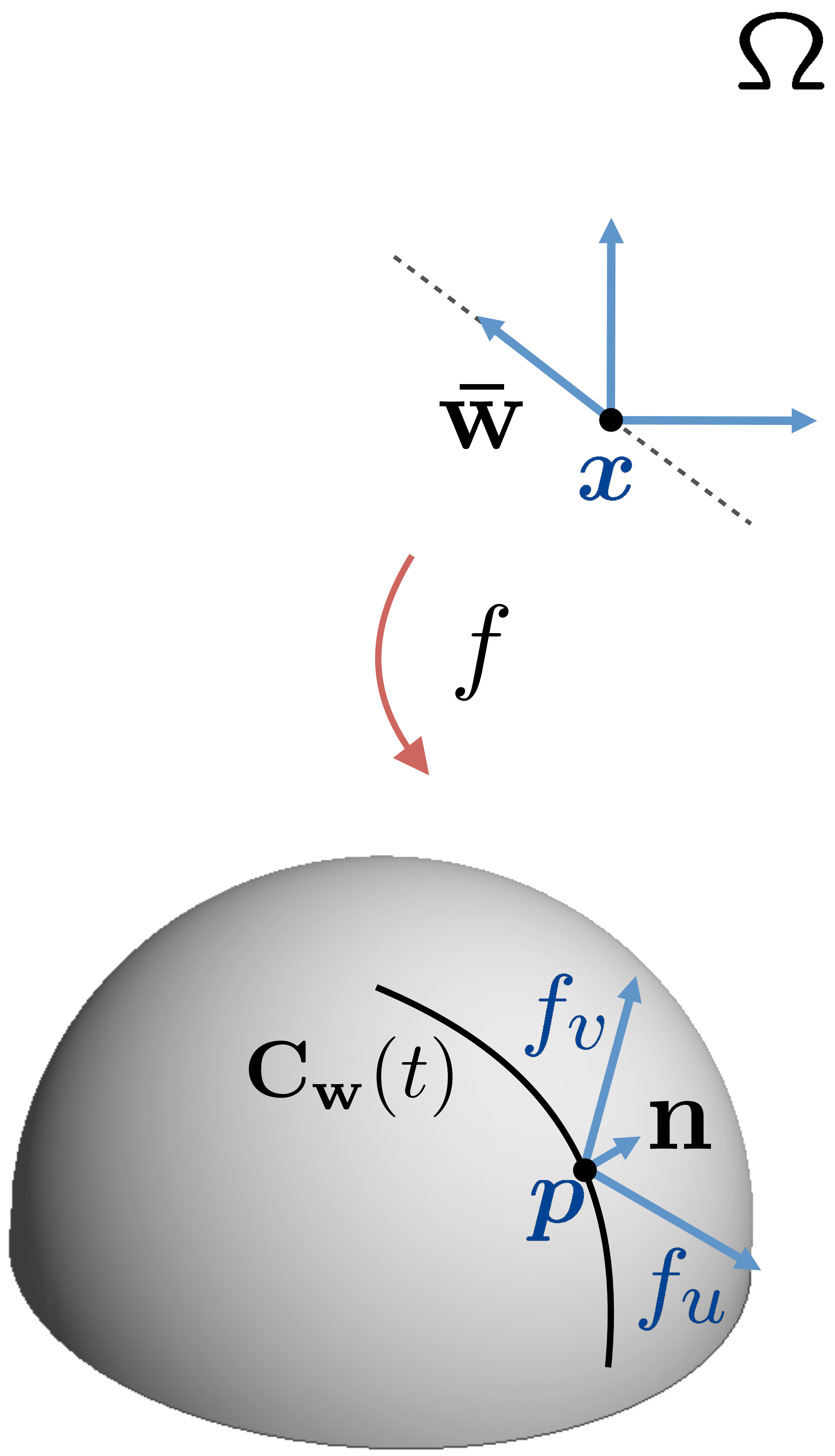
- Given a direction vector $\bar{\mathbf{w}} = (u_w, v_w)^T$

- Straight line in parameter space

$$(u, v) = (u_x, v_x) + t\bar{\mathbf{w}}$$

- Corresponds to a curve on S

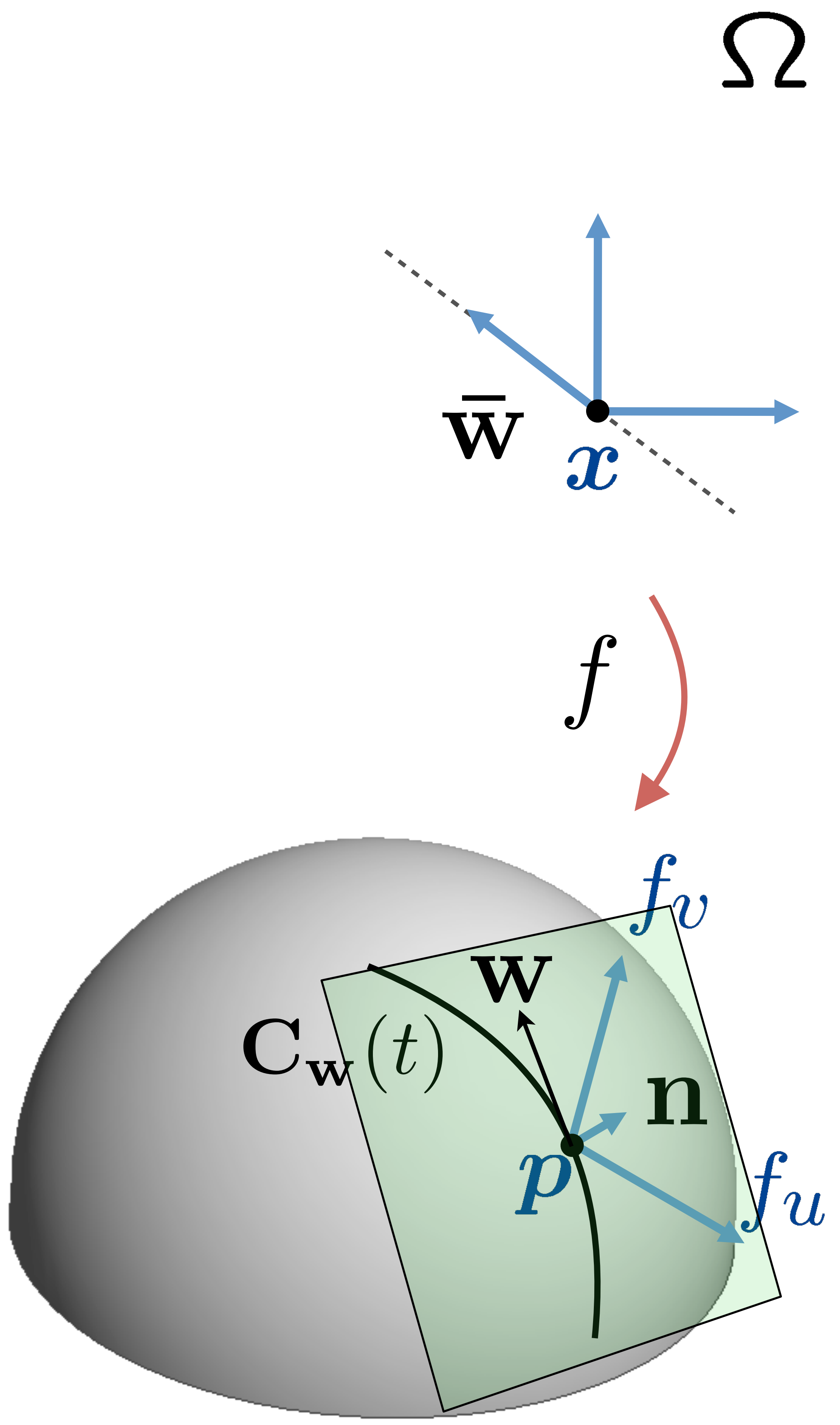
$$\mathbf{C}_\mathbf{w}(t) = f(u_x + tu_w, v_x + tv_w)$$



Directional derivatives

- Directional derivative $\mathbf{w}$ of f at (u_x, v_x) with respect to $\bar{\mathbf{w}}$ is the tangent to $\mathbf{C}_{\mathbf{w}}(t)$ computed at $t = 0$

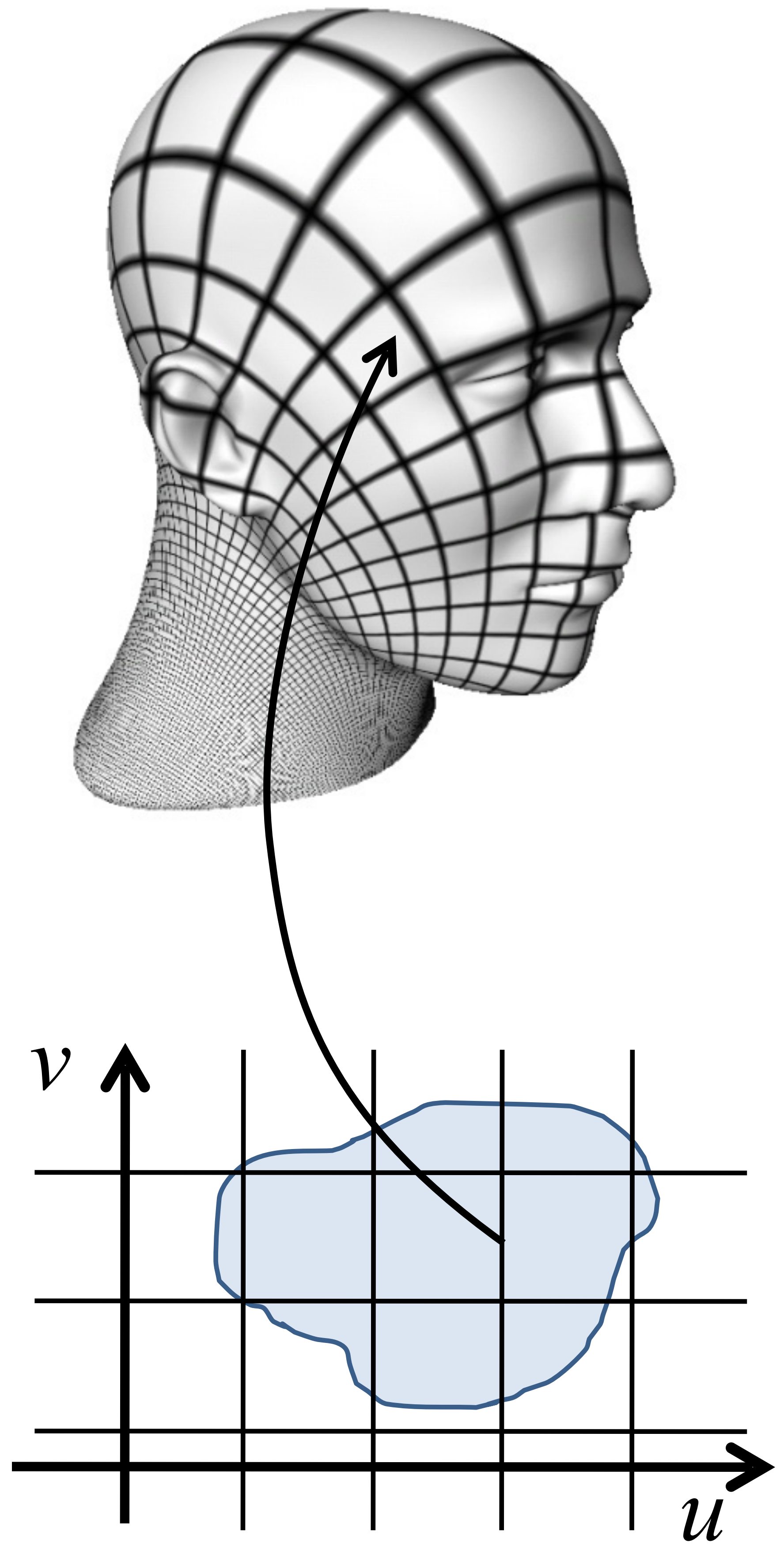
- By the chain rule $\mathbf{w} = J_f \bar{\mathbf{w}}$



