COMPETING MODELS WITH FEEDBACK

IN-KOO CHO AND KENNETH KASA

Abstract. This paper studies misspecified Bayesian learning with endogenous data when models compete via a process of model averaging. A decisionmaker forecasts an endogenous price sequence by averageing the forecasts from two models, one with constant parameters and one with time-varying parameters. The time-varying parameters model is misspecfied. Not only does it exclude a relevant explanatory variable, but it also fails to recognize the presence of model averaging. In contrast, the constant parameters model is correctly specified in the sense that it includes all relevant explanatory variables and recognizes that prices are generated by averaging. We show that when expectational feedback is sufficiently strong and the excluded funamentals are not too important, the under parameterized time-varying parameters model will survive in the limit, despite its handicaps. This is because time varying parameters allow the model to better respond to feedback in the data.
JEL Classification Numbers: C52, D80

What the government observes depends partly on what it believes. – Sargent (1999, p. 75)

1. Introduction

Model use and construction in economics confronts four key challenges: (1) Uncertainty, (2) Misspecification, (3) Feedback, and (4) Competition. In principle, dealing with model uncertainty is straightforward - specify a prior and apply Bayes Rule. Unfortunately, it's not quite that simple. In practice, parameter spaces are infinite dimensional, so priors are almost surely misspecified. Fortunately, the effects of prior misspecification are well understood. Berk (1966) shows that misspecified Bayesian learning converges to a model within the prior that minimizes the Kullback-Leibler distance to the unknown true model. The most serious challenge is *feedback*, i.e., the data are endogenous. They react to your own beliefs about the model. This is what sets economics apart from the natural sciences. As far as we know, the laws of physics do not respond to our own efforts to learn and apply them. By itself, feedback is not a problem. Dealing with feedback was what the Rational Expectations revolution was all about. However, the Rational Expectations hypothesis abstracts from model uncertainty and misspecification. In response, during the past decade researchers have begun studying misspecified learning with feedback. Esponda and Pouzo (2016) propose the concept of a Berk-Nash equilibrium. In a Berk-Nash equilibrium players use optimal strategies given their own beliefs, while at the same time each player's beliefs minimize an *endogenous* Kullback-Leibler distance to the true model. Esponda and Pouzo (2021) and Fudenberg, Lanzani, and Strack (2021) extend and refine the Berk-Nash equilibrium concept.

Date: March 3, 2025.

Our paper also studies misspecified Bayesian learning with feedback, but with an added twist. Here we assume models must *compete* via model averaging. This setting is particularly relevant for macroeconomic policy. Blinder (1998, p. 12) notes that model averaging is used at the Fed, albeit informally:

> *Parameter uncertainty, while difficult, is at least a relatively well defined problem. Selecting the right model from among a variety of non-nested alternatives is another matter entirely... My approach to this problem while on the Federal Reserve Board was relatively simple: Use a wide variety of models and don't ever trust any one of them too much....My usual procedure was to simulate a policy on as many of these models as possible, throw out the outliers, and average the rest to get a point estimate of a dynamic multiplier path.*

Despite its practical importance, there has been surprisingly little work done on how model competition interacts with misspecification and feedback.

Olea, Ortoleva, Pai, and Prat (2022) study model competition without feedback. They consider an environment in which a collection of (linear) models compete against each other on the basis of subjective mean-squared prediction error. Using a standard bias/variance decomposition of mean-squared error, they show that when data are scarce, small parsimonious models prevail. However, as data become more plentiful, the prevailing model becomes larger. Interestingly, they show that an overparameterized model has a positive probability of prevailing, even in the limit. Evans, Honkapohja, Sargent, and Williams (2013) and Cho and Kasa (2017) study model competition with feedback, but in a setting where models only differ in their assumptions about parameter stability. They consider a policymaker who must forecast a future endogenous price. He hires two competing forecasters to construct models and report forecasts. The policymaker forms his forecast as a recursively updated weighted average of the two forecasts. The true data-generating process is unknown, and relates the current price to the policymaker's forecast of next period's price and an exogenous fundamental. One forecaster thinks the environment is stationary, and so estimates (recursively) a constant parameters (CP) model. The other forecaster thinks the environment is nonstationary, and so estimates a time-varying parameters (TVP) model. Importantly, both forecasters have misspecified models of endogeneity. Since the coefficients are in fact constant, there is a sense in which the CP forecaster has the correct model. In a Rational Expectations equilibrium, the parameters *are* constant. His only misspecification is to neglect the transitional learning dynamics. If he were operating in isolation, these transitional dynamics would dissipate over time as long as standard 'E-stability' conditions are satisfied (Evans and Honkapohja (2001)). However, Cho and Kasa (2017) show that if the CP forecaster must compete in real-time against the TVP forecaster via a process of model averaging, then ultimately the TVP forecaster will prevail. As a result, the economy is more unstable than it otherwise would be. Hence, in this setting competition is *undesirable*. Competition is great when the rules are fixed and fair. But with feedback, the TVP model can effectively alter the rules of the game in its own favor. This is because the TVP model is better able to adapt to feedback induced time-variation in the data.

This paper revisits the analysis in Cho and Kasa (2017) by giving the CP model two extra advantages. First, we suppose the TVP model is (dogmatically) underparameterized. It excludes the exogenous fundamental, whereas the CP model not only includes it, but also knows its true coefficient value. We show that as long as the variance of the omitted fundamental is not too large, the TVP model continues to dominate in the long-run, despite being underparameterized. Its ability to react to feedback more than offsets the fact that it is underparameterized.

