06 - Elementary Differential Geometry

1 - Curves

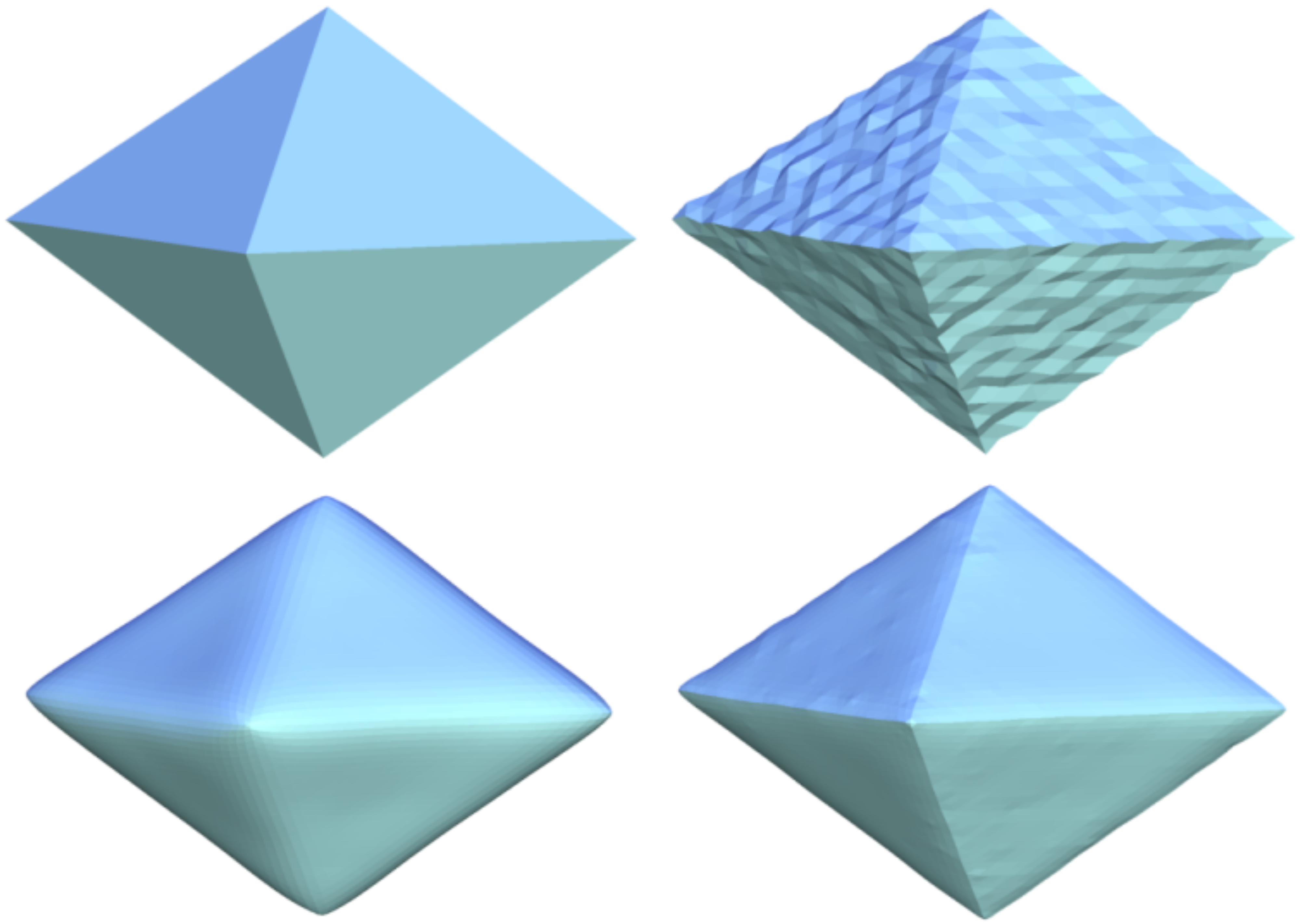
Acknowledgements: Daniele Panozzo, Olga Sorkine Hornung

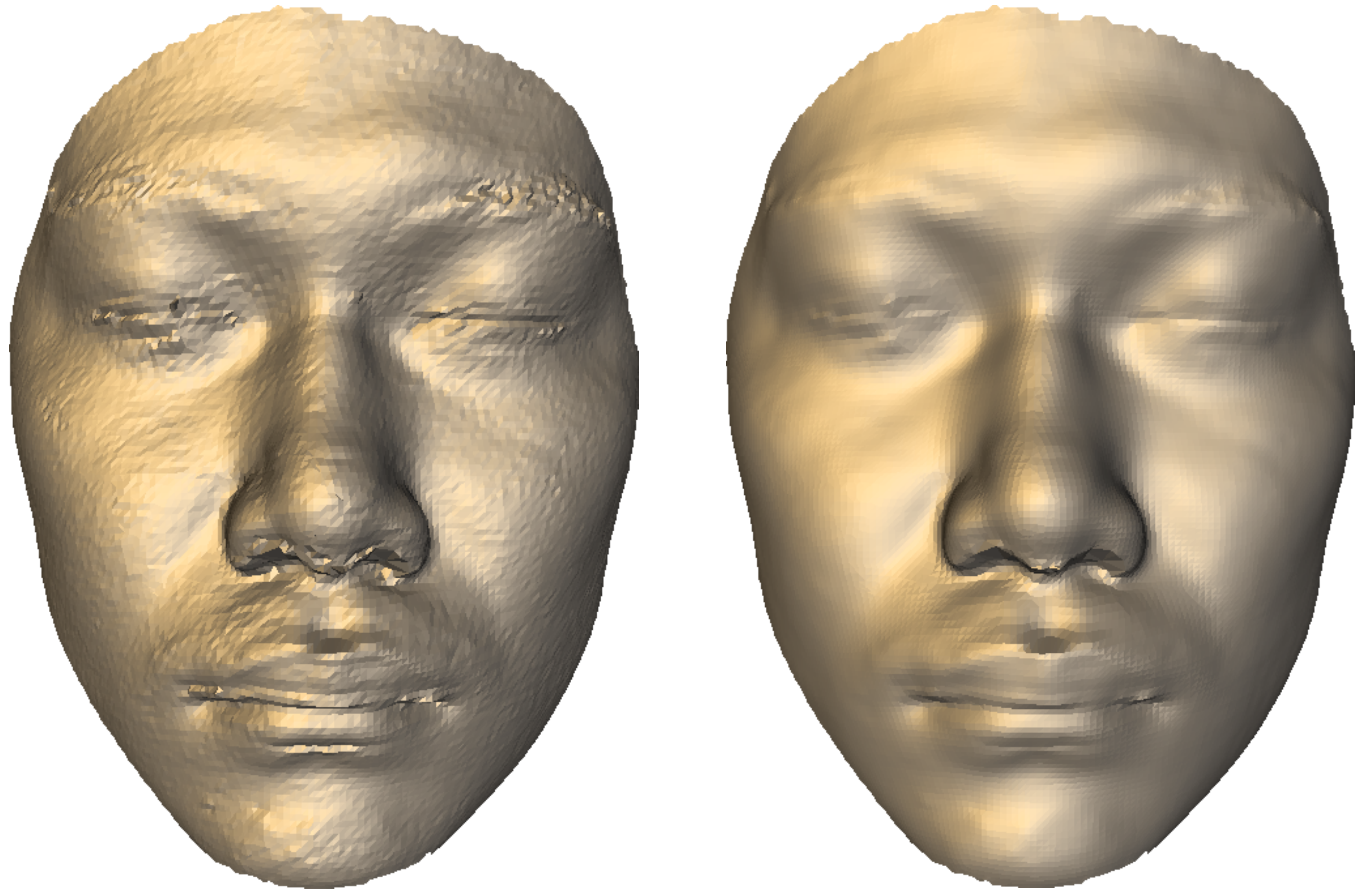
In this lecture

- Introduction to differential geometry
- Math of curves
  - general definitions and properties only
  - how to treat them in the discrete setting

Differential Geometry – Motivation

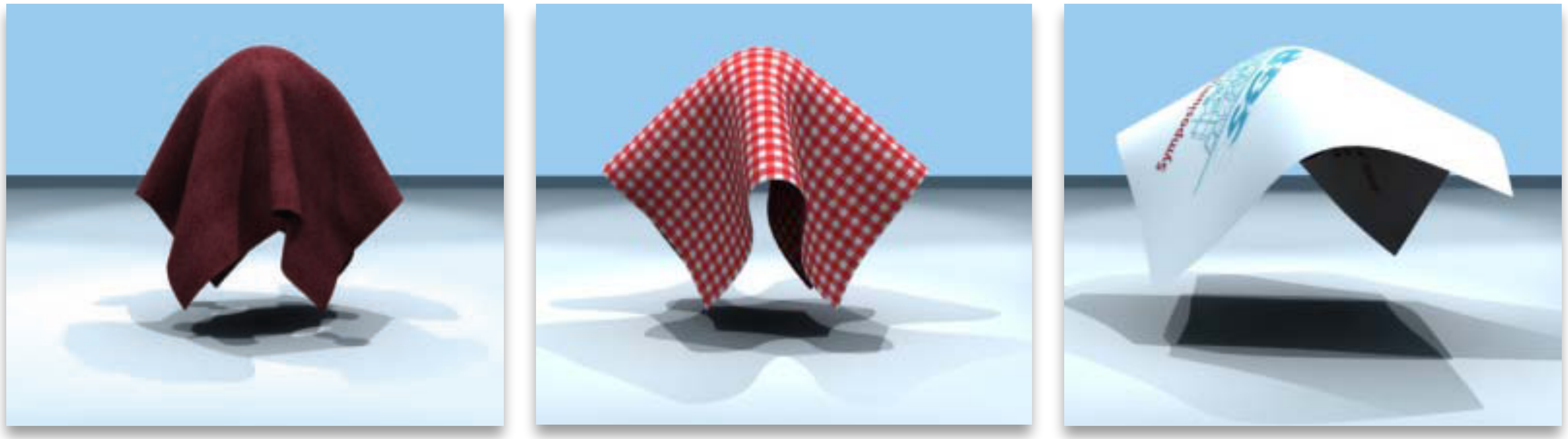
- Describe and analyze geometric characteristics of shapes
  - e.g. how smooth?

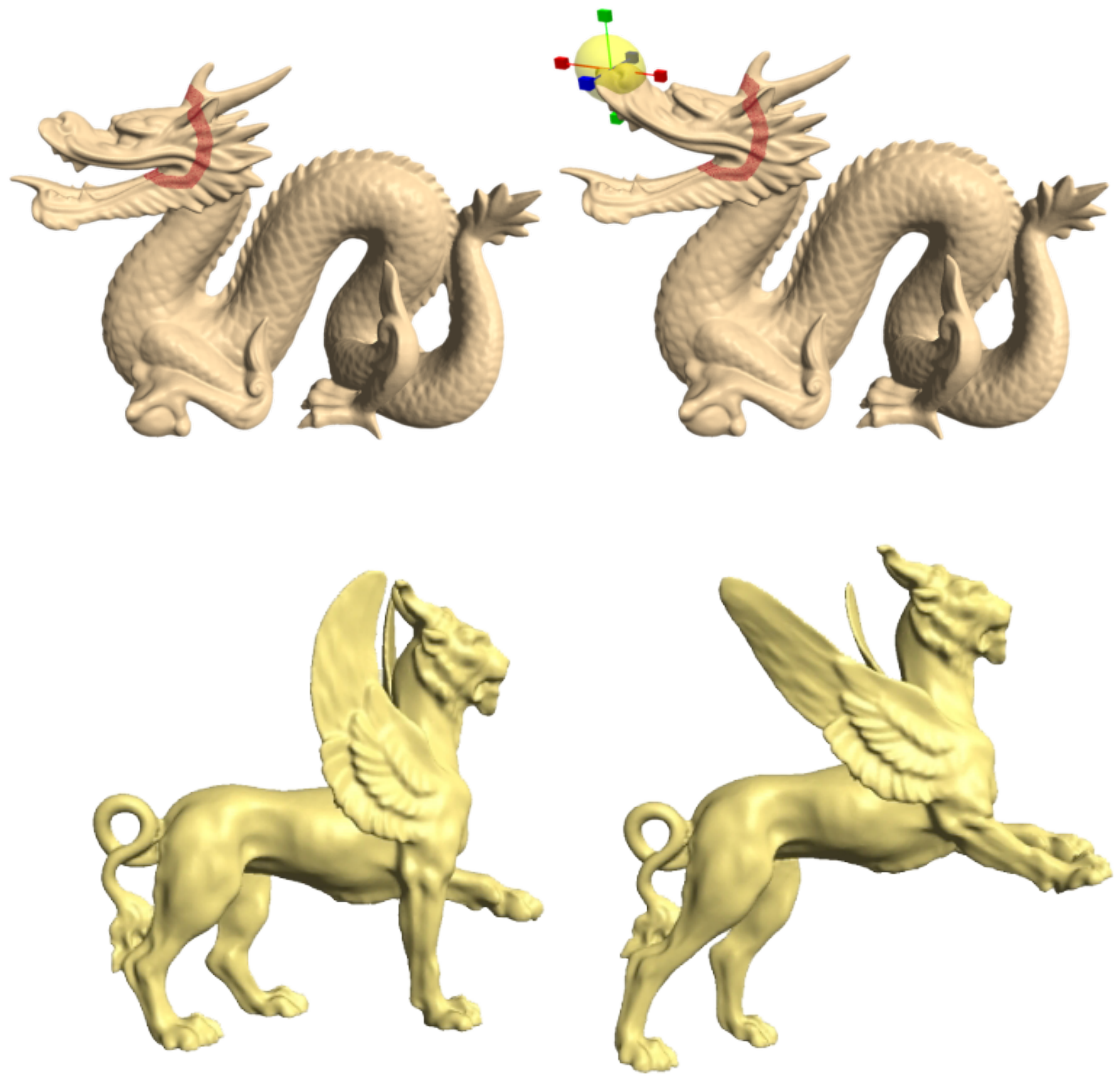




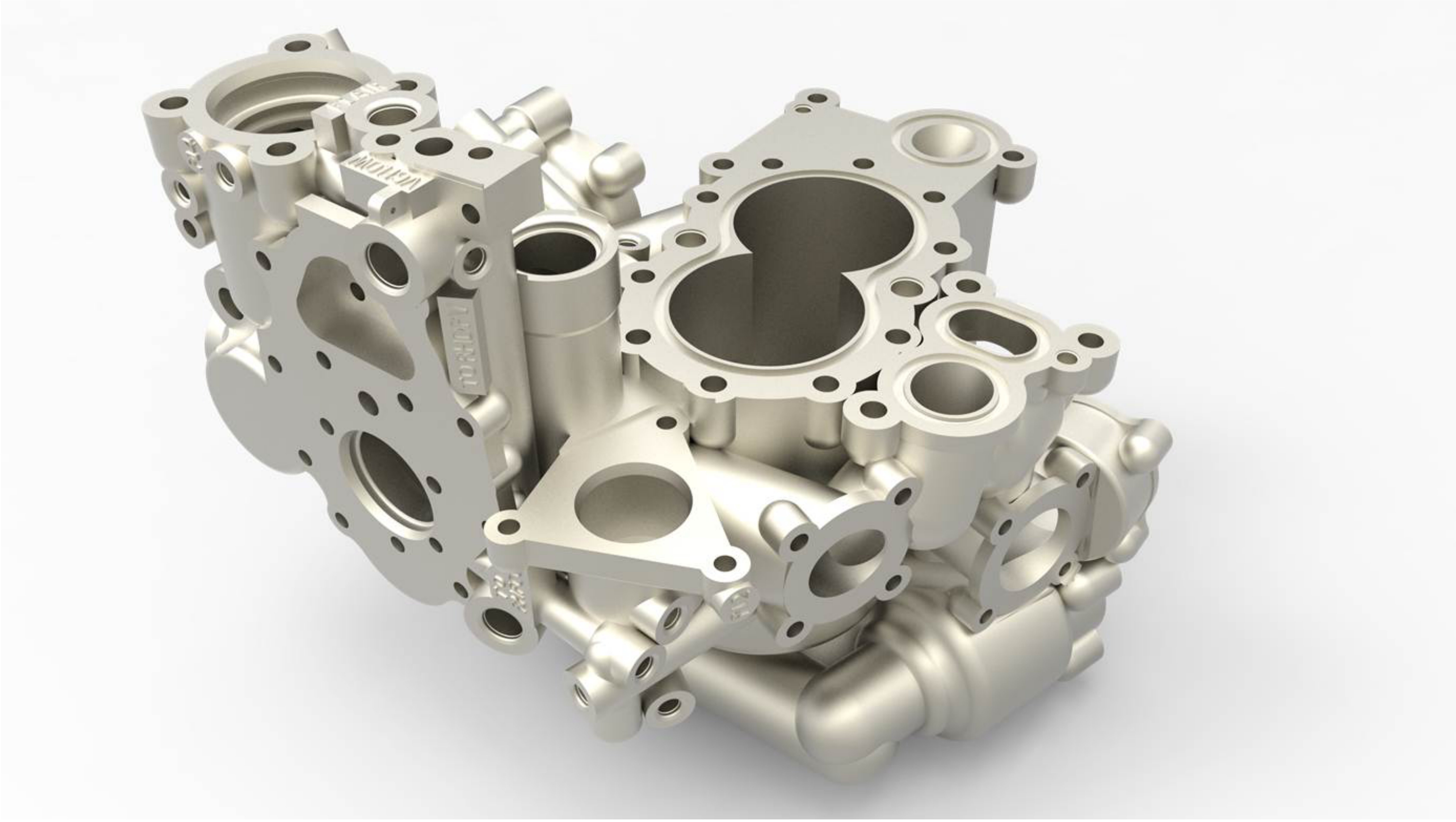
Differential Geometry – Motivation

- Describe and analyze geometric characteristics of shapes
  - e.g. how smooth?
  - how shapes deform





Reminder: how do we model shapes?



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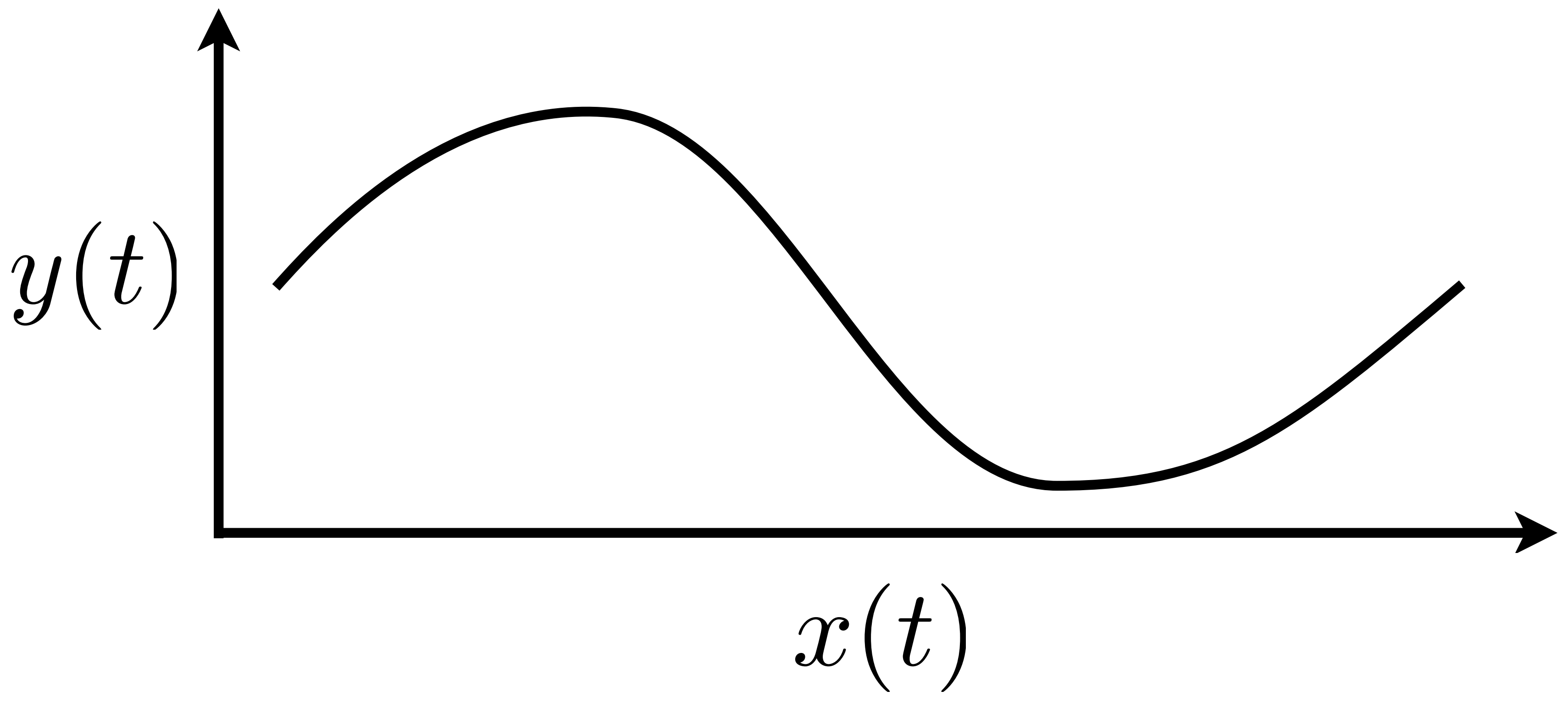
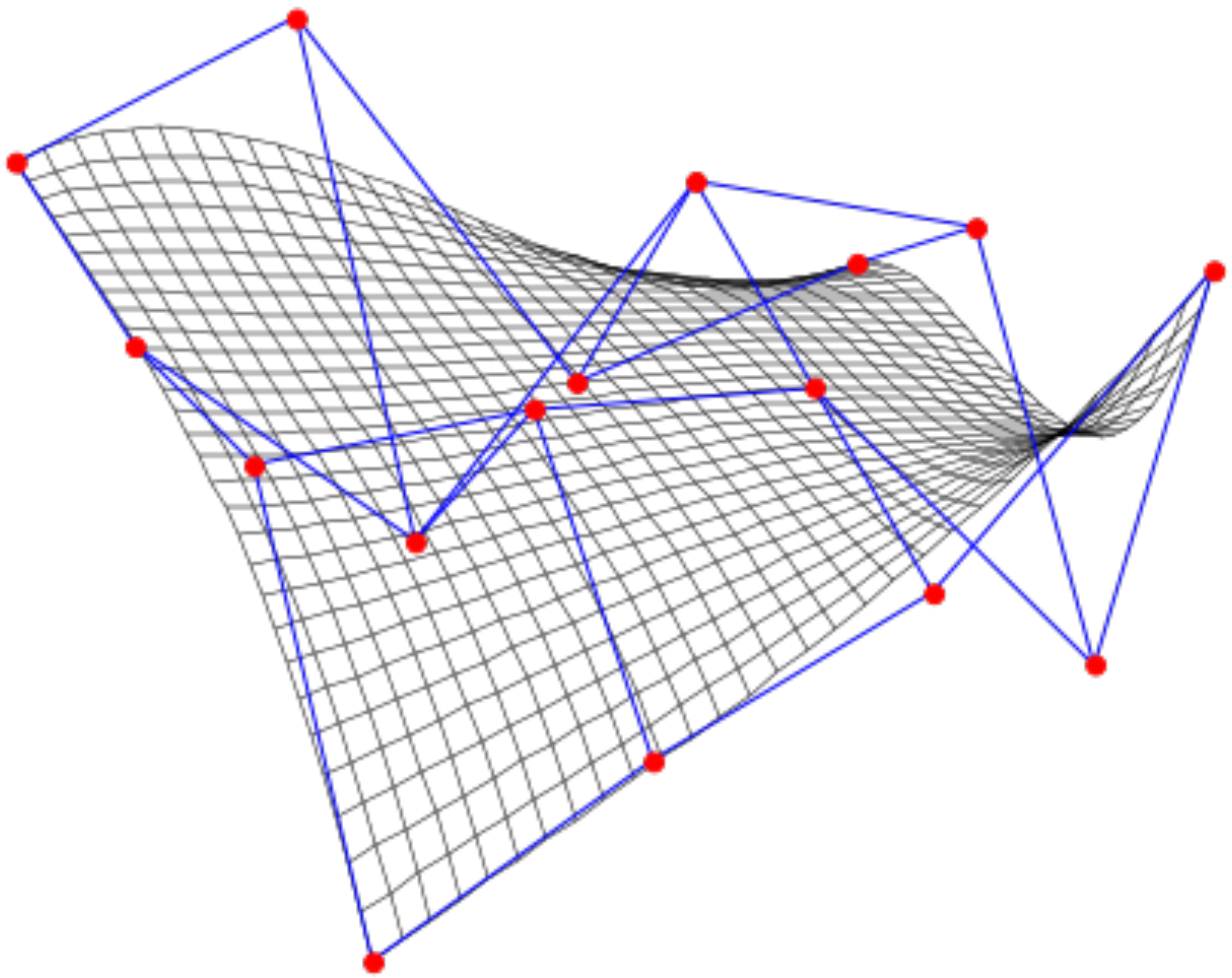
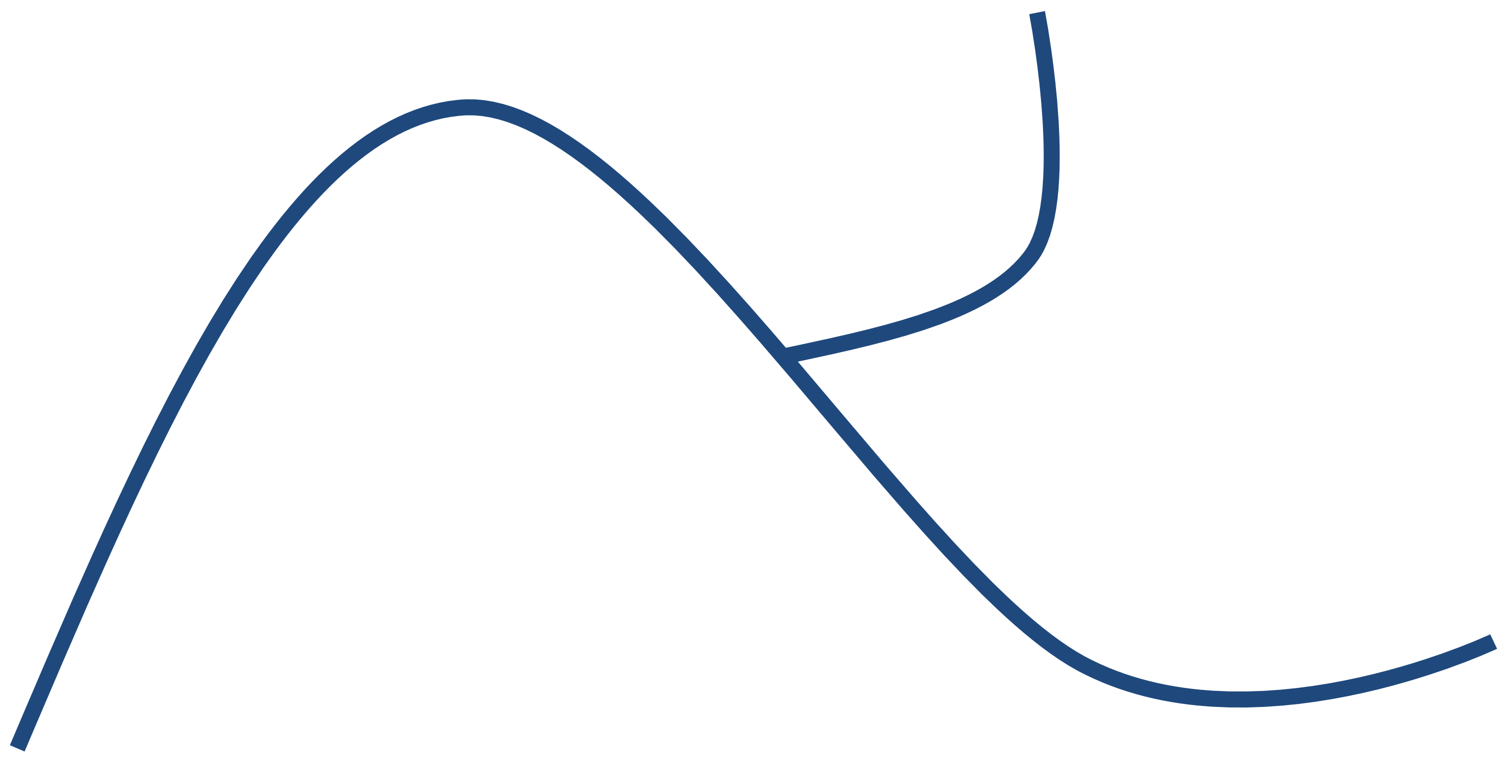
Building blocks: curves and surfaces





Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



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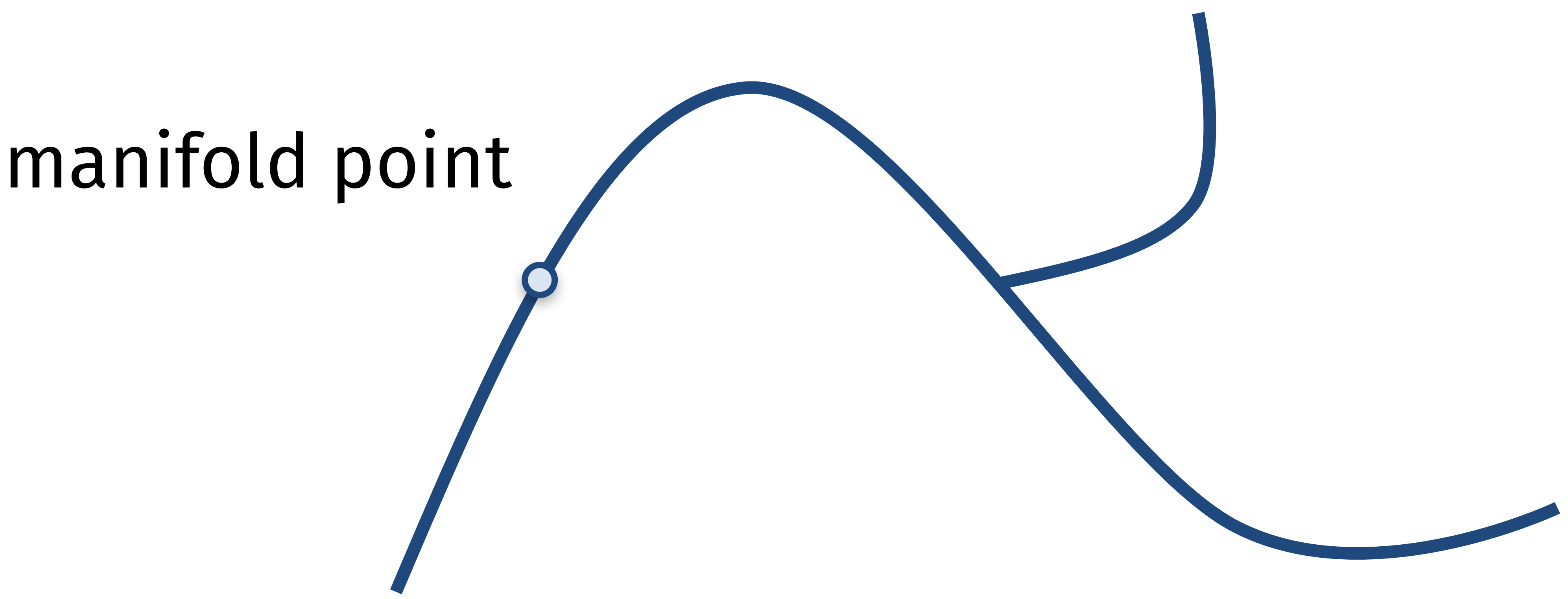