Isoparametric Lines

- Curves on the surface when keeping one parameter fixed
- Obtained by tracing one cardinal direction
- They form a "grid" on the surface

$$\gamma_{u_0} = f(u_0, v)$$
$$\gamma_{v_0} = f(u, v_0)$$



First fundamental form

- The Jacobian matrix encodes the *metrics* on the surface:

  we can measure distances, angles and areas

- Scalar product of two tangent vectors:

$$\mathbf{w}_1^T \mathbf{w}_2 = (J_f \bar{\mathbf{w}}_1)^T (J_f \bar{\mathbf{w}}_2) = \bar{\mathbf{w}}_1^T (J_f^T J_f) \bar{\mathbf{w}}_2$$

- *First fundamental form* defines the inner product on the tangent space

$$\mathbf{I} = J_f^T J_f = \left[\begin{array}{cc} E & F \\ F & G \end{array} \right] = \left[\begin{array}{cc} f_u^T f_u & f_u^T f_v \\ f_u^T f_v & f_v^T f_v \end{array} \right]$$

Measuring lengths

- The first fundamental form can be used to measure lengths on the surface:

  - Length of a tangent vector $\mathbf{w}$: $||\mathbf{w}||^2 = \bar{\mathbf{w}}^T \mathbf{I} \bar{\mathbf{w}}$

  - Length of a curve ***f(u(t))*** where ***u(t)*** is a curve in Ω:

$$l(a,b) = \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T} dt = \int_a^b \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2} dt$$

Measuring Areas

- The first fundamental form can be used to measure areas on the surface:
  - Area of a region ***f*(*U*)** where ***U*** is a region in Ω:

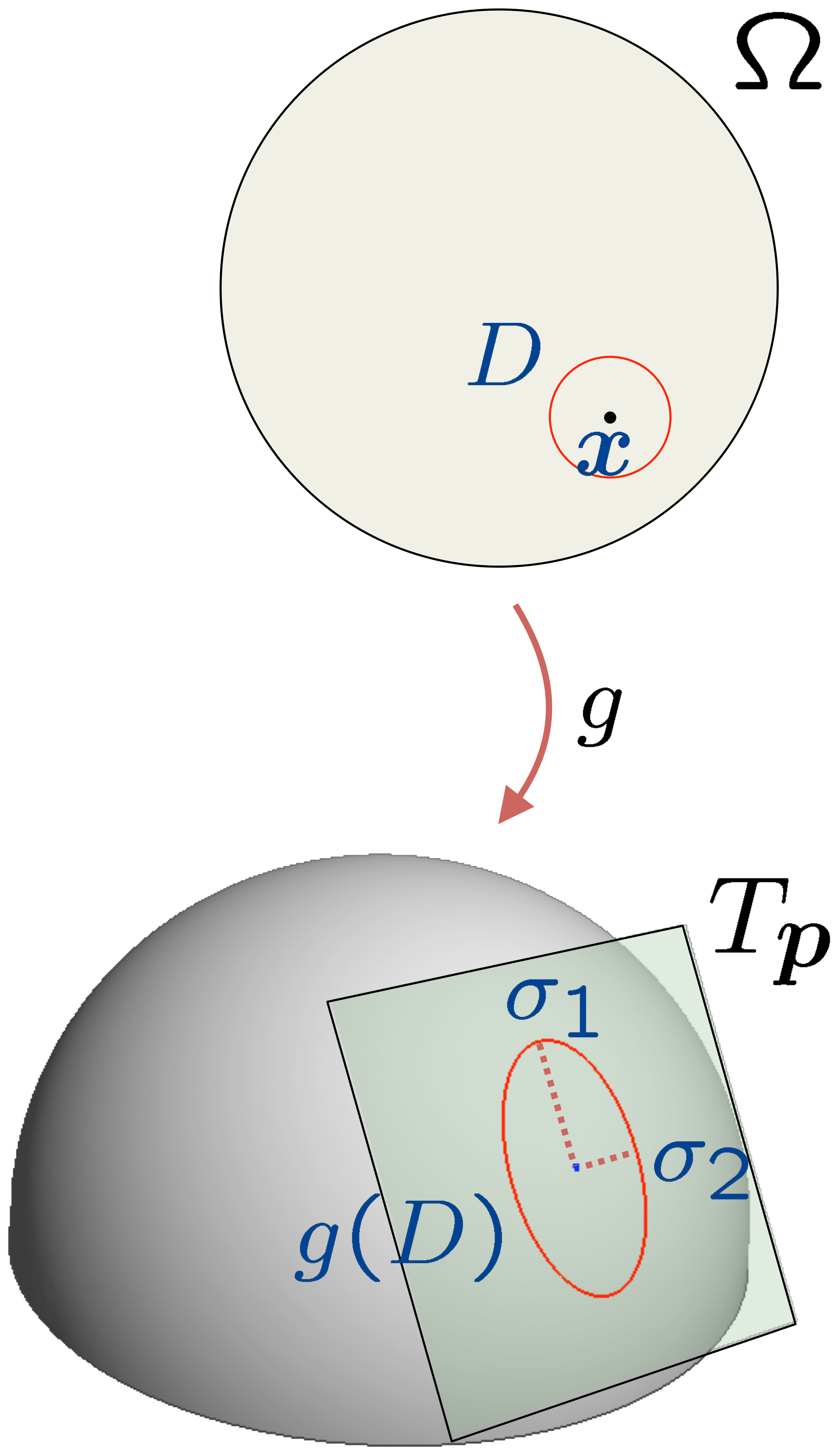
$$A = \iint_U \sqrt{\det(\mathbf{I})}dudv = \iint_U \sqrt{EG - F^2}dudv$$

Anisotropy ellipse

- small disk $D = D(\boldsymbol{x}, r)$ around $\boldsymbol{x}$
- image of D under g first order approximation of f

$$g(D) = \{g(y) : y \in D\} \subset T_{\boldsymbol{p}}$$

- shape of $g(D)$ on the tangent plane
  - ellipse
  - semiaxes $r\,\sigma_1$ and $r\,\sigma_2$
- behavior in the limit $\lim_{r \to 0} g(D) = f(D)$



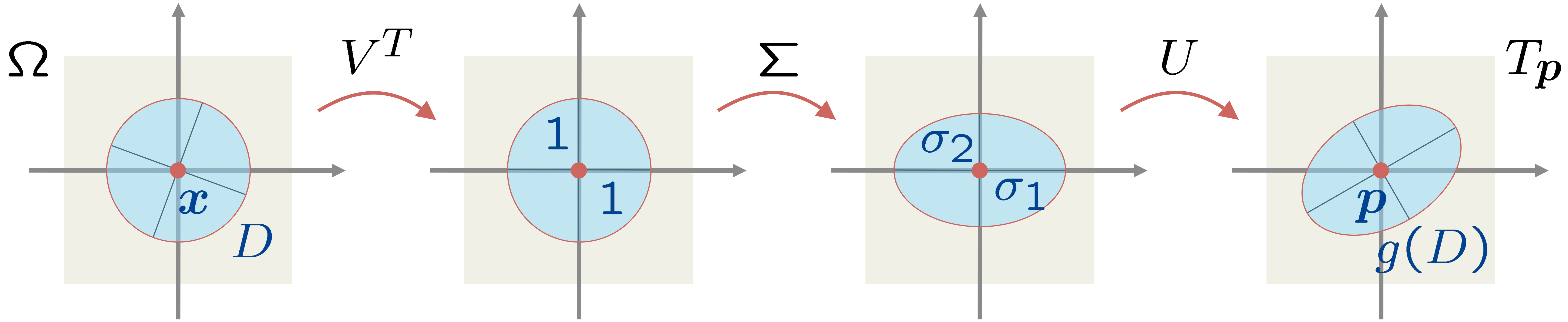
Anisotropy ellipse

- Singular Value Decomposition (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with rotations $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$

and scale factors (singular values) $\sigma_1 \geq \sigma_2 > 0$



Anisotropy ellipse

- From the first fundamental form $\mathbf{I}$:
  - The *eigenvectors* $\bar{\mathrm{e}}_1$ and $\bar{\mathrm{e}}_2$ of $\mathbf{I}$ map to the axes e_1 and e_2 of the ellipse
  - The *eigenvalues* λ_1 and λ_2 of $\mathbf{I}$ are the square lengths of the semiaxes of the ellipse

