14 - Geodesics on surfaces

Acknowledgements: Keenan Crane

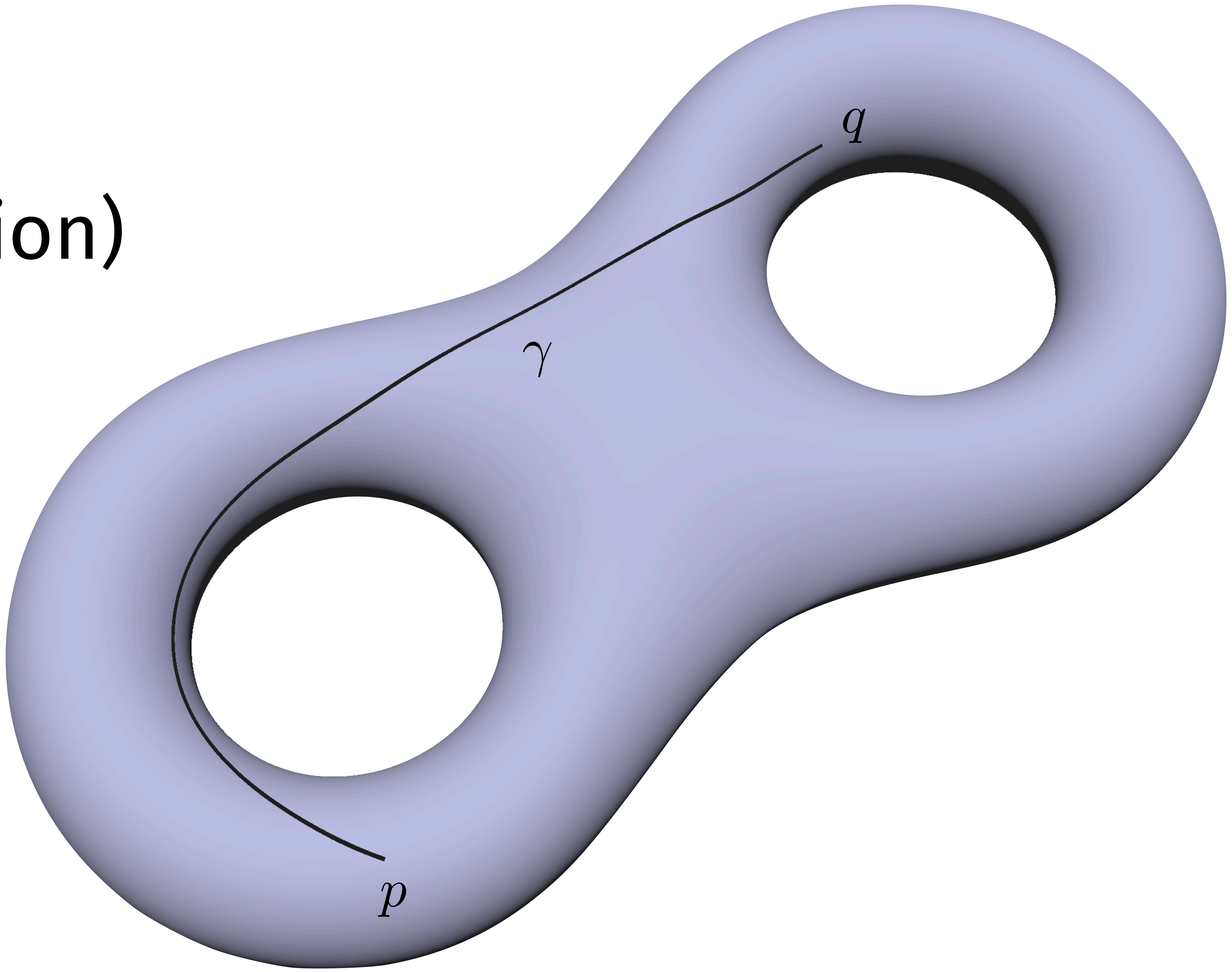
In this lecture

- Distances and curves on surfaces:
  - how to measure distances
  - how to go “straight” on a surface
  - how different this is from the plane

Curve on a surface

$\mathbf{M}$ surface, p, q points on $\mathbf{M}$, γ curve on $\mathbf{M}$ joining p to q:

- $\gamma : [0, l_\gamma] \longrightarrow \mathbf{M}$
- $\gamma[0] = p \quad \gamma[l_\gamma] = q$
- l_γ length of γ (arc-length parametrization)



Geodesic distance on a surface

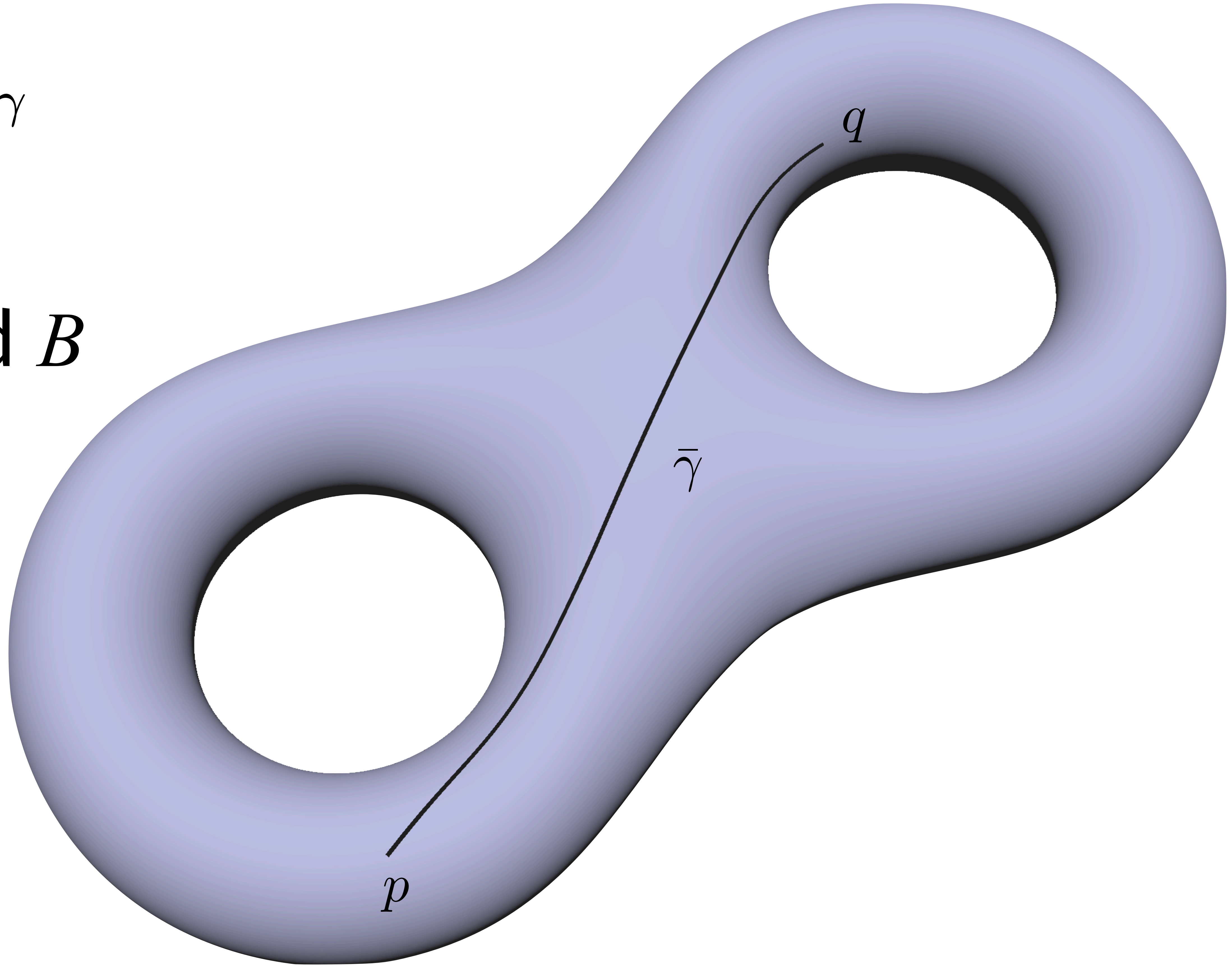
Distance between A and B:

length of the shortest curve joining A to B

$$d(A, B) = \min_{\gamma} l_{\gamma}$$

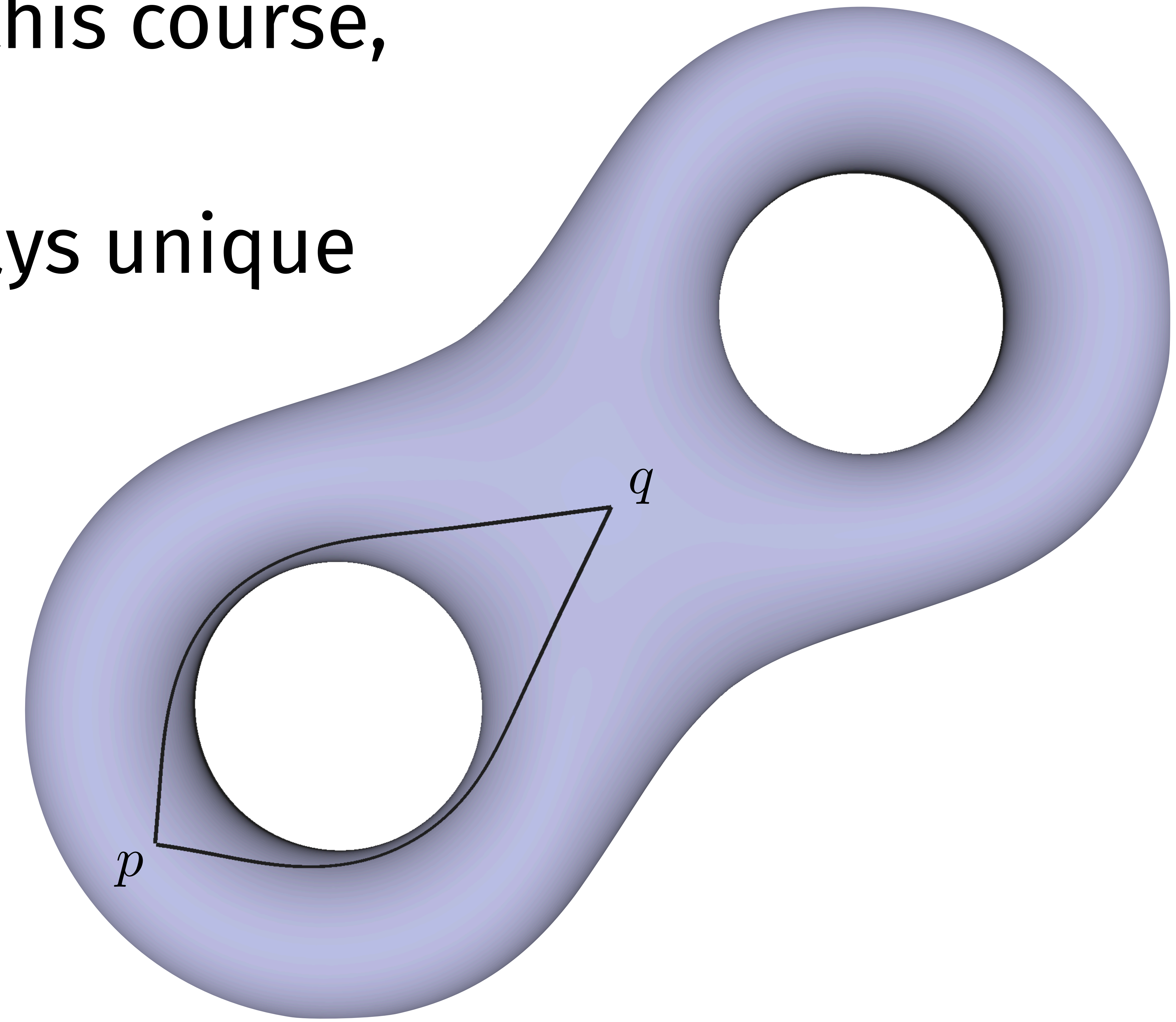
If $d(A, B) = l_{\bar{\gamma}}$ then $\bar{\gamma}$ is called a

shortest geodesic (path) between A and B



Geodesic distance on a surface

- The geodesic distance is a scalar function on the surface. In particular, in the cases of interest for this course, it is uniquely defined for each point.
- The shortest geodesic path is not always unique

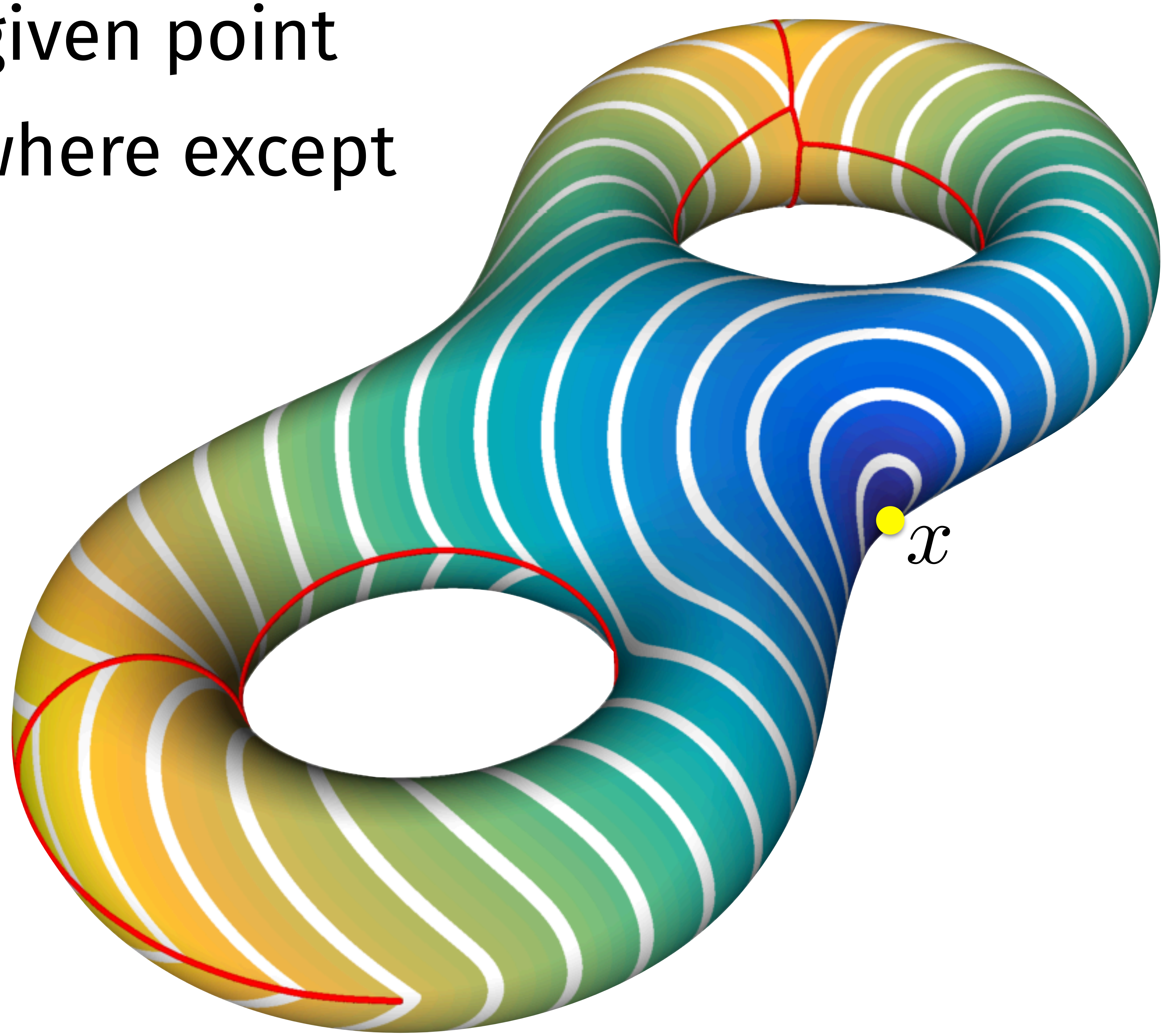


Cut locus

- The locus of points that have more than one shortest geodesic connecting to a given point
- The distance field is smooth everywhere except at the cut locus