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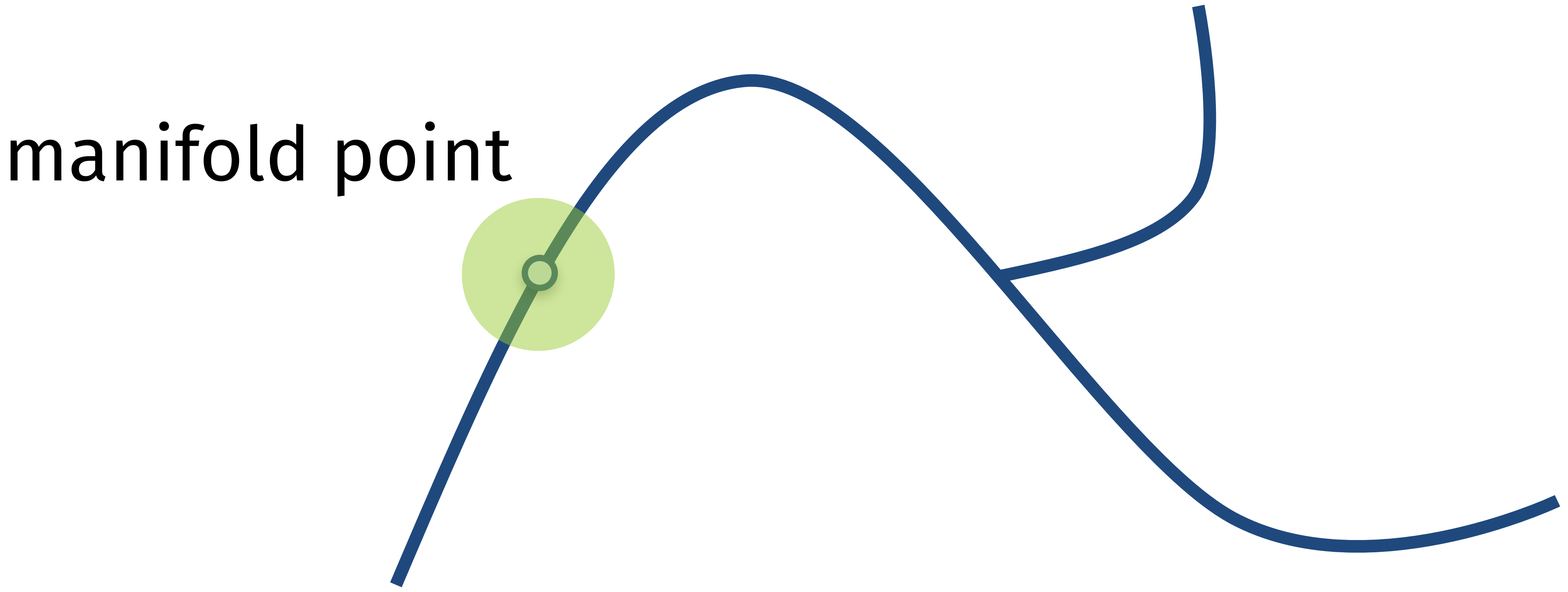

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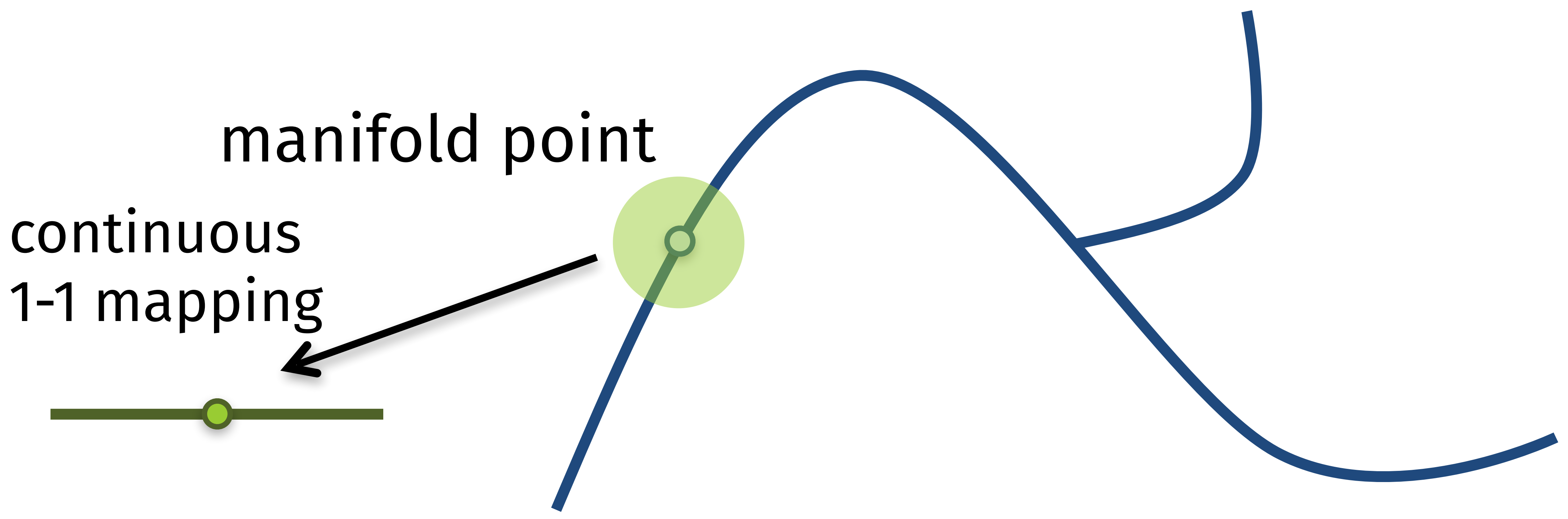

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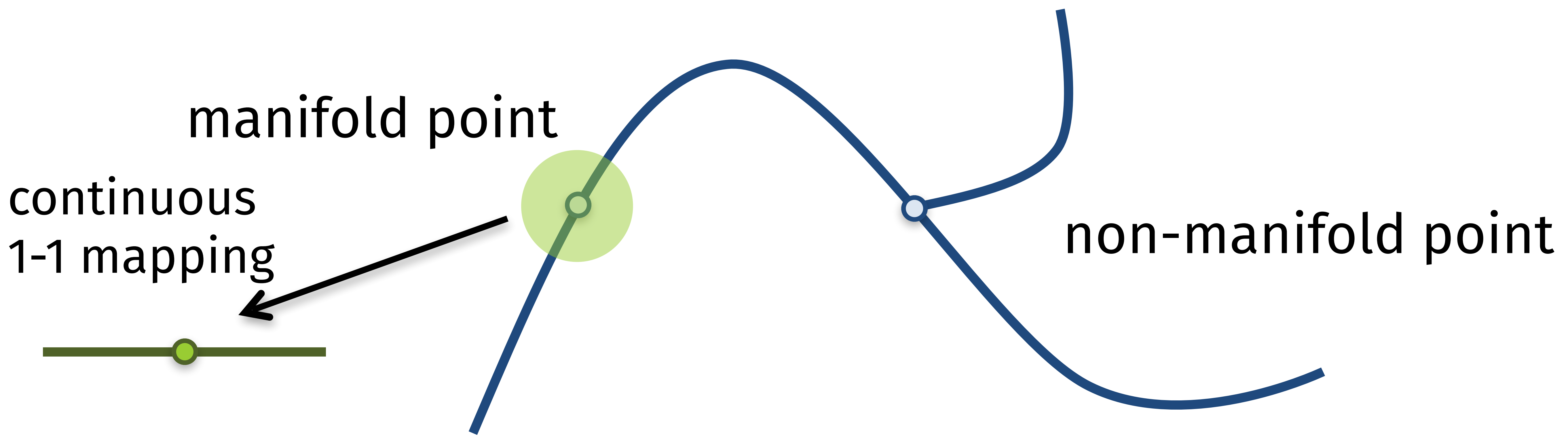

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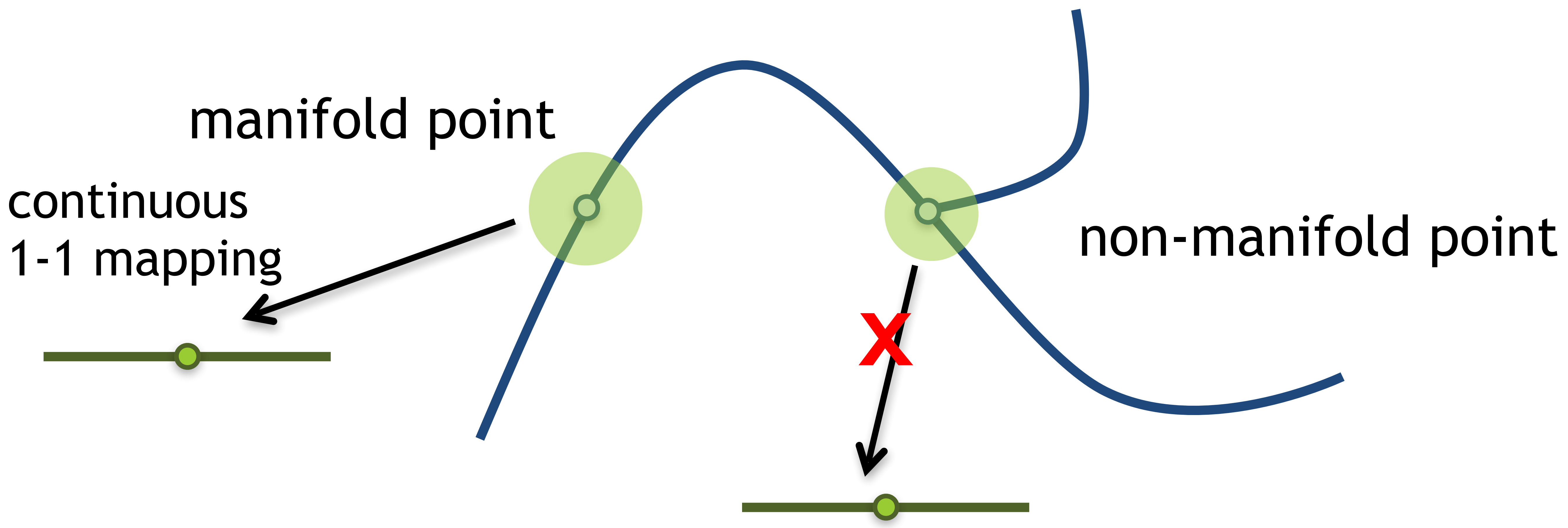

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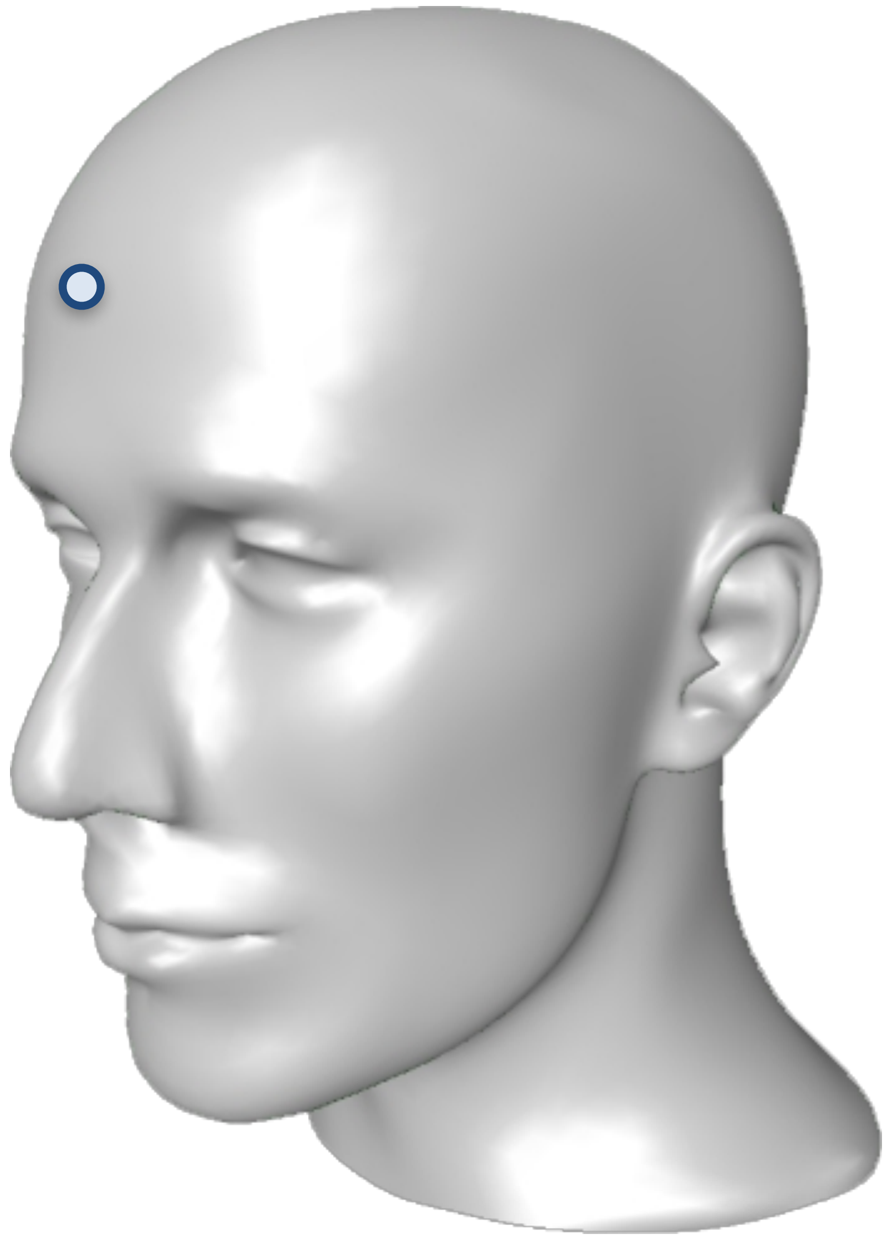

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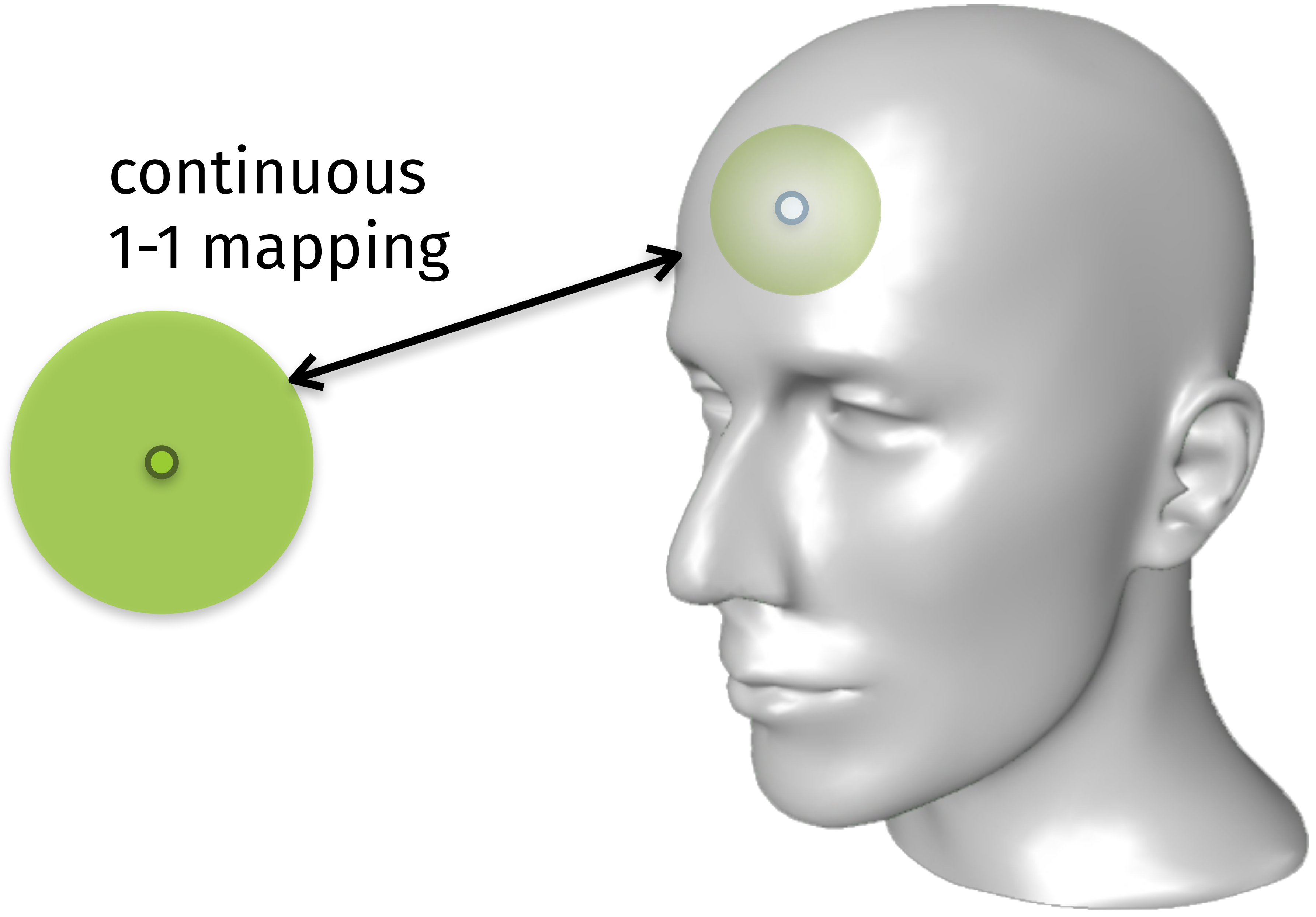
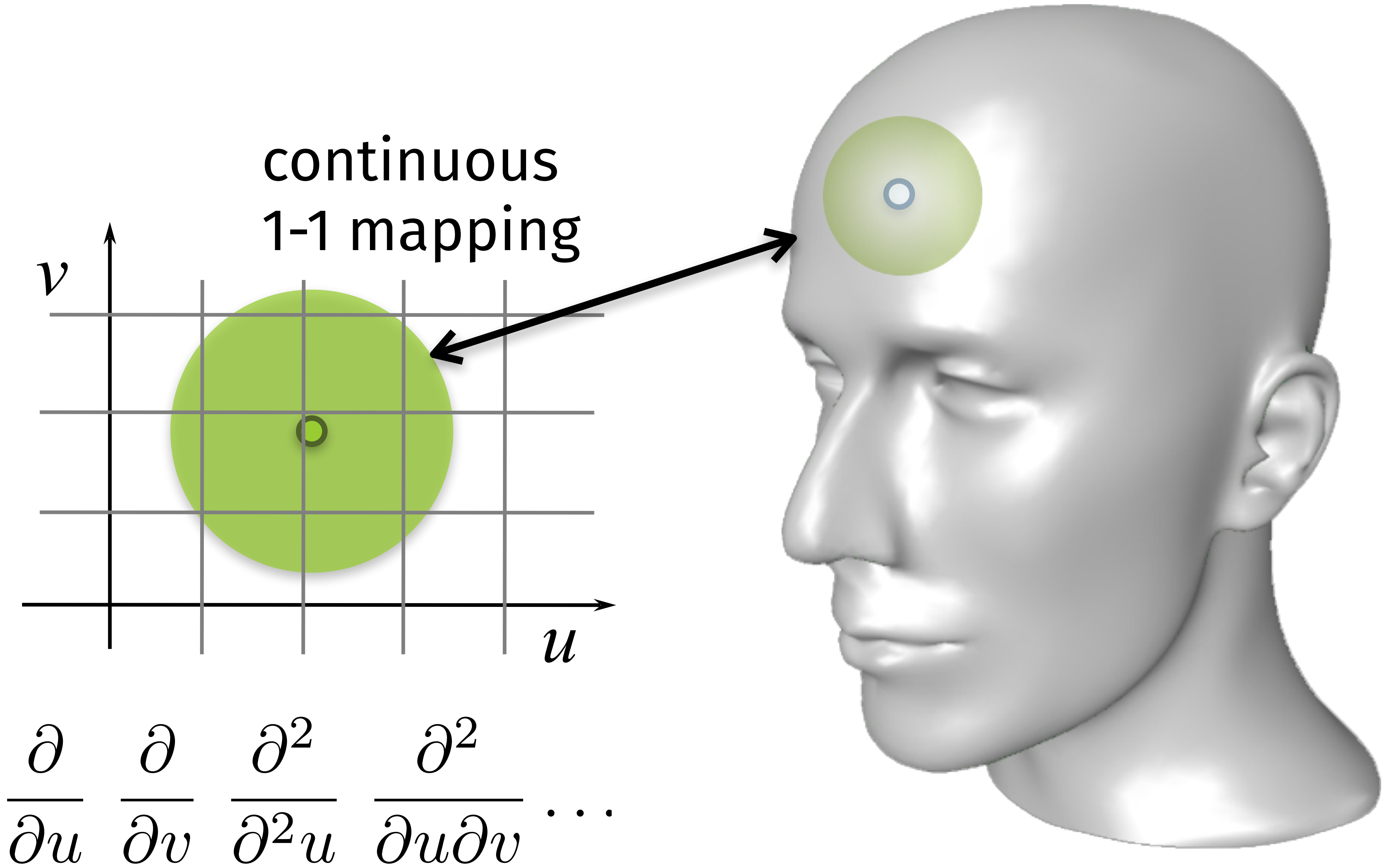


Differential Geometry Basics

- Geometry of manifolds
- Local observation: point + neighborhood



$$\frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial^2}{\partial^2 u} \quad \frac{\partial^2}{\partial u \partial v} \ldots$$

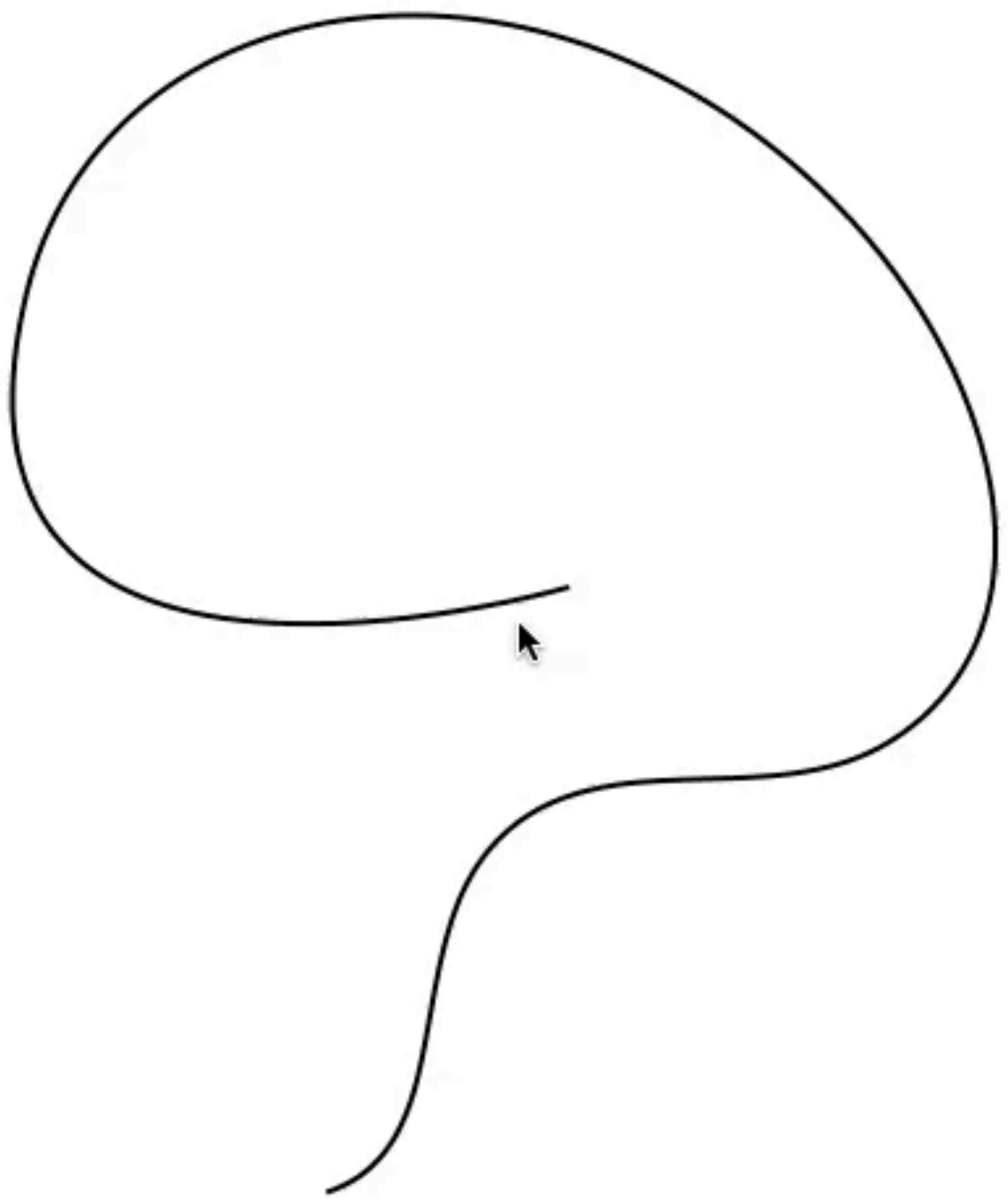
If a sufficiently smooth mapping can be constructed, we can look at its first and second derivatives to find:

Tangents, normals, curvatures, curve angles

Distances, topology

Curves

Keynote Demo



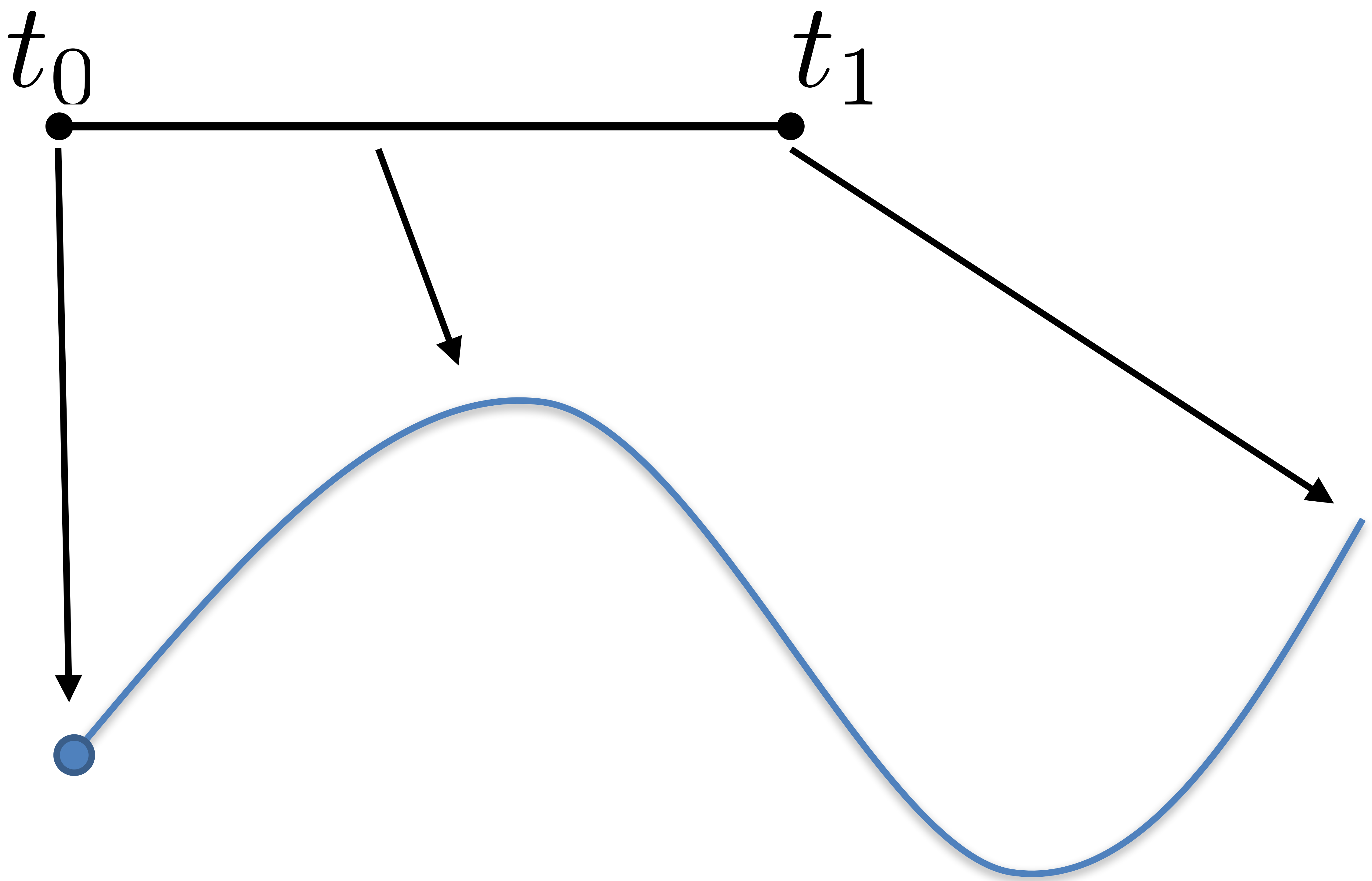
Modeling curves

- We need **mathematical concepts** to characterize the desired curve properties
- Notions from **curve geometry** help with designing user interfaces for curve creation and editing