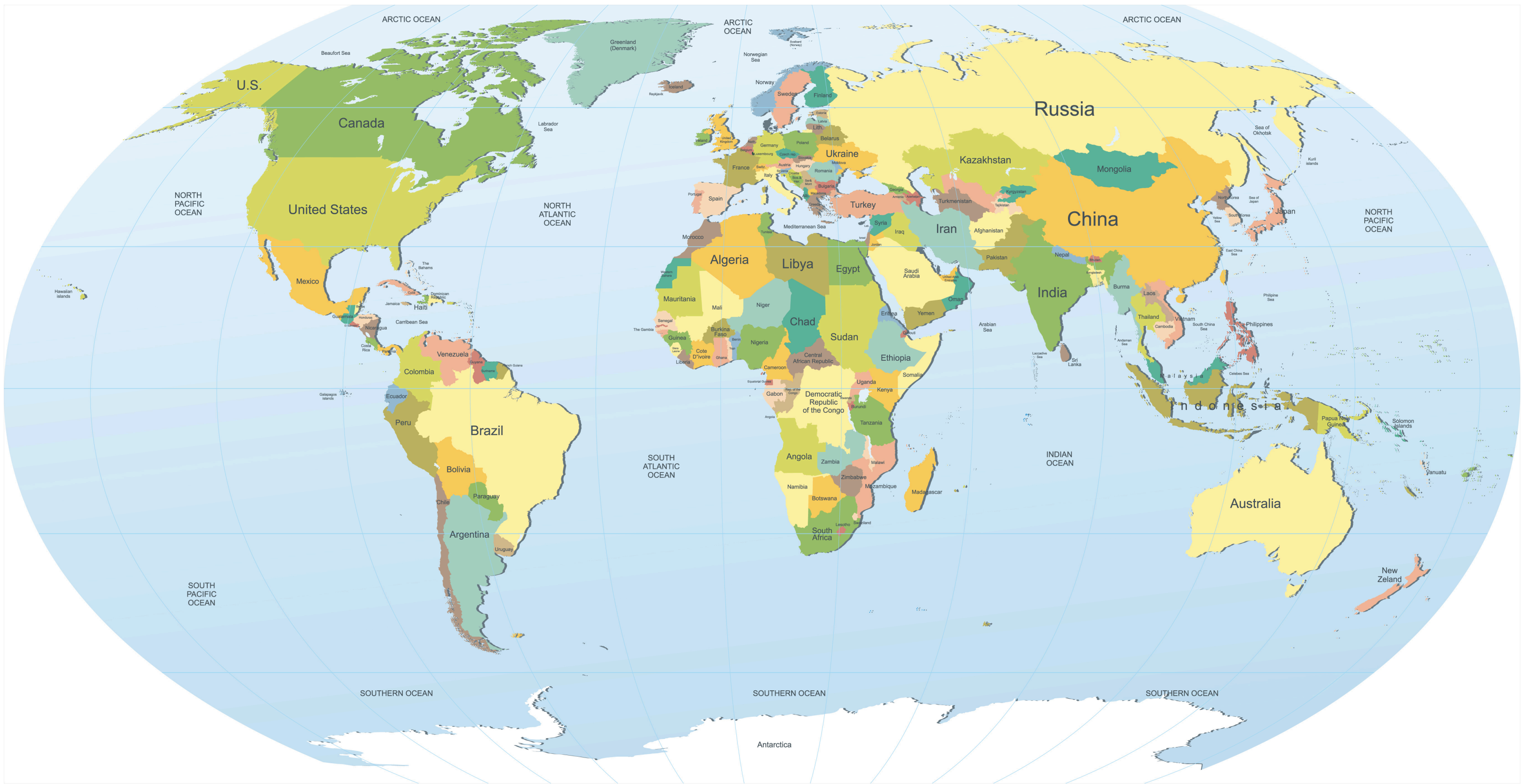
$$\sigma_1 = \sqrt{\lambda_1} \qquad \sigma_2 = \sqrt{\lambda_2}$$

Recap: first order differential properties

- Parametric function sends points to points: $p = f(x)$
- Jacobian sends vectors to vectors in the tangent plane: $\mathbf{w} = J_f \bar{\mathbf{w}}$
- With first differential form $\mathbf{I}$ we can measure:
  - Angles, lengths, areas on S
  - How parametric domain Ω is stretched (locally) to map it to surface S: anisotropy ellipse from eigenvalues and eigenvectors of $\mathbf{I}$

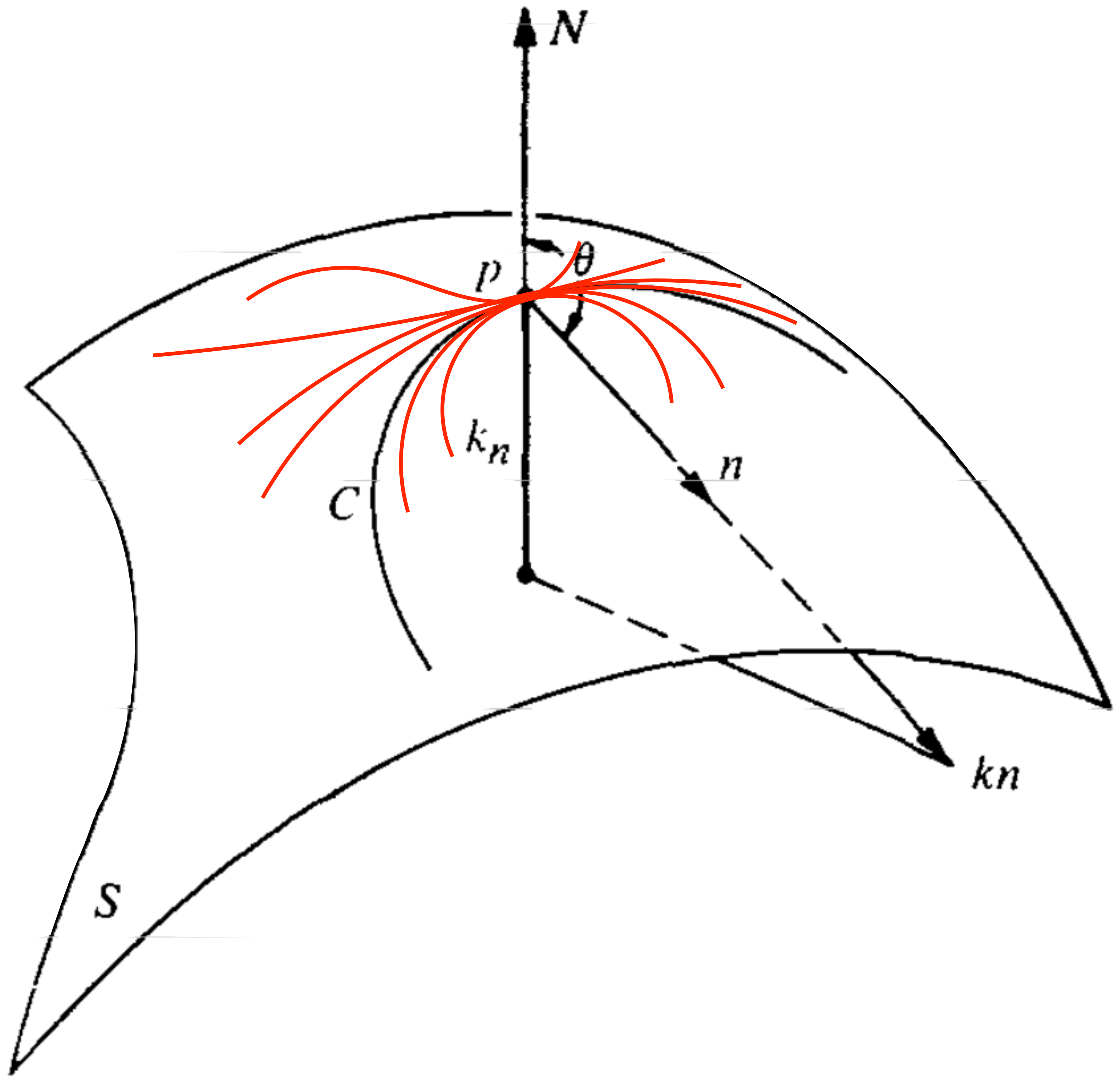
Intrinsic Geometry

- Properties of the surface that:
  - only depend on the first fundamental form
  - do not depend on the embedding of surface in space
    - length
    - angles
    - Gaussian curvature (Theorema Egregium)