One could argue that despite including all relevant explanatory variables, the CP model is at a serious disadvantage, since it fails to recognize that its forecast must compete with the forecast from the TVP model. Even if the CP forecaster is convinced that he is right, and the environment is stationary, he should realize that the policymaker's averaging strategy will indirectly induce nonstationarity. Thus, our second extension is to allow the CP forecaster to be aware of the competition. He effectively 'imports' some nonstationarity into his forecast by averaging his own forecast with that from the TVP model. In contrast, the TVP forecaster remains unaware of the competition. Surprisingly, we show that even now the TVP continues to survive. Interestingly, however, since the CP forecaster is increasingly 'copying' the TVP model as the weight on the TVP model increases, it is no longer the case that the TVP model dominates, in the sense that its weight converges to one. Instead, as long as feedback is strong enough, we obtain a strictly interior long-run equilibrium, where the superior adaptability of the TVP model exactly balances its underparameterization. This perhaps explains why parsimonious time-varying parameter models are so widely used in practice.[1]

The joint dynamics of the data, model coefficients, and model weights is a high dimensional, nonlinear, dynamic stochastic process. The key to making the analysis tractable is to exploit the fact that different processes evolve on different time-scales. This effectively reduces the dimensionality of the problem, by allowing us to study the interactions among smaller dimensional subsystems. This is an old idea, going all the way back to early 19th century celestial mechanics. Marcet and Sargent (1989) were the first to import time-scale separation methods into the economics literature. Our problem is more complex than the one studied by Marcet and Sargent, which just exploited a time-scale separation between the data and the coefficients of a single model. Here there are multiple models that evolve on different time-scales, as well as model weights that evolve on yet a different time-scale. Nonetheless, by taking limits in the appropriate order, we can study the dynamics of a *sequence* of Ordinary Differential Equations (Borkar (2008)).

The remainder ot the paper is organized as follows. Section 2 presents the data-generating process and the two competing models, and discusses how each model is updated using the Kalman filter. Section 3 discusses model averaging, and presents our main results concerning the asymptotics of the model weights. We consider two cases. One where the CP forecaster is aware of the policymaker's averaging, and one where he is not. In the first case, we show that if feedback is strong enough and the excluded

[1]Our result is related to a classic paper by Nelson (1972), who showed that simple univariate ARIMA models produced superior forecasts to the large FRB-MIT-Penn model. It is also related to the literature on Bayesian VARs, where it was discovered that the 'Minnesota prior' is an effective way of containing estimation variance. (The Minnesota prior postulates a nondogmatic diagonal VAR coefficient matrix with random walk parameter drift. See, e.g., Doan, Litterman, and Sims (1984)).

fundamental is not too important, then convergence to a unique, stable, interior steady state occurs, where both models continue to be used. In the second case, the TVP model dominates. Section 4 presents simulations that illustrate our theoretical results. This is useful since our results are asymptotic, and predict what happens as shock variances tend to zero in the appropriate order. The simulations show that convergence occurs continuously. Finally, Section 5 offers concluding remarks, and a technical appendix contains the proofs of our key convergence results.

2. Description

2.1. Rational Expectations. Consider the following workhorse asset pricing model, in which an asset price at time t, p_t, is determined according to

$$p_t = \gamma + \alpha \mathsf{E}_t p_{t+1} + \delta f_t + \sigma \epsilon_t \tag{2.1}$$

where $\alpha \in (0,1)$ is a (constant) discount rate, which determines the strength of expectational feedback. Empirically, it is close to one. The ϵ_t shock is standard Gaussian white noise. Fundamentals are assumed to evolve according to the AR(1) process

$$f_t = \rho f_{t-1} + \sigma_f \epsilon_{f,t} \tag{2.2}$$

for $\rho \in (0,1)$. The fundamentals shock, $\epsilon_{f,t}$, is standard Gaussian white noise, and is orthogonal to the price shock ϵ_t. The unique stationary rational expectations equilibrium is

$$p_t = \frac{\gamma}{1-\alpha} + \frac{\delta}{1-\alpha\rho} f_t + \sigma \epsilon_t. \tag{2.3}$$

2.2. Learning with a correct model. Suppose an agent knows the fundamentals process in (2.2), but does not know the structural price equation in (2.1). Instead, the agent postulates the following state-space model for prices

$$p_t(0) = \beta_t(0) + \frac{\delta}{1-\alpha\rho} f_t + \sigma \epsilon_t \tag{2.4}$$

$$\beta_t = \beta \tag{2.5}$$

for some β. To simplify the ensuing analysis, we assume the agent knows the coefficient on f_t, and must only estimate the constant term β. Requiring the agent to estimate the coefficient on f_t as well would strengthen our result, since we shall see that ultimately this model is dominated by the TVP model. Note that without model competition the Rational Expectations equilibrium is a special case of this agent's perceived model, with

$$\beta = \frac{\gamma}{1-\alpha}.$$

For now, suppose the agent adopts the dogmatic prior that parameters are constant.

$$\mathcal{M}_0 : \quad \beta_t = \beta \qquad \forall t \geq 1.$$

Let $\beta_t(0)$ be the conditional mean and $\Sigma_t(0)$ be the conditional variance of the posterior belief about the unknown β. Given this belief that the true model is $\mathcal{M}_0$, $(\beta_t(0), \Sigma_t(0))$

evolves according to Kalman filter algorithm:

$$\beta_{t+1}(0) = \beta_t(0) + \left(\frac{\Sigma_t(0)}{\sigma^2 + \Sigma_t(0)}\right)\left(p_t - \beta_t(0) - \frac{\delta}{1-\alpha\rho} f_t\right) \tag{2.6}$$

$$\Sigma_{t+1}(0) = \Sigma_t(0) - \frac{(\Sigma_t(0))^2}{\sigma^2 + \Sigma_t(0)}. \tag{2.7}$$

Following standard analysis (Evans and Honkapohja (2001)), we can prove the convergence and the stability properties of the learning dynamics under specification $\mathcal{M}_0$.

Proposition 2.1. *Given $\alpha \in (0,1)$ and the belief that parameters are constant, $\Sigma_t(0)$ converges to zero at rate t^{-1}, and*

$$\beta_t(0) \to \frac{\gamma}{1-\alpha}$$

with probability 1.

2.3. **Learning with a simpler but misspecified model.** Now suppose there is a second agent with different prior beliefs, described by the state space model:

$$p_t(1) = \beta_t(1) + \sigma\epsilon_t \tag{2.8}$$

$$\beta_t(1) = \beta_{t-1}(1) + \sigma_v v_t \tag{2.9}$$

where v_t is standard Gaussian white noise, which is orthogonal to all other variables. The innovation variance σ_v^2 reflects the agent's priors about parameter drift. The larger it is, the more aggressively will the agent revise his beliefs in response to forecast errors. We treat it as a fixed parameter. Cho and Kasa (2017, Section V) show that our results extend to the case where σ_v^2 is estimated.