$$d_x : \mathbf{M} \longrightarrow \mathbb{R}$$

$$d_x(y) = d(x, y)$$

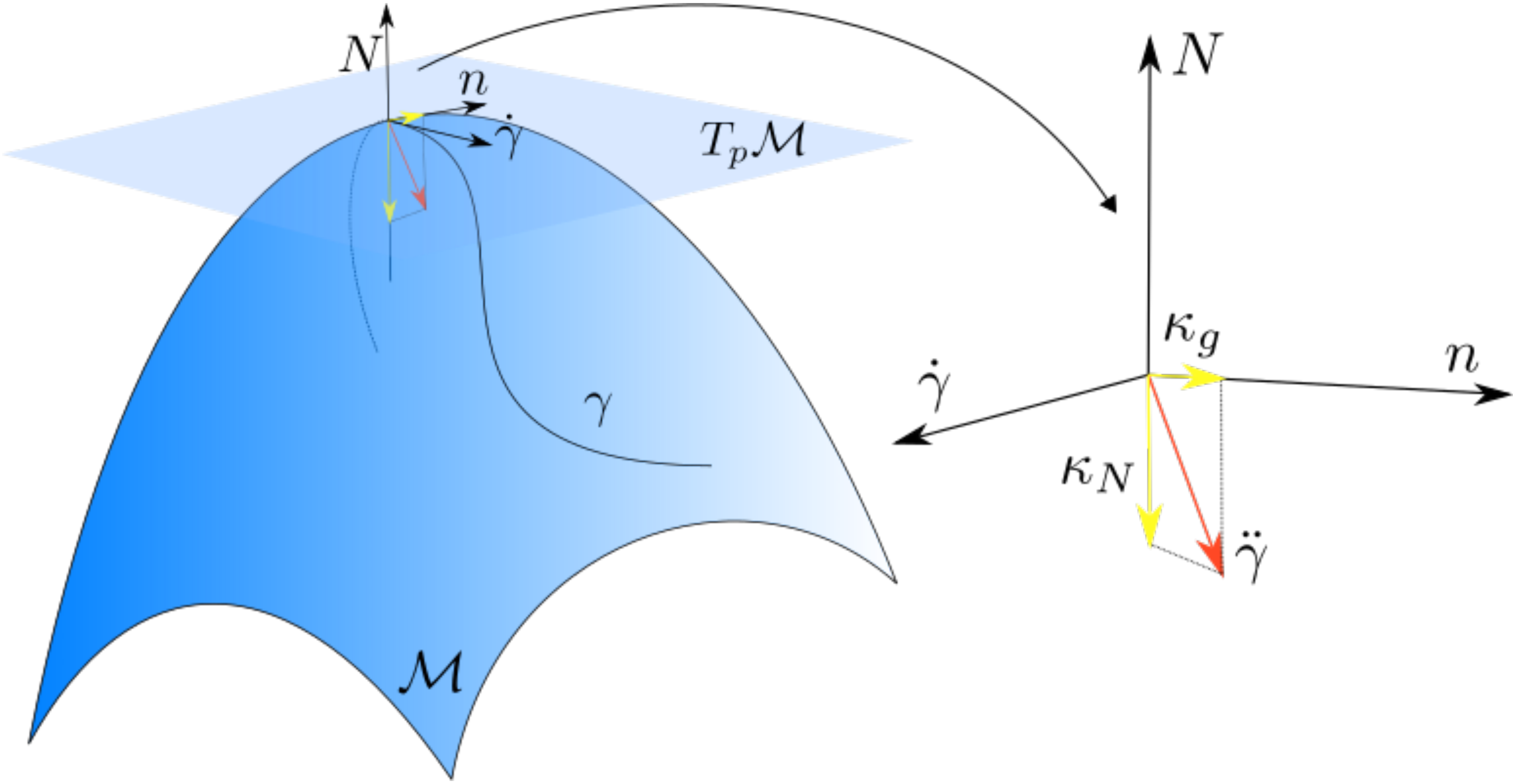


Geodesic lines

We know how to compute the tangent vector and the curvature of a curve

$$\gamma : I \to \mathbb{R}^3.$$

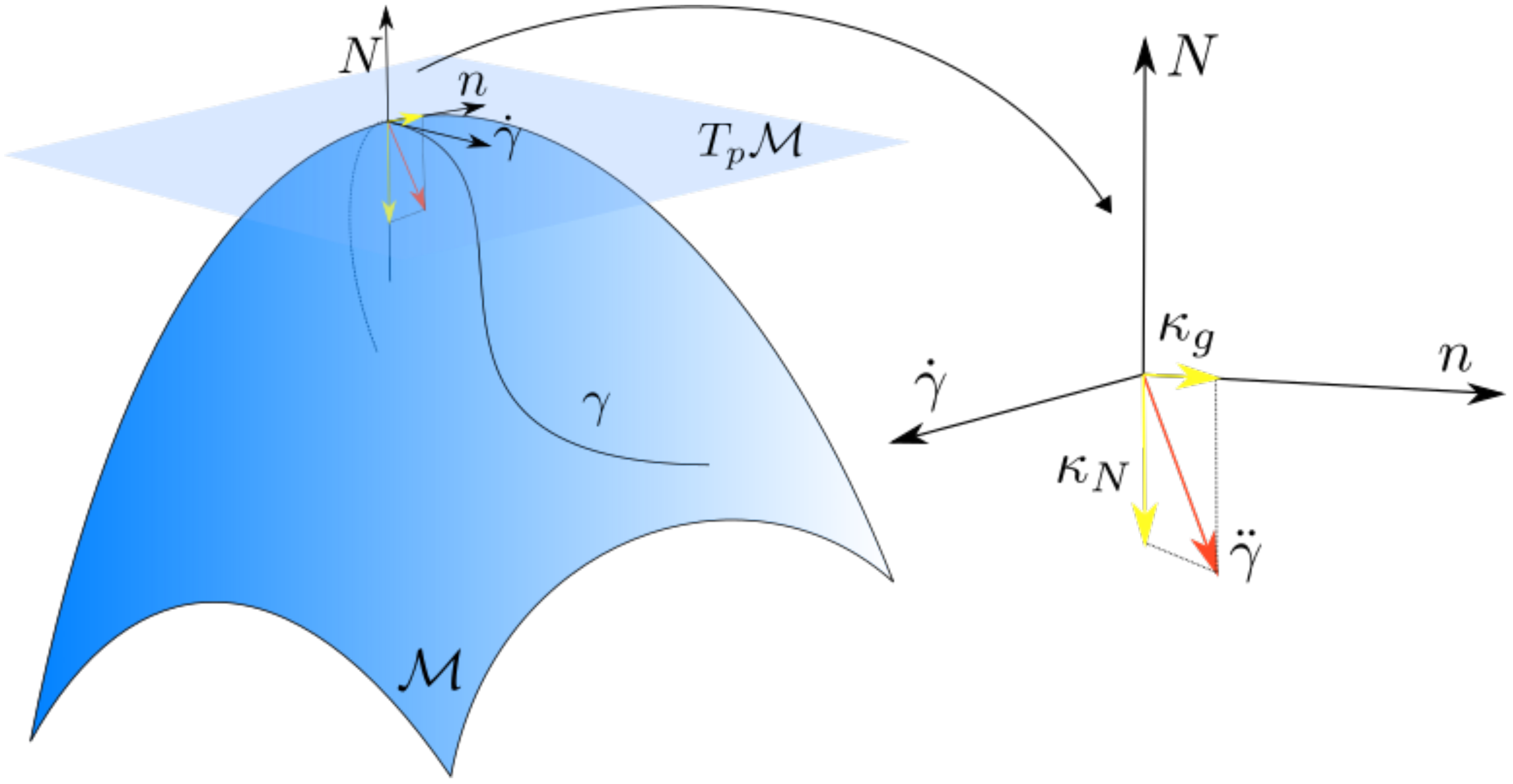
What if γ happens to lie on a surface $\mathscr{M}$?