Normal curvature

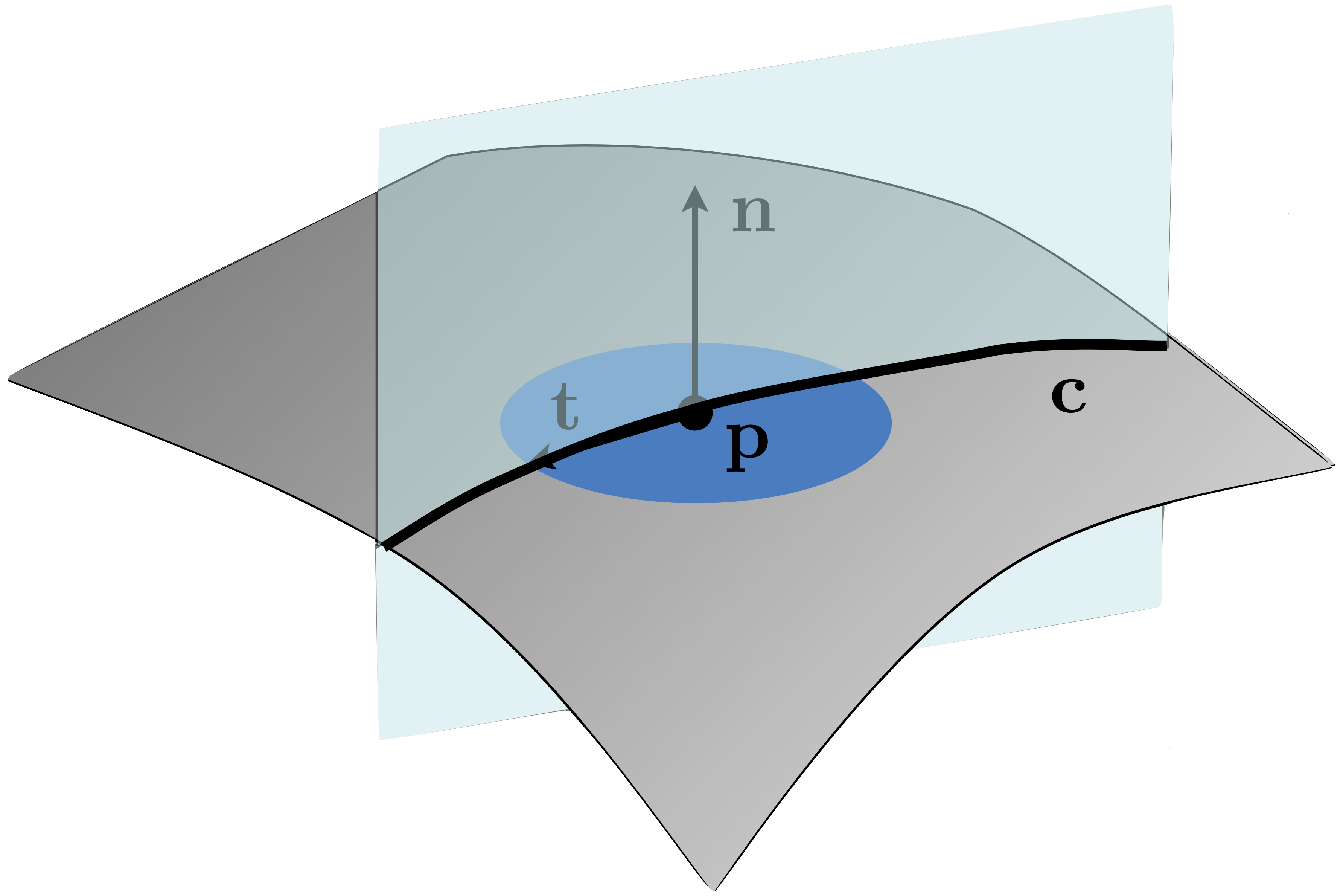
- Let $\mathbf{c}$ be a curve on S through p
- let k be the curvature of $\mathbf{c}$ at p
- the projection κ_n of the curvature κ onto the normal $\mathbf{N}$ is the *normal curvature* of $\mathbf{c}$ at p



- Proposition (Meusnier): all curves through p with the same tangent of $\mathbf{c}$ have the same normal curvature

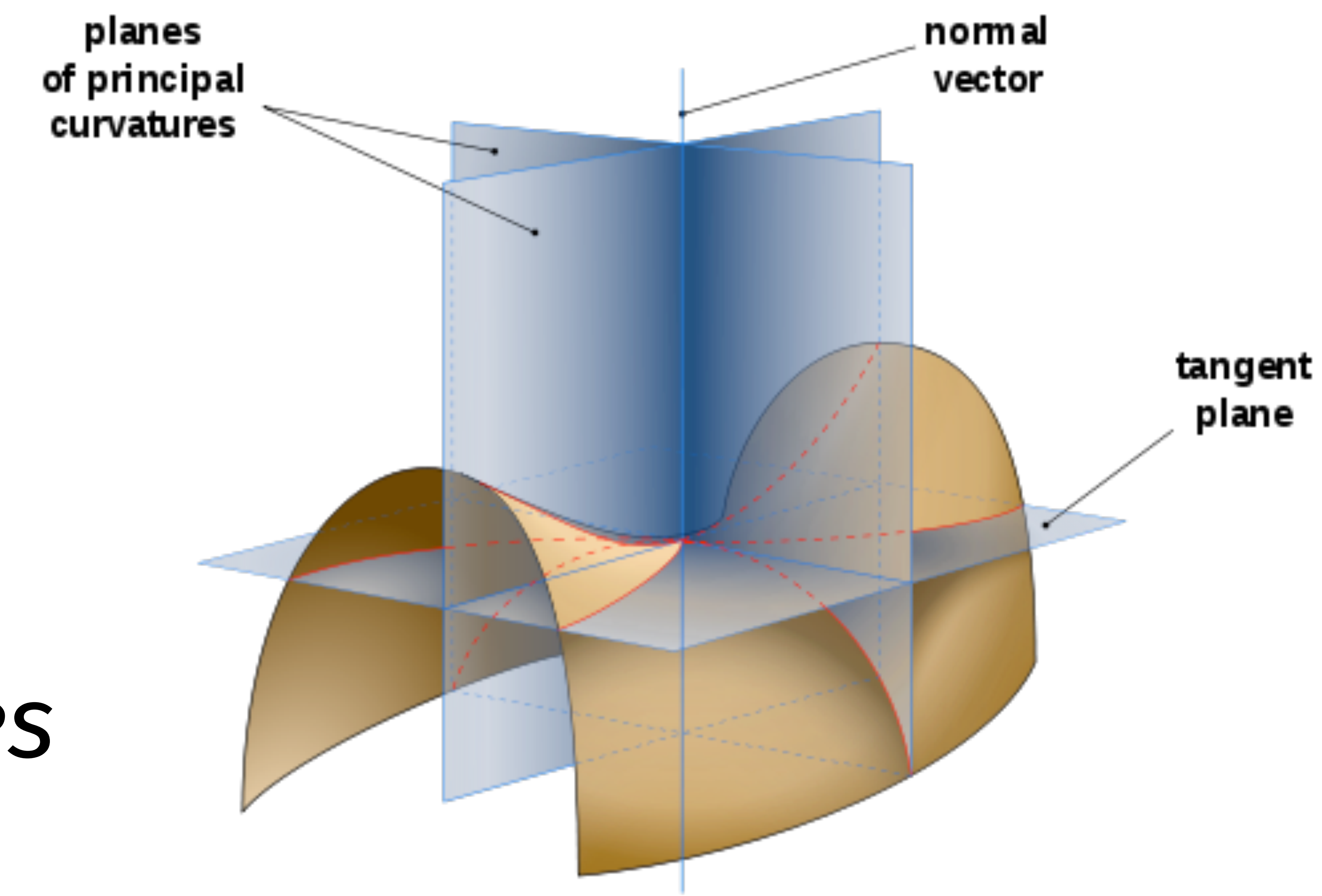
Normal curvature

- We can choose a specific curve $\mathbf{c}$ that has the normal aligned to $\mathbf{n}$
- A direction $\mathbf{t} = J_f \bar{\mathbf{t}}$ on the tangent plane at point p, together with the normal $\mathbf{n}$ at p define a plane that cuts S along a curve $\mathbf{c}$
- The curvature of $\mathbf{c}$ curve at p is the *normal curvature* w.r.t. $\mathbf{t}$



Principal curvatures

- Consider all possible directions $\mathbf{t}$ on the tangent plane at p
- Unless the normal curvature is equal at all $\mathbf{t}$'s, there exist exactly two directions:
  - $\mathbf{t}_1$ such that $\kappa_1 = \kappa(\mathbf{t}_1)$ is maximum
  - $\mathbf{t}_2$ such that $\kappa_2 = \kappa(\mathbf{t}_2)$ is minimum
- κ_1 and κ_2 are called the *principal curvatures*
- $\mathbf{t}_1$ and $\mathbf{t}_2$ are called the *principal directions of curvature*



Euler theorem

- The principal curvatures are mutually orthogonal and independent on parametrization

- for any direction $\mathbf{t}$ on the tangent plane

$$\kappa_n(\bar{\mathbf{t}}) = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi$$

  where ψ is the angle between $\mathbf{t}$ and $\mathbf{t_1}$

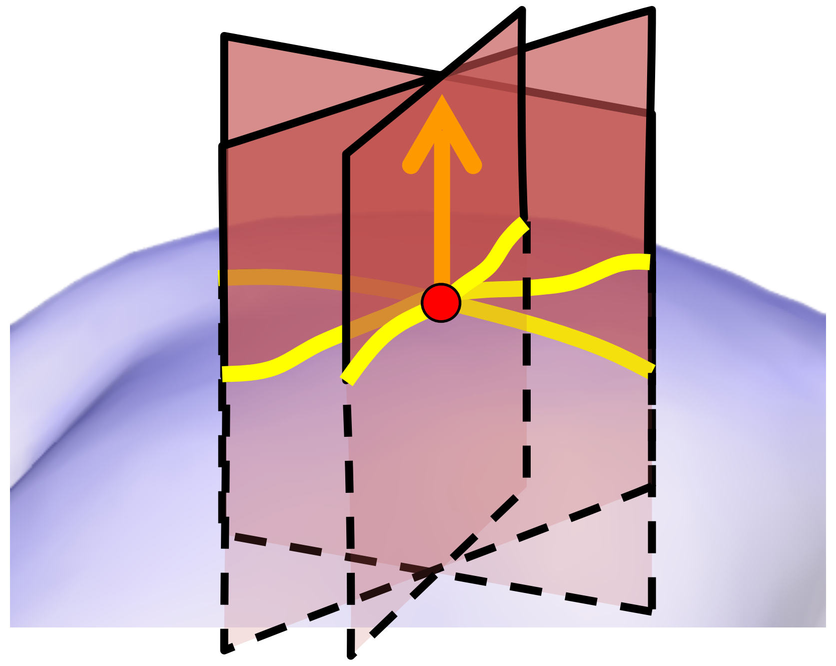
- *Mean curvature*: $$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\phi)d\phi$$