A potentially serious specification error here is that the agent's model excludes the fundamental f_t. To capture variation from the missing variable, the agent entertains a time varying parameters (TVP) model.

$$\mathcal{M}_1: \quad \sigma_v^2 > 0.$$

The missing varaible f_t produces a gap between the perceived law of motion and the actual law of motion of p_t. Let $\beta_t(1)$ and $\Sigma_t(1)$ be the mean and the variance of the posterior distribution about $\beta_t(1)$ conditioned on information at $t-1$, which is computed from a Gaussian prior. The evolution of $\beta_t(1)$ and $\Sigma_t(1)$ is dictated by the new Kalman filter:

$$\beta_{t+1}(1) = \beta_t(1) + \left(\frac{\Sigma_t(1)}{\sigma^2 + \Sigma_t(1)}\right)(p_t - \beta_t(1)) \tag{2.10}$$

$$\Sigma_{t+1}(1) = \Sigma_t(1) - \frac{(\Sigma_t(1))^2}{\sigma^2 + \Sigma_t(1)} + \sigma_v^2 \tag{2.11}$$

while the actual price p_t is driven by (2.1)

$$p_t = \gamma + \alpha\beta_t(1) + \delta f_t + \sigma\epsilon_t.$$

Since the Kalman gain no longer vanishes, we cannot obtain convergence with probability 1. Instead, we obtain convergence in distribution.

Proposition 2.2. *As $\sigma_v^2 \to 0$, $\beta_t(1)$ converges weakly to the solution of the following diffusion process*

$$d\beta = (1-\alpha)\left[\frac{\gamma}{1-\alpha} - \beta\right] d\tau + g\sqrt{\sigma^2 + \sigma_f^2/(1-\rho^2)}dW_\tau \tag{2.12}$$

where dW_τ is a standard Wiener process and $g = \sigma_v/\sigma$ is the steady state 'gain', which vanishes as $\sigma_v^2 \to 0$.

Proof. See Kushner and Yin (1997). □

Notice that (2.12) differs from the perceived law in (2.9) in two ways. First, it is not a random walk, it exhibits mean reversion. This inconsistency reflects the agent's neglect of the effects induced by his own learning. However, it will be quite small and difficult to detect when α is close to 1 and the innovation variance is small. Second, the prior volatility of $\beta_t(1)$ in (2.9) will be less than the steady state posterior volatility in (2.12) by a factor of $1 + \sigma_f^2/[\sigma^2(1-\rho^2)]$. This reflects the fact that the agent uses parameter drift to capture the dynamics of the omitted variable f_t. This inconsistency can be eliminated by allowing the agent to estimate σ_v.

3. Model averaging

Model competition can take many forms. Olea, Ortoleva, Pai, and Prat (2022) motivate their analysis in an auction setting. When studying competition with feedback, however, the most natural setting is one of model averaging. A decisionmaker mixes the predictions from two models because he wants to hedge his bets. Bayes rule provides an algorithm for updating the weights in response to new data.

Following Evans, Honkapohja, Sargent, and Williams (2013) and Cho and Kasa (2017) we consider a decentralized environment in which there is a separation between model construction and model use. This is particularly descriptive of macroeconomic policy. Letting π_t denote the current probability assigned by the policymaker to $\mathcal{M}_1$ (the TVP model), the policymaker's time-t forecast is becomes

$$\mathsf{E}_t p_{t+1} = \hat{p}_t = \pi_t p_t(1) + (1-\pi_t)p_t(0)$$

where $p_t(i)$ is the time-t forecast reported from $\mathcal{M}_i$. Since the TVP forecaster dogmatically omits the fundamental from his model, the forecast from $\mathcal{M}_1$ is just

$$p_t(1) = \beta_t(1)$$

where $\beta_t(1)$ is given by (2.10)-(2.11). For $p_t(0)$ we consider two cases, depending on whether the CP forecaster is aware of the policymaker's averaging. We can nest these two cases as follows

$$p_t(0) = (1-\hat{\pi}_t)\left[\beta_t(0) + \frac{\delta}{1-\alpha\rho} f_t\right] + \hat{\pi}_t\beta_t(1)$$

where $\hat{\pi}_t$ is the CP forecaster's current estimate of the policymaker's weight. It is given by the recursive least squares estimate

$$\hat{\pi}_t = \hat{\pi}_{t-1} + \frac{1}{t}\left(\pi_t - \hat{\pi}_{t-1}\right)$$

Note that we do not assume the CP forecaster knows the policymaker's current weight. Instead, he thinks it's an unknown constant, which he must estimate himself. This is in keeping with our assumption that model construction and model use is decentralized, and takes place among different agents. His assumption that the weight is constant will be correct in the limit, but neglects transition dynamics. The case where the CP forecaster is unaware of averaging can be studied by simply setting $\hat{\pi}_t = 0 \ \forall t$.

At this point it is useful to collect together the formulas that govern the evolution of the six endogenous stochastic processes: $(\pi_t, \hat{\pi}_t, \beta_t(0), \Sigma_t(0), \beta_t(1), \Sigma_t(1))$.

$$\beta_{t+1}(0) = \beta_t(0) + \left(\frac{\Sigma_t(0)}{\sigma^2 + \Sigma_t(0)}\right)(p_t - p_t(0)) \tag{3.13}$$

$$p_t(0) = (1-\hat{\pi}_t)\left[\beta_t(0) + \frac{\delta}{1-\alpha\rho} f_t\right] + \hat{\pi}_t \beta_t(1) \tag{3.14}$$

$$\hat{\pi}_t = \hat{\pi}_{t-1} + \frac{1}{t}(\pi_t - \hat{\pi}_{t-1}) \tag{3.15}$$

$$\Sigma_{t+1}(0) = \Sigma_t(0) - \frac{(\Sigma_t(0))^2}{\sigma^2 + \Sigma_t(0)} \tag{3.16}$$

$$\beta_{t+1}(1) = \beta_t(1) + \left(\frac{\Sigma_t(1)}{\sigma^2 + \Sigma_t(1)}\right)(p_t - p_t(1)) \tag{3.17}$$

$$p_t(1) = \beta_t(1) \tag{3.18}$$

$$\Sigma_{t+1}(1) = \Sigma_t(1) - \frac{(\Sigma_t(1))^2}{\sigma^2 + \Sigma_t(1)} + \sigma_v^2 \tag{3.19}$$

$$\hat{p}_t = (1-\pi_t)p_t(0) + \pi_t p_t(1) \tag{3.20}$$

$$\begin{aligned} p_t &= \gamma + \alpha \mathsf{E}_t \hat{p}_{t+1} + \delta f_t + \sigma \epsilon_t \\ &= \gamma + \alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)\beta_t(1) + \alpha(1-\pi_t)(1-\hat{\pi}_t)\beta_t(0) \\ &\quad + \left(\alpha(1-\pi_t)(1-\hat{\pi}_t)\frac{\delta\rho}{1-\alpha\rho} + \delta\right) f_t + \epsilon_t. \end{aligned} \tag{3.21}$$

$$\frac{1}{\pi_{t+1}} - 1 = \frac{A_{t+1}(0)}{A_{t+1}(1)}\left(\frac{1}{\pi_t} - 1\right) \tag{3.22}$$

where $A_t(i)$ is the policymaker's perceived likelihood function of $\mathcal{M}_i$

$$A_t(i) = \frac{1}{\sqrt{2\pi(\sigma^2 + \Sigma_t(i))}} \exp\left[-\frac{(p_t - p_t(i))^2}{2\pi(\sigma^2 + \Sigma_t(i))}\right] \tag{3.23}$$