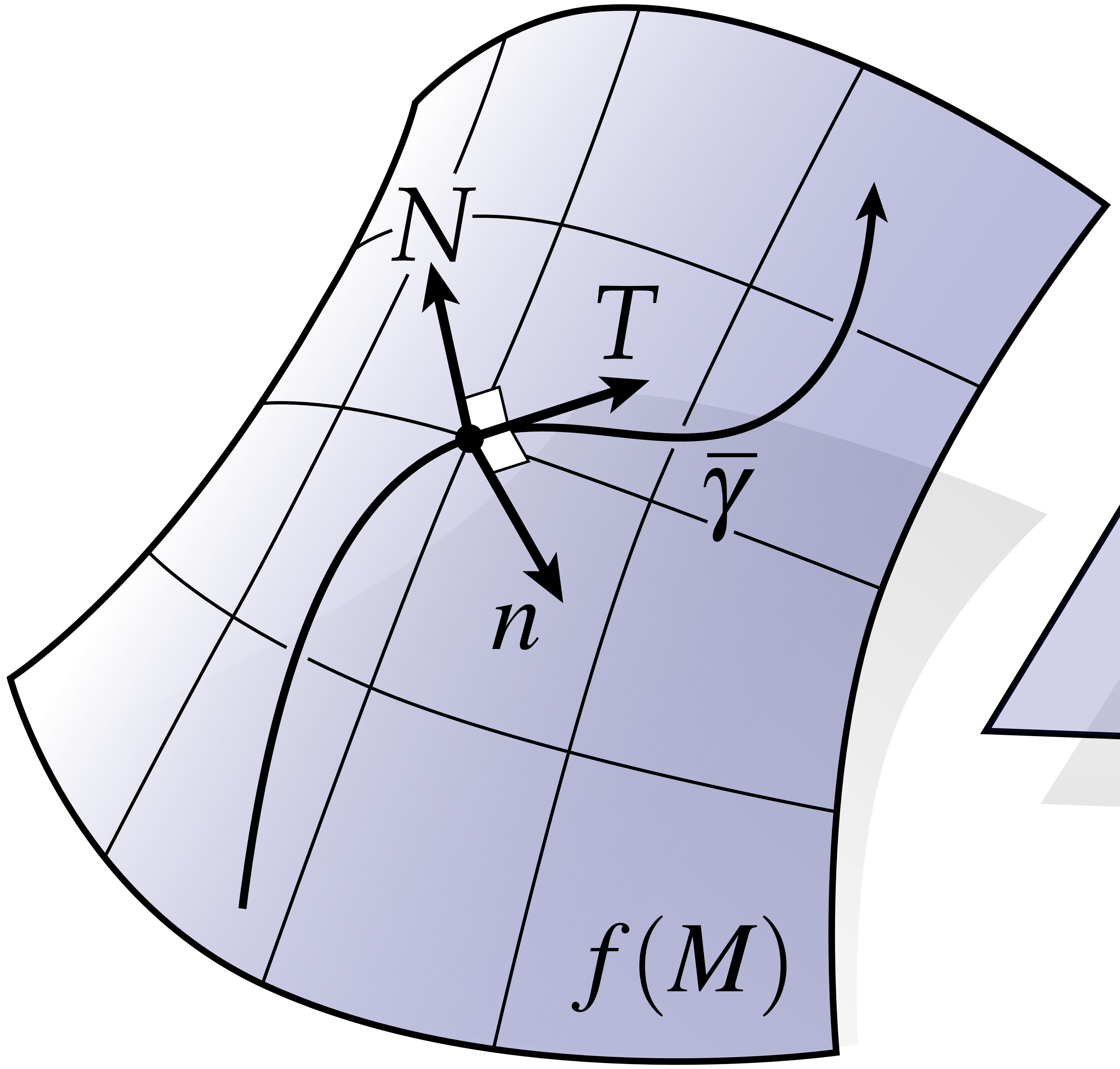
Geodesic lines

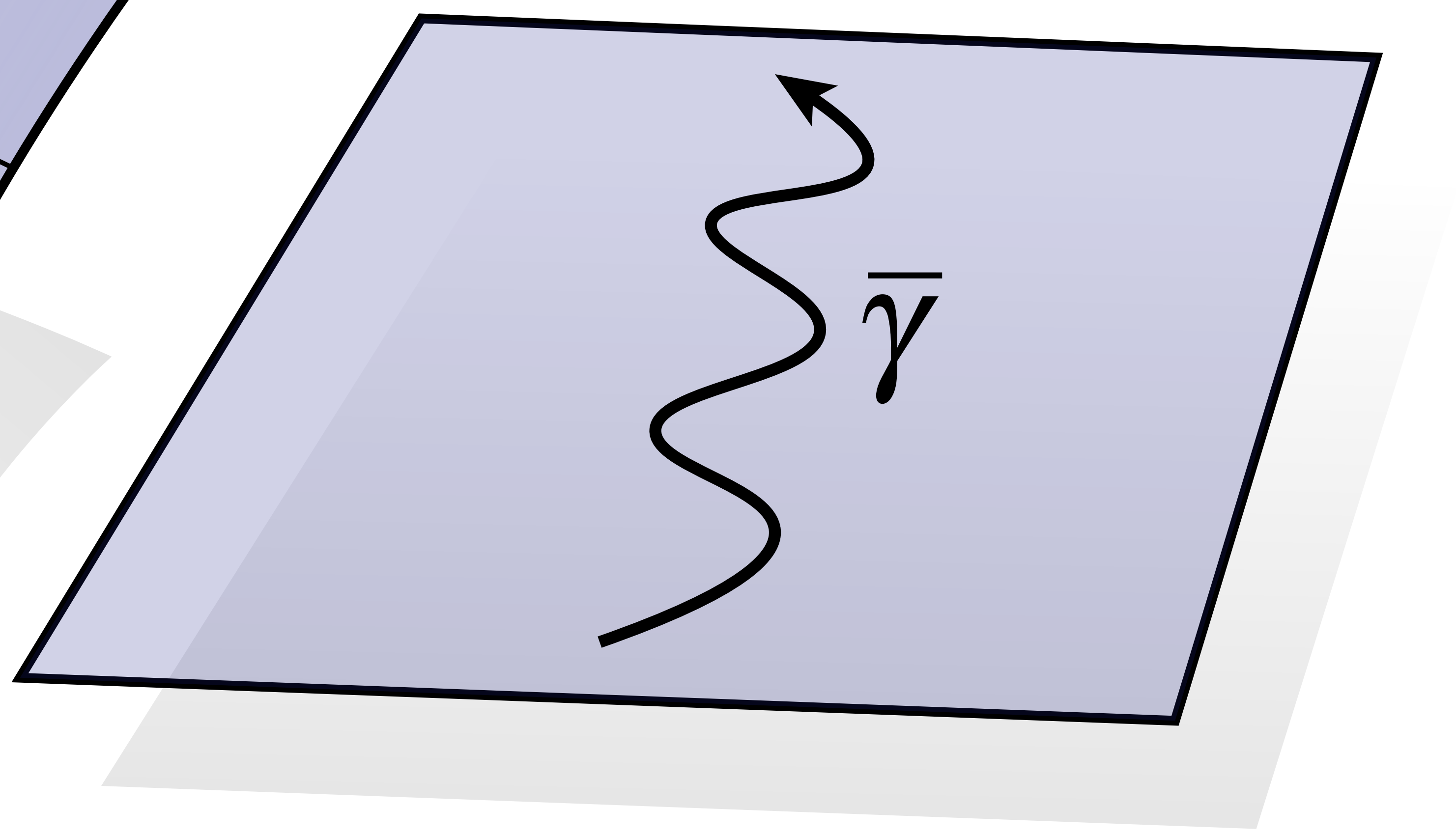
A curve γ is a *geodesic line* if it bends *with* $\mathbf{M}$ but it does not bend *on* $\mathbf{M}$

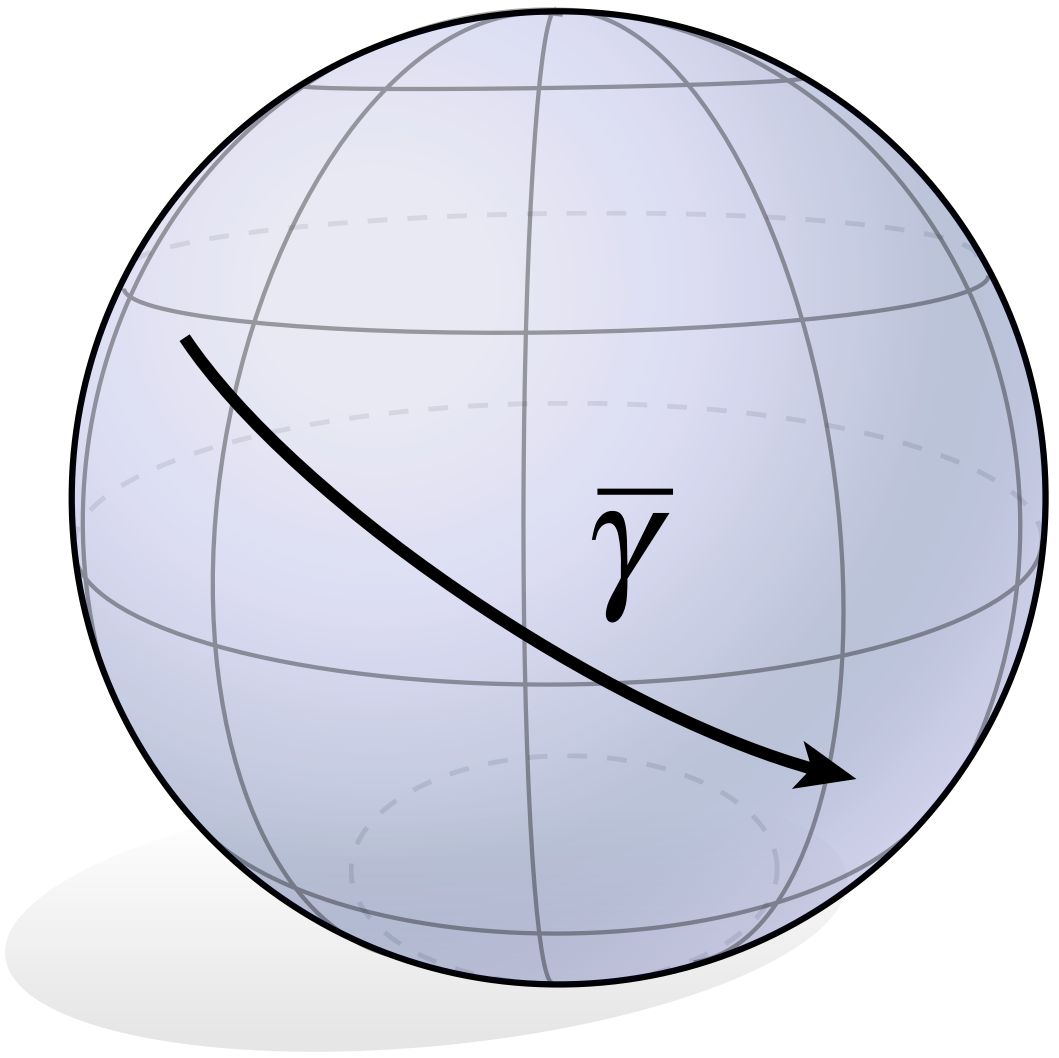
- Tangent of γ: $T(s) = \frac{d}{ds}\gamma(s)$ where s is the arc-length parameter
- Normal of γ: $n = T \times N$ where N is the normal to the surface
- Normal curvature: $\kappa_N = N \cdot \frac{d}{ds}T$
- Geodesic curvature: $\kappa_g = n \cdot \frac{d}{ds}T$
- A curve is a geodesic line iff $\kappa_g = 0$