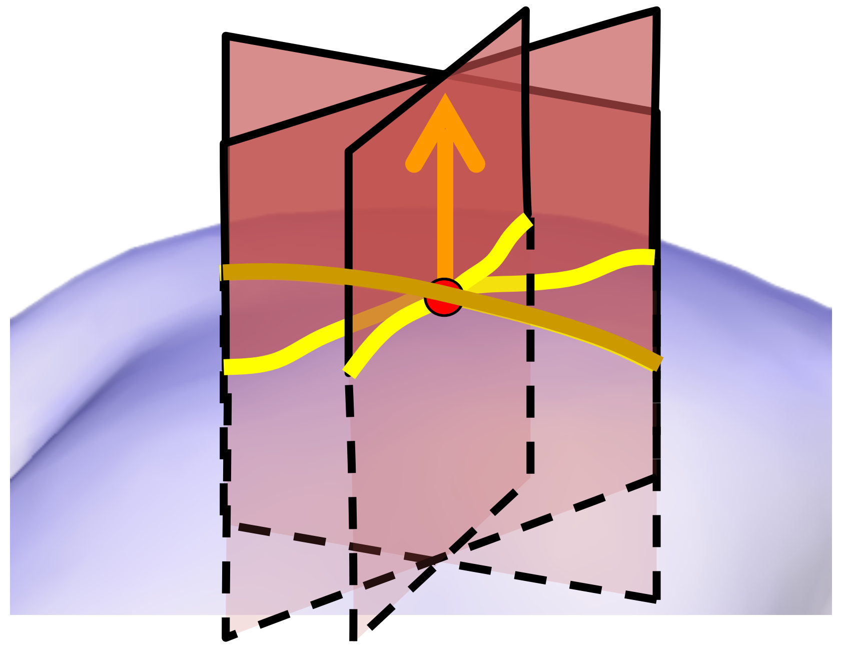
- *Gaussian curvature*: $$K = \kappa_1 \cdot \kappa_2$$

Mean Curvature

- Intuition for mean curvature

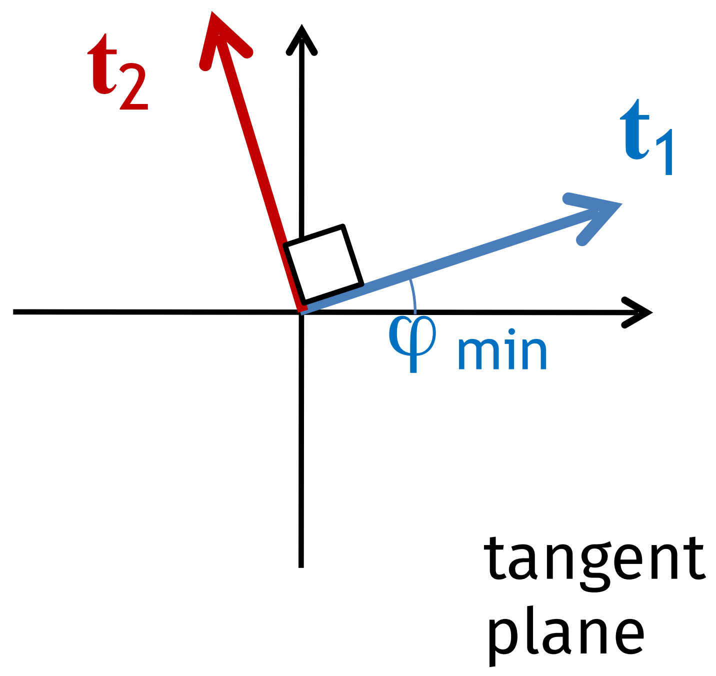
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi) d\varphi$$

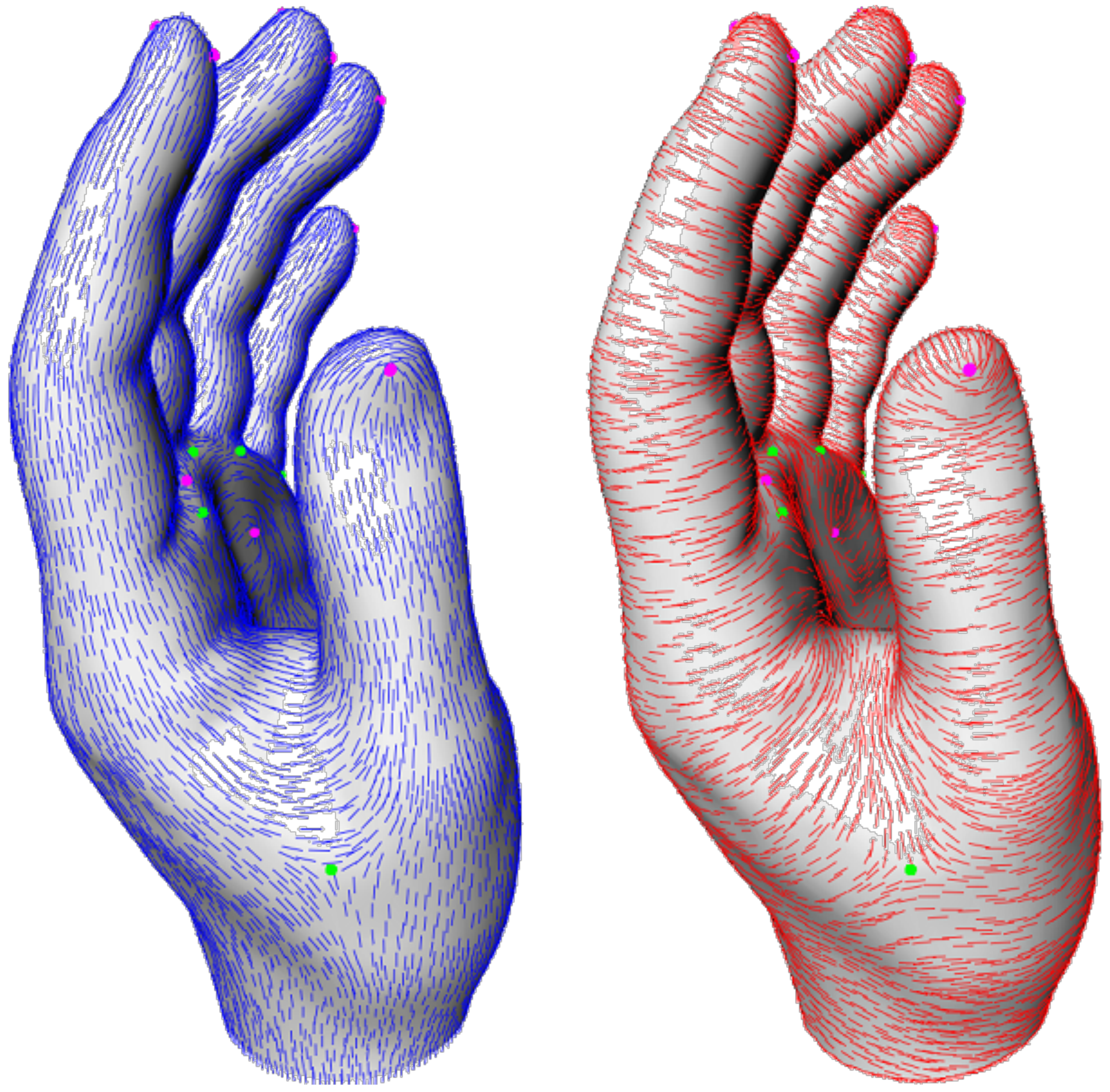




Principal Directions

- Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}

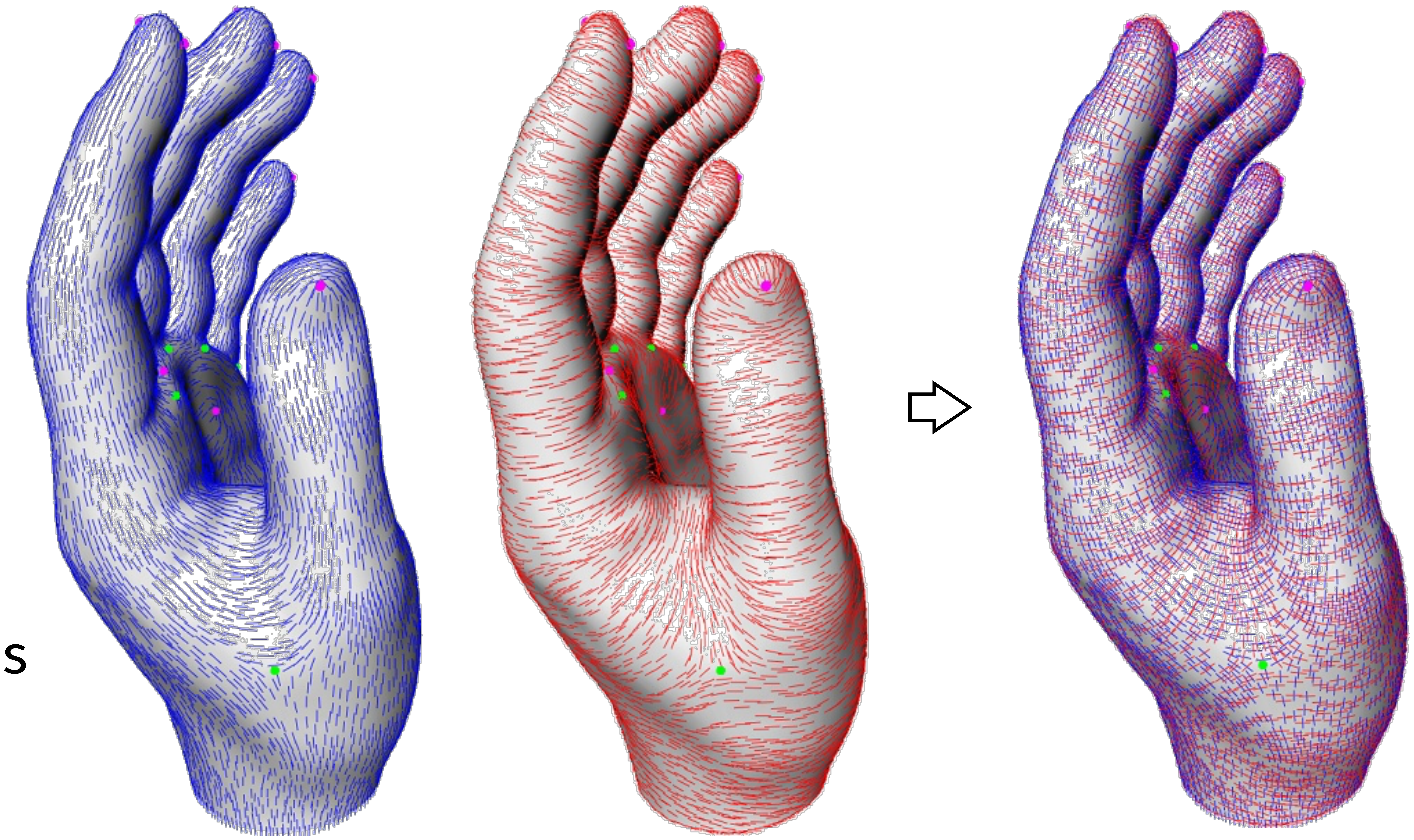




min curvature   max curvature

Pierre Alliez, David Cohen-Steiner, Olivier Devillers, Bruno Lévy, and Mathieu Desbrun. 2003. Anisotropic polygonal remeshing. *ACM Trans. Graph.* 22, 3 (July 2003), 485-493. DOI: http://dx.doi.org/10.1145/882262.882296