Note that in applying Bayes rule, (3.22), the policymaker is unaware that the CP modeler is responding to his own averaging efforts.

Equations (3.13)-(3.23) represent a high dimensional, nonlinear, dynamic stochastic system. One might think that analytic results are out of reach. Fortunately, we can exploit a couple of key features to dramatically simplify the analysis. First, since the models themselves are linear and Gaussian, the forecaster's confidence, as given by the conditional variances, $\Sigma_t(i)$, evolve *deterministically* and *exogenously*. This would not be the case in general. Second, since $\Sigma_t(0)$ converges to 0 at rate t^{-1} whereas $\Sigma_t(1)$ converges to a strictly positive limit, the two model estimates, $(\beta_t(0), \beta_t(1))$, evolve on different time-scales. $\beta_t(1)$ evolves 'faster' than $\beta_t(0)$. This opens the door to a classical time-scale separation strategy, in which we can fix $\beta_t(0)$ at a constant when studying the dynamics of $\beta_t(1)$. This simplifies the analysis to studying the interactions among smaller dimensional subsystems.

The mathematical details are contained in the Appendix. Here we merely summarize the results. There are two main results. The first applies to the case when the CP forecaster ignores model averaging, $\hat{\pi}_t = 0 \ \forall t$. One might think that since his model includes f_t while the TVP model mistakenly excludes it, the CP model would outperform the TVP model, and $\pi_t \to 0$. Although this would certainly be the case if the variance of f_t is large, this is not a particularly interesting case, since such obvious misspecification would likely be easily detected. We are more interested in smaller, more subtle, misspecifications. Interesingly, our first result shows that when f_t is not too important, the TVP model will dominate, despite its under-parameterization.

Theorem 3.1. *Suppose that $\alpha\rho > \frac{1}{2}$. $\forall \epsilon > 0$, define*

$$T^{\epsilon}(T, \beta(0), \pi) = \#\left\{t \le T \mid (\beta_t(0), \pi_t) \in \mathcal{N}_{\epsilon}(\beta(0), \pi)\right\}$$

as the number of periods $\beta_t(0)$ and π_t stay in an ϵ-neighborhood of $(\beta(0), \pi)$. Then

$$\lim_{\sigma_f \to 0} \lim_{\sigma_v \to 0} \lim_{T \to 0} \mathsf{E} \frac{T^{\epsilon}(T, \frac{\gamma}{1-\alpha}, 1)}{T} = 1.$$

Note that the order of limits here is crucial. By letting $T \to \infty$ first, we are charactering how the long-run stationary distribution of π_t behaves as the parameters change. Theorem 3.1 says that as long as the excluded fundamental is not too important, the TVP model ultimately prevails. By driving the variances to 0, we obtain a law of large numbers sort of characterization in which mean occupancy time converges to 1. Section 4 shows what happens for strictly positive (σ_f, σ_v), so that each model continues to be used with positive probability.

Our second main result addresses the concern that $\mathcal{M}_0$ is being driven out because its assumption that the data are stationary is seriously misspecified when it must compete with a TVP model. So we now consider the case when $\hat{\pi}_t$ obeys (3.15). Interestingly, we now find that $\mathcal{M}_1$ no longer dominates. This is because as $\pi_t \to 1$, $\mathcal{M}_0$ is increasingly mimicking the forecast from $\mathcal{M}_1$. Instead, we now find that there exists an interior, stable, long-run equilibrium in which the adaptability advantage of $\mathcal{M}_1$ exactly matches the better fit of $\mathcal{M}_0$. However, for this to occur, feedback must be stronger than before.

Theorem 3.2. *Suppose that $\alpha\rho > \frac{2+\sqrt{3}}{4}$. Then*

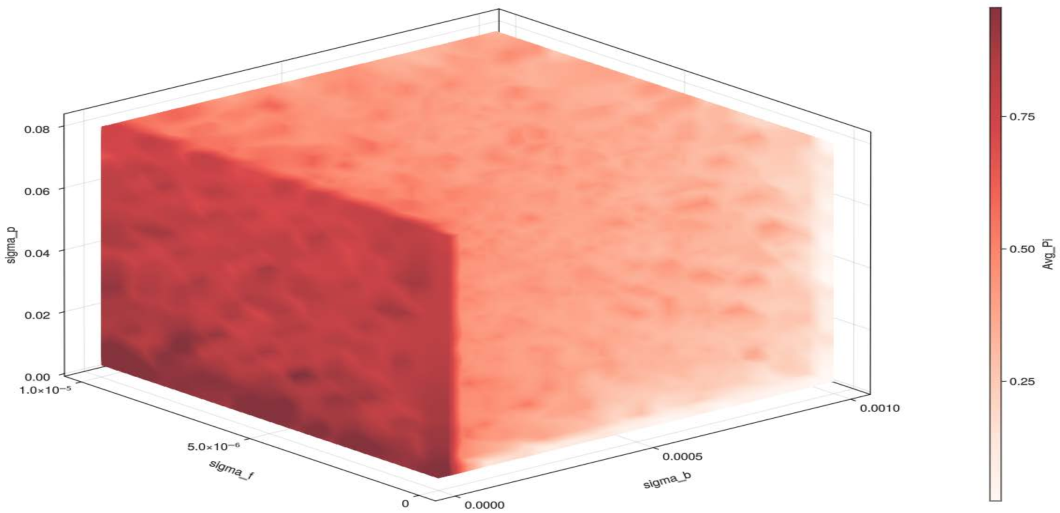
$$\lim_{\sigma_f \to 0} \lim_{\sigma_v \to 0} \lim_{T \to \infty} \mathsf{E} \frac{T^{\epsilon}\left(T, \frac{\gamma}{1-\alpha}, \bar{\pi}\right)}{T} = 1. \tag{3.24}$$

where

$$\bar{\pi} = \frac{4\alpha\rho - 1 + \sqrt{1 - 16\alpha\rho(1-\alpha\rho)}}{4\alpha\rho} < 1 \tag{3.25}$$