Geodesic lines







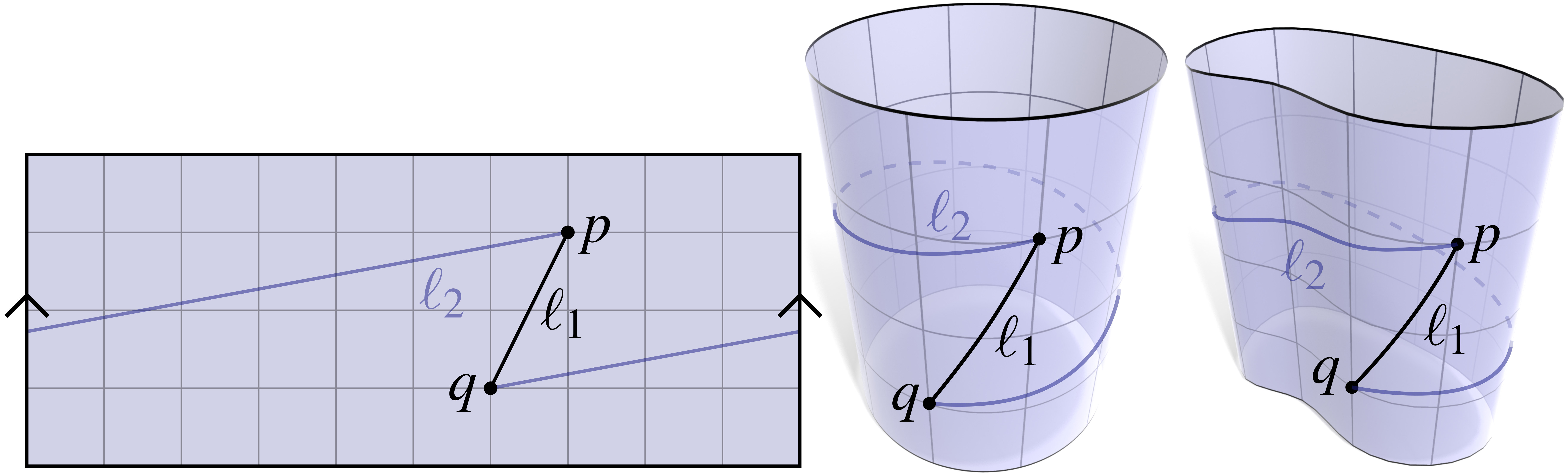
$$\kappa_N = 0 \qquad \kappa_g \neq 0$$

$$\kappa_N \neq 0 \qquad \kappa_g = 0$$

Geodesic lines

In some sense, a geodesic line is *straight* on $\mathbf{M}$

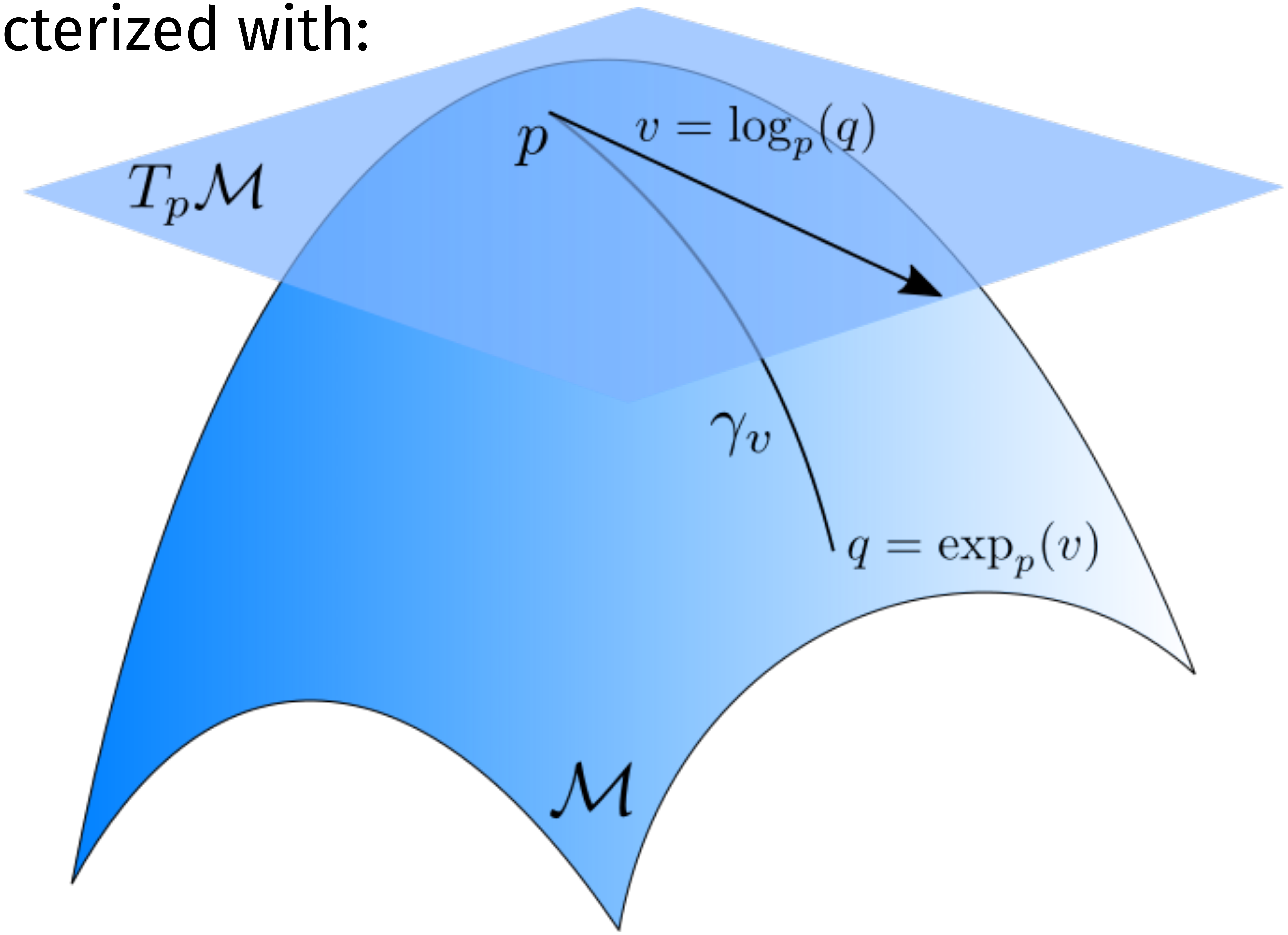
- Shortest paths are geodesic lines
- Not all geodesic lines are shortest paths
- Geodesic lines are *locally* shortest



Geodesic lines

A geodesic line is fully characterized with:

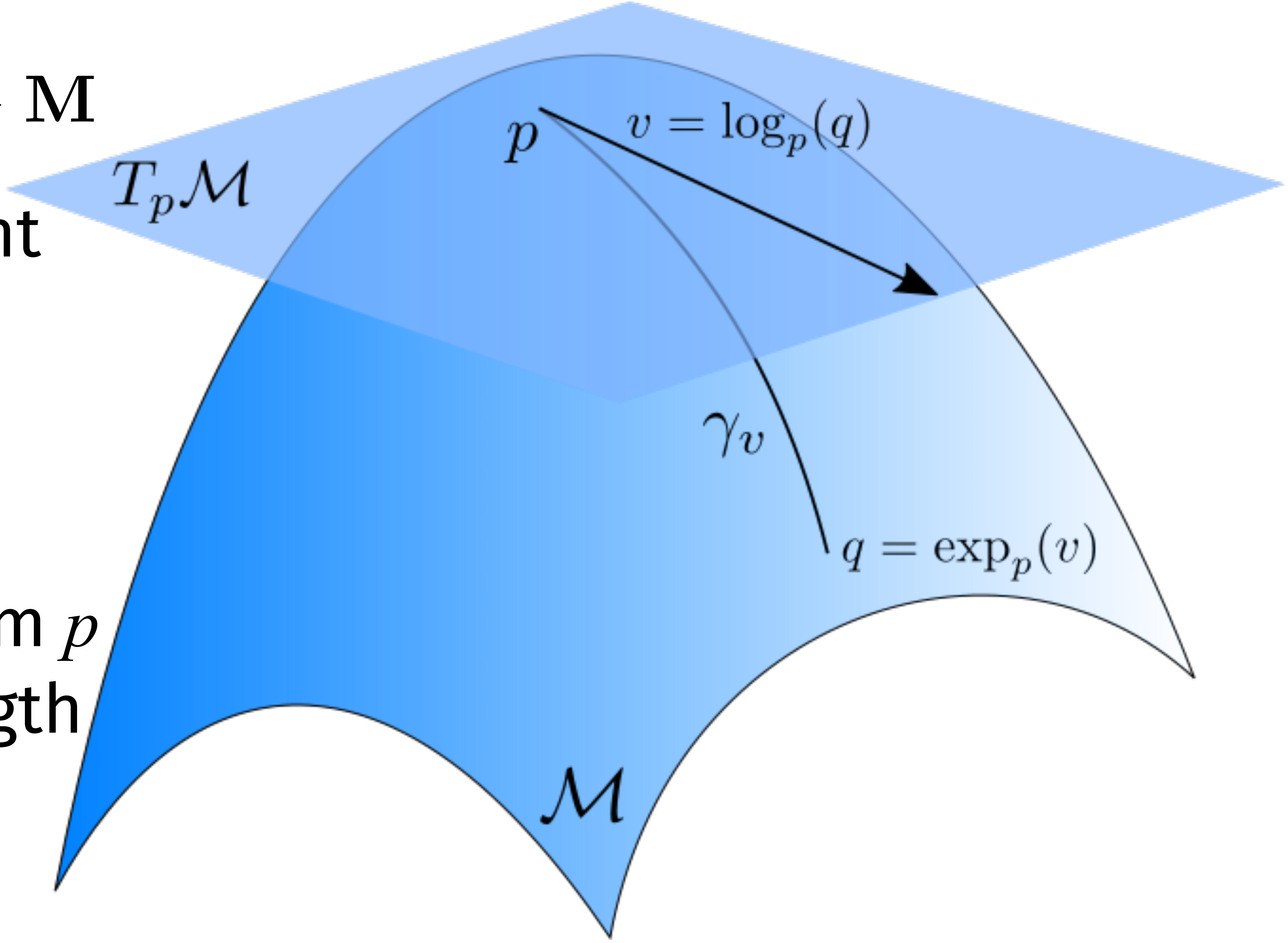
- Its starting point p
- Its tangent direction at p
- Its length