Principal Directions



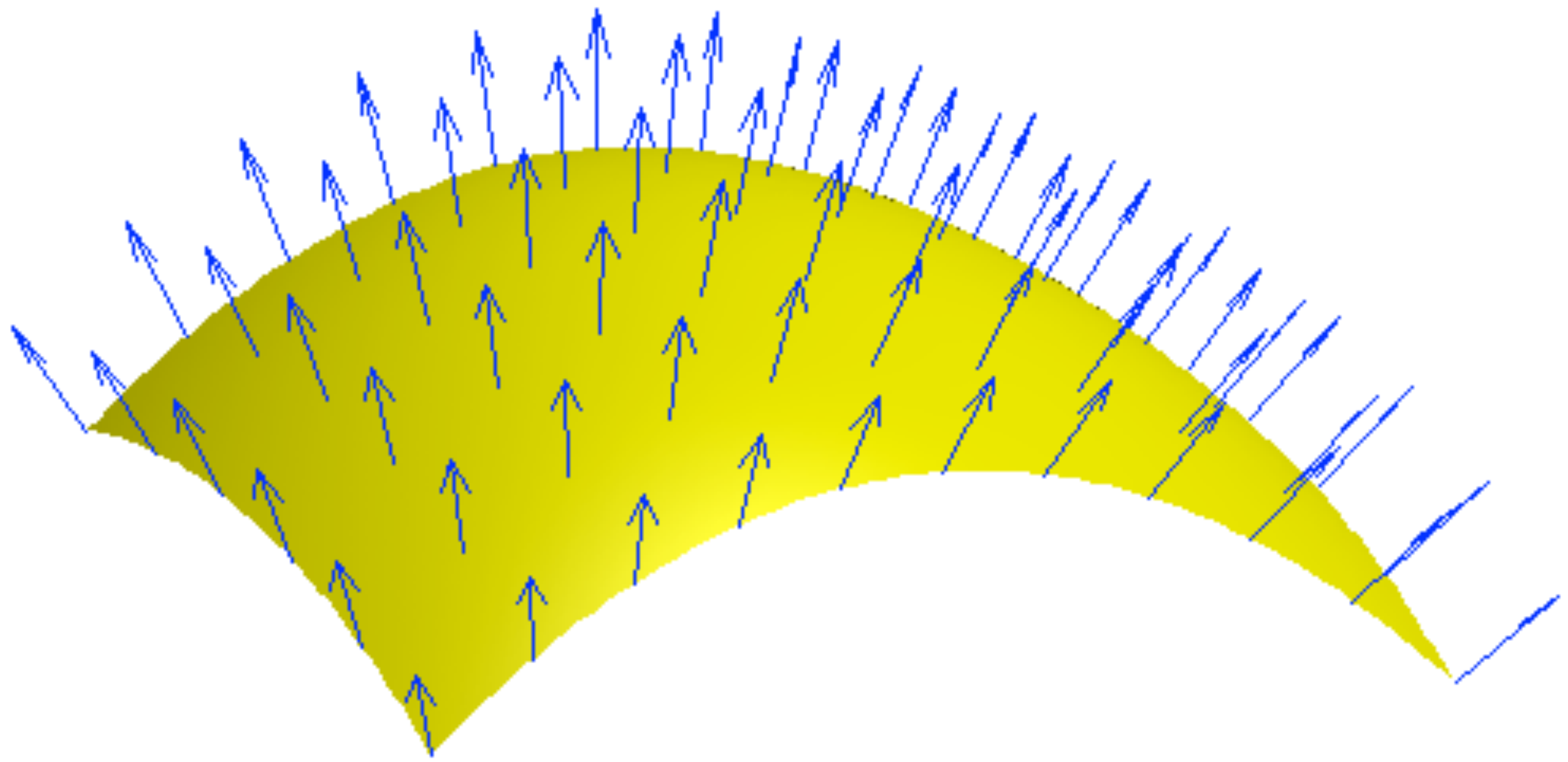
Principal direction lines form an "infinite grid" on the surface

Pierre Alliez, David Cohen-Steiner, Olivier Devillers, Bruno Lévy, and Mathieu Desbrun. 2003. Anisotropic polygonal remeshing. *ACM Trans. Graph.* 22, 3 (July 2003), 485-493. DOI: http://dx.doi.org/10.1145/882262.882296

Gauss map

- The Gauss map sends a point on the surface to the outward pointing unit normal vector:

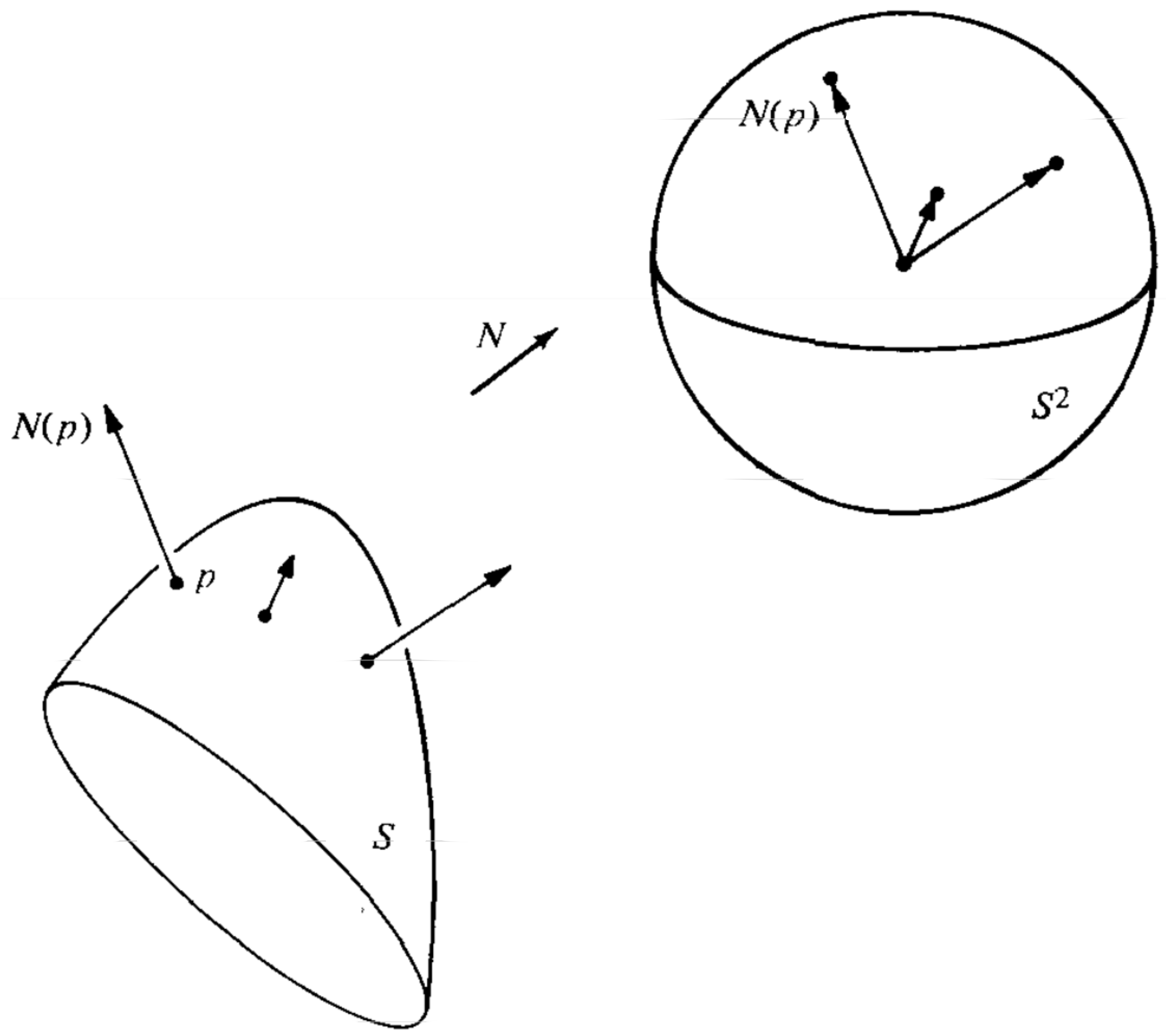
$$N(p) = N(f(x)) = \frac{f_u(x) \times f_v(x)}{||fu(x) \times fv(x)||} = \mathbf{n}$$



Gauss map

- The Gauss map can be seen as a map from the surface to the unit sphere:

$$N : \underset{\text{surface}}{S} \longrightarrow \underset{\text{sphere}}{S^2}$$



Gauss map

- The differential of the Gauss map dN tell us how the normal bends while moving away from a point p

- The displacement of $\mathbf{n}$ is a vector in the tangent plane $T_p(S)$

- So dN can be seen as an automorphism on the tangent bundle of the surface:

$$dN_p : T_p(S) \longrightarrow T_p(S)$$

$$dN : T(S) \longrightarrow T(S)$$

Second fundamental form

- The second fundamental form is a quadratic form defined on the tangent plane $T_p(S)$

- It measures the component to the differential of the Gauss map in a given direction along the same direction:

$$II_p(v) = - < dN_p(v), v >$$

- If v is a unit vector, then $II_p(v)$ is equal to the normal curvature k_n of a curve passing through p and tangent to v.