4. Simulations

Figure 4 reports numerical simulations that illustrate the convergence described by Theorem 3.2. We run the model for 10,000 periods and record $\pi_{10,000}$. A typical distribution of $\pi_{10,000}$ over the sample paths is bimodal, concentrating at either 1 or 0. We take the average of $\pi_{10,000}$ over 64 sample paths and plot the results. We fix $\sigma_v = 0.0005$, $\alpha = .92$ and $\rho = .99$ throughout, and then plot $\pi_{10,000}$ as σ_f^2 varies from 0.0001 to 0.08. The left panel plots the results for $\sigma^2 = .75$ and the right panel plots the results for $\sigma^2 = .0035$.



There are two points to notice. First, in both cases the probability that π_t converges to 1 increases as σ_f^2 gets smaller. This is not surprising, since the misspecification of $\mathcal{M}_1$ is less severe when σ_f^2 is smaller, and we know from Cho and Kasa (2017) that $\mathcal{M}_1$ dominates when f_t is absent. Second, $\mathcal{M}_1$ does better when σ^2 is smaller. The right panel shows that convergence to $\pi = 1$ is virtually certain for a strictly positive σ_v^2. This is because the relative importance of feedback induced time variation increases as σ^2 gets smaller, and capturing this feedback is the comparative advantage of $\mathcal{M}_1$.

5. Concluding Remarks

This paper has extended the results of OOPP to include expectational feedback and real-time learning. It has been shown that a misspecified/underparameterized model can not only survive in the limit, it can actually dominate. This surprising result happens because it can better respond to the endogeneity of the (unknown) data-generating process.

We should emphasize that we present these results as merely a useful warning to the conclusions of OOPP, one that we thnk is relevant to real world policy environments, where misspecified endogeneity is endemic. We certainly do not mean to suggest that underparameterized models are a good idea! For example, it seems clear that a TVP model that *included* the fundamentals would dominate both of the models considered here.

Appendix A. Preliminaries

A complete description of the evolution of the varaibles is as follows.

$$
\begin{aligned}
\beta_{t+1}(0) &= \beta_t(0) + \left(\frac{\Sigma_t(0)}{\sigma^2+\Sigma_t(0)}\right)(p_t - p_t(0)) && \text{(A.26)}\\
p_t(0) &= (1-\hat{\pi}_t)\left[\beta_t(0) + \frac{\delta}{1-\alpha\rho} f_t\right] + \hat{\pi}_t\beta_t(1) && \text{(A.27)}\\
\Sigma_{t+1}(0) &= \Sigma_t(0) - \frac{(\Sigma_t(0))^2}{\sigma^2+\Sigma_t(0)} && \text{(A.28)}\\
\beta_{t+1}(1) &= \beta_t(1) + \left(\frac{\Sigma_t(1)}{\sigma^2+\Sigma_t(1)}\right)(p_t - p_t(1)) && \text{(A.29)}\\
p_t(1) &= \beta_t(1) && \text{(A.30)}\\
\Sigma_{t+1}(1) &= \Sigma_t(1) - \frac{(\Sigma_t(1))^2}{\sigma^2+\Sigma_t(1)} + \sigma_v^2 && \text{(A.31)}\\
\hat{p}_t &= (1-\pi_t)p_t(0) + \pi_t p_t(1) && \text{(A.32)}\\
p_t &= \gamma + \alpha \mathsf{E}_t \hat{p}_{t+1} + \delta f_t + \sigma\epsilon_t \\
&= \gamma + \alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)\beta_t(1) + \alpha(1-\pi_t)(1-\hat{\pi}_t)\beta_t(0) \\
&\quad + \left(\alpha(1-\pi_t)(1-\hat{\pi}_t)\frac{\delta\rho}{1-\alpha\rho} + \delta\right) f_t + \epsilon_t. && \text{(A.33)}\\
\frac{1}{\pi_{t+1}} - 1 &= \frac{A_{t+1}(0)}{A_{t+1}(1)}\left(\frac{1}{\pi_t} - 1\right) && \text{(A.34)}
\end{aligned}
$$

where $A_t(i)$ is the "perceived" likelihood function of $\mathcal{M}_i$

$$
A_t(i) = \frac{1}{\sqrt{2\pi(\sigma^2+\Sigma_t(i))}} \exp\left[-\frac{(p_t - p_t(i))^2}{2\pi(\sigma^2+\Sigma_t(i))}\right] \tag{A.35}
$$

$\forall i, t$.

(A.23)-(A.25) summarize the behavior of $M(0)$ forecater. His forecasting rule (A.24) is "augmented" by including two state variables: [1] $p_t(1) = \beta_t(1)$, acknowledging its impact on the actual law of motion of p_t, and [2] $\hat{\pi}_t$, acknowledging the process that the decision maker aggregates the two different forecasts, $p_t(0)$ and $p_t(1)$. $M(0)$ forecaster believes that π_t follows a stationary process, but knows that $\beta_t(1)$ does not follow a stationary process. Thus, $M(0)$ forecaster takes the most recent $\beta_t(1)$ to his forecasting rule, while estimates π_t according to the simple average.

(A.26)-(A.28) describe the behavior of $M(1)$ forecaster, who believes $\sigma_v > 0$ and updates the unobservable $\beta_t(1)$ accordingly. (A.29)-(A.31) describe the behavior of the decision maker, who aggregates the two forecasts according to (A.32). (A.30) shows how the actual price is realized. The decision maker does not observe the stationarity of the dynamic system and relies on the likelihood ratio to determine the odd ratio about the stationarity of the system.

The augmented $M(0)$ contains the key elements that determine the asymptotic properties of the dynamic system, and therefore, considered "correctly specified" according to Esponda and Pouzo (2016). Under the definition of Esponda and Pouzo (2016), a correctly specified model is the actual data generating process. We argue that a correct specification according to Esponda and Pouzo (2016) does not guaratee to generate the data, if the expectational feedback is strong. In our case, the strength of the feedback is quantified by α, which determines the strength of the expectation to the actual price.