2D parametric curve

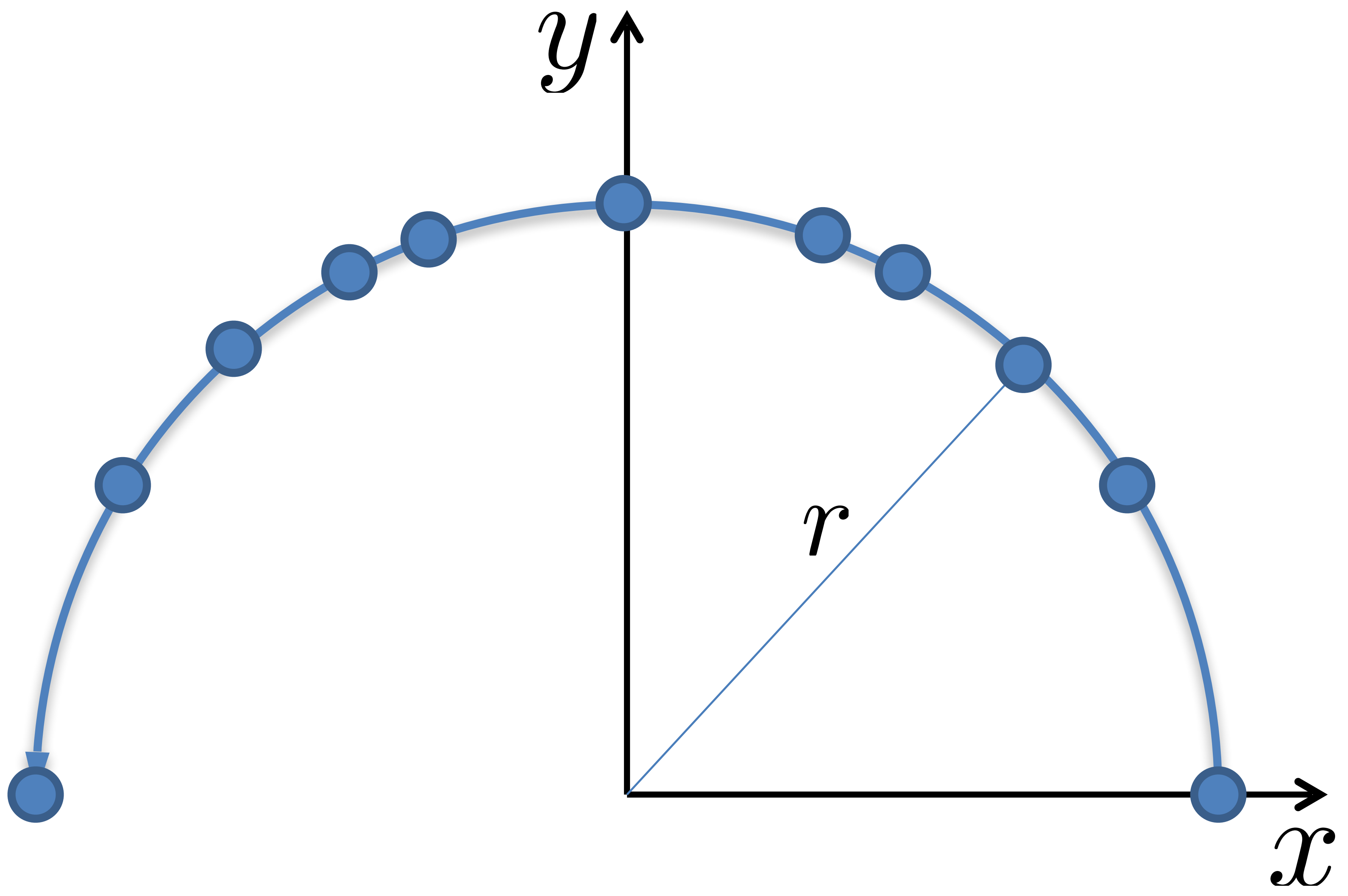
$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ t \in [t_0, t_1]$$

$\mathbf{p}(t)$ must be continuous



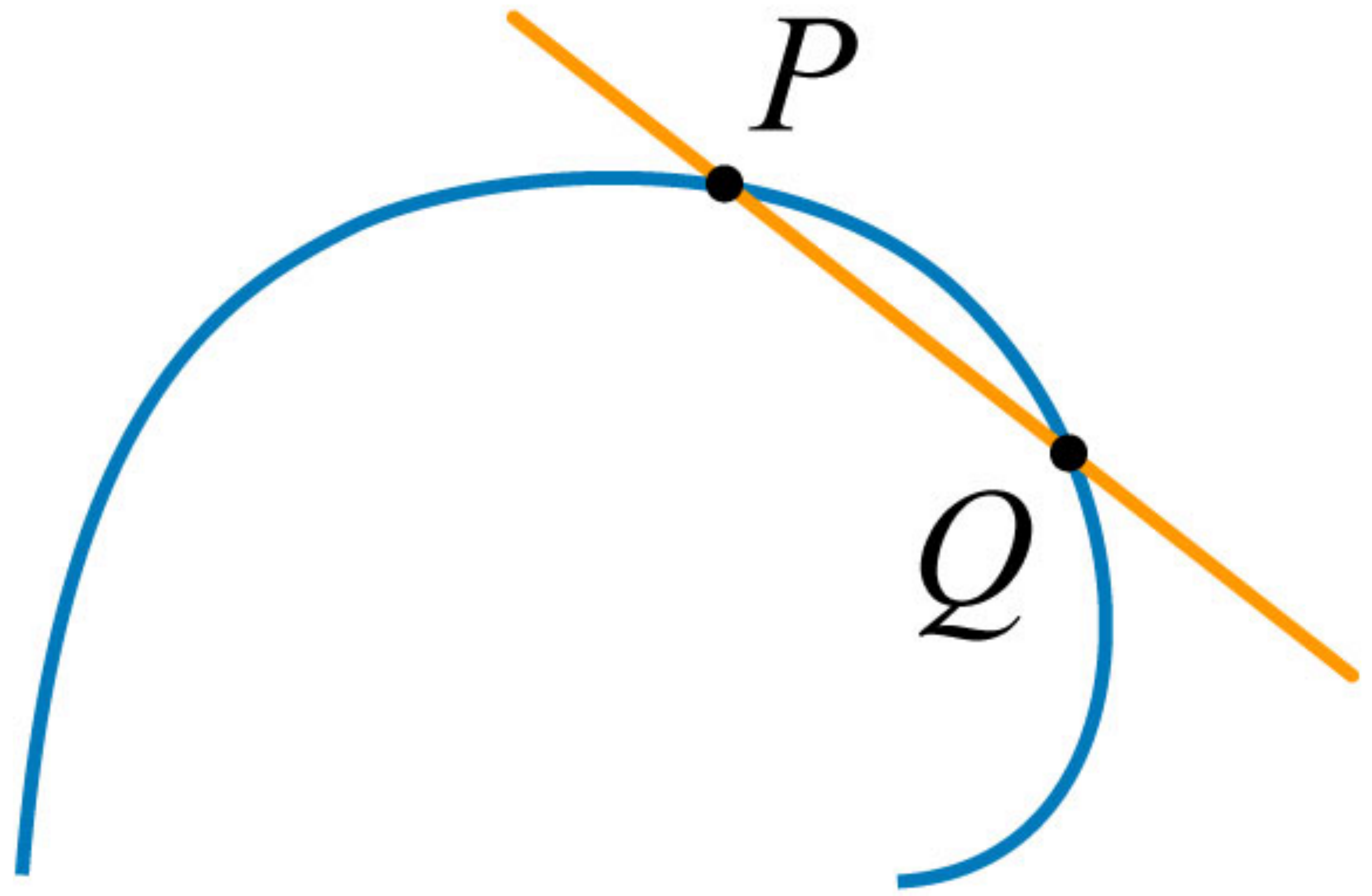
A curve can be parameterized in many different ways

$$\begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}, t \in [0, \pi] \qquad \begin{pmatrix} -rt \\ r\sqrt{1-t^2} \end{pmatrix}, t \in [-1, 1]$$



Tangent: geometric construction

- Secant: a line through two points on the curve



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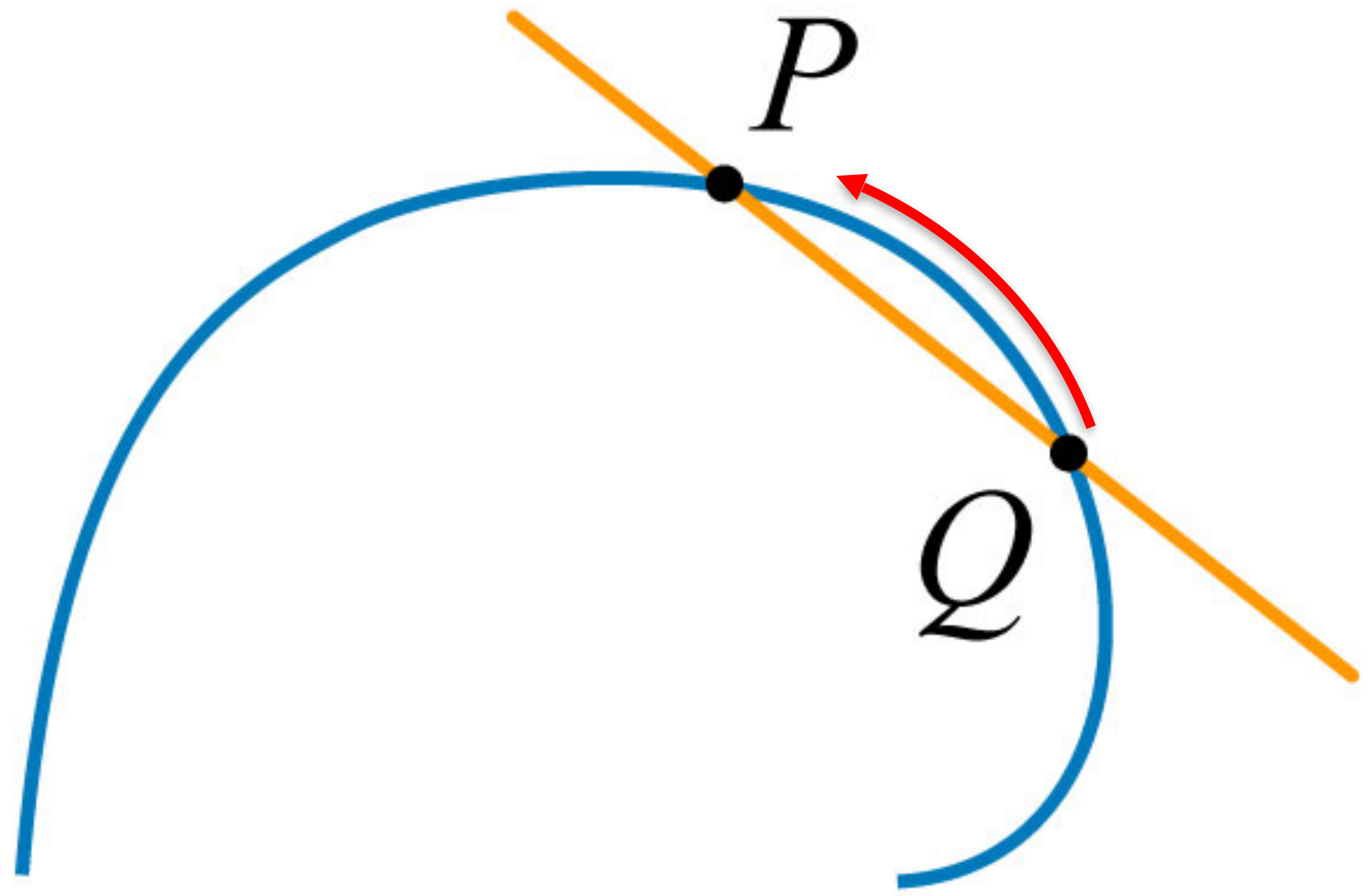
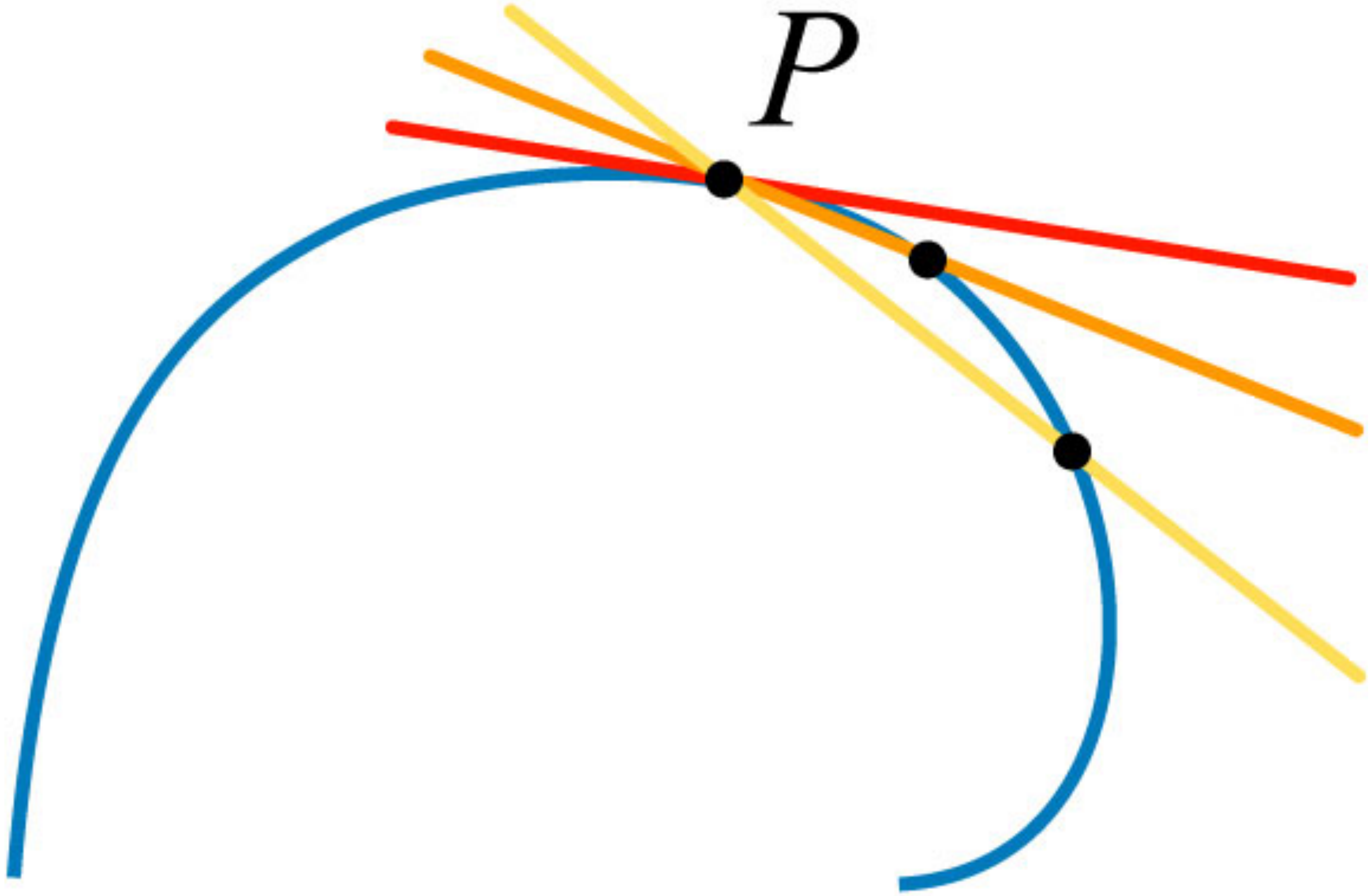
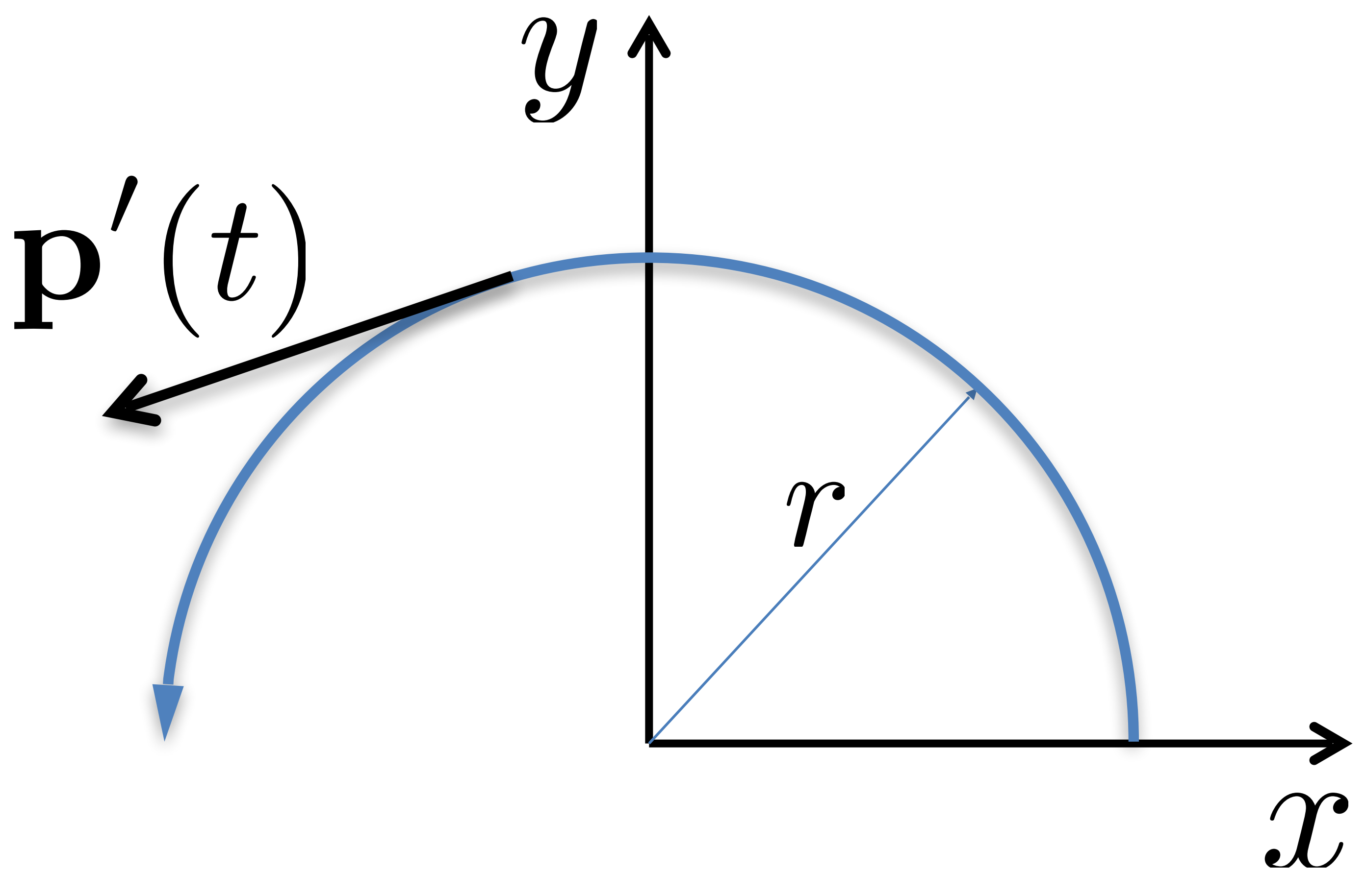


Tangent: geometric construction

- Tangent: the limiting secant as the two points come together



Tangent vector: analytic construction



$$\mathbf{p}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

$$\mathbf{p}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix}$$

$$\|\mathbf{p}'(t)\| = \text{speed}$$

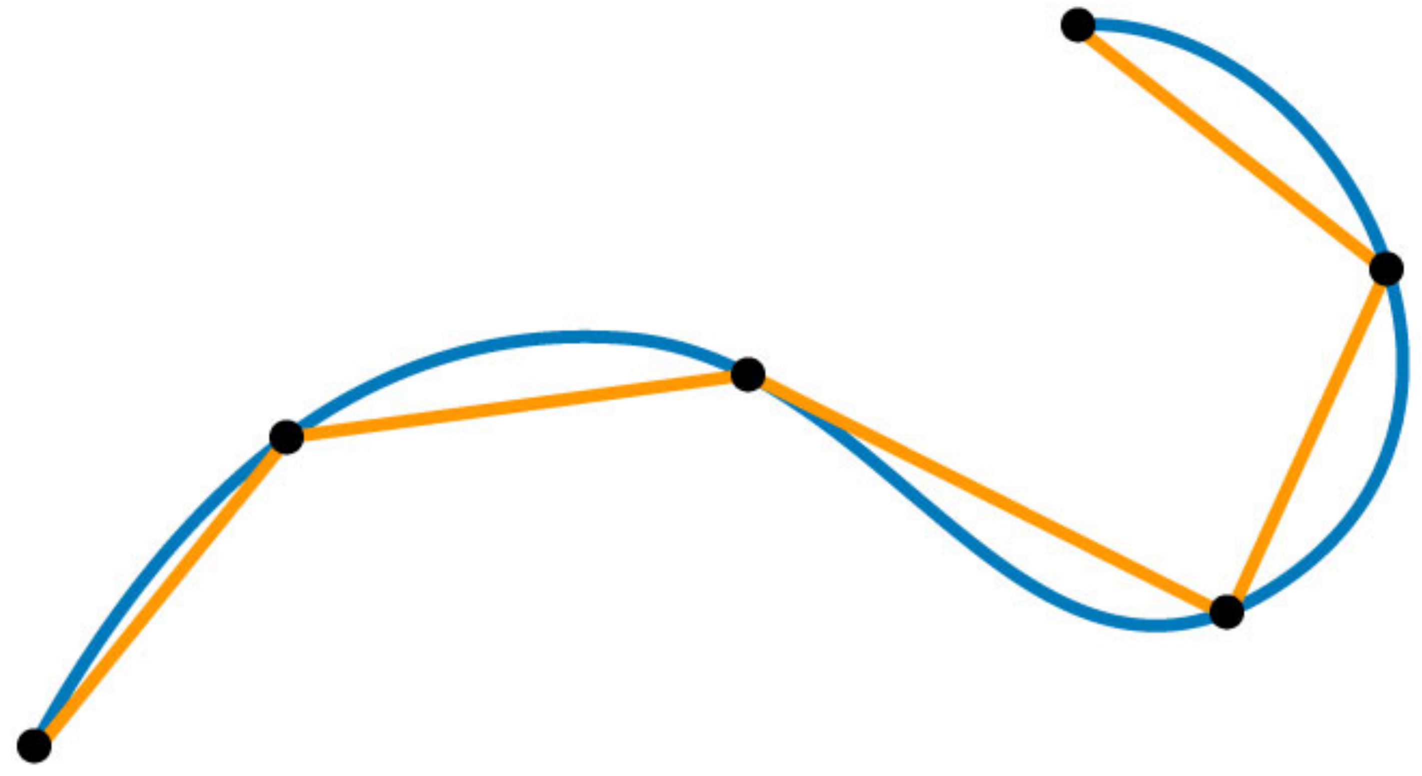
Parametrization-independent!

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t) = \text{unit tangent}$$

Arc length

- How long is the curve between t_0 and t? How far does the particle travel?
- Speed is $\|\mathbf{p}'(t)\|$, so:

$$s(t) = \int_{t_0}^{t} \|\mathbf{p}'(t)\| dt$$

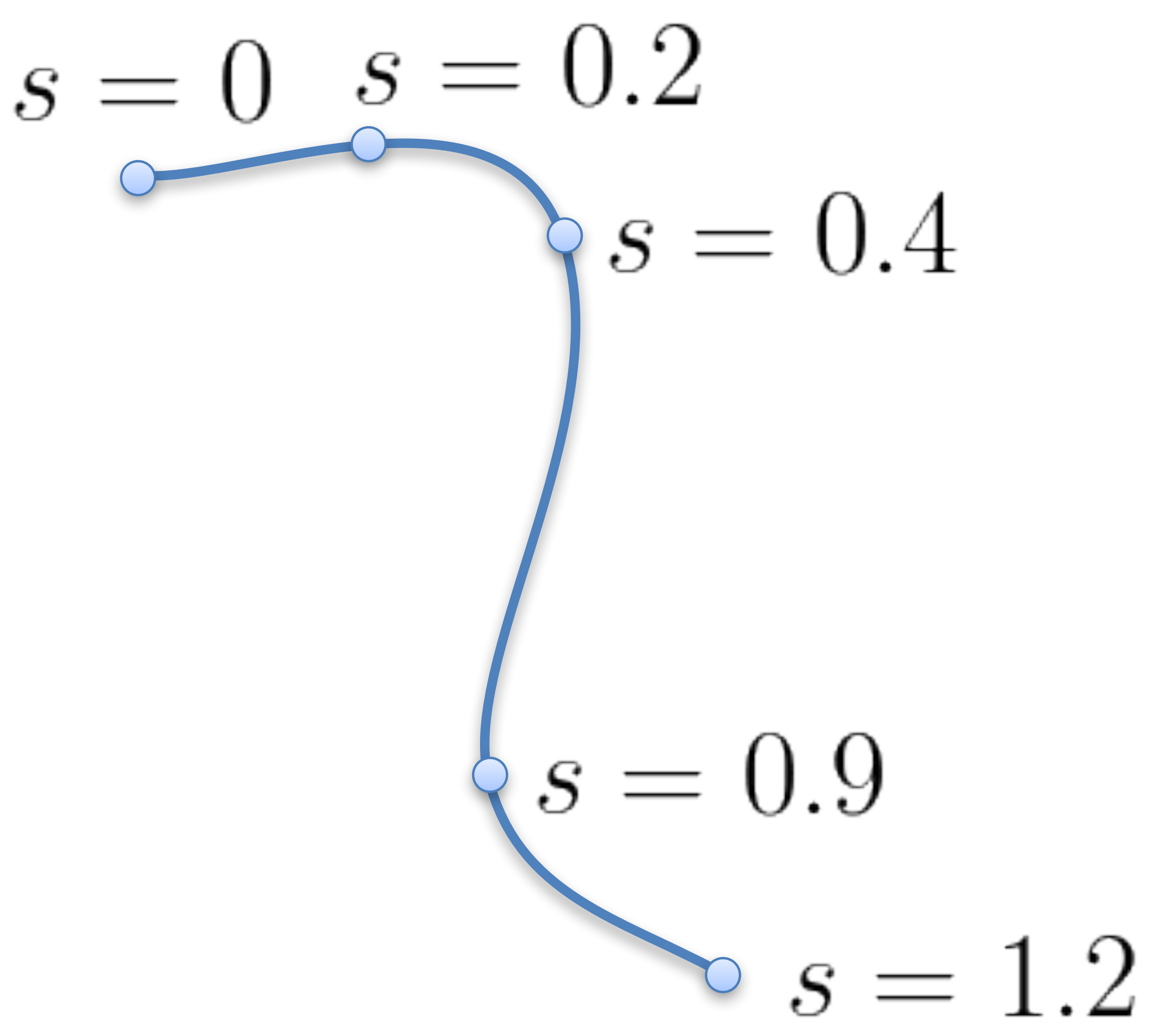


- Speed is non-negative, so $s(t)$ is non-decreasing

Arc length parameterization

- Every curve has a natural parameterization:

$$\mathbf{p}(s), \text{ such that } \|\mathbf{p}'(s)\| = 1$$



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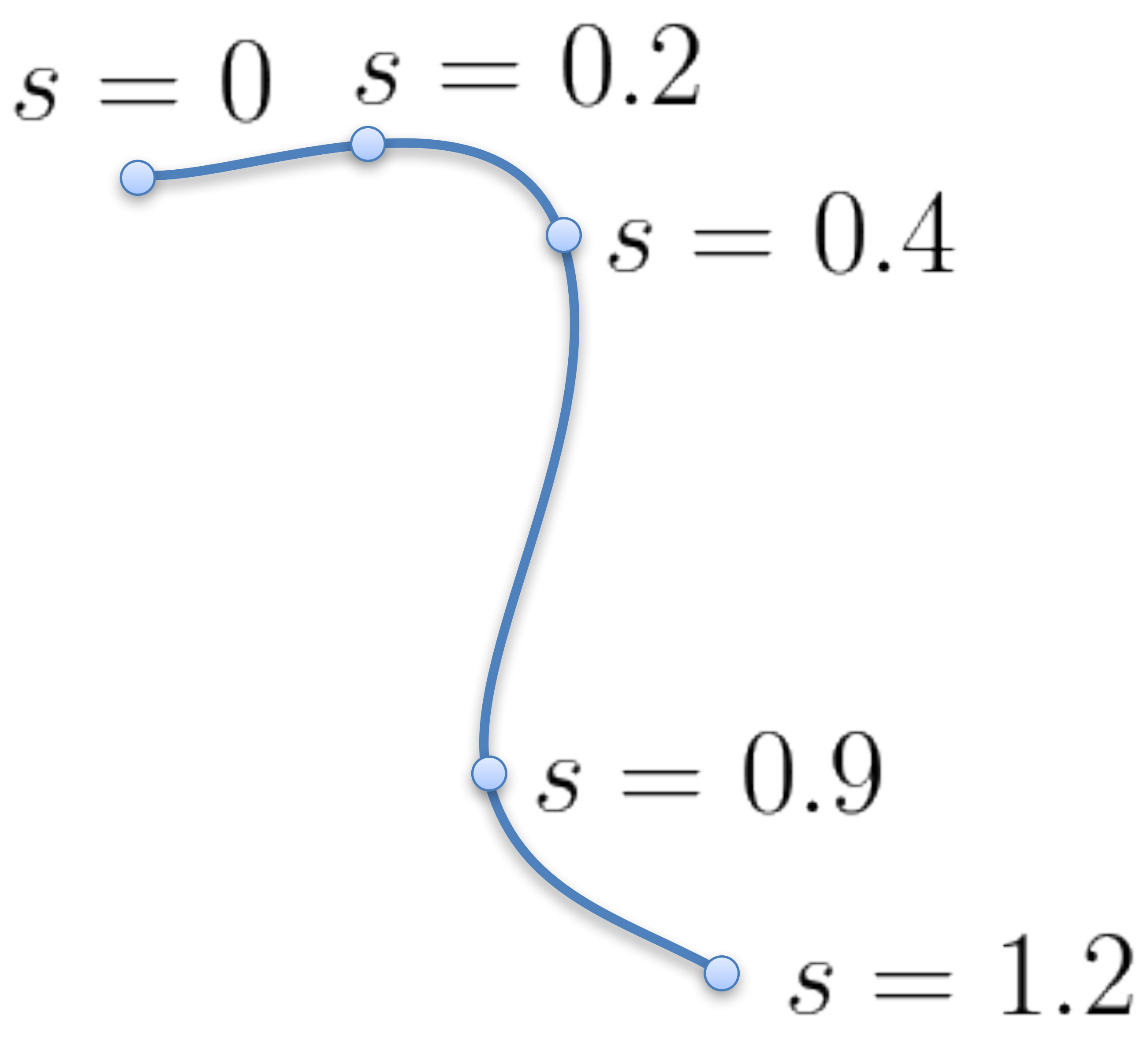
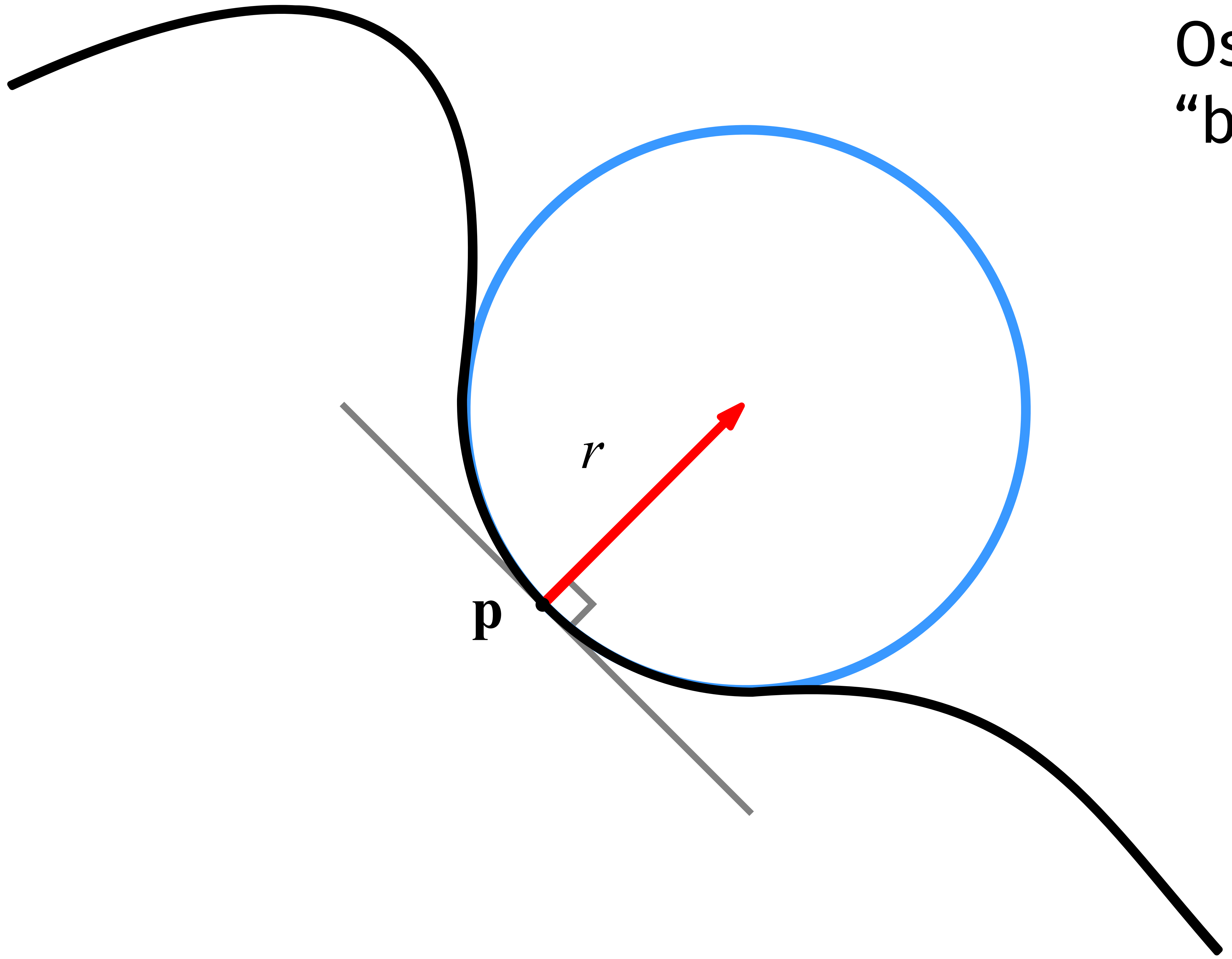