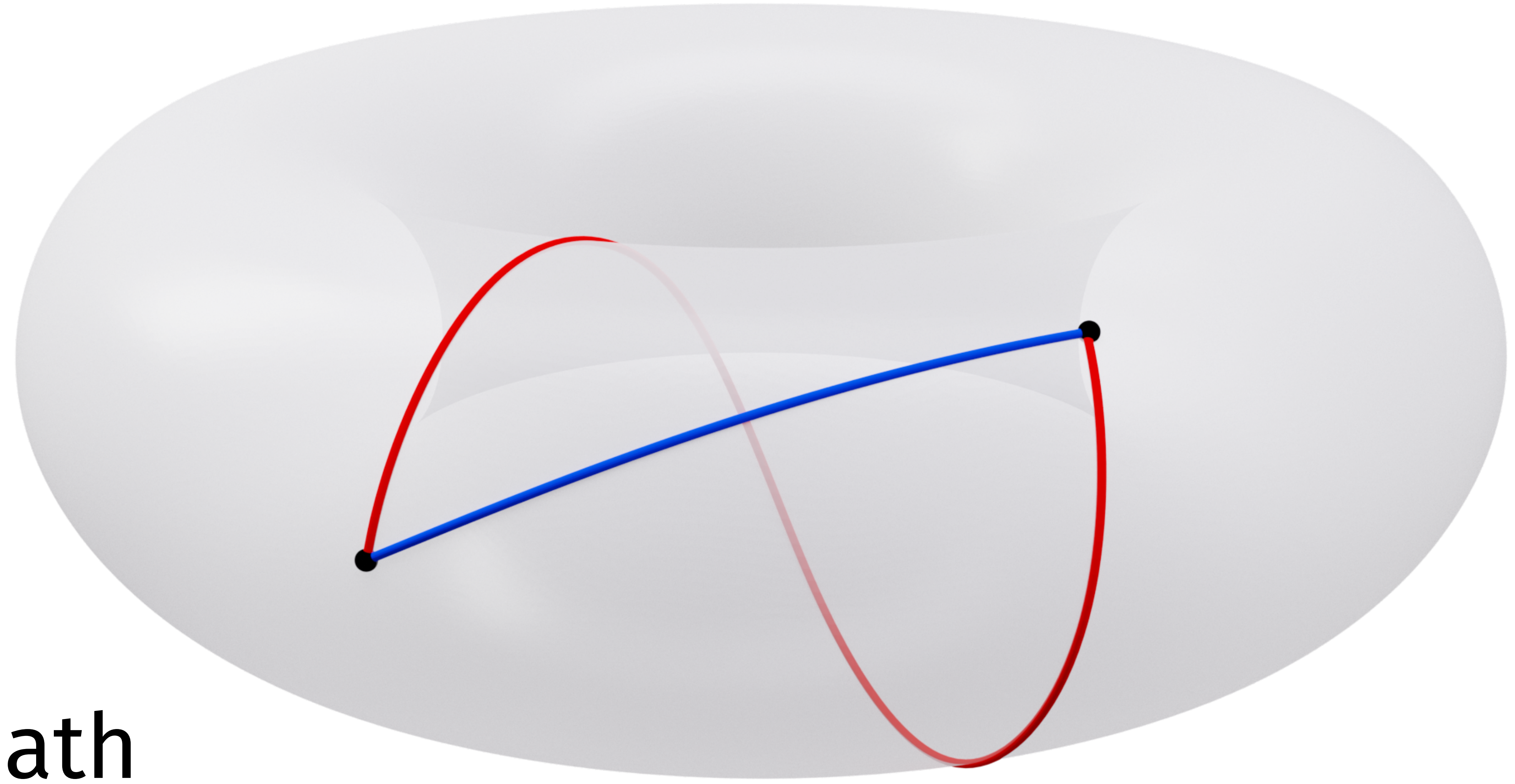
Exponential map

Exponential map: $\exp_p : T_p\mathbf{M} \longrightarrow \mathbf{M}$

- it maps vectors from the tangent plane at a given point p into points on $\mathbf{M}$
- the exp map of a vector $\mathbf{t}$ is the endpoint of a geodesic cast from p in the direction and for the length of $\boldsymbol{t}$



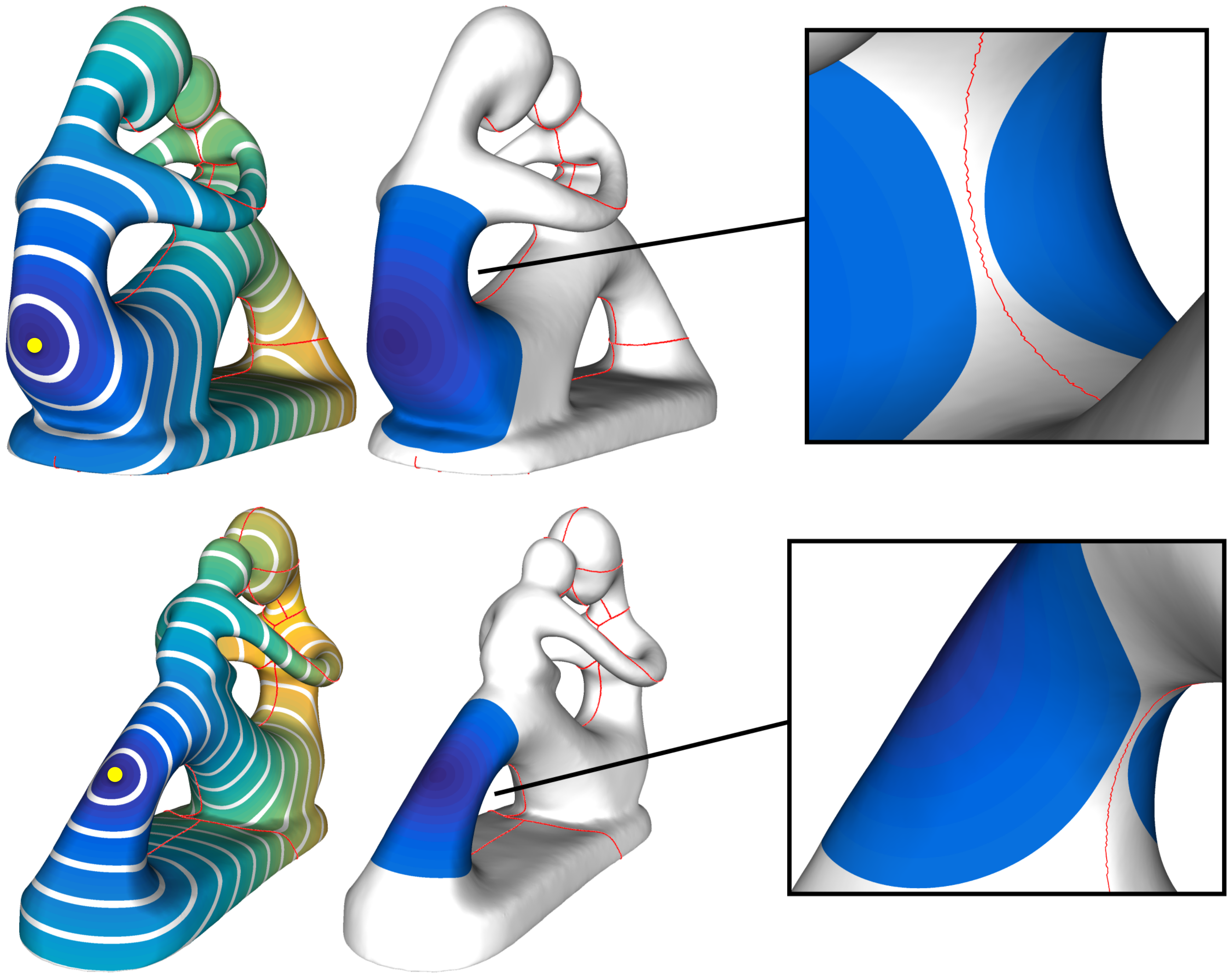
Geodesic lines and shortest paths



- A shortest path on $\mathbf{M}$ is a geodesic line
- A geodesic line is *locally* shortest:
  - given γ geodesic line and $p = \gamma(s)$
  - there exist $\varepsilon>0$ such that $\gamma|_{[s-\varepsilon,s+\varepsilon]}$ is a shortest path
- Intuitively:
  - a short enough geodesic is also a shortest path
  - but if you take it long enough, there might be a totally different path that reaches the same point at a shorter length
  - a long enough geodesic may even self-intersect

Exponential map and cut locus

- the exp map is injective only up to the cut locus
- *injectivity radius*: distance from p of its closest point on the cut locus