A.1. **Preliminaries.** The analysis generally follows the same steps as the proof of Theorem 3.1. We describe the key steps, while referring to the proof of Theorem 3.1 along the way.

Since $\beta_t(1)$ evolves at a faster time scale than $\beta_t(0)$, we fix $\beta_t(0)$ to calculate the limit of the mean of $\beta_t(1)$ that moves according to (A.29). Given that the decision maker aggregates $p_t(1)$ and $p_t(0)$ acording to (A.32), (A.33) becomes

$$\begin{aligned} p_t &= \gamma + \alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)\beta_t(1) + \alpha(1-\pi_t)(1-\hat{\pi}_t)\beta_t(0) \\ &\quad + \left(\alpha(1-\pi_t)(1-\hat{\pi}_t)\frac{\delta\rho}{1-\alpha\rho} + \delta\right) f_t + \epsilon_t. \end{aligned} \tag{A.36}$$

For a fixed $\beta_t(0) = \beta(0)$, $\beta_t(1)$ converges to satisfy

$$\mathsf{E}\left[p_t - p_t(1)\right] = 0$$

as $t \to \infty$ so that

$$\beta_t(1) = \frac{\gamma + \alpha(1-\pi_t)(1-\hat{\pi}_t)\beta_t(0)}{1 - \alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)} + \xi_t \tag{A.37}$$

implying that

$$\beta_t(1) - \beta_t(0) = \frac{\gamma - (1-\alpha)\beta_t(0)}{1 - \alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)} + \xi_t \tag{A.38}$$

where $\sigma_\xi^2 \to 0$ as $\sigma_v \to 0$. Thus, the variance of the forecasting error of $p_t(1)$ converges to

$$\begin{aligned} &\mathsf{E}(p_t - p_t(1))^2 \\ &\qquad = A\sigma_\xi^2 + \sigma^2\left(\alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)\right)^2\Sigma(1) + \left((1-\pi_t)(1-\hat{\pi}_t)\frac{\alpha\delta\rho}{1-\alpha\rho} + \delta\right)^2 \sigma_f^2 \end{aligned}$$

for some $A < 0$. After substituting (A.37) into (A.36), we have

$$p_t = \frac{\gamma + \alpha(1-\pi_t)^2\beta_t(0)}{1 - \alpha(\pi_t + (1-\pi_t)\pi_t)} + \left(\alpha(1-\pi_t)^2\frac{\delta\rho}{1-\alpha\rho} + \delta\right) f_t + \epsilon_t. \tag{A.39}$$

$\beta_t(0)$ converges to a value satisfying

$$\mathsf{E}p_t - \beta_t(0) = 0$$

implying that

$$\beta_t(0) \to \frac{\gamma}{1-\alpha}$$

with probability 1. A simple calculation shows that

$$\beta_t(1) \to \frac{\gamma}{1-\alpha}$$

weakly. As $t \to \infty$, $\Sigma_t(0) \to 0$. Thus, after setting $\Sigma_t(0) = 0$, we can write

$$\mathsf{E}(p_t - p_t(0))^2 \tag{A.40}$$

$$= \left(\frac{(1-\alpha)(1-\hat{\pi}_t)}{1-\alpha(\pi_t + (1-\pi_t)\hat{\pi}_t)}\right)^2 \left(\frac{\gamma}{1-\alpha} - \beta_t(0)\right)^2 \tag{A.41}$$

$$+ \left((1-\pi_t)(1-\hat{\pi}_t)\frac{\alpha\rho\delta}{1-\alpha\rho} - (1-\hat{\pi}_t)\frac{\delta}{1-\alpha\rho} + \delta\right)^2 \frac{\sigma_f^2}{1-\delta^2} \tag{A.42}$$

$$+\sigma^2 + B\sigma_\xi^2. \tag{A.43}$$

for some $B > 0$. Since π_t evolves according to (A.34), we can write

$$\phi_t = \phi_{t-1} + \frac{1}{t}\left(\log\frac{A_t(0)}{A_t(1)} - \phi_{t-1}\right)$$

where

$$\phi_t = \frac{1}{t}\log\left[\frac{1-\pi_t}{\pi_t}\right].$$

The associate ODE is

$$\dot{\phi} = \mathsf{E}\log\frac{A_t(0)}{A_t(1)} - \phi.$$

Note that if $\mathsf{E}\log\frac{A_t(0)}{A_t(1)} > 0$, $\pi_t \to 0$, and if $\mathsf{E}\log\frac{A_t(0)}{A_t(1)} < 0$, $\pi_t \to 1$. We identify a condition that $\forall\sigma^2 > 0$,

$$\lim_{\sigma_f \to 0}\lim_{\sigma_v \to 0}\lim_{t \to \infty}\mathsf{E}\log\frac{A_t(0)}{A_t(1)} > 0$$

which is a critical step to analyze the limit points of $(\hat{\pi}_t, \pi_t)$. Note

$$\mathsf{E}\log\frac{A_t(0)}{A_t(1)} = \frac{\mathsf{E}(p_t - p_t(0))^2}{\Sigma_t(0)+\sigma^2} - \frac{\mathsf{E}(p_t - p_t(1))^2}{\Sigma_t(1)+\sigma^2} + \frac{1}{2}\log\frac{\Sigma_t(1)+\sigma^2}{\Sigma_t(0)+\sigma^2}. \tag{A.44}$$

Since $t\to\infty$ and $\Sigma_t(0)\to 0$, we set $\Sigma_t(0)=0$ to simplify the expression of (A.44). Then, we let $\sigma_v \to 0$ so that $\Sigma_t(1), \sigma_\xi^2 \to 0$ to further simply the expression. We can show that (A.44) is positive if and only if

$$\begin{aligned} -&\left[\left(1-(1-\pi_t)(1-\hat{\pi}_t)\left(\frac{\alpha\rho\delta}{1-\alpha\rho}-(1-\hat{\pi}_t)\frac{\delta}{1-\alpha\rho}+\delta\right)\right)^2\frac{\sigma_f^2}{1-\delta^2}+\sigma^2\right.\\ &\left.+\left(\frac{(1-\alpha)(1-\hat{\pi}_t)}{1-\alpha(\pi_t+(1-\pi_t)\hat{\pi}_t)}^2\left(\frac{\gamma}{1-\alpha}-\beta_t(0)\right)^2\right)\right]\\ &+\left[+\left((1-\pi_t)(1-\hat{\pi}_t)\frac{\alpha\delta\rho}{1-\alpha\rho}+\delta\right)^2\frac{\sigma_f^2}{1-\delta^2}+\sigma^2\right]. \end{aligned} \tag{A.45}$$