Second fundamental form

- In local coordinates (parametric form): the second fundamental form is given by the projections of second order derivatives of the parametric function onto the normal line

- *Second fundamental form:*

$$\mathbf{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} f_{uu}^T \mathbf{n} & f_{uv}^T \mathbf{n} \\ f_{uv}^T \mathbf{n} & f_{vv}^T \mathbf{n} \end{bmatrix}$$

Normal curvature

- Normal curvature of a parametric surface can be computed by means of the first and second fundamental forms

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{Lu_t^2 + 2Mu_t v_t + Nv_t^2}{Eu_t^2 + 2Fu_t v_t + Gv_t^2}$$

- The denominator accounts for the fact that vector $\mathbf{t}$ is not necessarily a unit vector

Shape operator

- The *shape operator* Sh is the *differential* of the Gauss map in the tangent bundle

- It is a linear operator that, when applied to a tangent vector $\mathbf{t}$ tells how the surface normal varies when moving along $\mathbf{t}$ on S

- In local coordinates given by frame (f_u, f_v), it is represented by a 2×2 matrix defined at each point p through the coefficients of the first and second fundamental form:

$$Sh = (EG - F^2)^{-1} \begin{pmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{pmatrix}$$

Shape operator

- The principal directions of curvature are the *eigenvectors* of Sh
- The principal curvatures are the *eigenvalues* of Sh
- The shape operator is described by a diagonal matrix of principal curvatures in the frame $(\mathbf{t}_1,\mathbf{t}_2)$

Curvature tensor

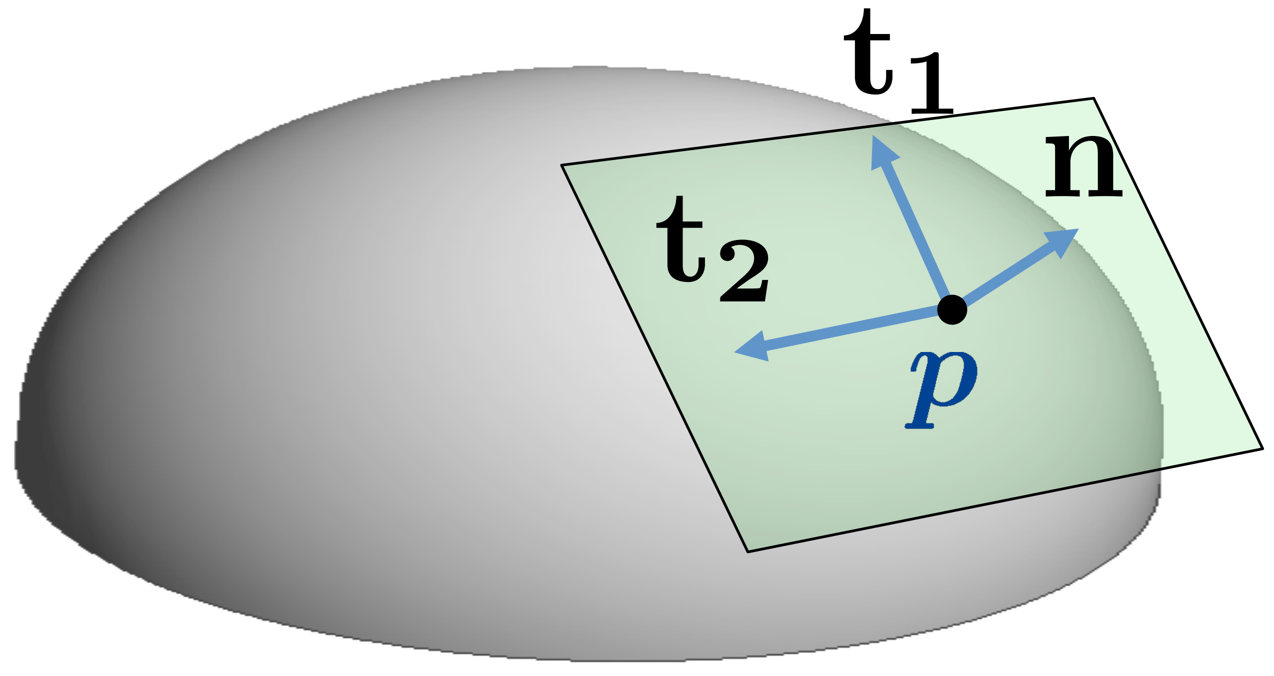
- The *curvature tensor* $\mathbf{C}$ is a symmetric 3×3 matrix with
  - eigenvalues κ_1, κ_2 and 0
  - eigenvectors $\mathbf{t_1}$, $\mathbf{t_2}$ and $\mathbf{n}$

$$\mathbf{C} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$
$$\mathbf{P} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}]$$
$$\mathbf{D} = \mathrm{diag}(\kappa_1, \kappa_2, 0)$$

- It defines a tensor field on S that can be used to compute all curvatures
  - Extrinsic, but independent of parametrization

Darboux frame

- Principal directions have arbitrary orientations
- Orientations can be selected such that $(\mathbf{t_1},\mathbf{t_2},\mathbf{n})$ form a right-handed 3D frame with origin at p and $\mathbf{n}$ points outward
- The surface in a sufficiently small neighborhood of p can be expressed in explicit form $n = f(t_1,t_2)$ with respect to the Darboux frame

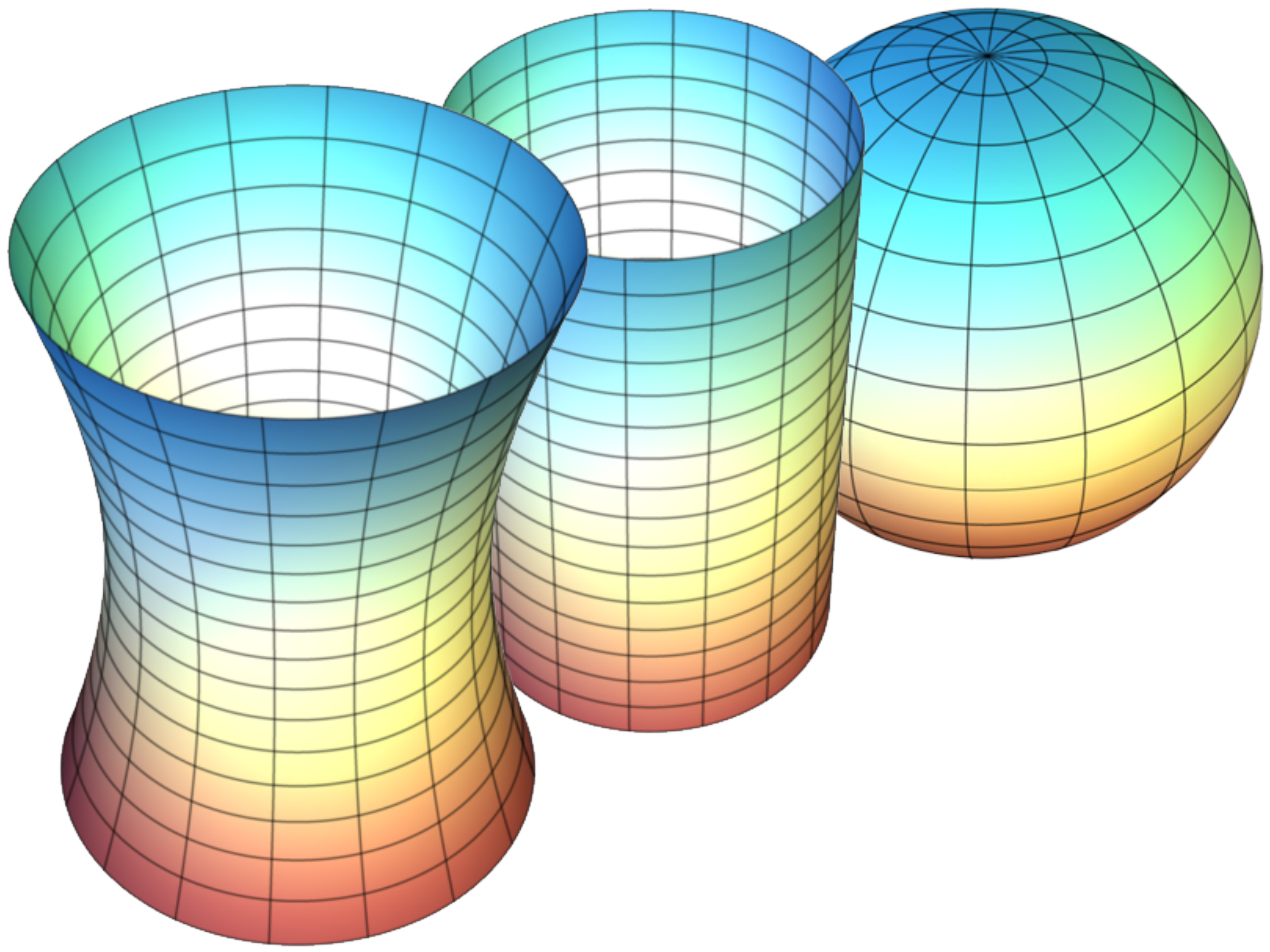


Recap: second order differential properties

- With second differential form $\mathbf{II}$ we can measure:
  - Normal curvature for any direction $\mathbf{t}$: $\kappa_n(\bar{\mathbf{t}}) = \dfrac{\bar{\mathbf{t}}^T \mathbf{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \dfrac{Lu_t^2 + 2Mu_t v_t + Nv_t^2}{Eu_t^2 + 2Fu_t v_t + Gv_t^2}$
  - Gaussian curvature (intrinsic property): $K = \kappa_1 \cdot \kappa_2 = \dfrac{\det(\mathbf{II})}{\det(\mathbf{I})}$
- Shape operator Sh is built from coefficients of fundamental forms and provides:
  - Principal curvatures (eigenvalues)
  - Principal curvature directions (eigenvectors)
  - Gaussian curvature (determinant) and mean curvature (half trace)
  - Curvature tensor (independent of parametrization)

Classification through Gaussian curvature

- A point $\mathbf{p}$ on the surface is called
  - Elliptic, if $K > 0$
  - Parabolic, if $K = 0$
  - Hyperbolic, if $K < 0$