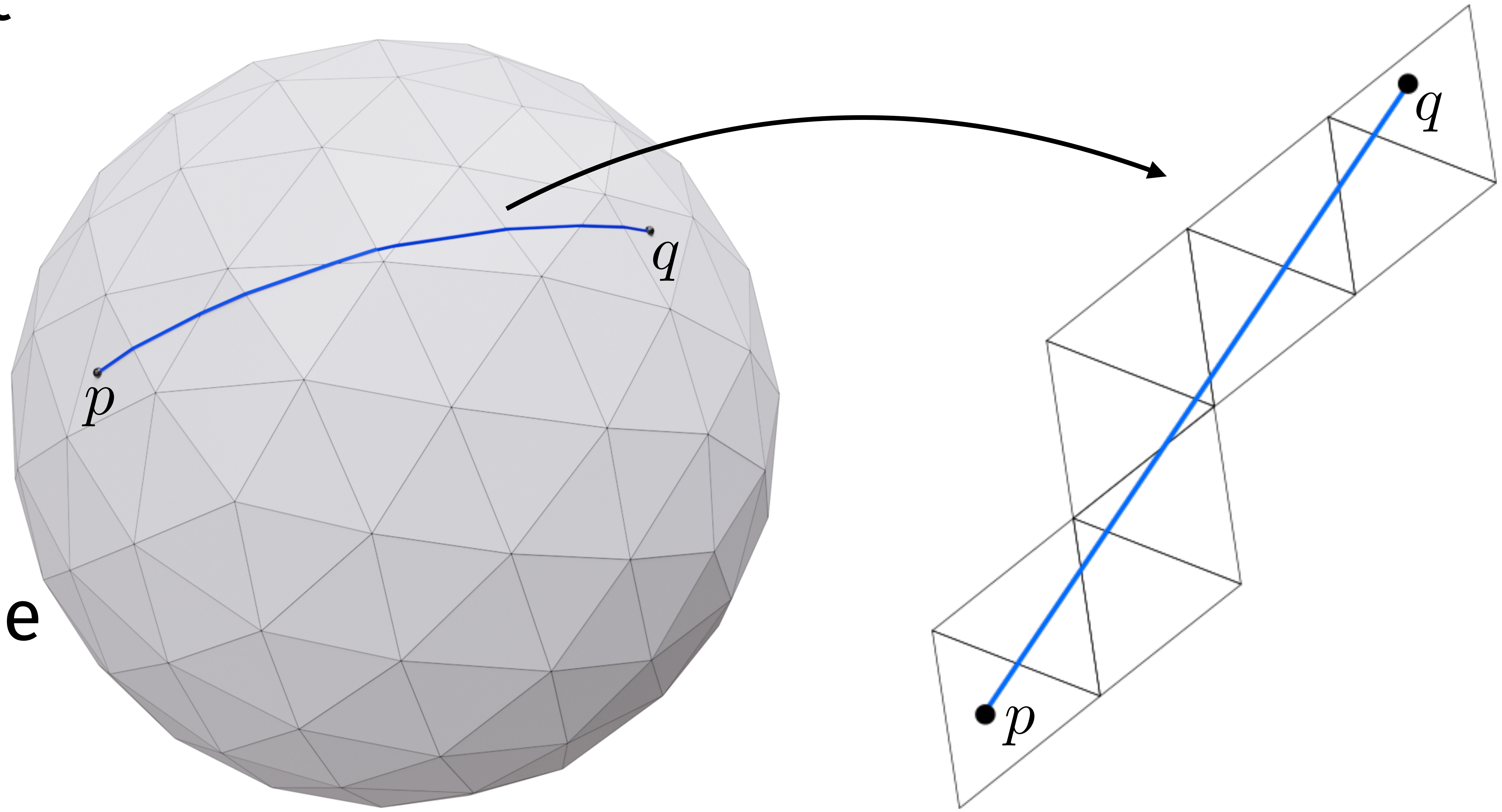


Normal and convex sets

- A submanifold $\mathbf{B} \subset \mathbf{M}$ is
  - A *normal neighborhood* of point x if for each $y \in \mathbf{B}$ there exists a unique shortest path on $\mathbf{M}$ from x to y
    - The exp map is injective on normal neighborhoods
  - A *totally normal set* if it is an open set that is a normal neighborhood of all its points
  - *Strongly convex* if for each pair x and y of its points there exists a unique shortest path on $\mathbf{M}$ connecting x to y and it is contained in $\mathbf{B}$
- Normal/convex sets cannot cross the cut locus/loci

Polyhedral setting

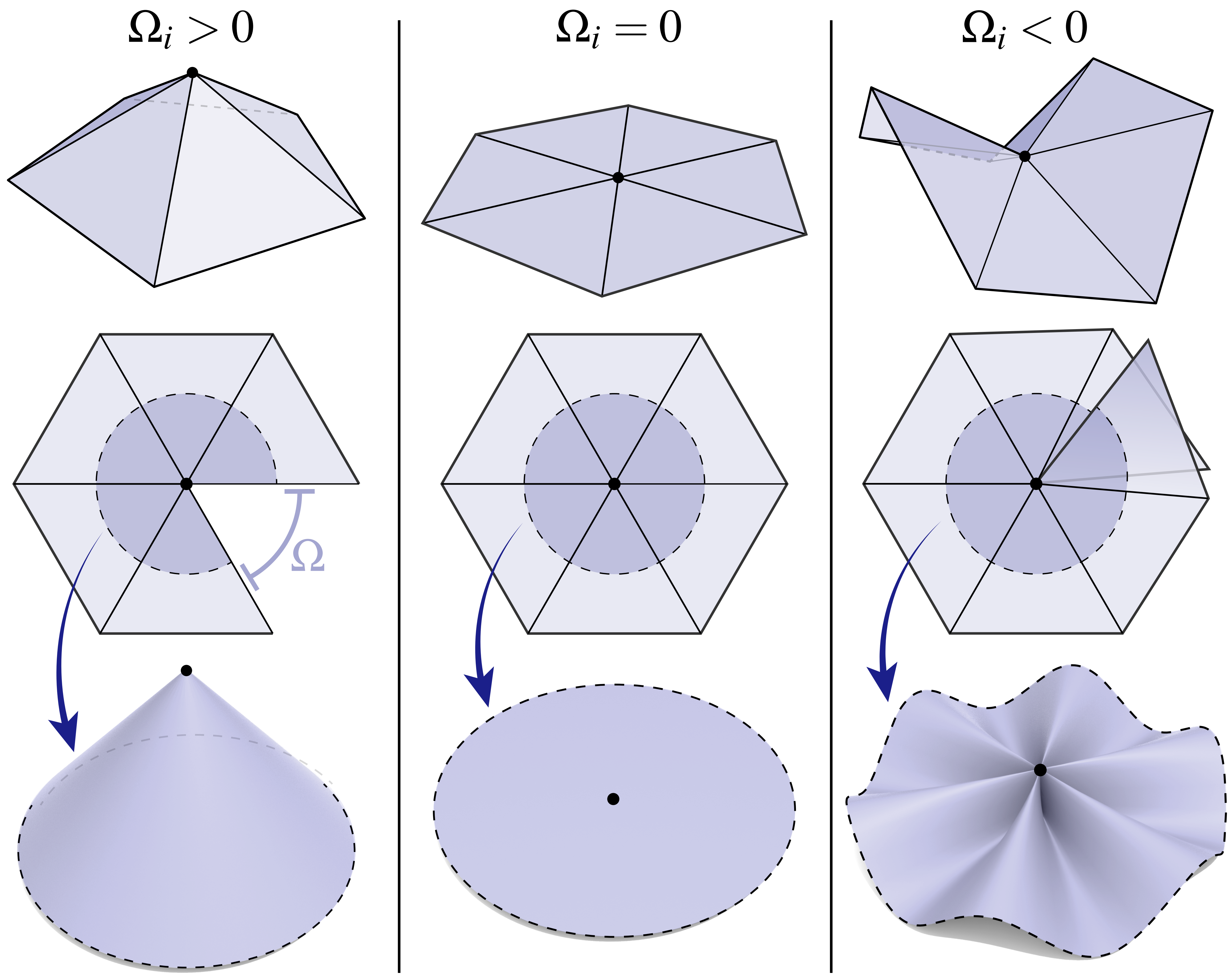
- On a polygonal mesh, shortest paths are polylines
- One straight-line segment per face crossed
- The shortest path line is a straight line once the crossed strip of faces is unfolded to the plane
- The path can turn at saddle vertices (see next)



Polyhedral setting

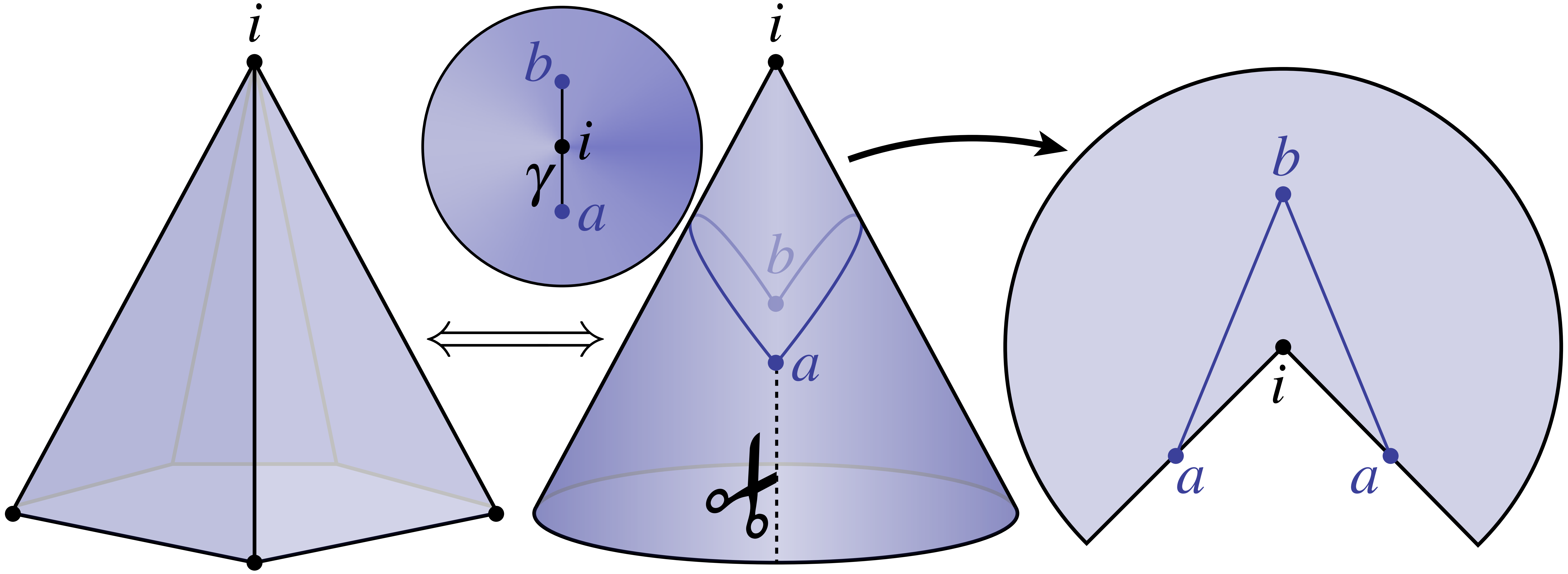
Classification of vertices by total angle: $\Omega_i = 2\pi - \sum_{ijk} \theta_i^{jk}$