- Isometry between parameter domain and curve
- Tangent vector is unit-length: $\mathbf{p}'(s) = \mathbf{T}(s)$

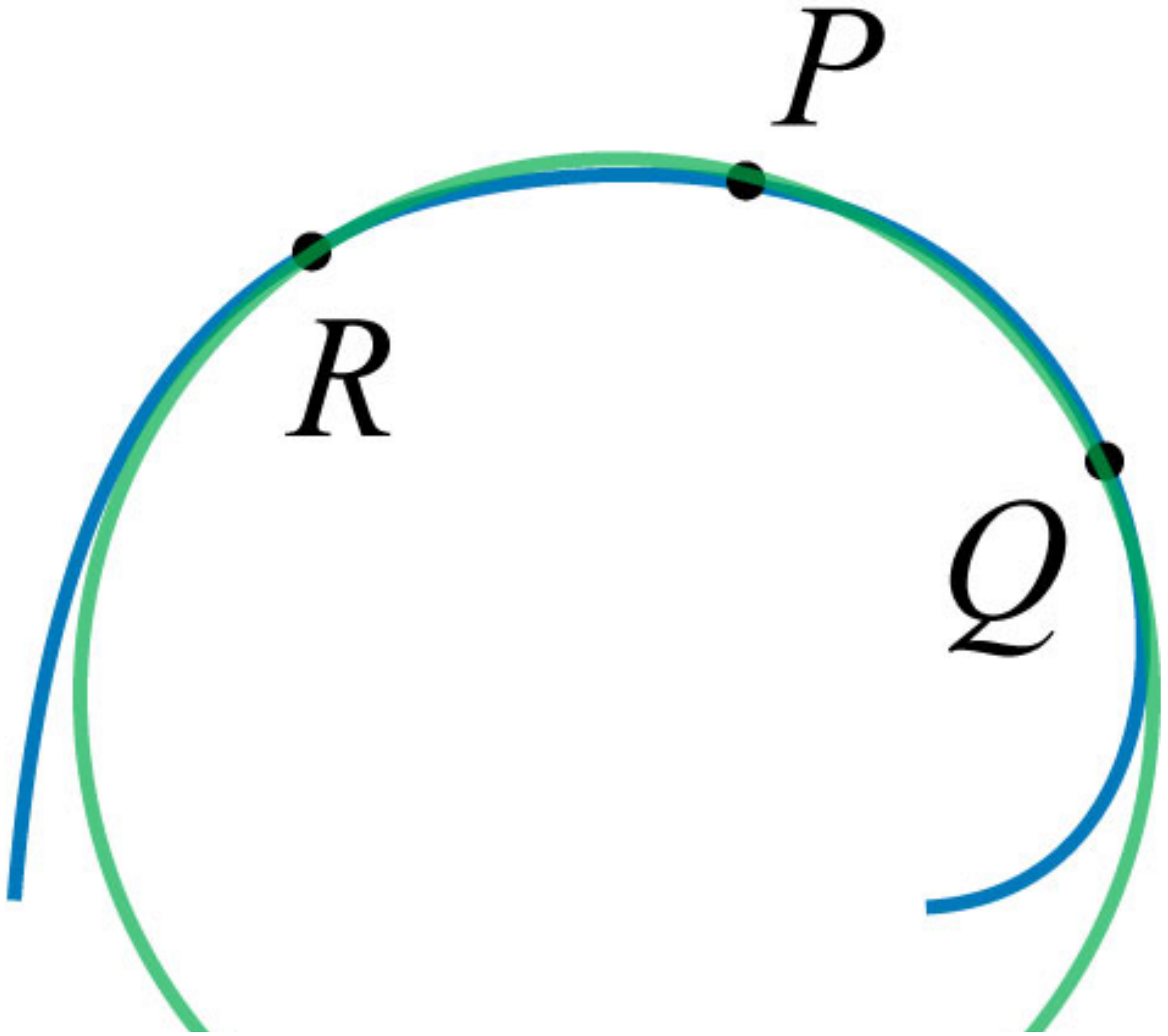
Curvature: geometric construction



Osculating circle
“best fitting circle”

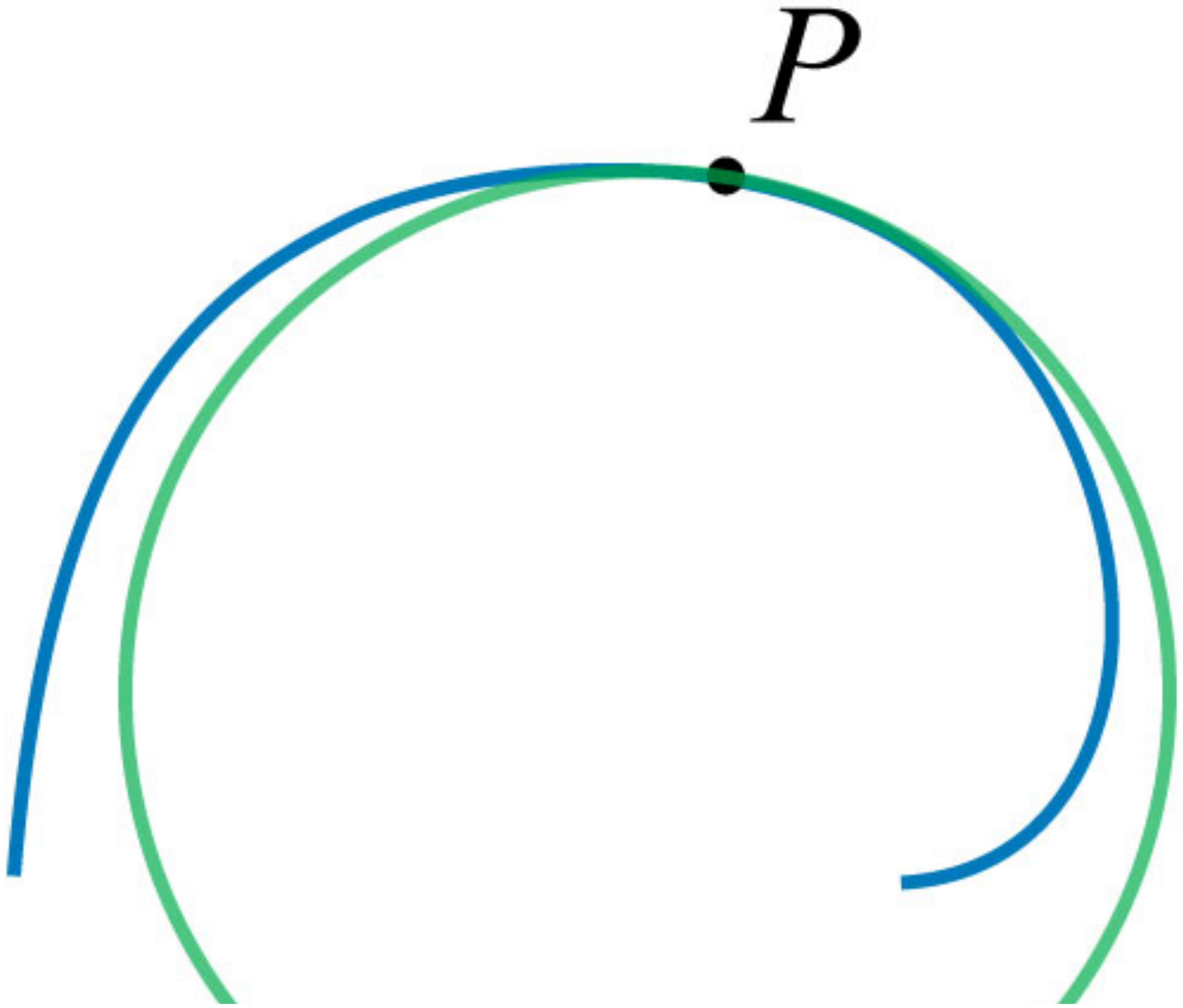
Curvature: geometric construction

- Consider the circle passing through three points on the curve…



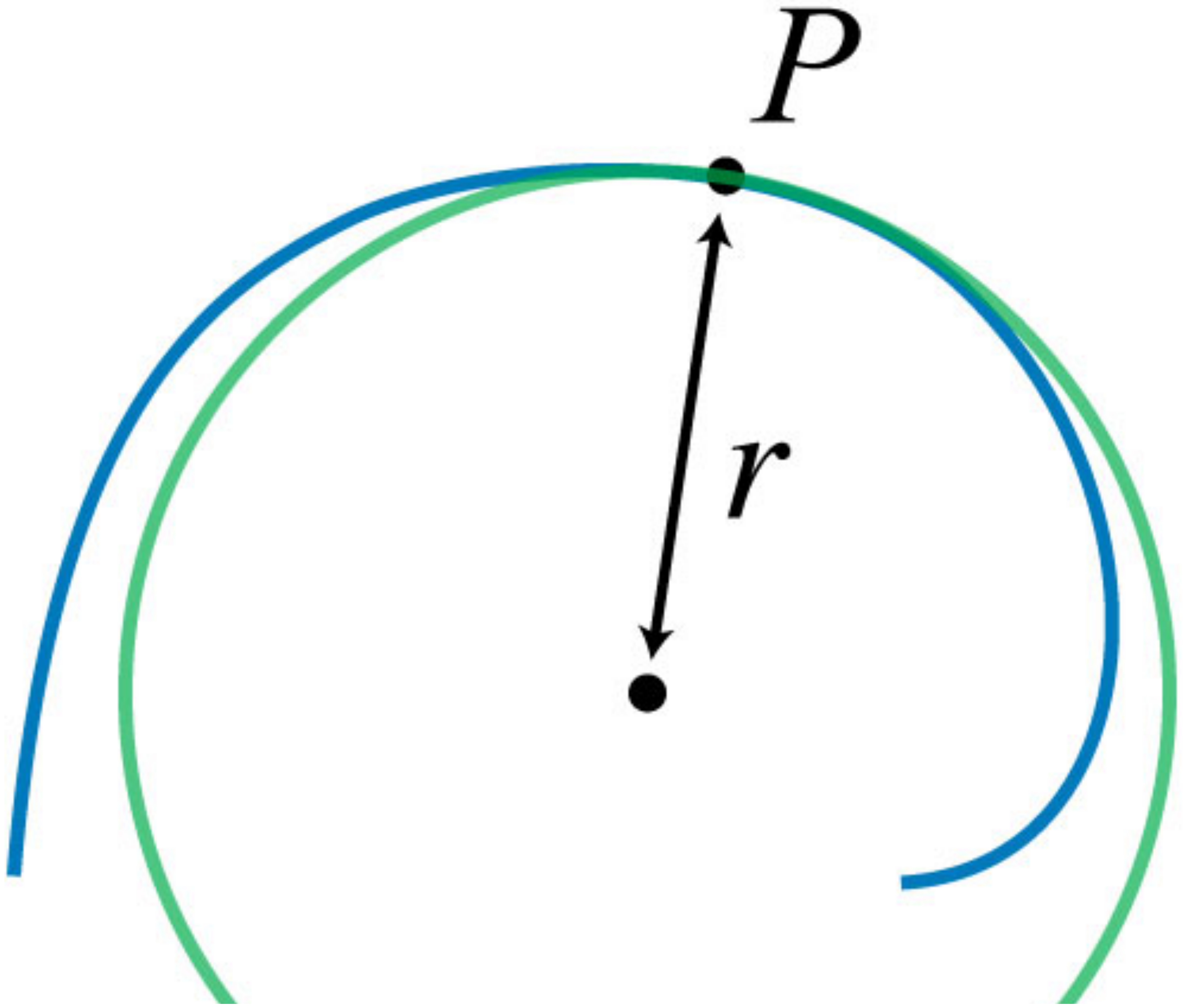
Curvature: geometric construction

- ...the limiting circle as $\boldsymbol{Q}$ and $\boldsymbol{R}$ move to $\boldsymbol{P}$: **osculating circle**



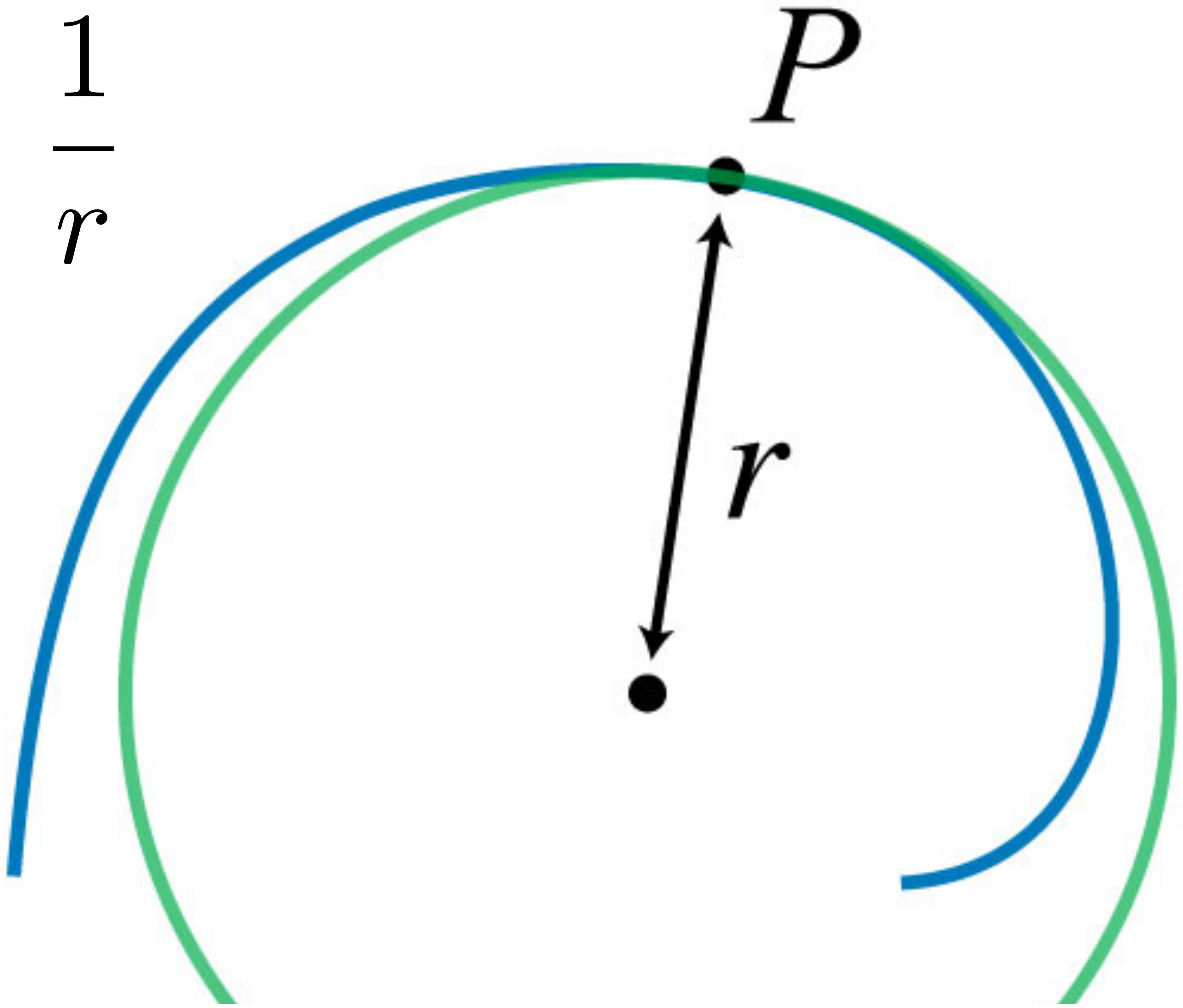
Curvature: geometric construction

- …take the radius of the osculating circle



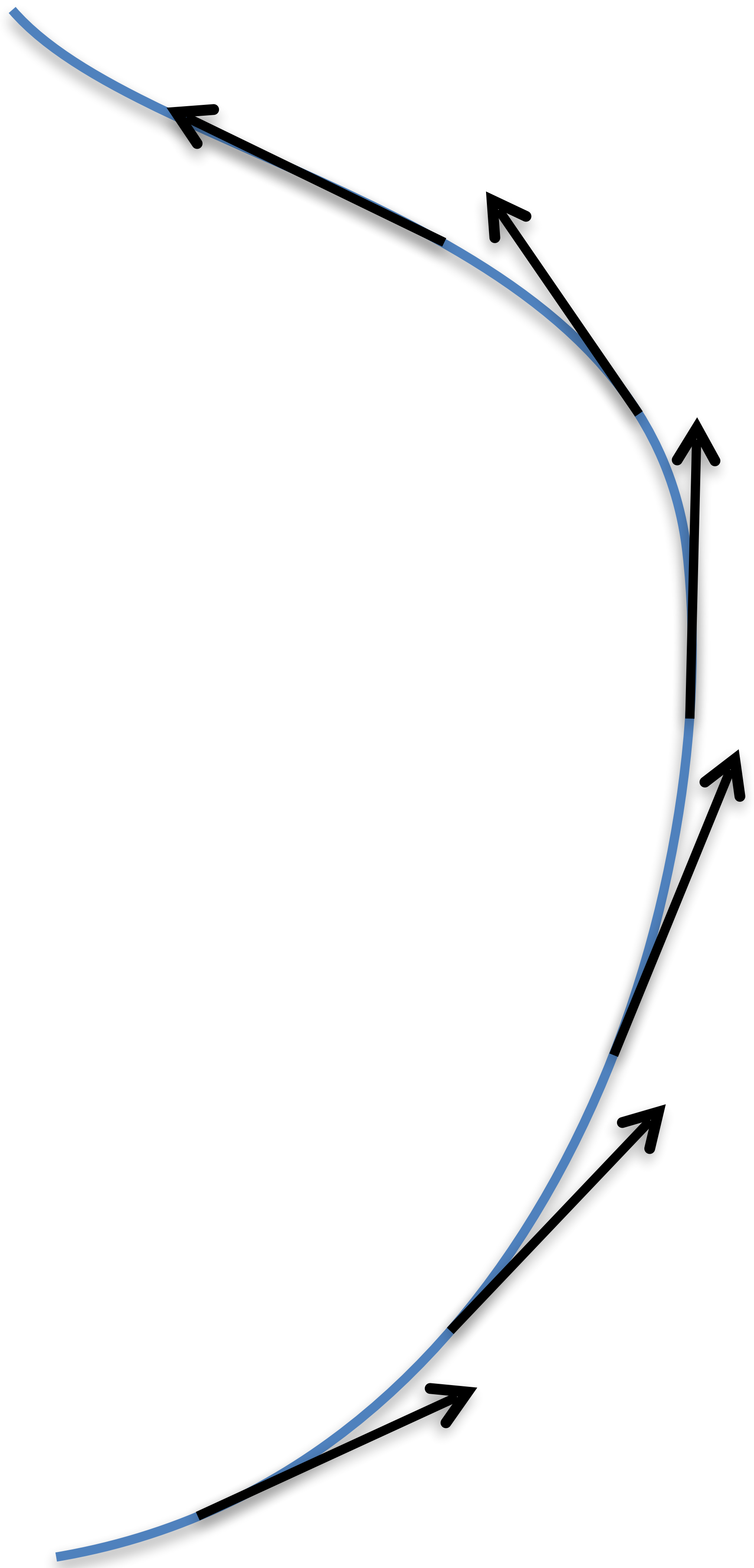
Curvature: geometric construction

Curvature $\kappa = \dfrac{1}{r}$

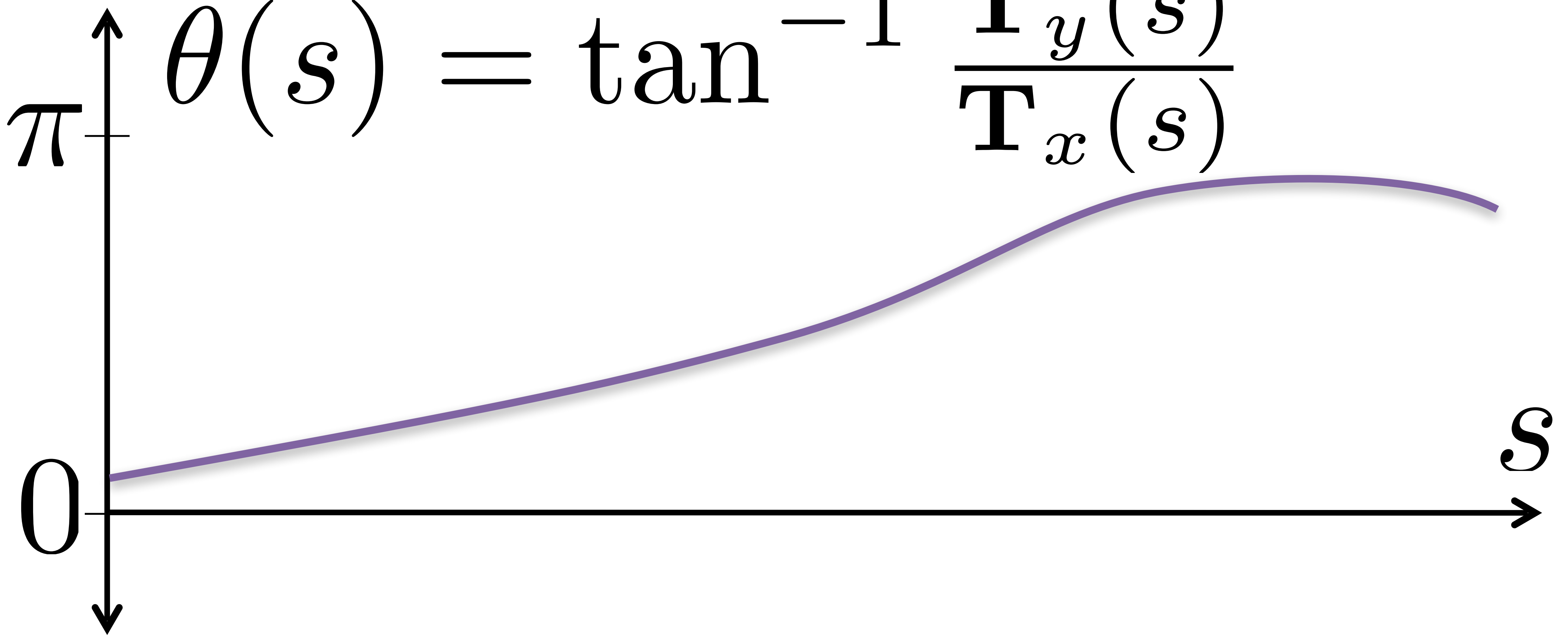


Curvature: analytic construction

- How much does the curve turn per unit s?



$$\theta(s) = \tan^{-1} \frac{\mathbf{T}_y(s)}{\mathbf{T}_x(s)}$$



Curvature

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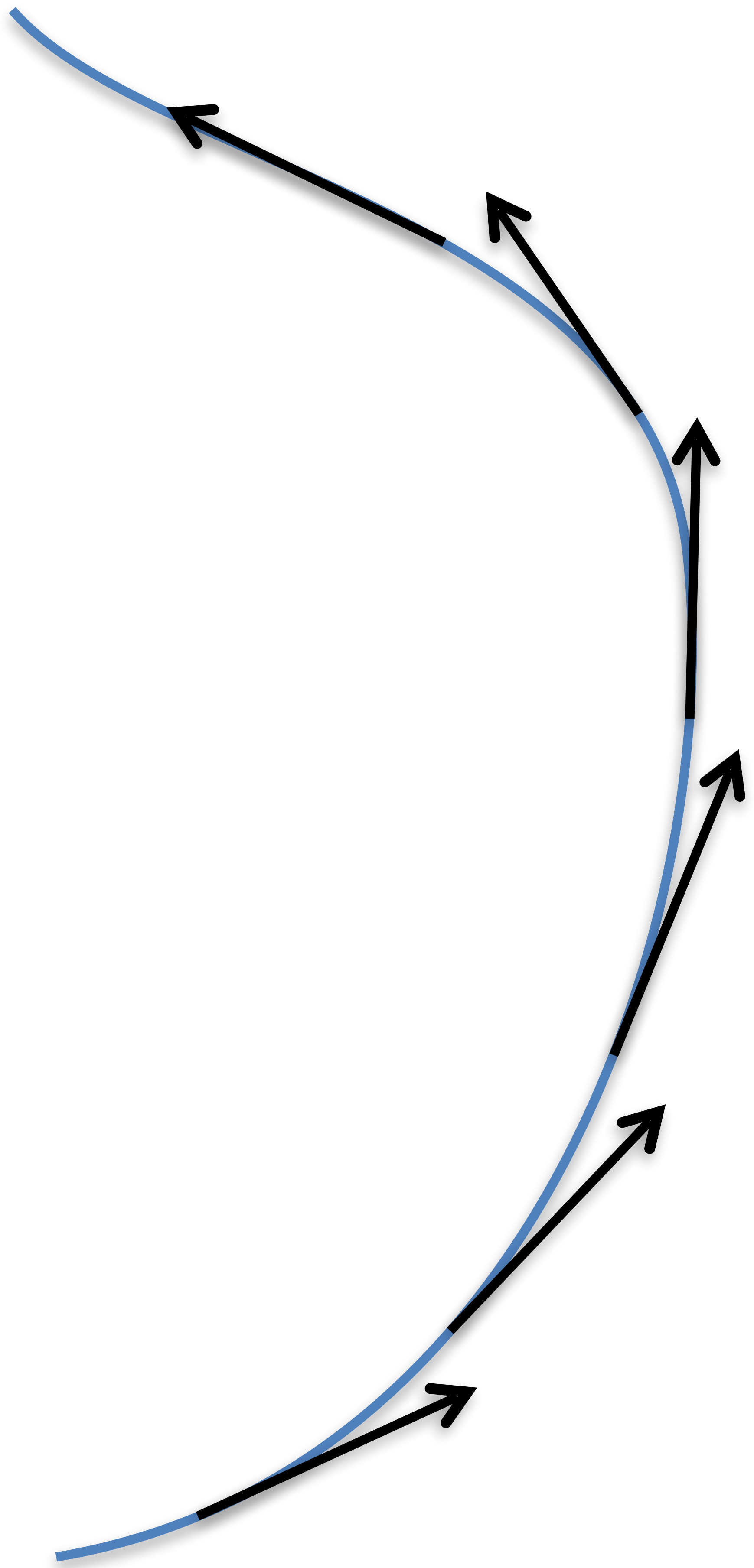

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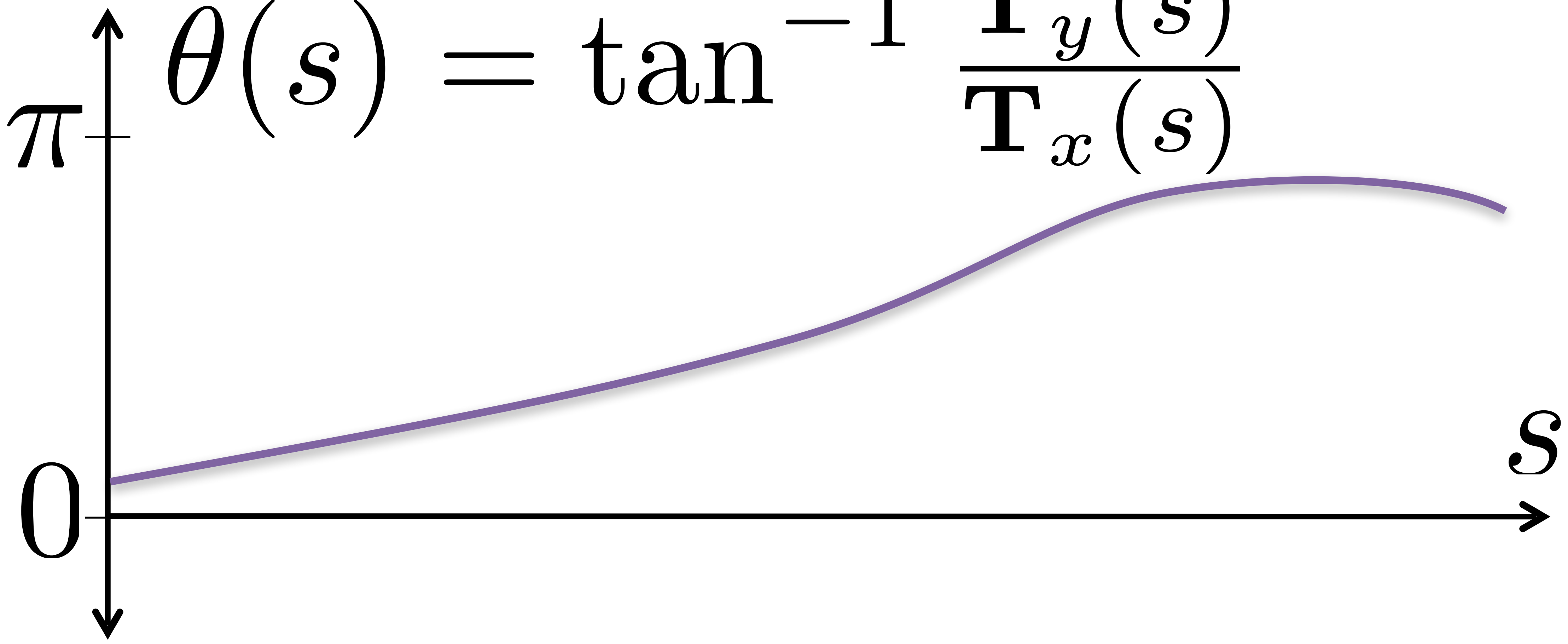
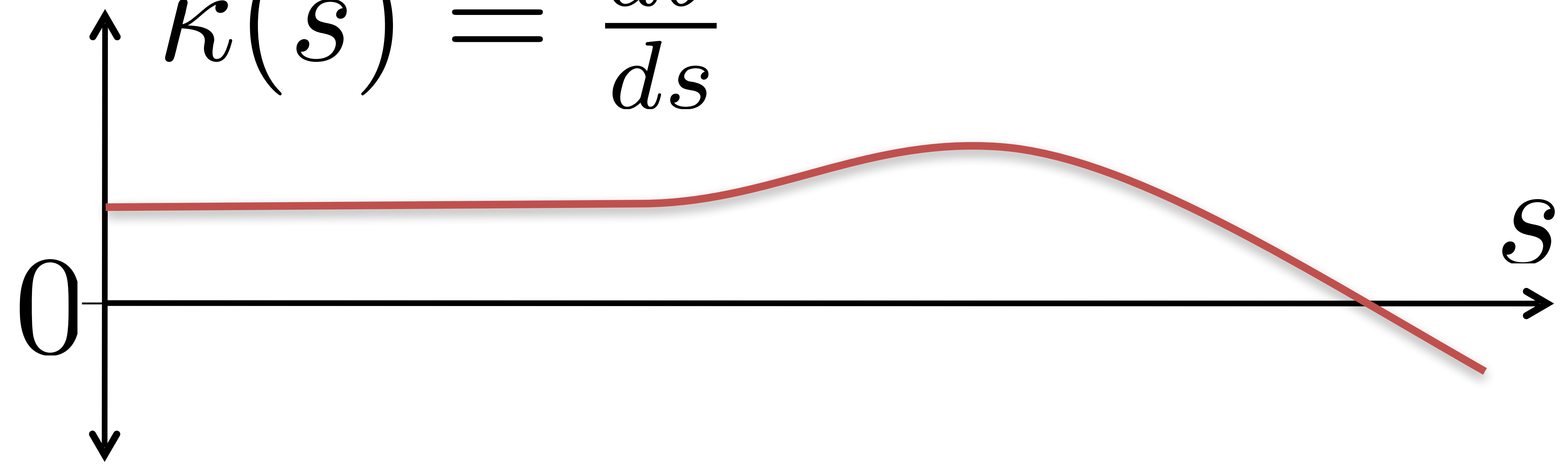


$$\kappa(s) = \frac{d\theta}{ds}$$



Curvature of a circle

- Curvature of a circle:

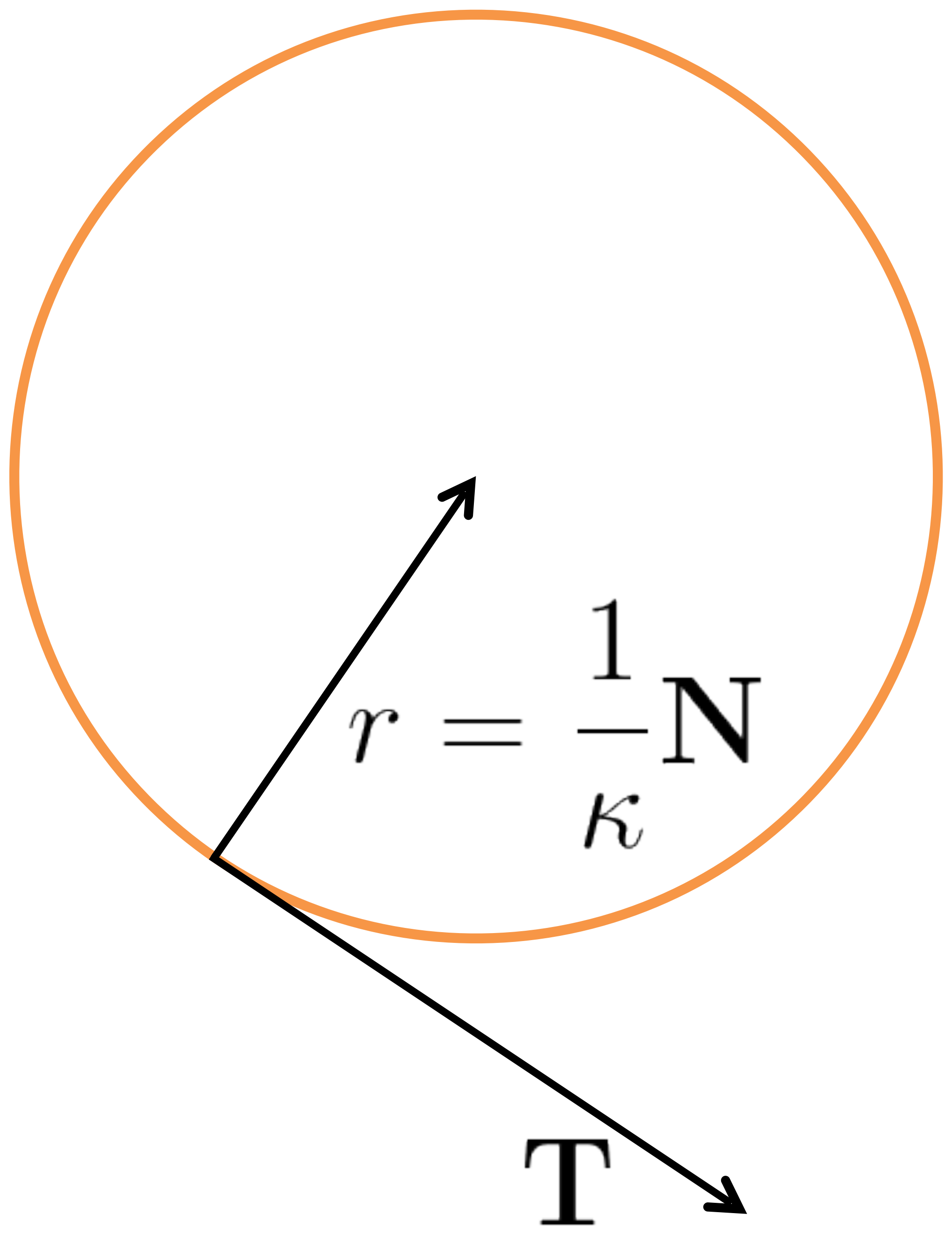
$$\mathbf{p}(s) = \begin{pmatrix} r\cos(s/r) \\ r\sin(s/r) \end{pmatrix}$$

$$\mathbf{p}'(s) = \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \end{pmatrix}$$

$$\theta(s) = \tan^{-1}\frac{\cos(s/r)}{-\sin(s/r)}$$
$$= s/r - \pi/2$$

$$\kappa(s) = 1/r$$

$$\mathbf{N}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T}(t)$$ = Unit Normal



Curvature profile

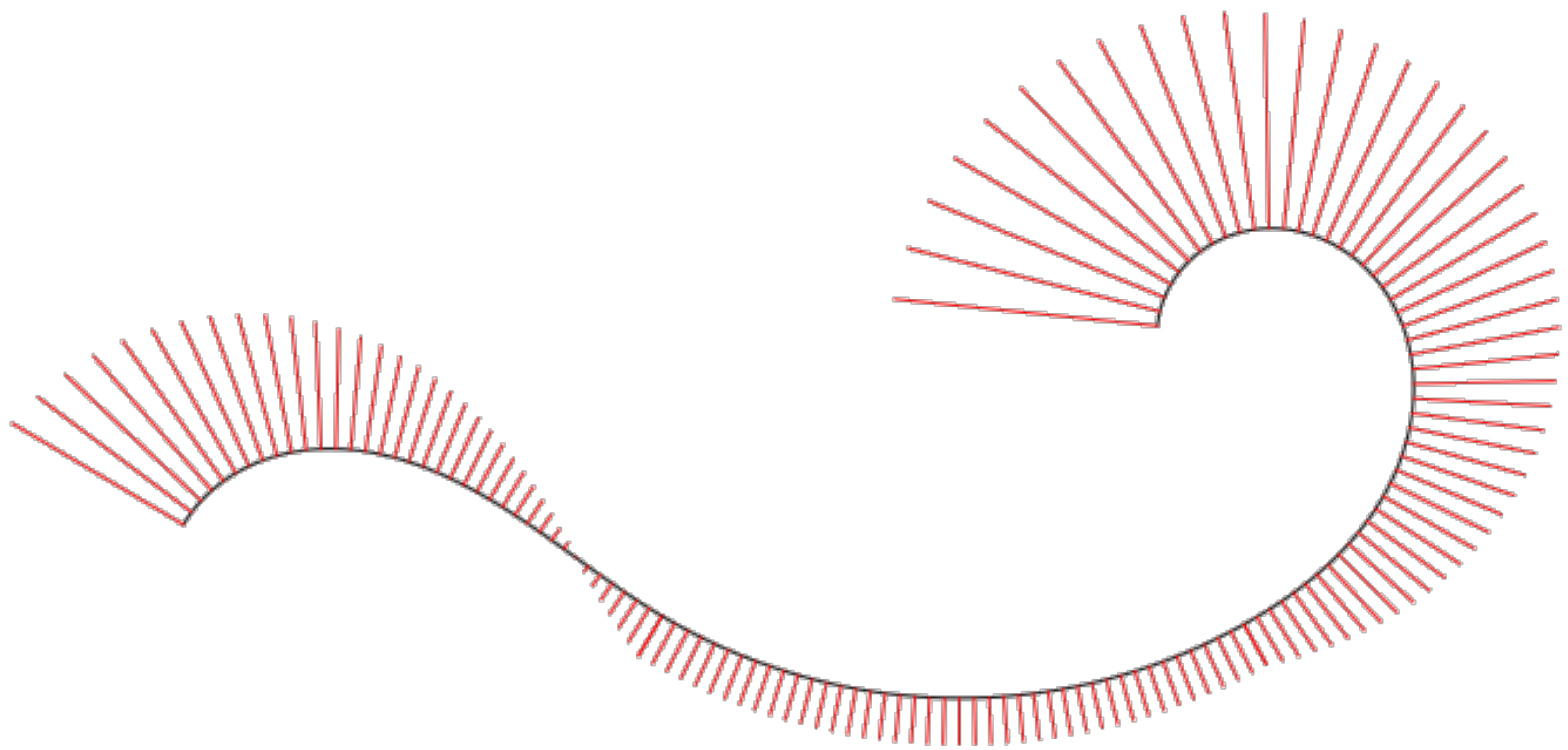
- Given $\kappa(s)$, we can get $\theta(s)$ up to a constant by integration.
- Integrating

$$\mathbf{p}(s) = \mathbf{p}_0 + \int_{s_0}^{s_1} \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} ds$$

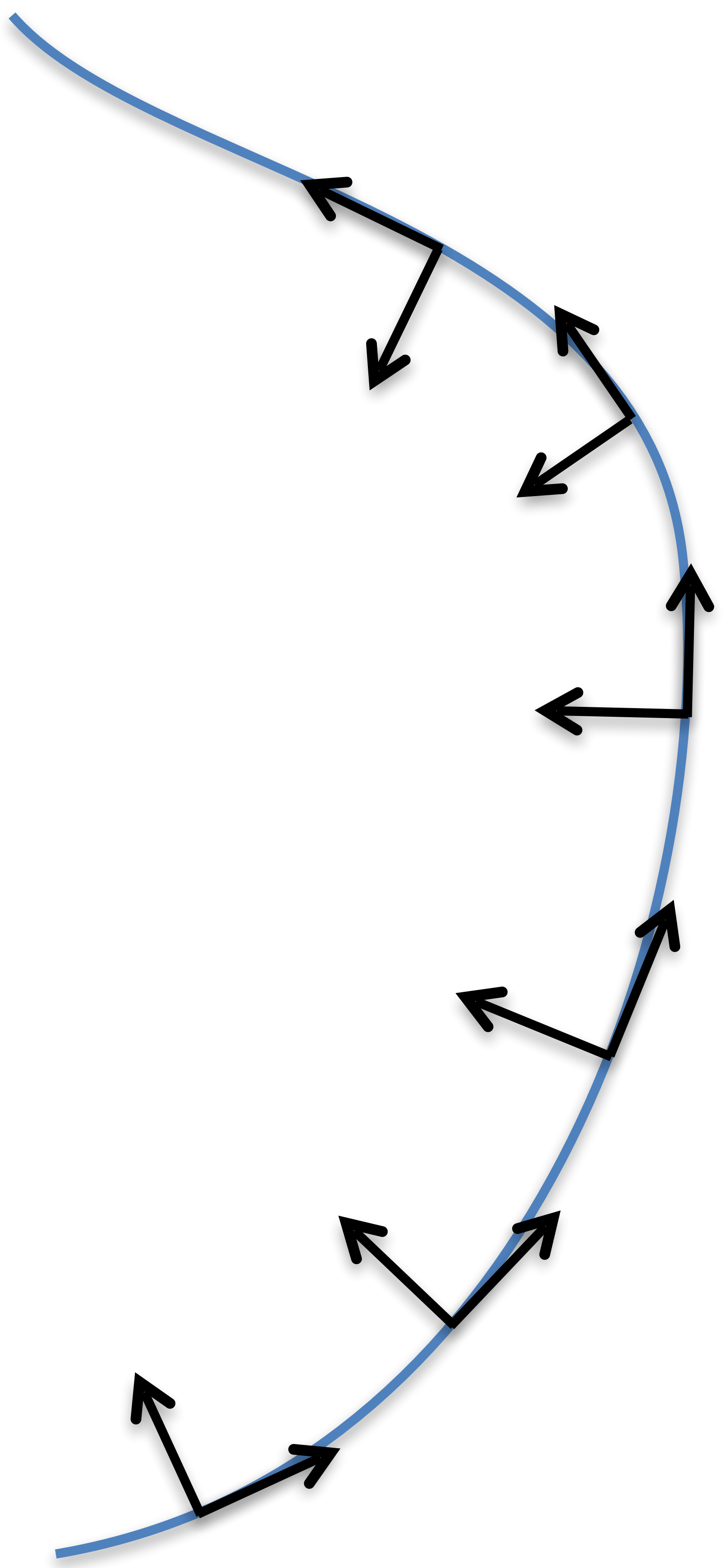
reconstructs the curve up to rigid motion

Curvature normal

- Points inward $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$
- $-\kappa(s)\mathbf{N}(s)$ useful for evaluating curve quality



Frénet Frame



$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$
$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}$$

Smoothness of curves

Two kinds, parametric and geometric:

$$C^1: \mathbf{p}(t) \text{ is continuously differentiable}$$

$$G^1: \mathbf{p}(s) \text{ is continuously differentiable}$$

Parametrization-Independent - Geometric property

$$C^1 \qquad \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

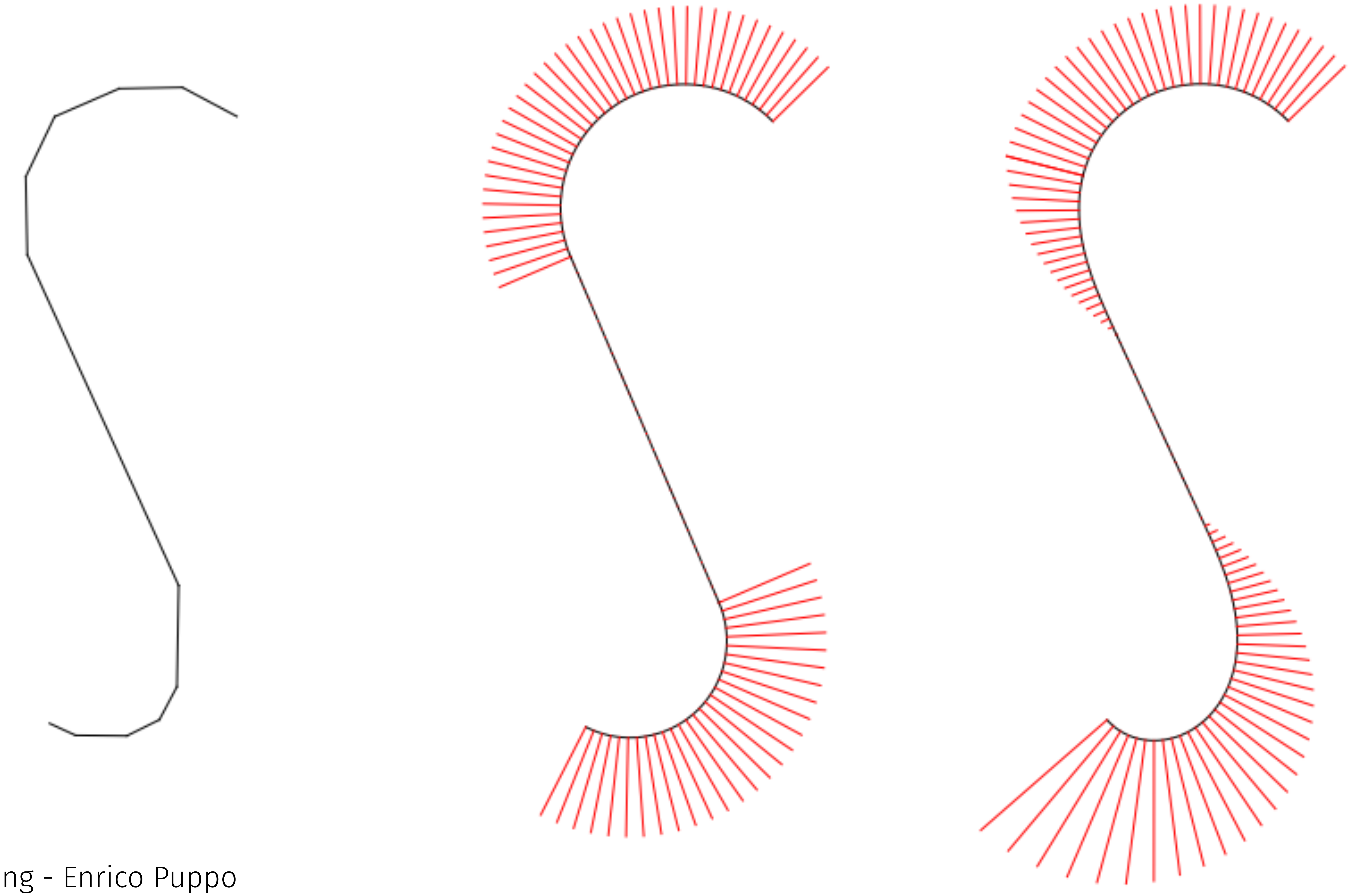
Same curve

Different parametrization, not smooth at t=1

$$G^1 \qquad \begin{pmatrix} \cos \hat{t} \\ \sin \hat{t} \end{pmatrix}, \; \hat{t} = \begin{cases} t+1 & \text{if } t < 1 \\ 2t & \text{if } t \geq 1 \end{cases}$$

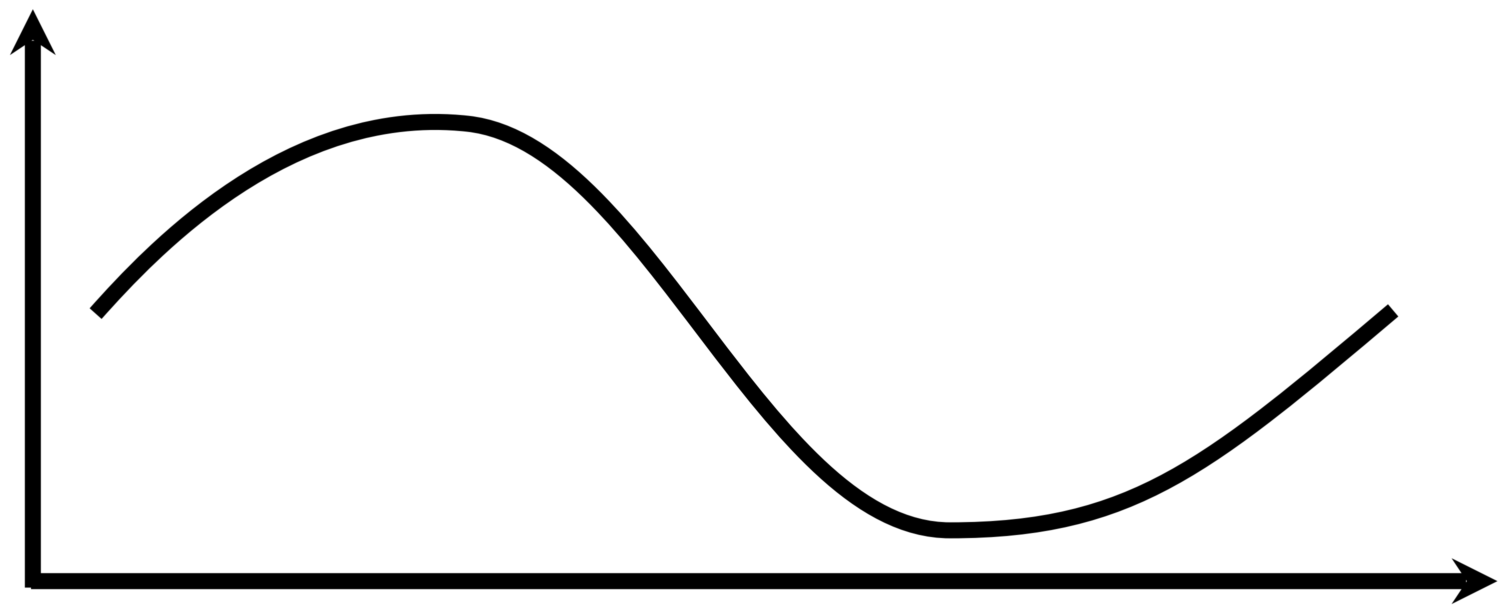
Smoothness example

G^0 G^1 G^2



Recap on parametric curves

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \; t \in [t_0, t_1]$$



$$\|\mathbf{p}'(t)\| = \text{speed}$$

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t)$$ tangent

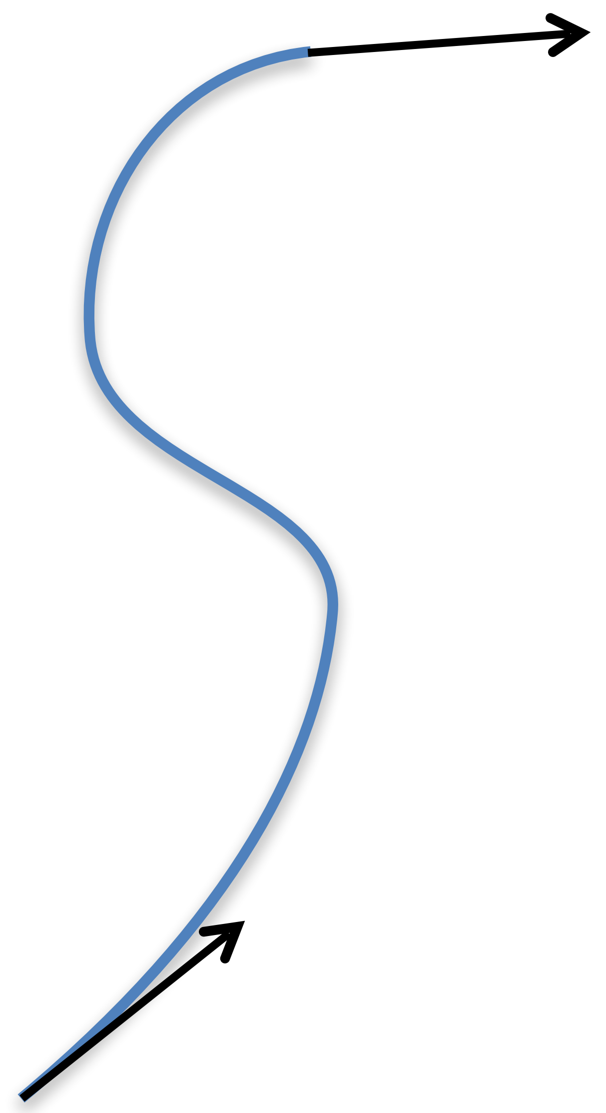
$$s(t) = \int_{t_0}^{t} \|\mathbf{p}'(t)\| dt$$ arc length

$$\kappa(s) = \frac{d\theta}{ds} = \mathbf{T}'(s) \cdot \mathbf{N}(s)$$ curvature

Turning

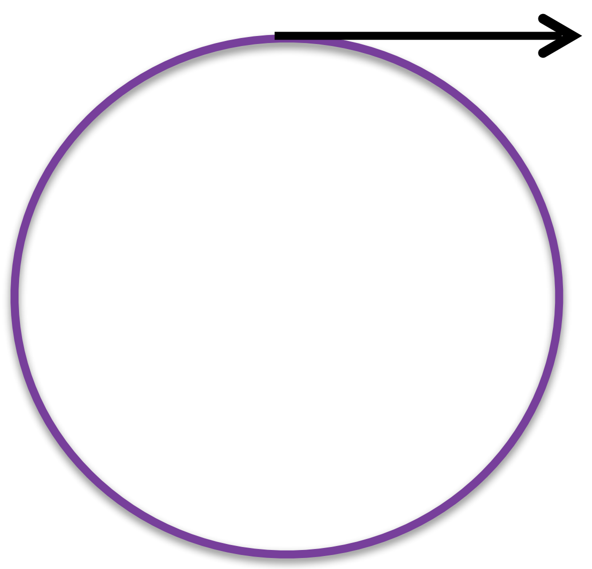
- Angle from start tangent to end tangent:

$$\int_{s_0}^{s_1} \kappa(s)ds = \int_{t_0}^{t_1} \kappa(t)\|\mathbf{p}'(t)\|dt$$



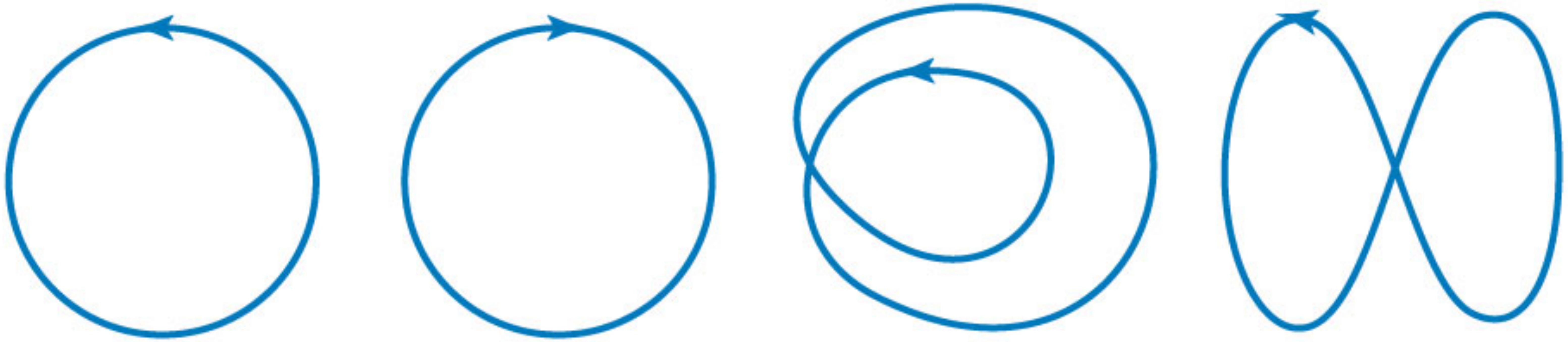
- If curve is closed, the tangent at the beginning is the same as the tangent at the end

$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$



Turning numbers

$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$

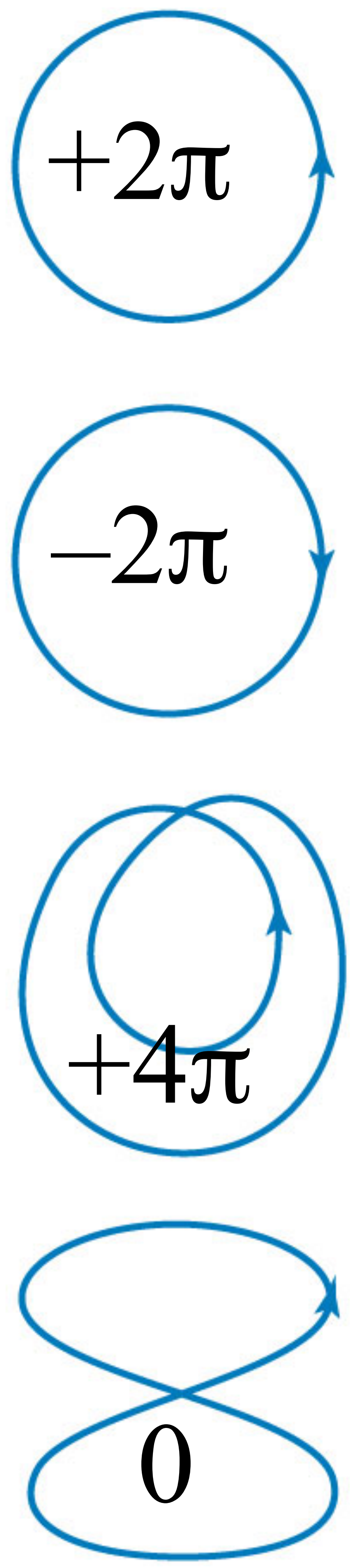


$n =$ 1 –1 2 0

- n measures how many full turns the tangent makes

Turning Number Theorem

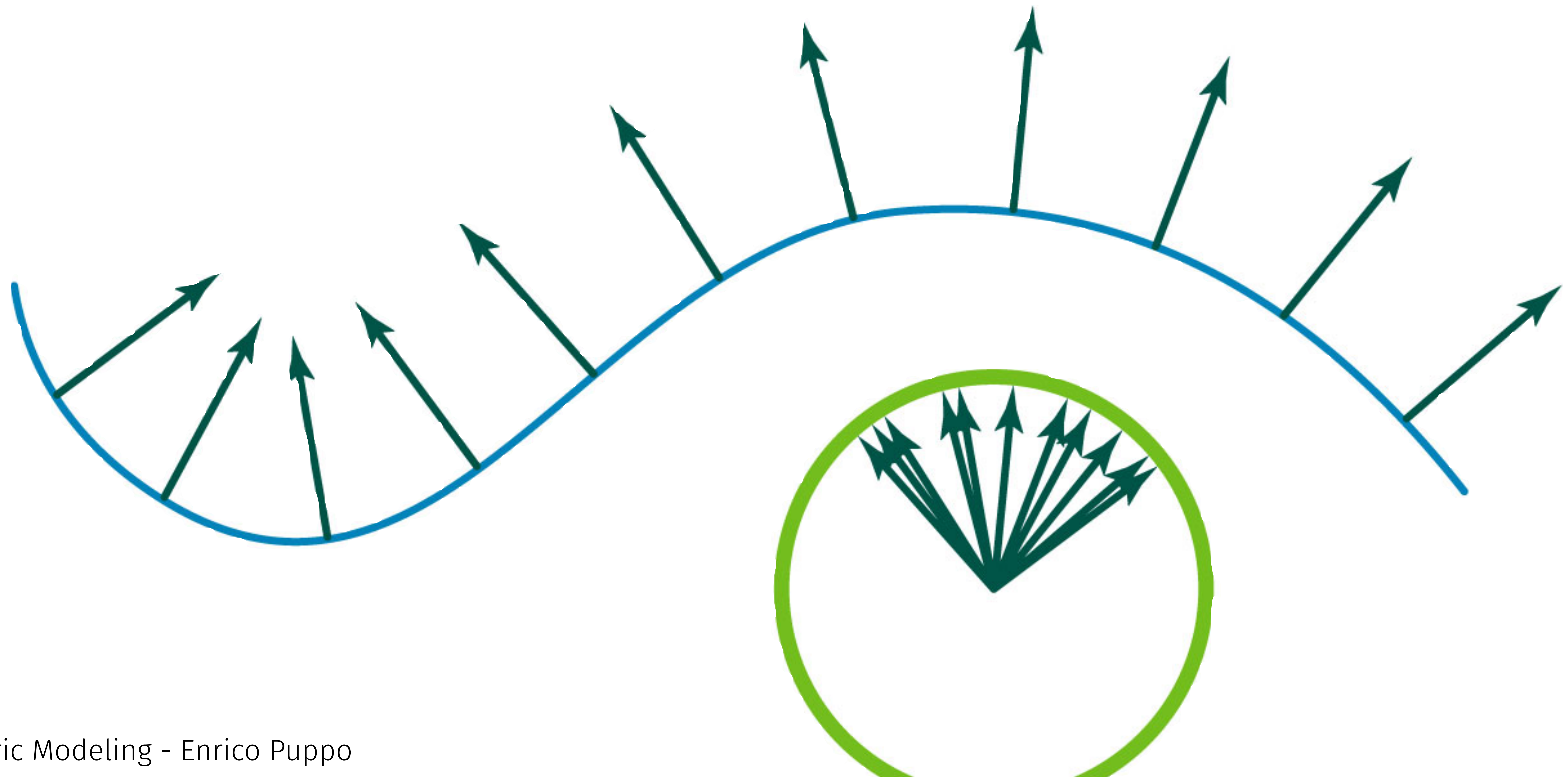
$$\int_\gamma \kappa \, dt = 2\pi k$$



- For a closed curve, the integral of curvature is an integer multiple of 2π

Gauss map $\hat{\mathbf{n}}(\mathbf{p})$

- Point on curve maps to point on unit circle.