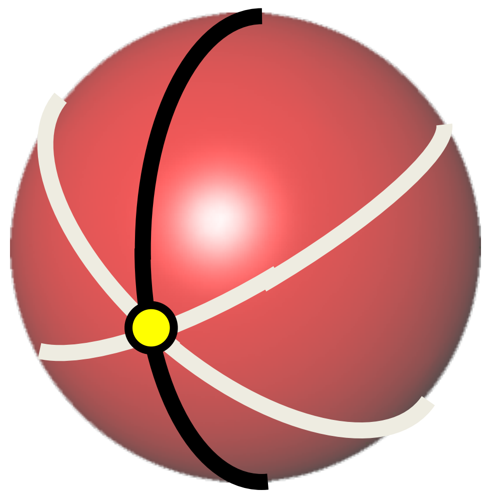


- Developable surface iff $K = 0$ at all points

Local Surface Shape by Curvatures

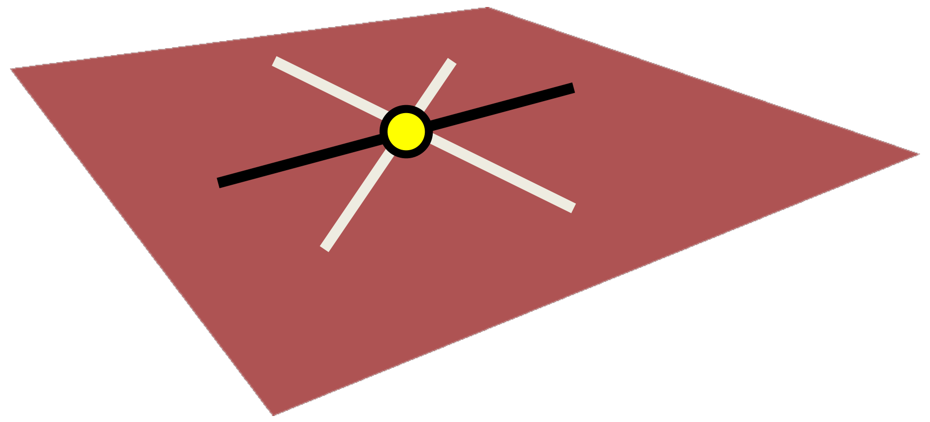
Isotropic:
all directions are principal directions

$K > 0,\ \kappa_1 = \kappa_2$



spherical (umbilical)

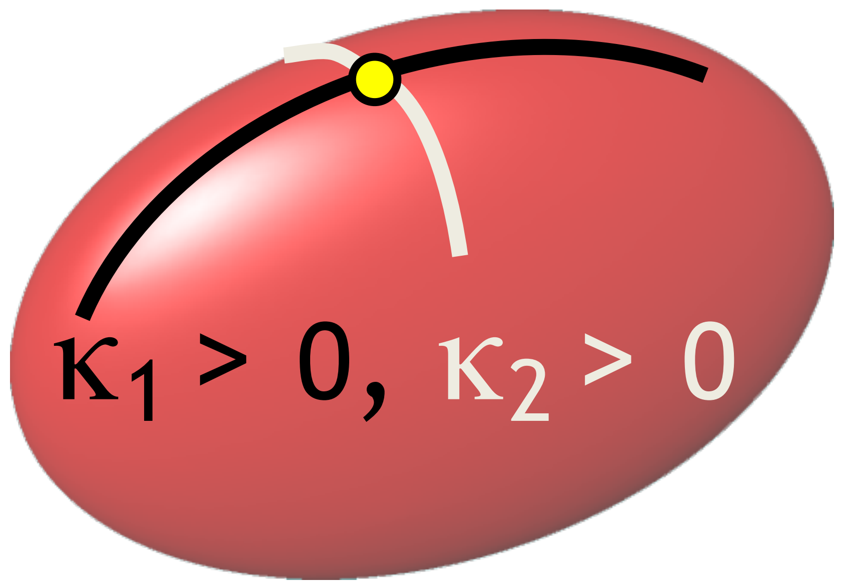
$K = 0$



planar

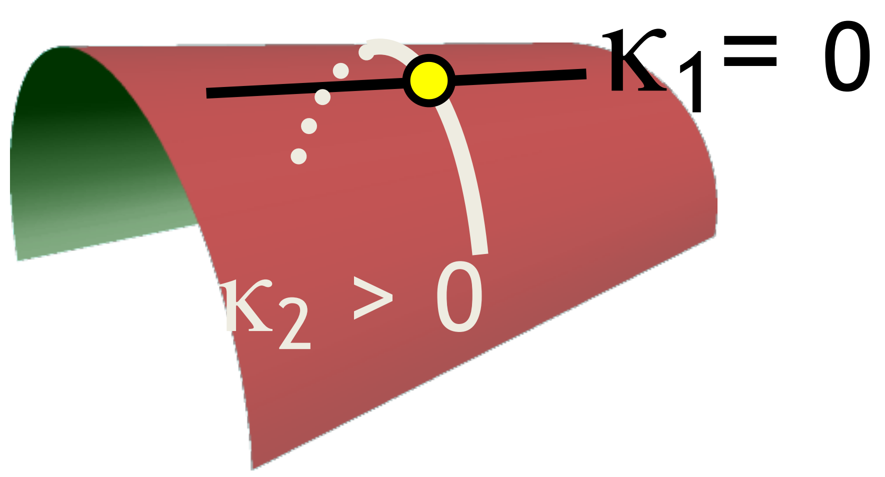
Anisotropic:
2 distinct principal directions

$K > 0$



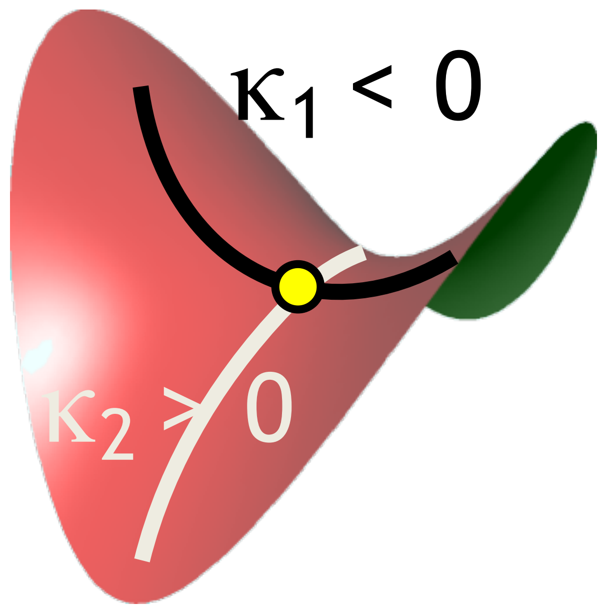
elliptic

$K = 0$



parabolic

$K < 0$



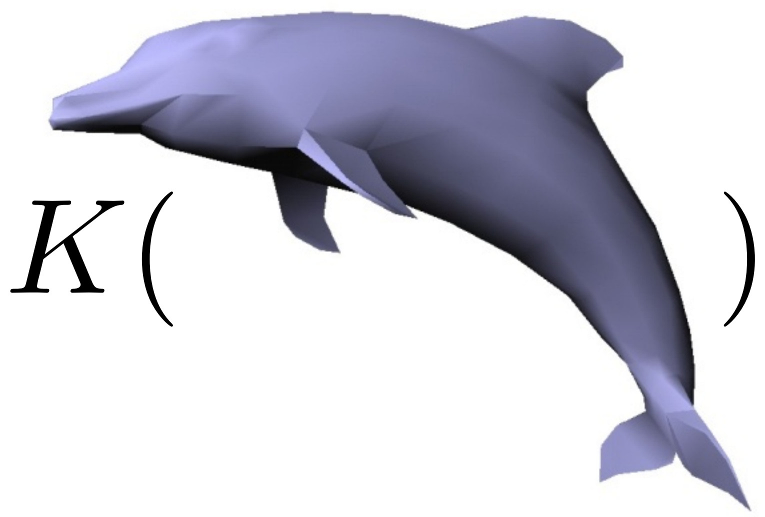
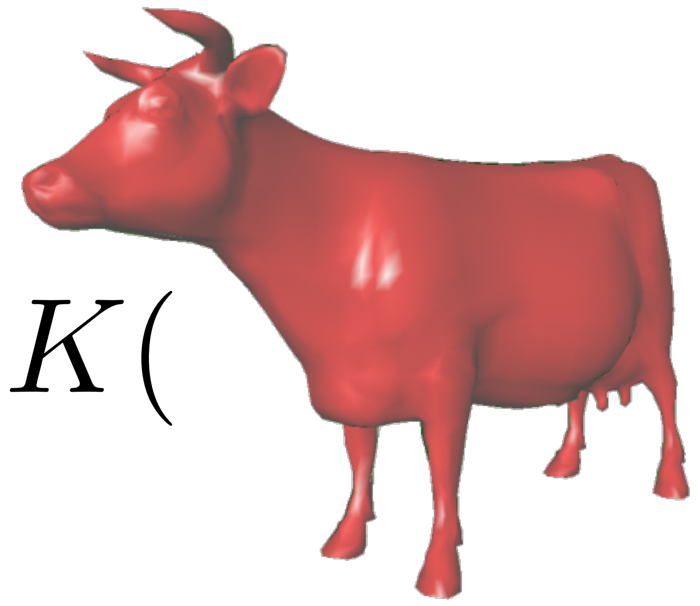
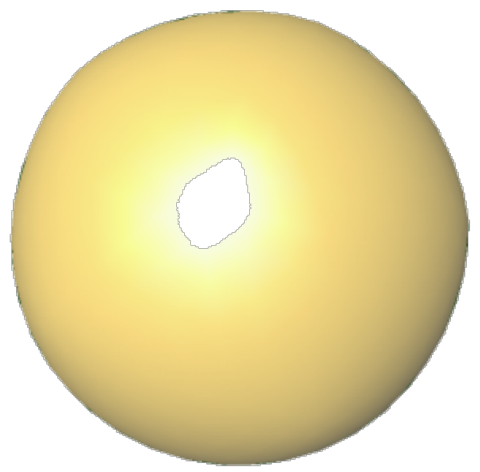
hyperbolic

Gauss-Bonnet Theorem

- For a closed surface M:

Surface characteristic
from Euler-Poincaré formula

$$\int_{\mathcal{M}} K \, dA = 2\pi \chi(\mathcal{M})$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

Gauss-Bonnet Theorem

- For a closed surface M:

$$\int_{\mathcal{M}} K \, dA = 2\pi \, \chi(\mathcal{M})$$

- Compare with planar curves:

$$\int_{\gamma} \kappa \, ds = 2\pi \, k$$

Thank you