Note

$$\mathsf{E}\log\frac{A_t(0)}{A_t(1)} > 0$$

if and only if

$$\begin{aligned} &\frac{(1-\alpha)(1-\hat{\pi}_t)}{1-\alpha(\pi_t+(1-\pi_t)\hat{\pi}_t)}^2\left(\beta_t(0)-\frac{\gamma}{1-\alpha}\right)^2\\ \leq\ &\frac{\sigma_f^2}{1-\delta^2}\left[\left((1-\hat{\pi}_t)(1-\pi_t)\frac{\alpha\rho\delta}{1-\alpha\rho}+\delta\right)^2\right.\\ &\left.-\left((1-\pi_t)(1-\hat{\pi}_t)\left(\frac{\alpha\rho\delta}{1-\alpha\rho}-(1-\hat{\pi}_t)\frac{\delta}{1-\alpha\rho}+\delta\right)\right)^2\right]\\ =\ &\frac{\sigma_f^2}{1-\delta^2}\left[(1-\hat{\pi}_t)\frac{\delta}{1-\alpha\rho}\right]\left[2(1-\pi_t)(1-\hat{\pi}_t)\left(\frac{\alpha\rho\delta}{1-\alpha\rho}+\delta\right)-(1-\hat{\pi}_t)\frac{\delta}{1-\alpha\rho}\right]. \end{aligned}$$

Since the left hand side of the inequality is positive, it is necessary that the right hand side must be positive, implying that the term inside of the last bracket on the right hand side of the inequality must be positive. The term in the last bracket is positive if and only if

$$\alpha\rho < \frac{1+\hat{\pi}_t}{2(1-(1-\hat{\pi}_t)(1-\pi_t))}. \tag{A.46}$$

A.2. $\hat{\pi}_t \equiv 0$**.** Suppose that $\hat{\pi}_t = 0$. The right hand side becomes $\frac{1}{2}$ if $\pi_t = 1$. Since $\alpha\rho > \frac{1}{2}$. The inequality must fail around a small neighborhood of $\pi_t = 1$, implying that (A.44) must be negative. Thus, $\pi_t = 1$ is a locally stable point in a small neighborhood of $\pi_t = 1$. Following the same logic as in Cho and Kasa (2017), we can show that as $\sigma_f \to 0$, the duration time around $\pi_t = 1$ converges to 1 so that $M(1)$ dominates $M(0)$. Recall that $\hat{\pi}_t$ is a sample average of π_t. $\forall\epsilon > 0$, define

$$T^\epsilon(T, \beta(0), \hat{\pi}) = \#\left\{t \leq T \mid (\beta_t(0), \hat{\pi}_t) \in \mathcal{N}_\epsilon(\beta(0), \hat{\pi})\right\}$$

as the number of rounds when $\beta_t(0)$ and the sample average of π_t stays in ϵ neighborhood of $(\beta(0), \hat{\pi})$. For reference, let us re-state Theorem 3.1.

Proposition A.1. *Suppose that* $\alpha\rho > \frac{1}{2}$.

$$\lim_{\sigma_f\to 0}\lim_{\sigma_v\to 0}\lim_{T\to 0}\mathsf{E}\frac{T^\epsilon(T, \frac{\gamma}{1-\alpha}, 1)}{T} = 1.$$

A.3. **Adaptive Learning.** Proposition A.1 is striking in that the decision maker ends up believing in the incorrectly specified forecasting rule $M(1)$ if the expectational feedback is sufficiently strong. Yet, the forecasting rule of $M(0)$ is also severely misspecified in the sense that the $M(0)$ forecaster ignores the fact that the decision maker aggregates the two forecasts.

A natural response would be to augment $M(0)$ forecasting rule by assumign that $\hat{\pi}_t$ follows (**??**). That is, the $M(0)$ forecaster is aware that the decision maker aggregates the two forecasts by assgining probability weight π_t to the $M(1)$ forecast. Since the $M(0)$ forecaster does not observe π_t in period t, he estimates π_t according to (**??**) under the assumption that π_t follows a stationary process. This assumption is consistent with his belief that β is an unknown constant so that he estimates according to (A.26).

The augmented $M(0)$ forecasting rule is correctly specified according to Esponda and Pouzo (2016) but is very sensible for three important reasons. First, the $M(0)$ forecaster correctly models how the decision maker aggregates the two forecasts. Second, since the $M(0)$ forecaster cannot observe π_t, he estimates π_t by the sample average of the past values of π_t, which is a very sensible estimator under a broad set of conditions. Third, the $M(0)$ forecast is based on the belief that the underlying state is stationary with a possible exception of $\beta_t(1)$, which the $M(0)$ forecaster treats as an exogenous stochastic process. If $M(0)$ forecast prevails, then the result stochastic process converges to a stationary process around a rational expectations outcome, thus self-confirming the belief of the $M(0)$ forecaster.

To analyze the dynamic system, we need to consider $\hat{\pi}_t$ in addition to $(\beta_t(0), \pi_t)$, while assuming that $\beta_t(1)$ evolves accordign to (A.38). Note that $\beta_t(0)$ and $\hat{\pi}_t$ evolves according to the time scale of the sample average. Also, note that π_t evolves at the faster rate than $\beta_t(0)$ and $\hat{\pi}_t$ in the interior of $[0,1]$, but in a small neighborhood of the boundary of $[0,1]$, π_t evolves according to a slower time scale than the other two variables.

Consider $(\beta_t(0), \hat{\pi}_t, \pi_t) \in \mathbb{R} \times [0,1]^2$. Define

$$\mathcal{R}_+ = \left\{ (\beta(0), \hat{\pi}, \pi) \mid \mathsf{E} \log \frac{A(0)}{A(1)} \geq 0 \right\} \tag{A.47}$$

as the area in $\mathbb{R} \times [0,1]^2$ where $\pi_t \to 0$ very quickly. Similarly, define

$$\mathcal{R}_- = \left\{ (\beta(0), \hat{\pi}, \pi) \mid \mathsf{E} \log \frac{A(0)}{A(1)} < 0 \right\} \tag{A.48}$$

as the area in $\mathbb{R} \times [0,1]^2$ where $\pi_t \to 1$ very quickly for given $(\beta_t(0), \hat{\pi}_t)$.