- Elliptic: $\Omega_i > 0$
- Euclidean: $\Omega_i = 0$
- Hyperbolic (saddle): $\Omega_i < 0$



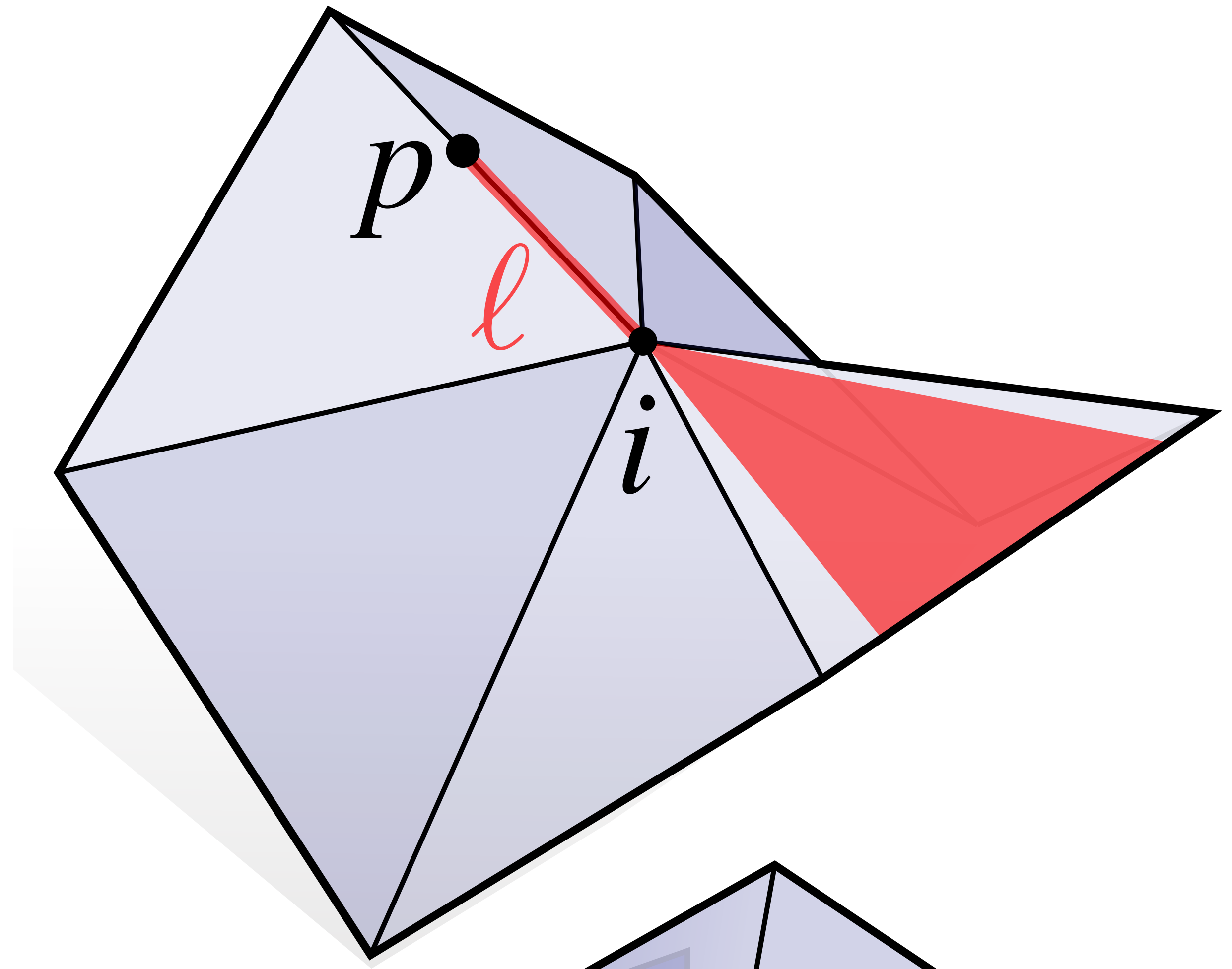
Polyhedral setting

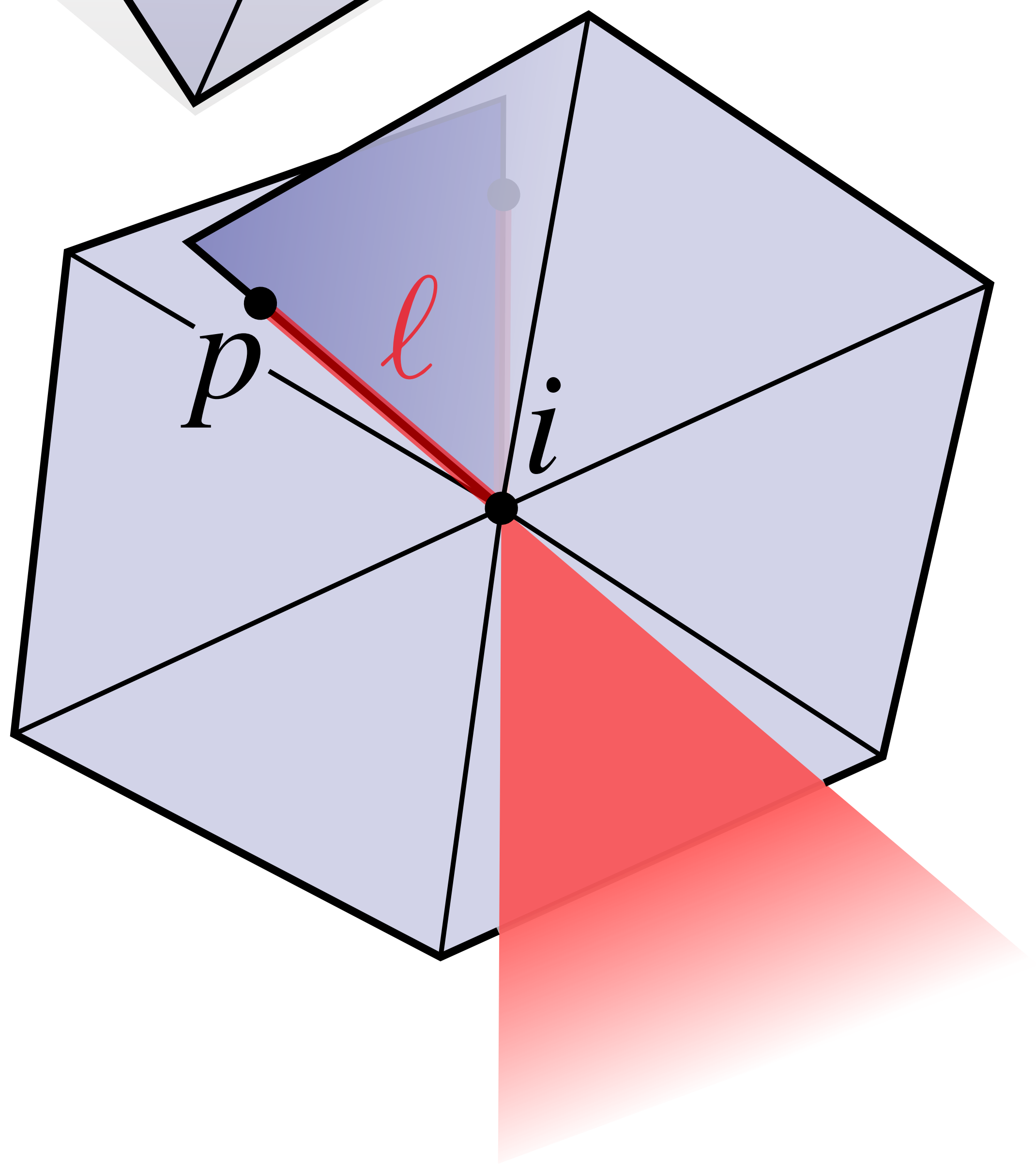
- An elliptic vertex is locally like the apex of a cone
- No geodesic shortest path through elliptic vertices
- It is always shorter to go around the apex than going through it



Polyhedral setting

- There are infinitely many geodesics coming from a direction and through a saddle vertex
- Such geodesics form a *funnel*:
  - Cut 1-ring along incoming direction ℓ
  - Unfolding 1-ring to the plane creates overlap
  - Extending the two copies of ℓ with straight lines forms a wedge
  - All directions within wedge extend the geodesic ℓ





Geodesic queries

- Geodesic Tracing (GT, a.k.a. initial value problem):
  - given point p and tangent direction $\mathbf{t}$, trace a geodesic line starting at p along $\mathbf{t}$ and for the length of $\mathbf{t}$
  - explicit (pointwise) computation of $\exp_p(\mathbf{t})$
  - easiest query on meshes

- Solutions follow from the property of geodesics of being

the straightest possible curves

Geodesic queries

- Point-to-Point Geodesic Path (PPGP, a.k.a. boundary value problem):
  - Find a (shortest) geodesic connecting two points p and q
  - Not easy in general
  - On a mesh, if a triangle strip connecting p to q is given, then the shortest path can be found by unfolding it
  - The shortest path can “turn” at saddle vertices
- Solutions follow from the fact that *shortest paths are geodesics*

Geodesic queries

- Single Source Geodesic Distance/path (SSGD/SSGP):
  - Given a point x evaluate its distance field $d_x(y)$ at all points y on $\mathbf{M}$
  - Not easy in general; on meshes, often evaluated just at vertices
  - Different techniques working on meshes
  - In the SSGP version, also connecting paths can be computed explicitly

- Solutions may define $d_x(y)$ in a continuous or discrete way

Next time

Algorithms for resolving the geodesic queries on meshes

Thank you