Space curves (3D)

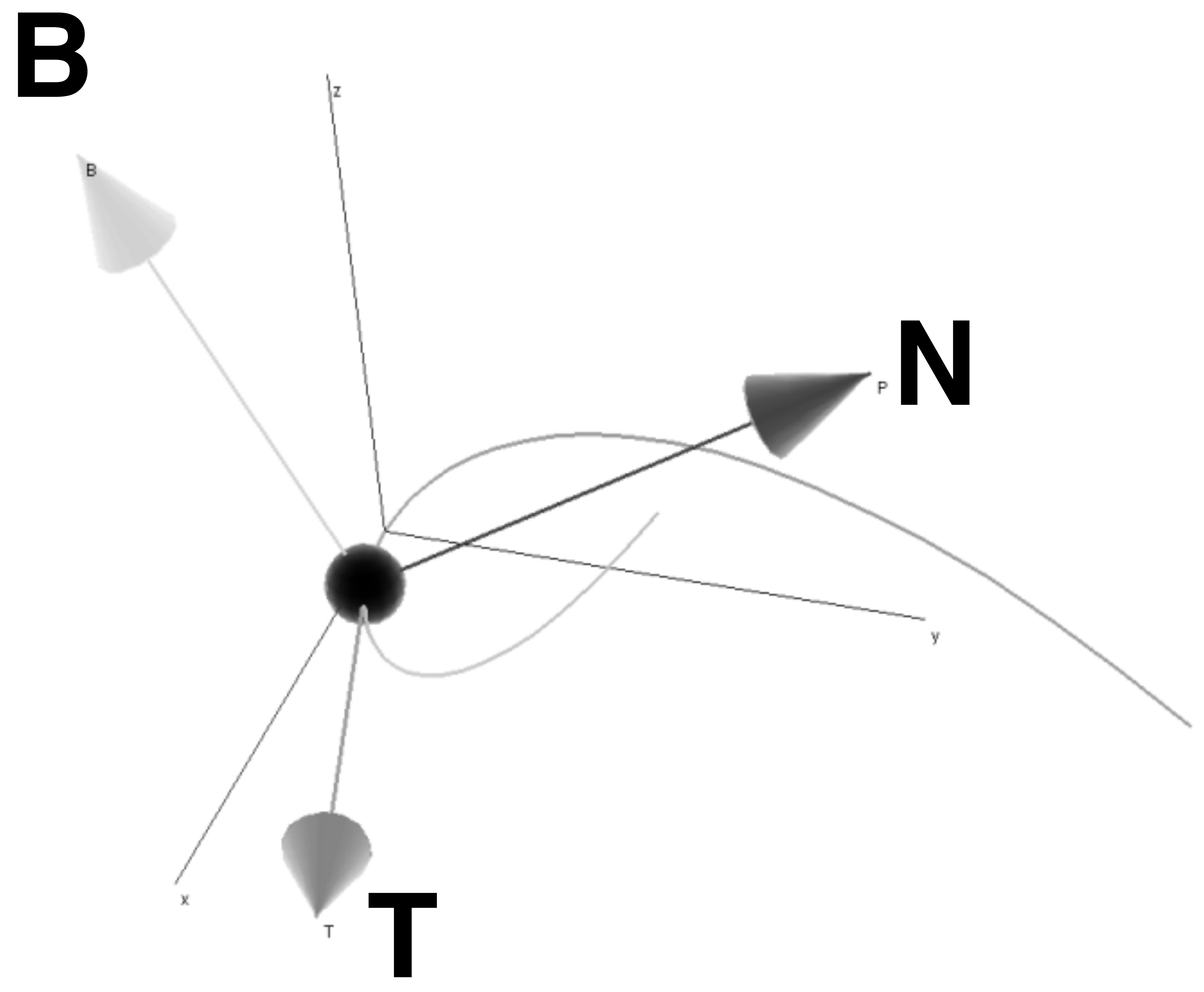
- In 3D, many vectors are orthogonal to **T**

$$\mathbf{N}(s) := \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$$
$$\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$$

- $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form the "Frénet frame"

- τ is torsion: non-planarity

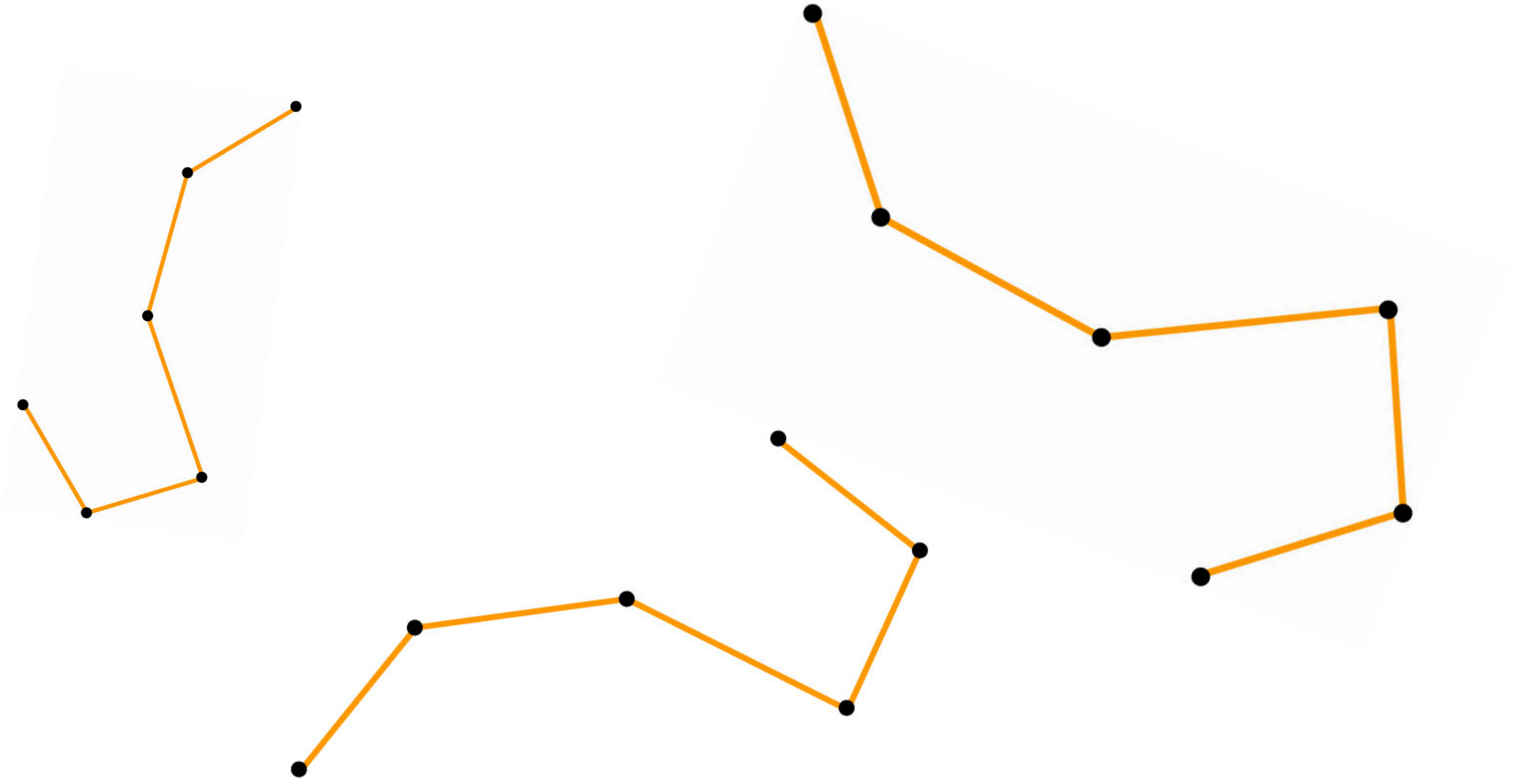
$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$



Discrete Differential Geometry of Curves

Some references: see
http://ddg.cs.columbia.edu/

Discrete Planar Curves

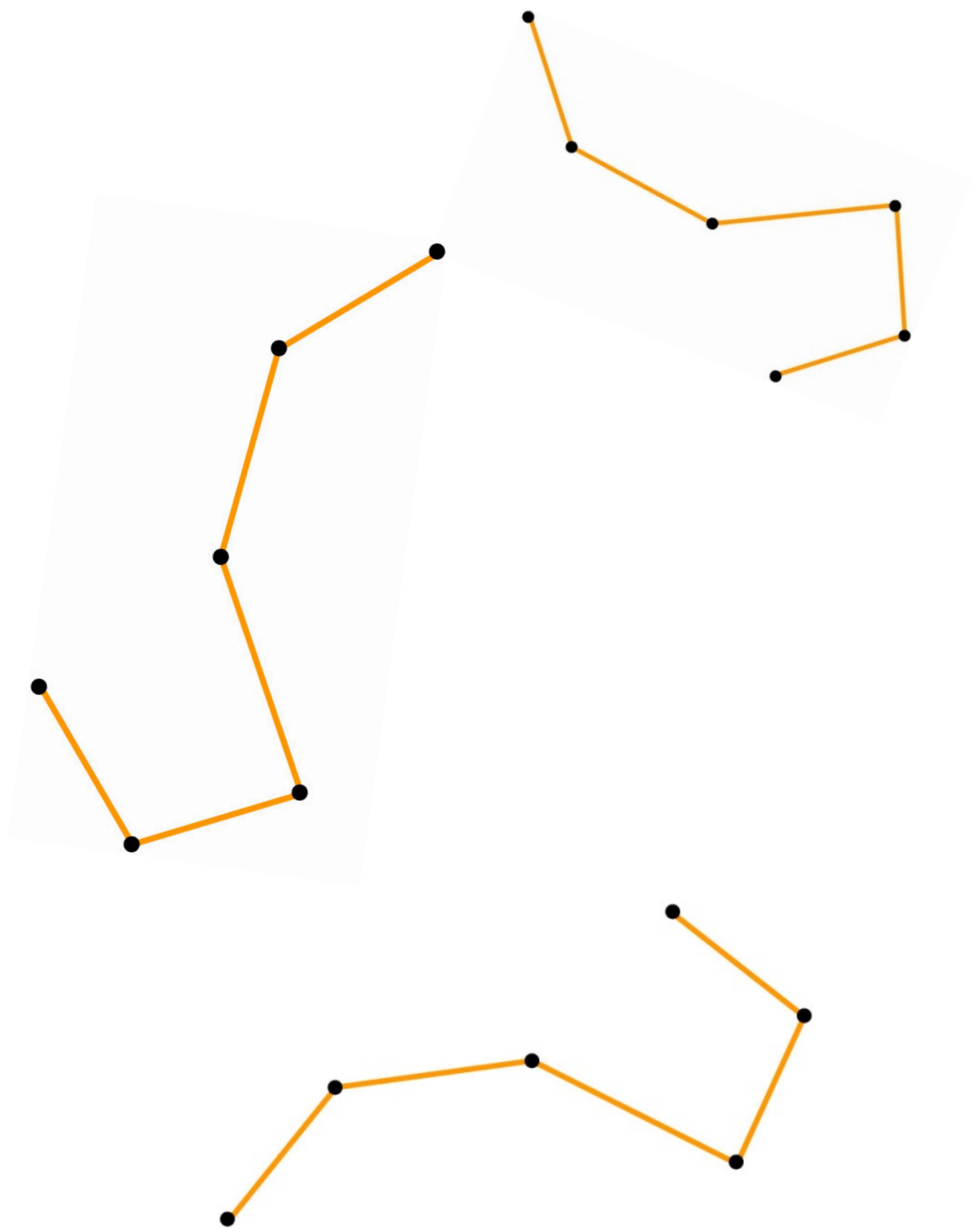


Discrete Planar Curves

- Piecewise linear curves
- Not smooth at vertices
- Can’t take derivatives

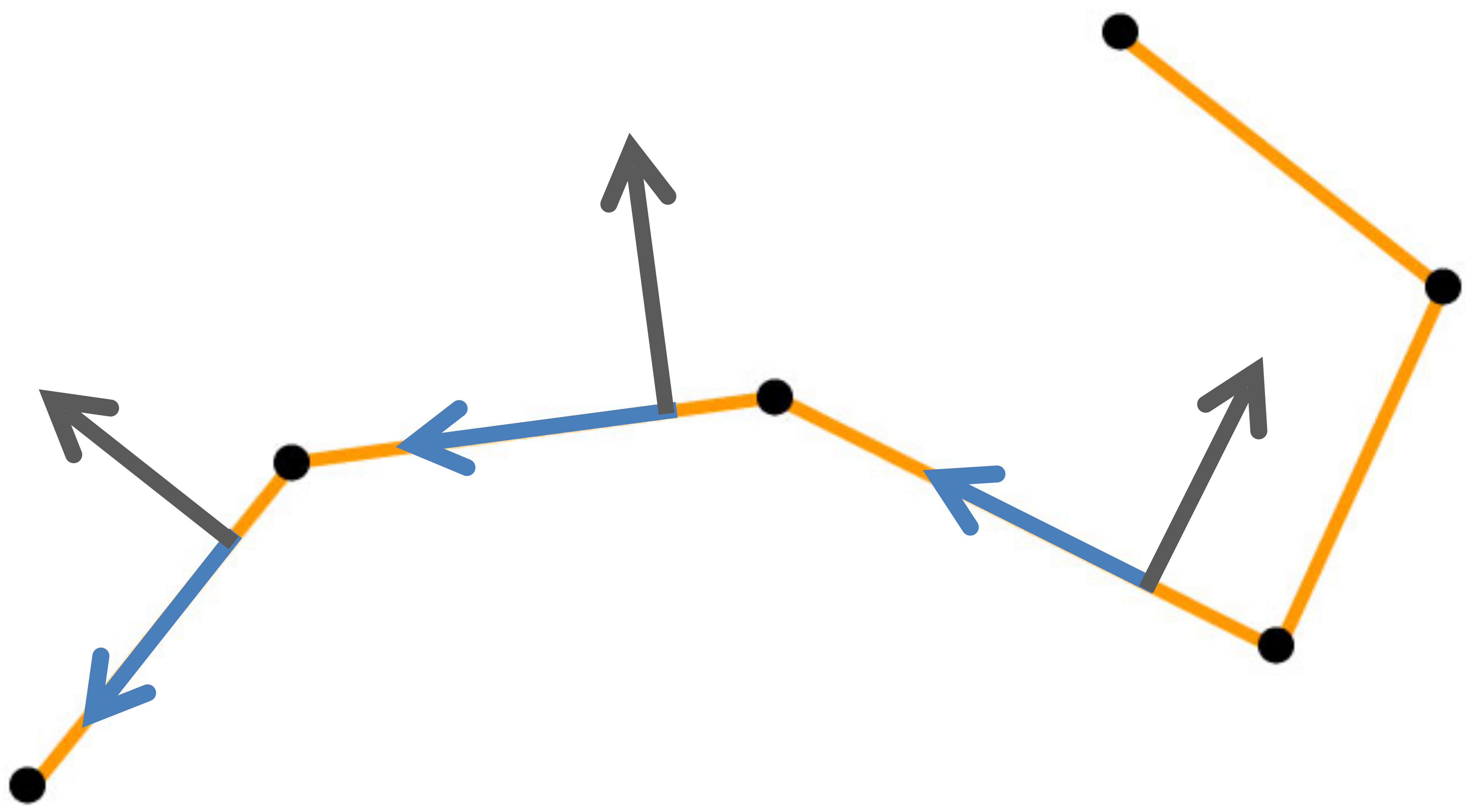
- Generalize notions from the smooth world for the discrete case!

- **There is not one single way**



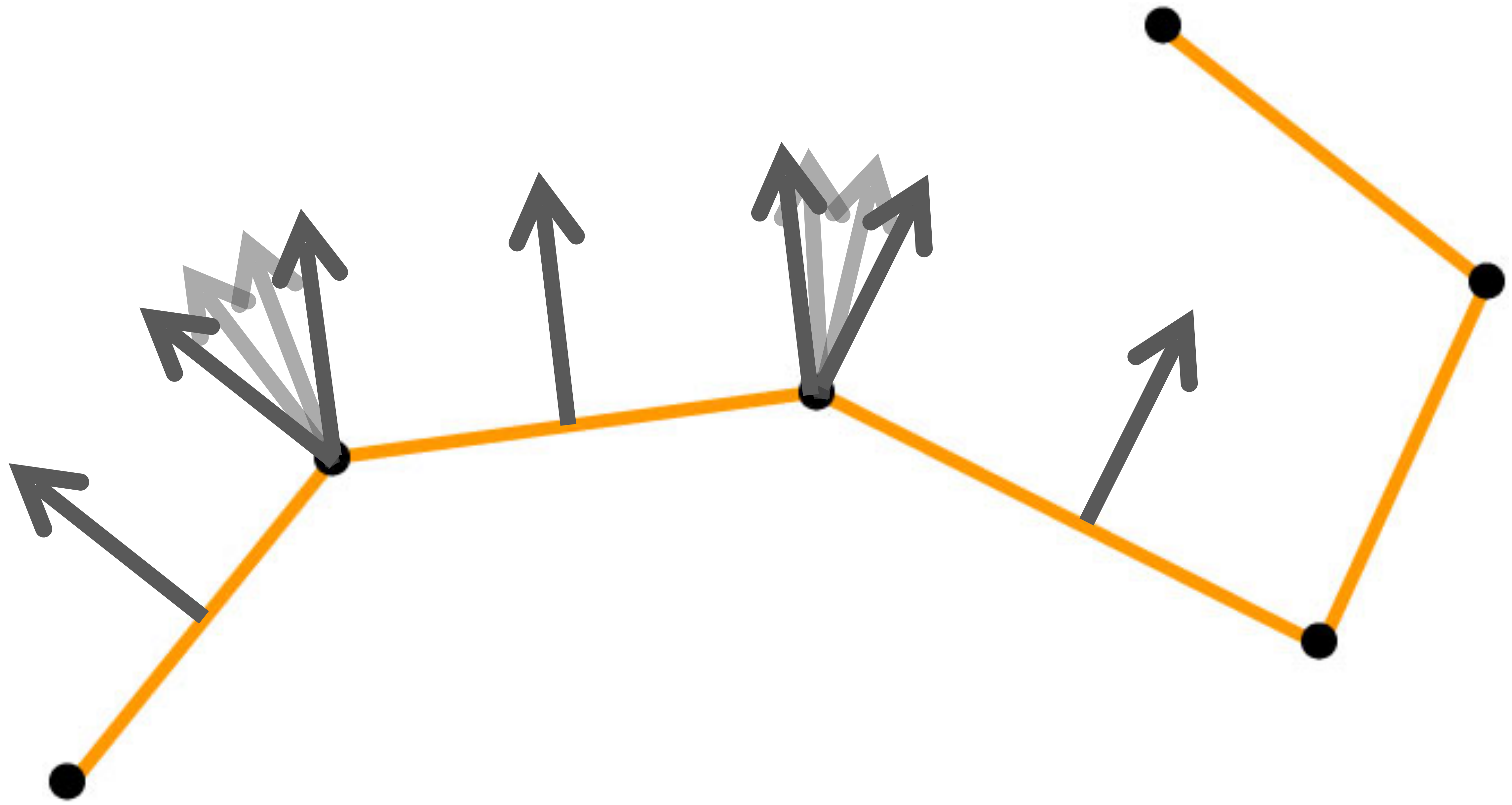
Tangents, Normals

- For any point on the edge, the tangent is simply the unit vector along the edge and the normal is the perpendicular vector



Tangents, Normals

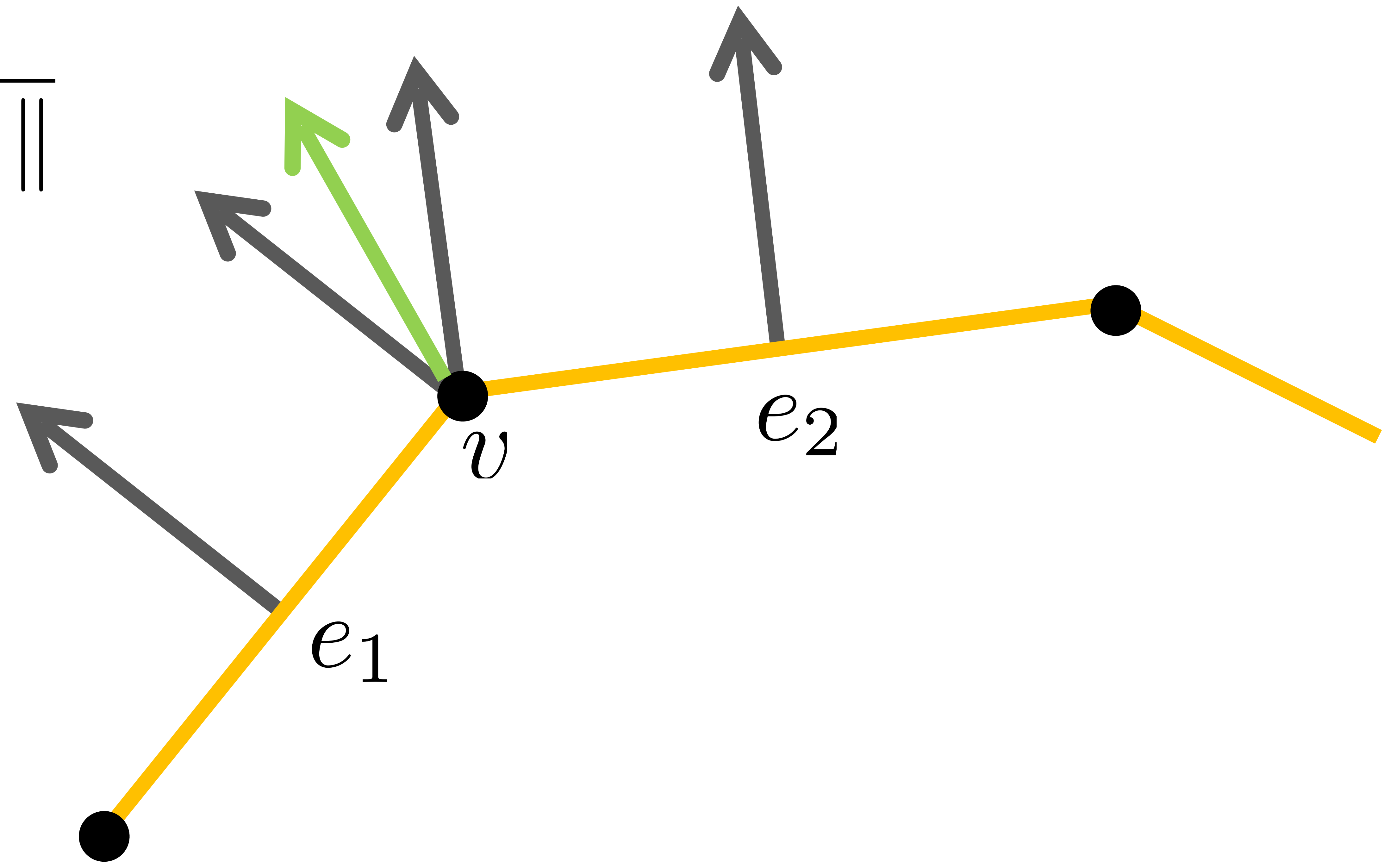
- For vertices, we have many options



Tangents, Normals

- Can choose to average the adjacent edge normals

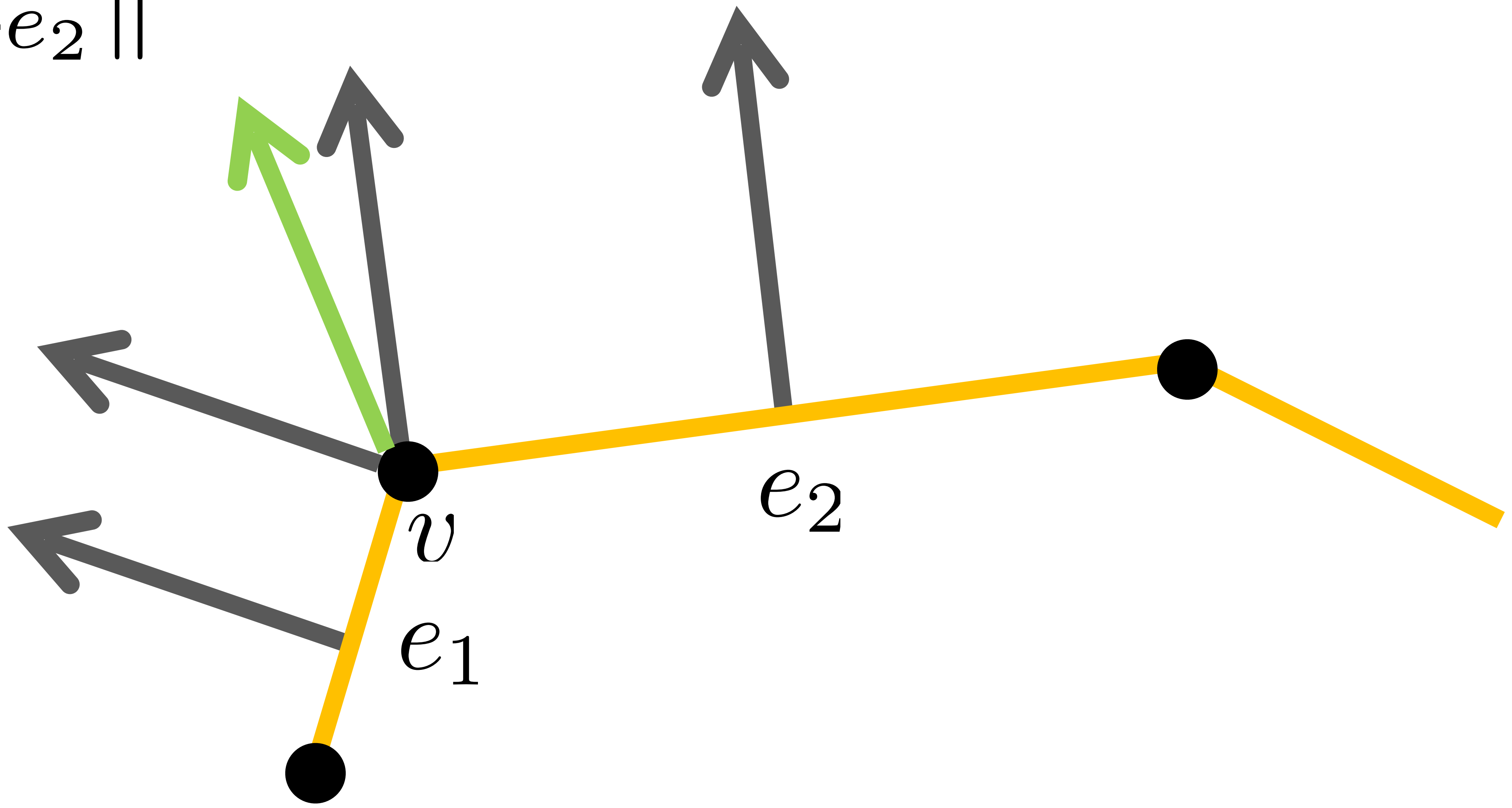
$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

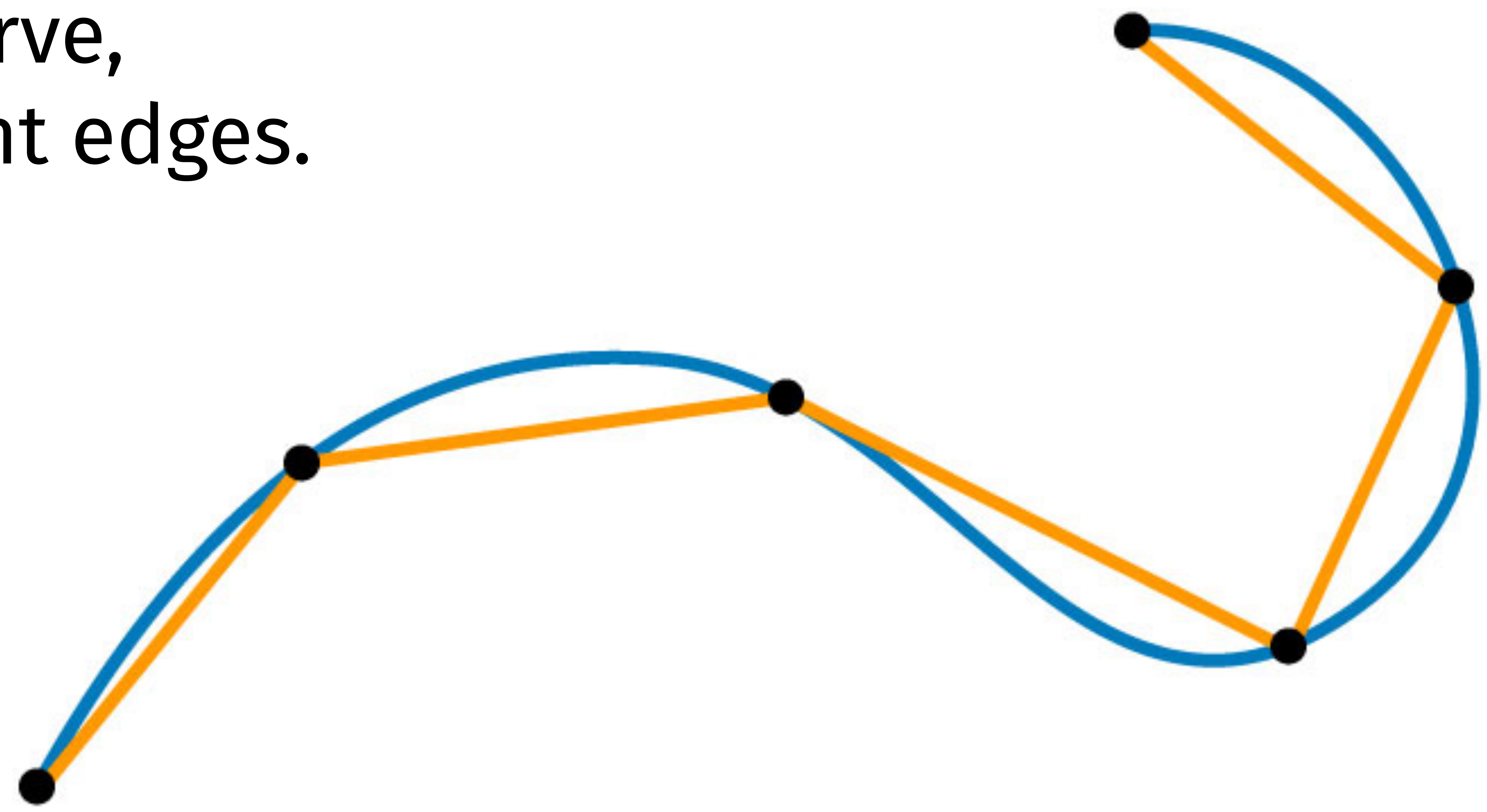
- Weight by edge lengths

$$\hat{\mathbf{n}}_v = \frac{|e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}}{\||e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}\|}$$



Inscribed Polygon, p

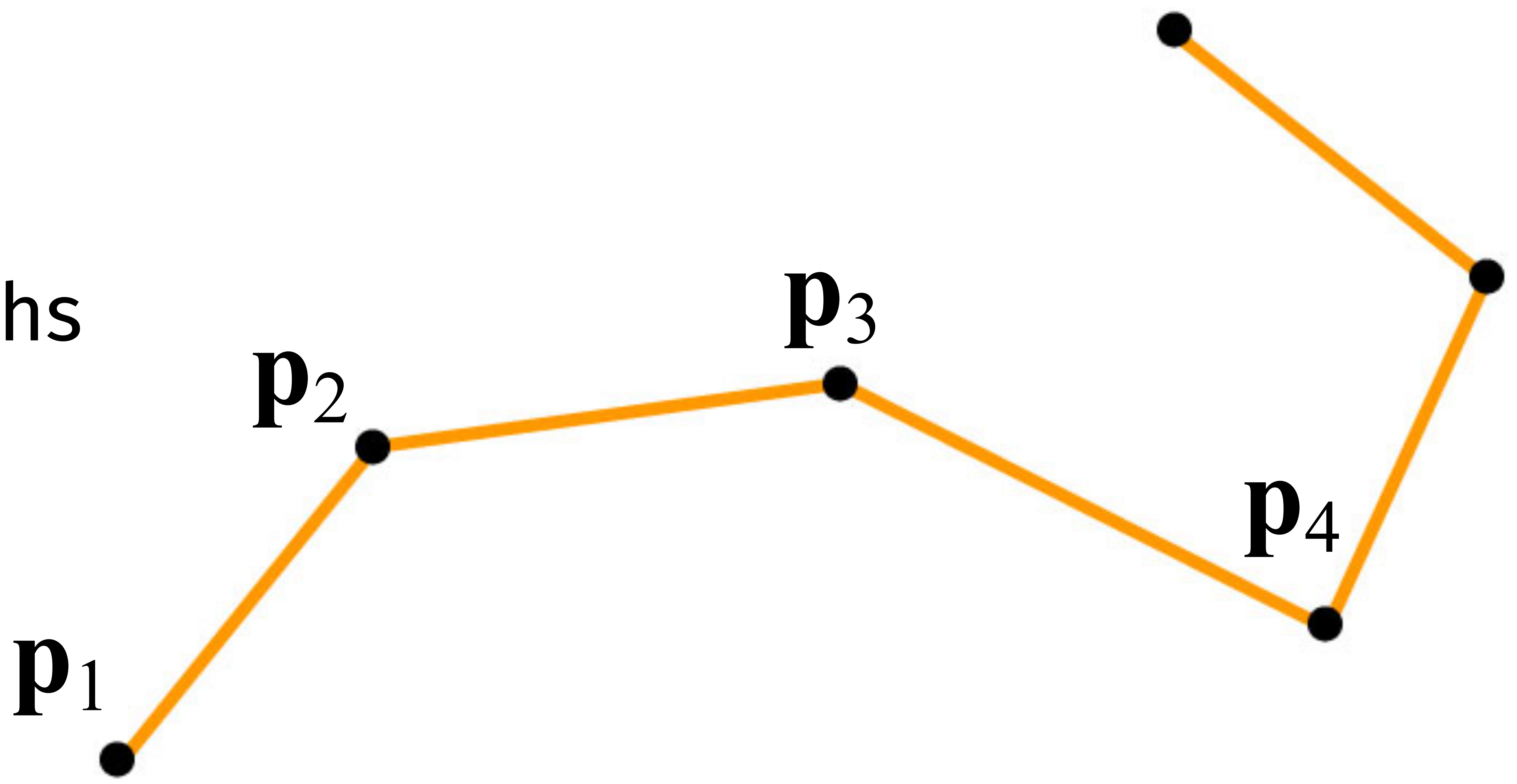
- Connection between discrete and smooth
- Finite number of vertices
  each lying on the curve,
  connected by straight edges.



The Length of a Discrete Curve

$$\text{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

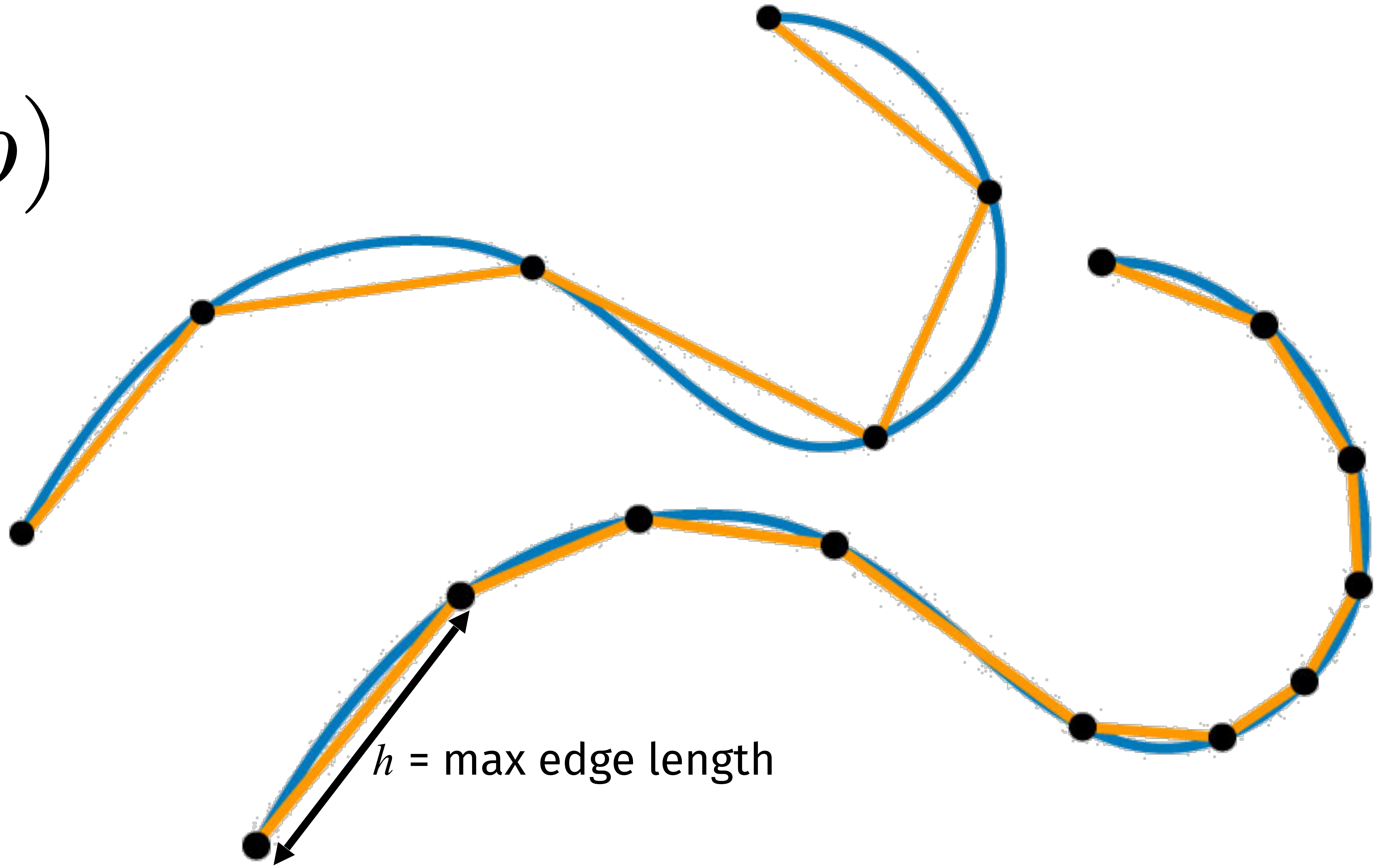
- Sum of edge lengths



The Length of a Continuous Curve

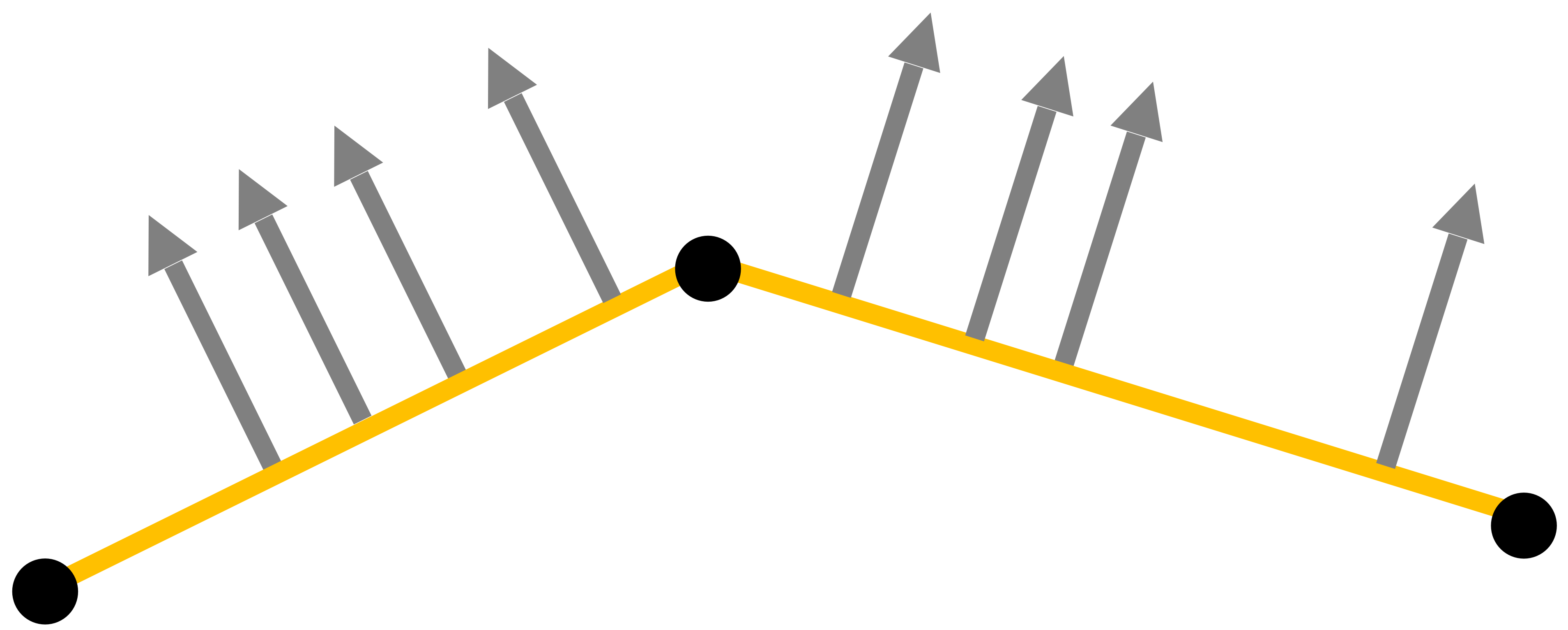
- Take limit over a refinement sequence

$$\lim_{h \to 0} \text{len}(p)$$



Curvature of a Discrete Curve

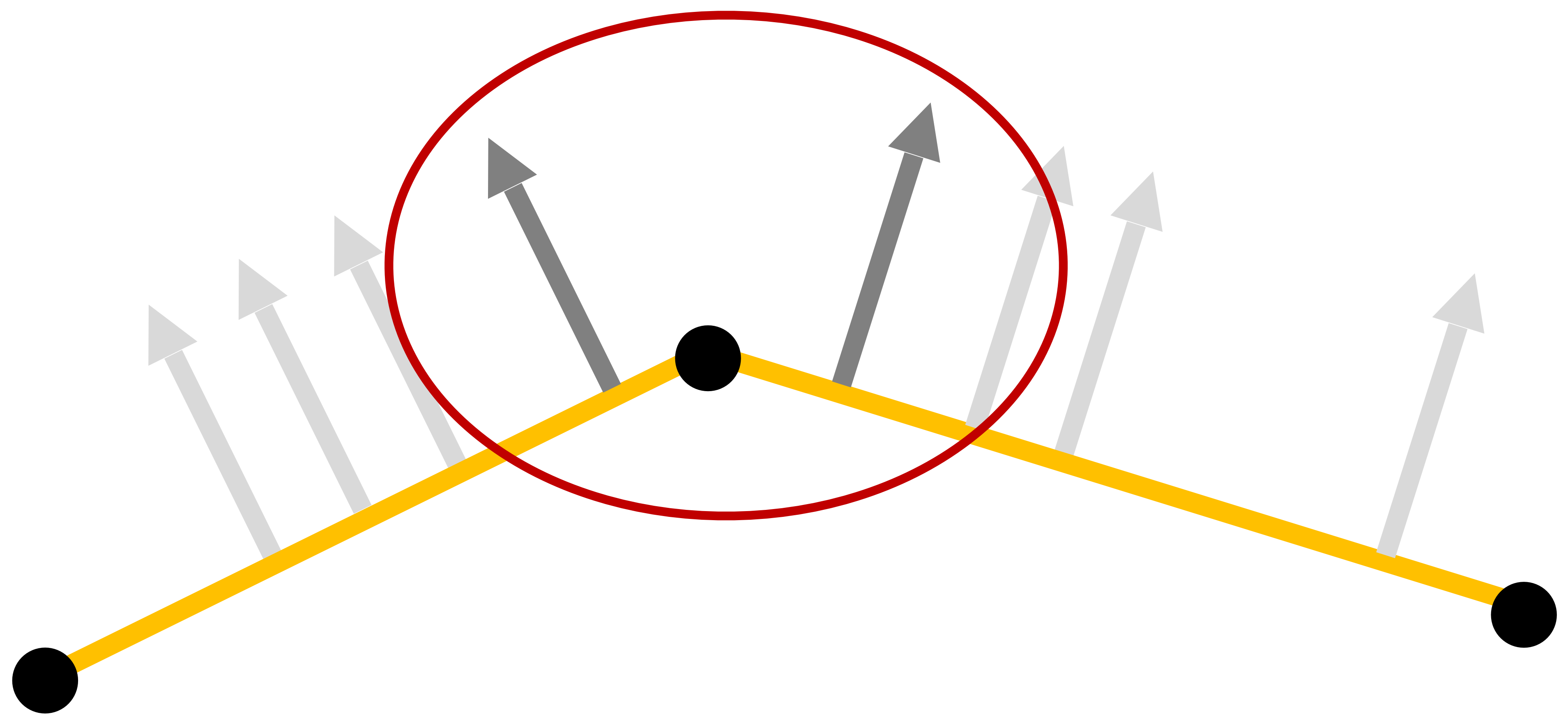
- Curvature is the change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges

Curvature of a Discrete Curve

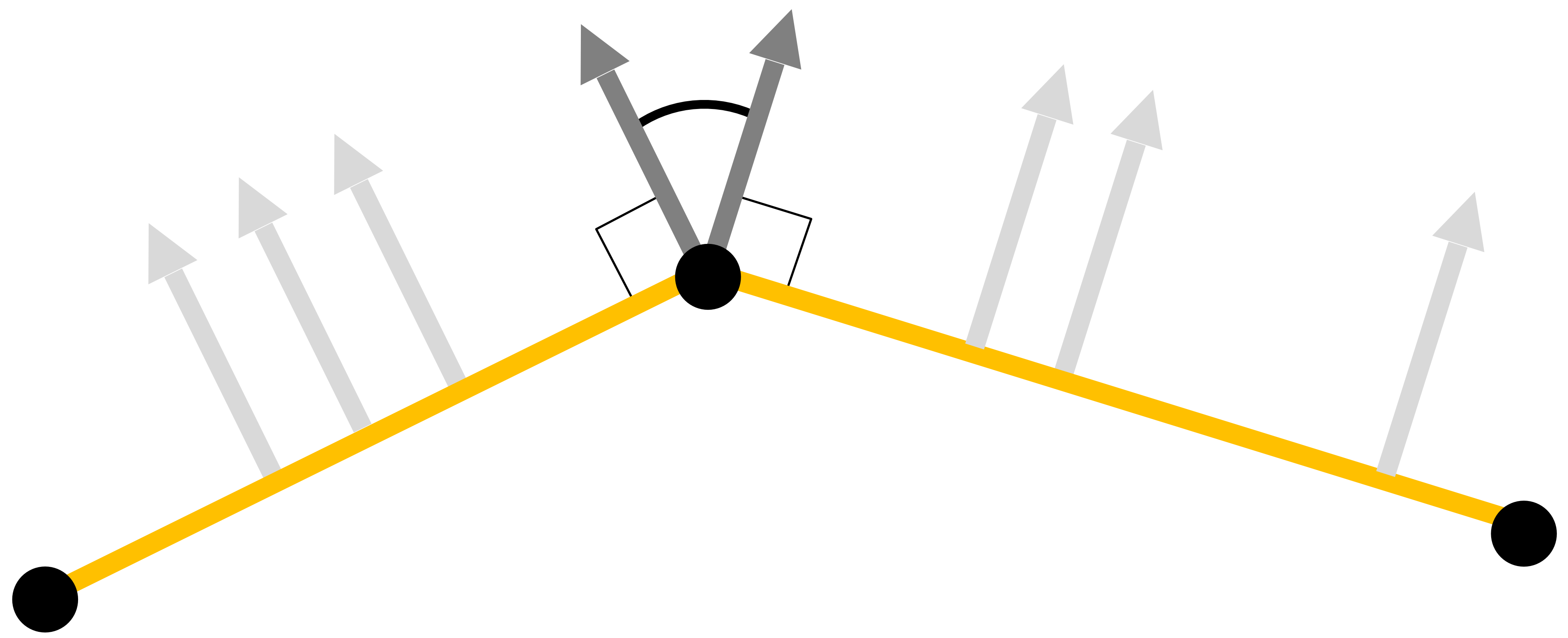
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

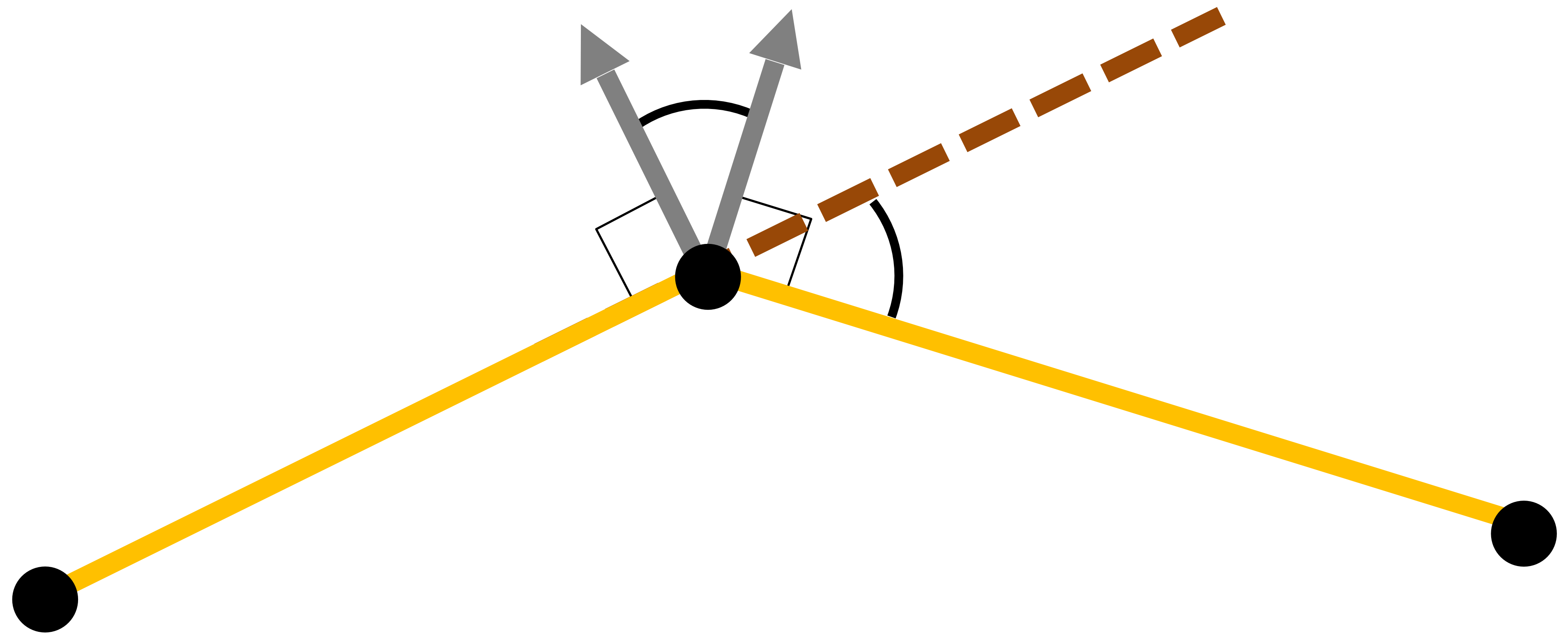
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

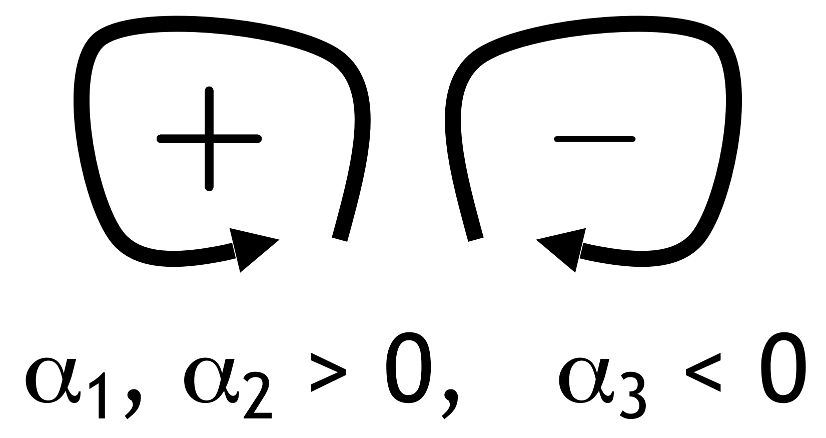
- Curvature is the change in normal direction as we travel along the curve

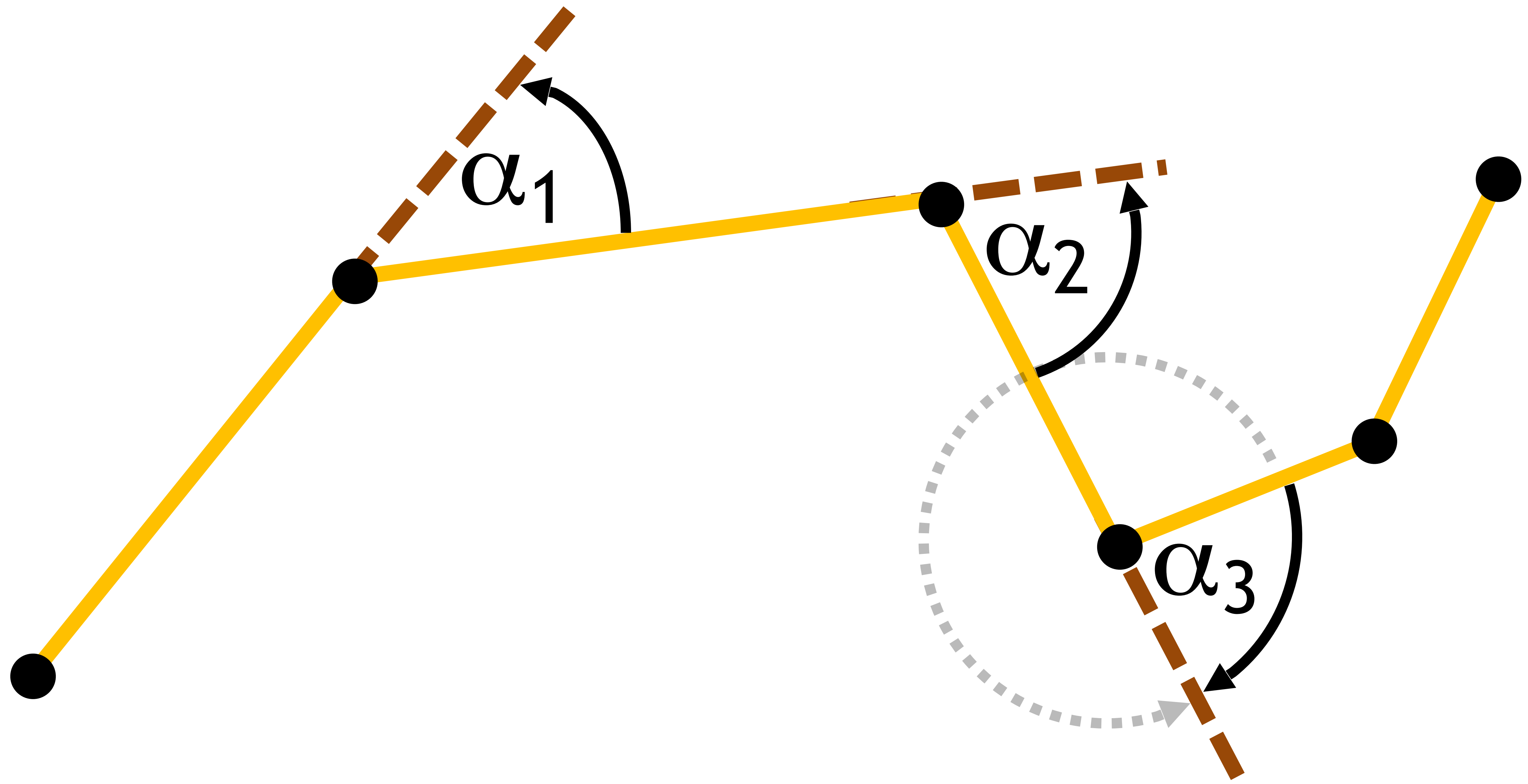


same as the turning angle
between the edges

Curvature of a Discrete Curve

- Zero along the edges
- Turning angle at the vertices
  = the change in normal direction

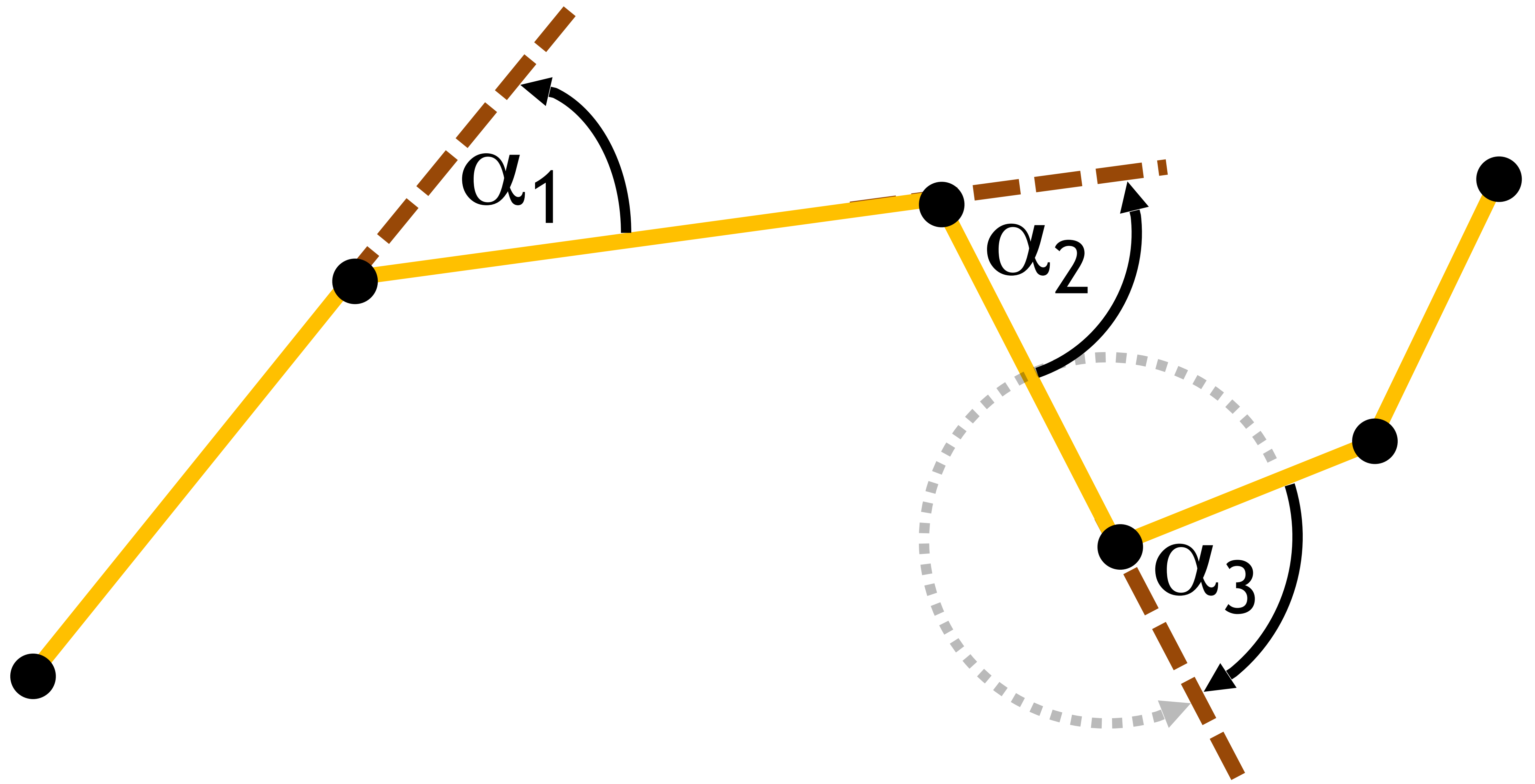




Total Signed Curvature

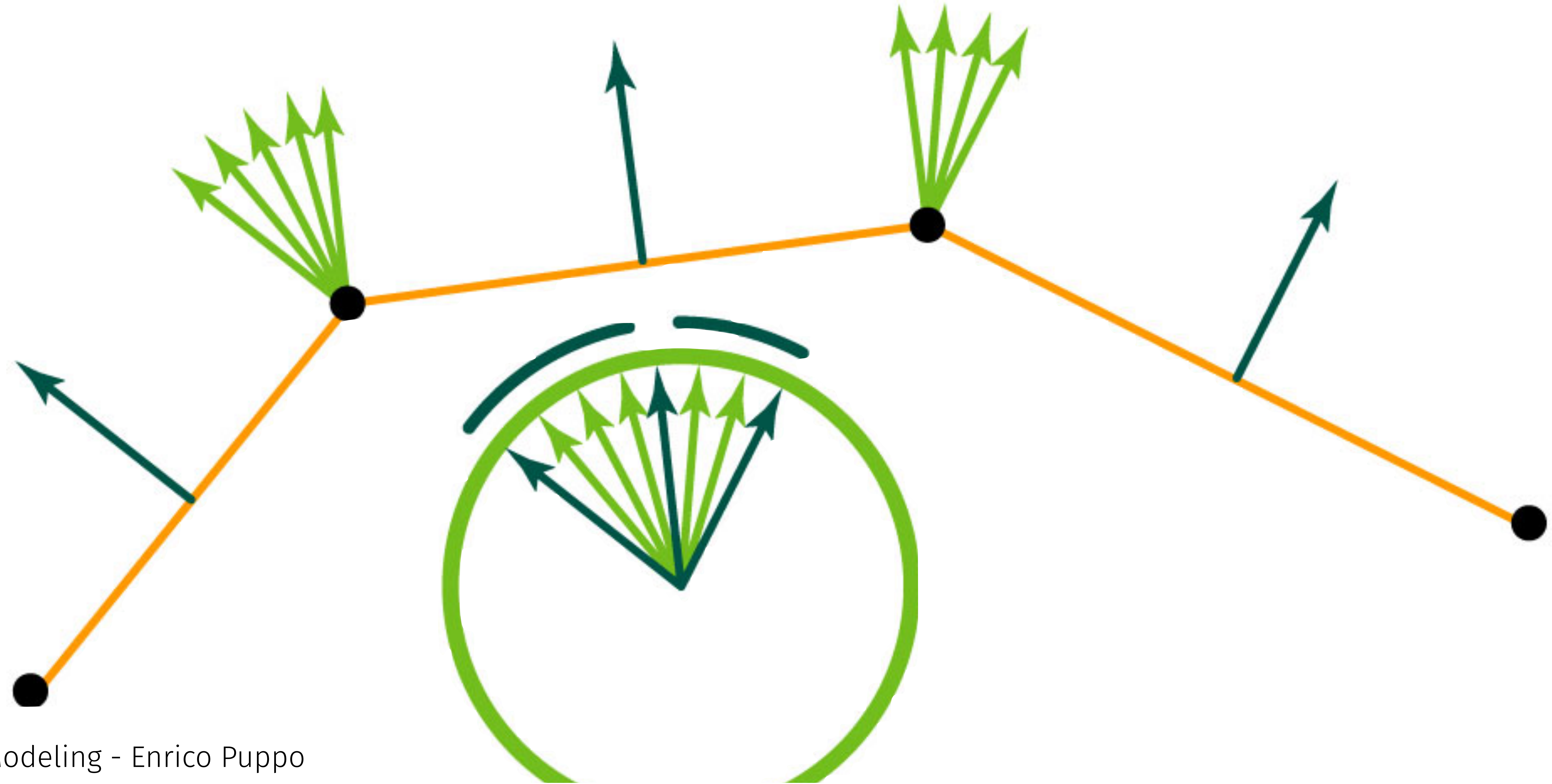
$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i$$

- Sum of turning angles



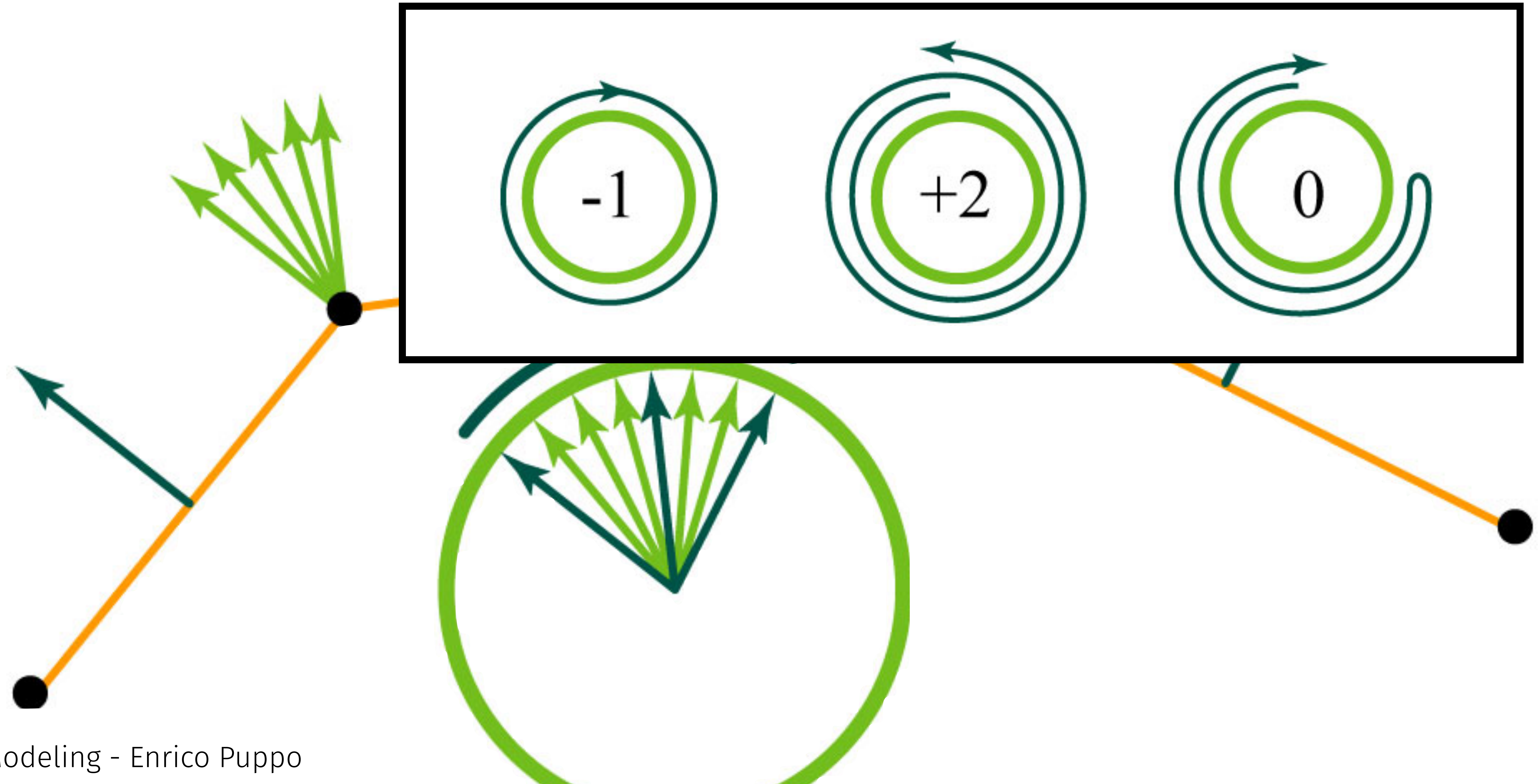
Discrete Gauss Map

- Edges map to points, vertices map to arcs



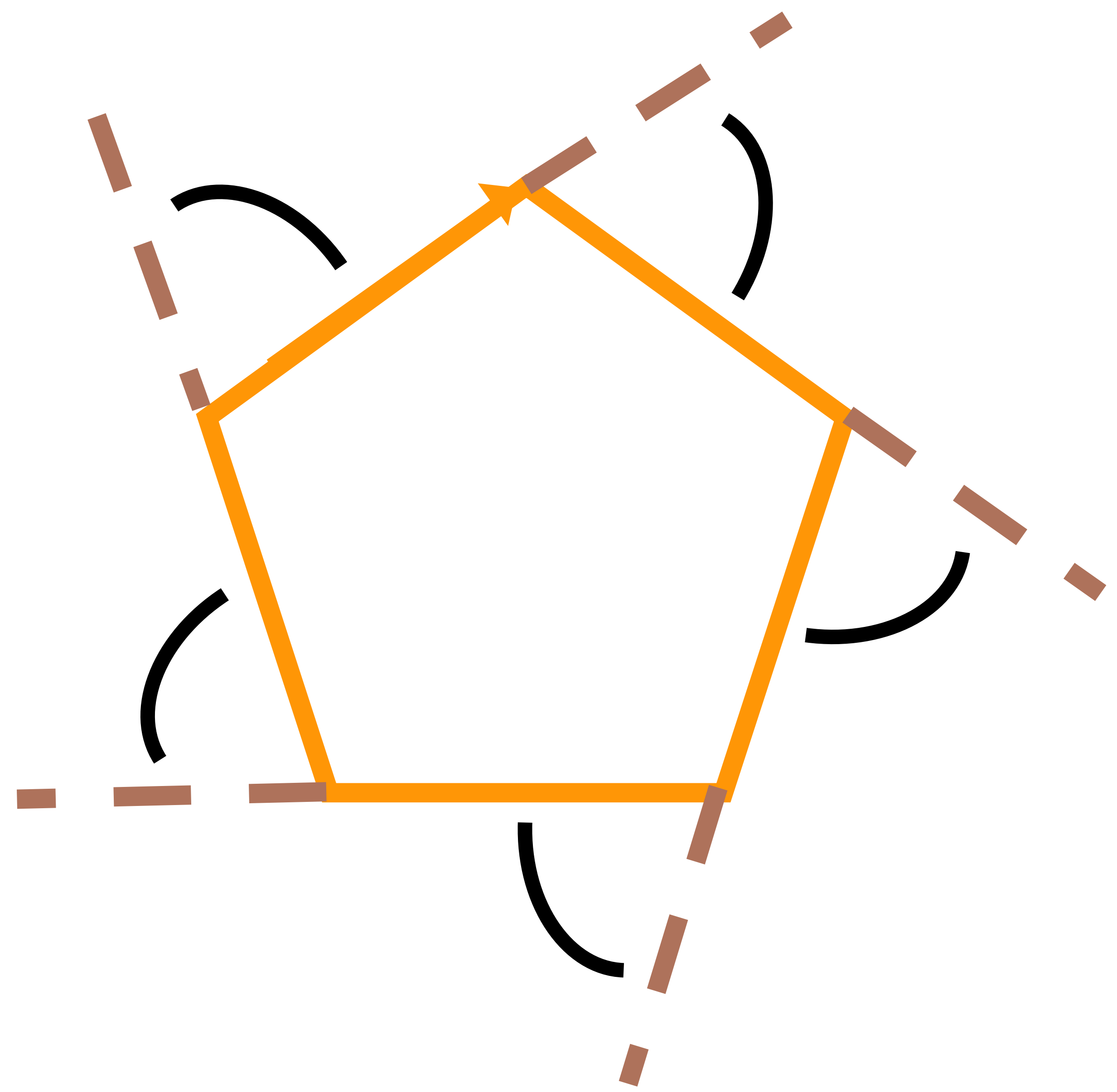
Discrete Gauss Map

- Turning number well-defined for discrete curves



Discrete Turning Number Theorem

$$\operatorname{tsc}(p) = \sum_{i=1}^{n} \alpha_i = 2\pi k$$



- For a closed curve,
  the total signed curvature is
  an integer multiple of 2π
  - proof: sum of exterior angles

Turning Number Theorem

Continuous world

$$\int_\gamma \kappa \, dt = 2\pi k$$

Discrete world

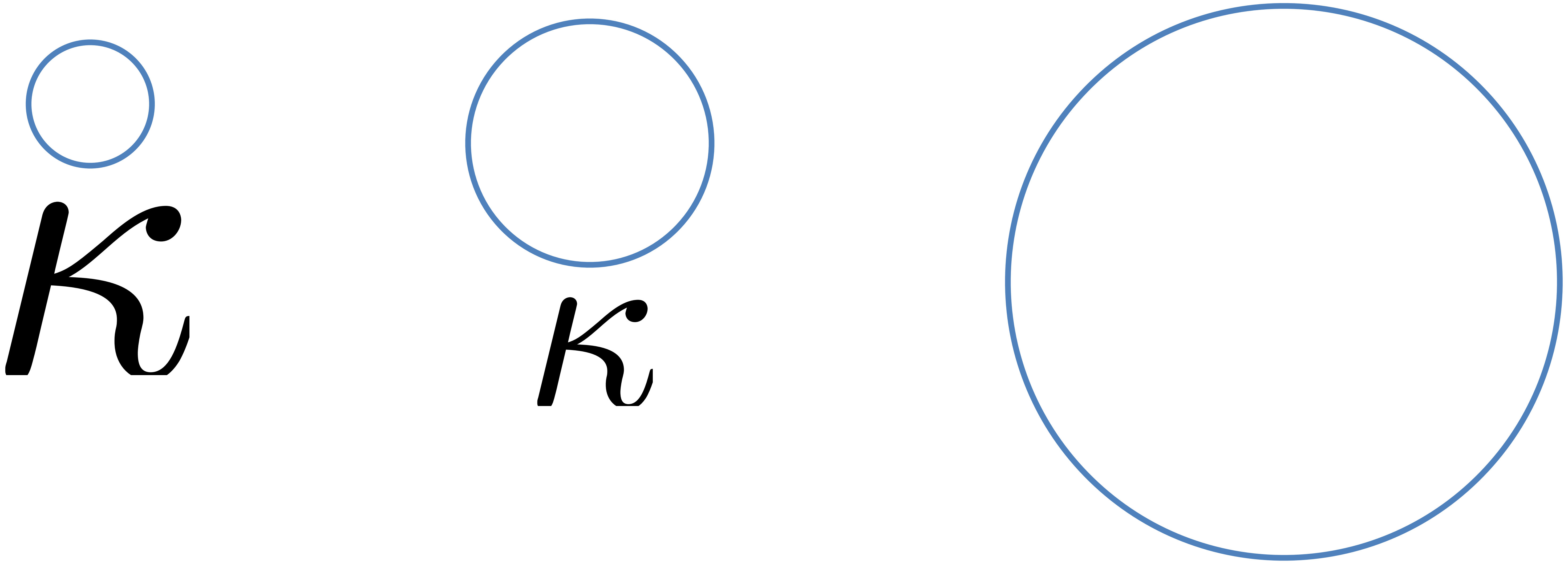
$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

k:

$$\kappa = \alpha_i \quad ??$$

Curvature is scale dependent

$$\kappa = \frac{1}{r}$$

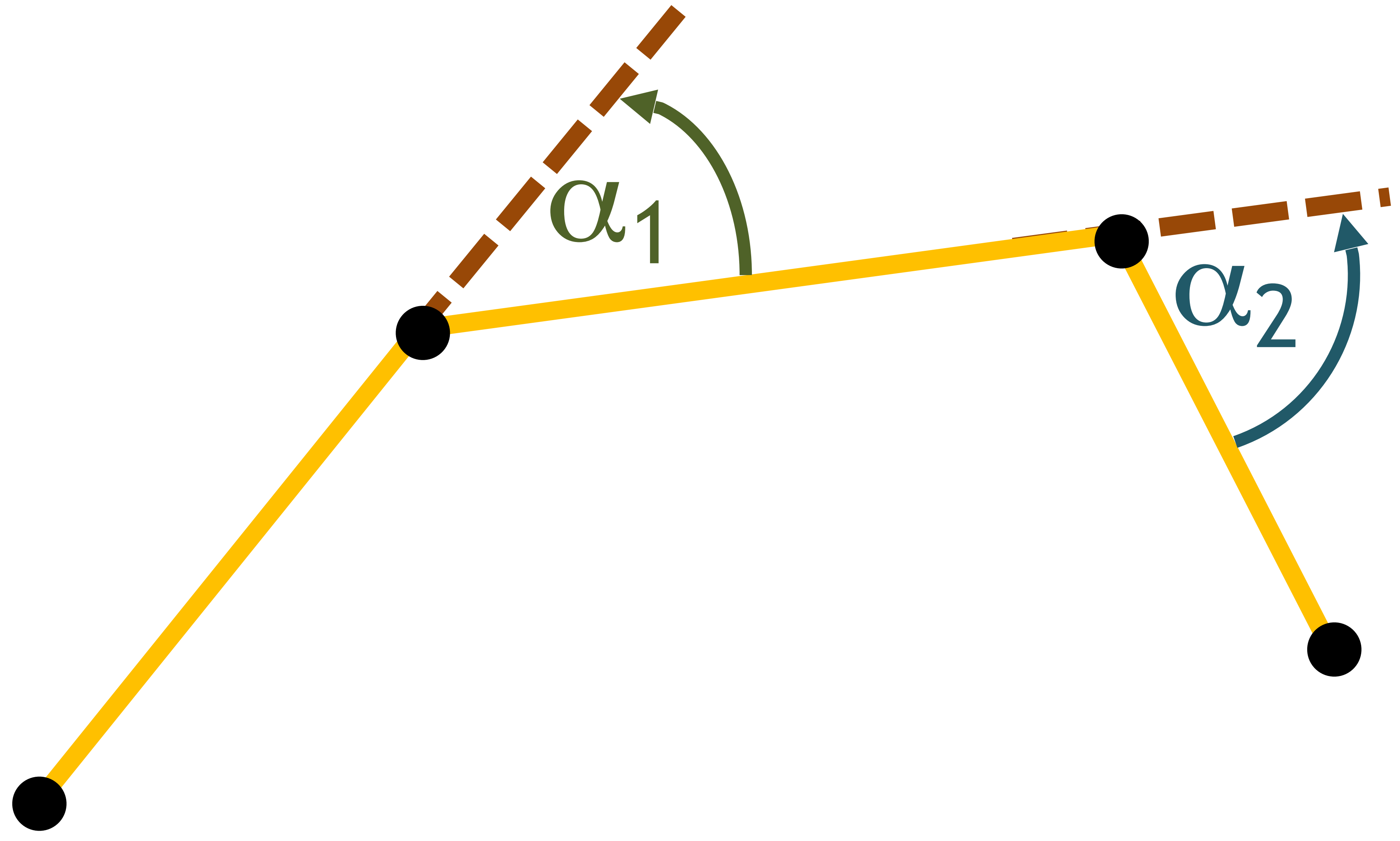


κ κ κ

α_i is scale-independent

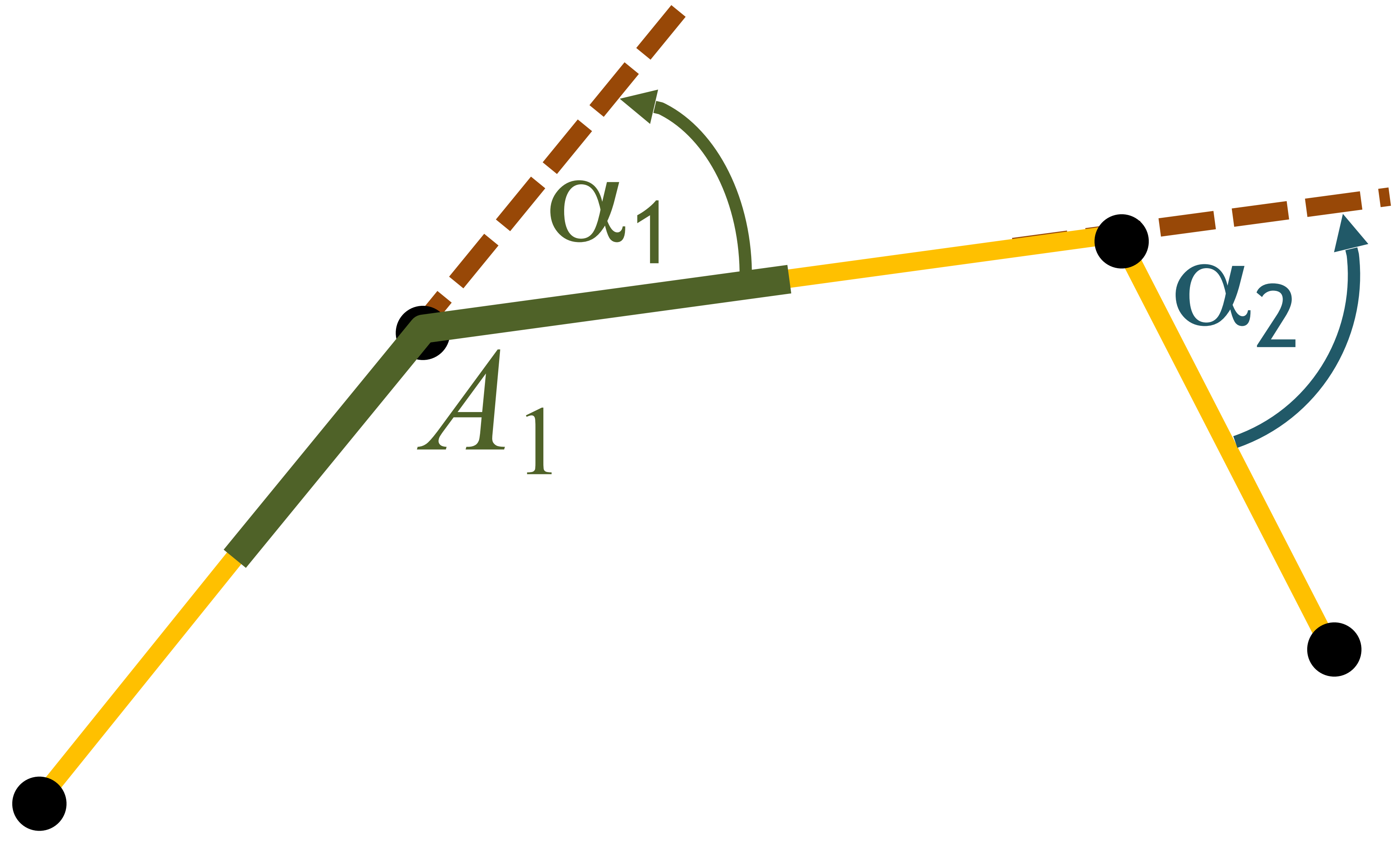
Discrete Curvature – Integrated Quantity!

- Cannot view α_i as pointwise curvature
- It is *integrated curvature* over a local area associated with vertex i



Discrete Curvature – Integrated Quantity!

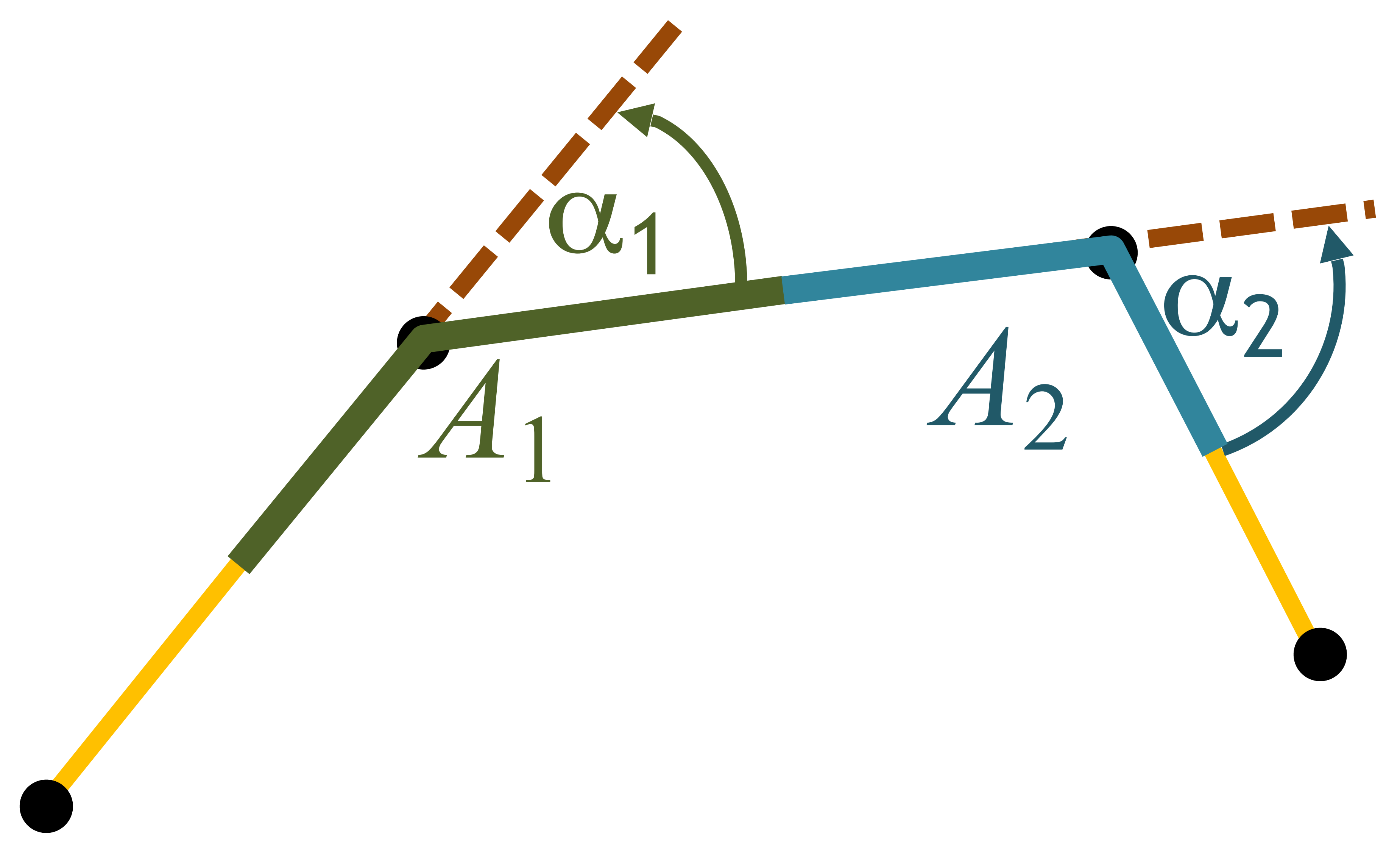
- Integrated over a local area associated with vertex i



$$\alpha_1 = A_1 \cdot \kappa_1$$

Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i



$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

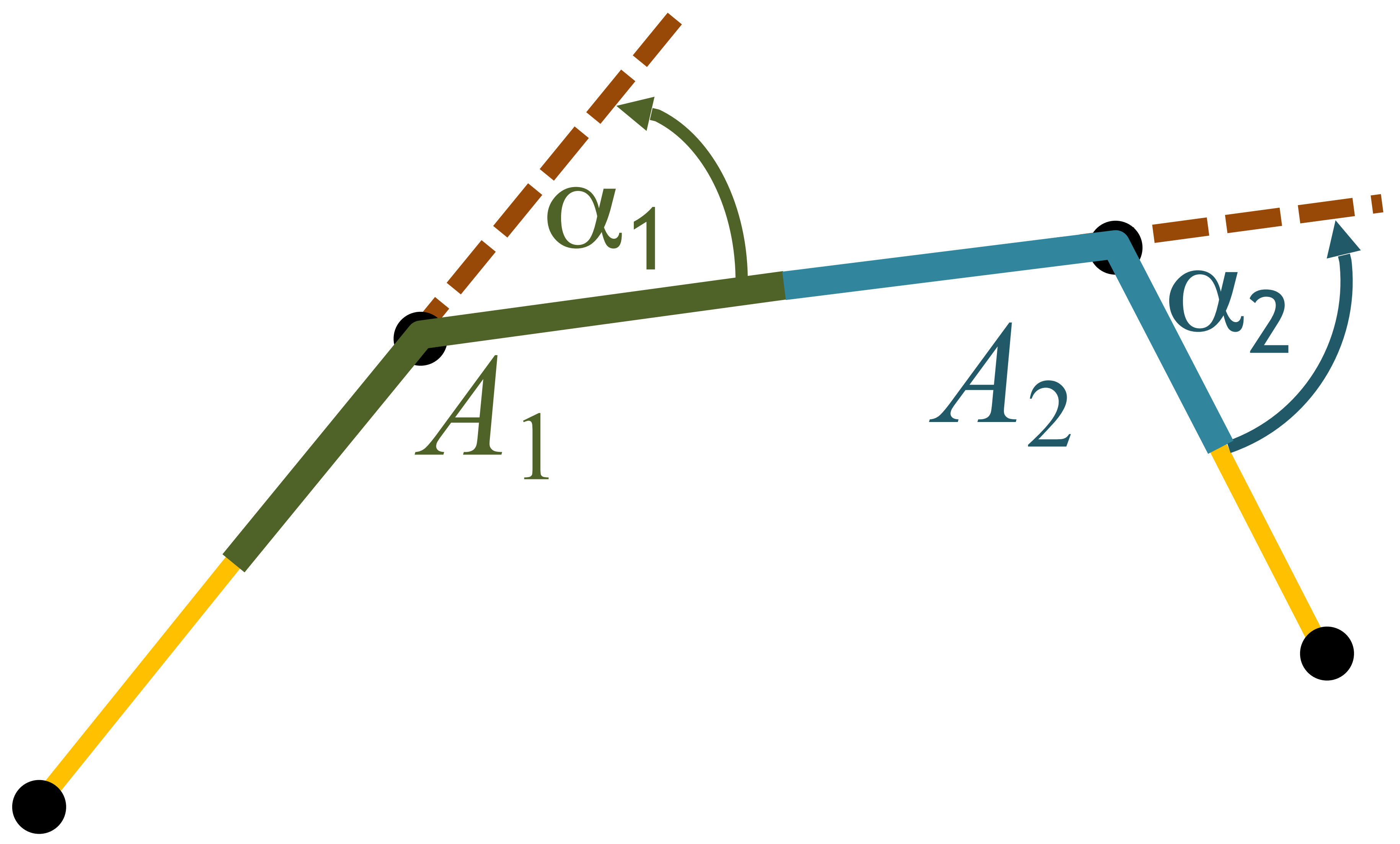
Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

$$\sum A_i = \mathrm{len}(p)$$



The vertex areas A_i form a covering of the curve.
They are pairwise disjoint (except endpoints)

Structure Preservation

- Arbitrary discrete curve
  - total signed curvature obeys discrete turning number theorem
  - $\boxed{\text{even on a coarse polyline}}$
  - which continuous theorems to preserve?
    - that depends on the application…

discrete analogue of continuous theorem

Convergence

- Consider refinement sequence
  - length of inscribed polygon approaches length of smooth curve
  - in general, a discrete measure approaches its continuous analogue
  - which refinement sequence?
    - depends on discrete operator
    - pathological sequences may exist
  - in what sense does the operator converge?
    (pointwise, L_2; linear, quadratic)

Recap

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.

Thank you