For a small value of $\alpha\rho < 1$, $(\beta(0), \hat{\pi}, \pi) = (\frac{\gamma}{1-\alpha}, 0, 0)$ is the globally stable point, implying that the decision maker believes in $M(0)$ with probability 1, if the feedback is not strong. However, if $\alpha\rho$ is sufficiently close to 1, the dynamic system has another stable point where the decision maker assigns probability weight close to 1 to $M(1)$, even though the forecasting rule of $M(1)$ is "flawed" for important reasons.

To see this, consider a hyperplane

$$\mathsf{H} = \{(\beta(0), \hat{\pi}, \pi) \mid \pi = \hat{\pi}\} \tag{A.49}$$

where the sample average of π_t coincides with the expected value of π_t. Define

$$\mathsf{H}_+ = \{(\beta(0), \hat{\pi}, \pi) \mid \pi \geq \hat{\pi}\}$$

where the expected value of π_t is larger than $\hat{\pi}_t$. It typically happens if $\pi_t = 1$. Similarly, define

$$\mathsf{H}_- = \{(\beta(0), \hat{\pi}, \pi) \mid \pi \leq \hat{\pi}\}$$

as the closed half space above and below H where the expected value of π_t is larger than $\hat{\pi}_t$. It typically happens if $\pi_t = 0$.

Lemma A.2. *$\mathcal{R}_+ \cap \mathsf{H}_-$ is a connected set if and only if $\alpha\rho < \frac{2+\sqrt{3}}{4} < 1$.*

Proof. Consider the "ridge line" of $\mathcal{R}_+$. The cross section of $\mathcal{R}_+$ at each $\hat{\pi}_t$ is peaked if $\beta_t(0) = \frac{\gamma}{1-\alpha}$. Let $\mathcal{R}_+(\hat{\pi})$ be the cross section of $\mathcal{R}_+$ at $\hat{\pi}$. Define

$$\pi^*(\hat{\pi}) = \max_{\pi} \left\{ \left(\frac{\gamma}{1-\alpha}, \hat{\pi}, \pi \right) \in \mathcal{R}_+ \right\}.$$

By the "ridge line" of $\mathcal{R}_+$, we mean the curve

$$\left\{(\beta(0), \hat{\pi}, \pi) \mid \beta(0) = \frac{\gamma}{1-\alpha}, \pi = \pi^*(\hat{\pi}), \hat{\pi} \in [0,1]\right\}. \tag{A.50}$$

One can easily show that $\mathcal{R}_+ \cap \mathsf{H}$ is connected if and only if (A.50) does not intersect with H, which is equivalent to

$$2((1-\pi)^2\alpha\rho + 1 - \alpha\rho) - (1-\pi) = 0$$

has no real solution. A simple calculation shows that the quadratic equation has no real valued solution if and only if

$$\alpha\rho < \frac{2+\sqrt{3}}{4}.$$

If

$$\alpha\rho \geq \frac{2+\sqrt{3}}{4},$$

$$\bar{\pi} = \frac{4\alpha\rho - 1 + \sqrt{1 - 16\alpha\rho(1-\alpha\rho)}}{4\alpha\rho} \tag{A.51}$$

and

$$\underline{\pi} = \frac{4\alpha\rho - 1 - \sqrt{1 - 16\alpha\rho(1-\alpha\rho)}}{4\alpha\rho} \tag{A.52}$$

are the two solutions. □

Proposition A.3. *Suppose that $t \to \infty$ and then $\sigma_v^2 \to 0$. If*

$$\alpha\rho < \frac{2+\sqrt{3}}{4},$$

then

$$(\beta(0), \hat{\pi}, \pi) = \left(\frac{\gamma}{1-\alpha}, 0, 0\right)$$

is the globally stable solution of of the mean dynamics of $(\beta_t(0), \hat{\pi}_t, \pi_t)$ so that

$$(\beta_t(0), \hat{\pi}_t, \pi_t) \to \left(\frac{\gamma}{1-\alpha}, 0, 0\right)$$

weakly. If

$$\alpha\rho > \frac{2+\sqrt{3}}{4},$$

so that $\bar{\pi} > \underline{\pi}$, then

$$(\beta(0), \hat{\pi}, \pi) = \left(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi}\right)$$

and

$$(\beta(0), \hat{\pi}, \pi) = \left(\frac{\gamma}{1-\alpha}, 0, 0\right)$$

are two locally stable stationary solutions of the mean dynamics of $(\beta_t(0), \hat{\pi}_t, \pi_t)$.

Proof. Observe that π_t evolves at a faster time scale than $\beta_t(0)$ and $\hat{\pi}_t$ in the interior of $[0,1]$. If $(\beta_t(0), \hat{\pi}_t, \pi_t) \in \mathcal{R}_-$, then $\pi_t \to 1$ (or close to 1, because $\pi_t = 1$ is not reachable), and if $(\beta_t(0), \hat{\pi}_t, \pi_t) \in \mathcal{R}_+$, then $\pi_t \to 0$ (or close to 0, because $\pi_t = 0$ is not reachable). From the perspective of $(\beta_t(0), \hat{\pi}_t)$, $\pi_t \to 1$ and $\pi_t \to 0$ occurs almost instantaneously. Thus, the mean dynamics of $\hat{\pi}_t$ can be approximated by

$$\dot{\hat{\pi}} = \begin{cases} 1 - \hat{\pi}_t & \text{if } (\beta_t(0), \hat{\pi}_t, \pi_t) \in \mathcal{R}_-; \\ -\hat{\pi}_t & \text{if } (\beta_t(0), \hat{\pi}_t, \pi_t) \in \mathcal{R}_+. \end{cases}$$

Let us project the gradient vector field to the space of $(\beta_t(0), \hat{\pi}_t)$ where the variables evolve according to the same time scale as the sample average.

Suppose $\alpha\rho < \frac{2+\sqrt{3}}{4}$. Since $\beta_t(0) \to \frac{\gamma}{1-\alpha}$ with probability 1. It suffices to check whether there is a stable point along the hyperplane where $\beta_t(0) = \frac{\gamma}{1-\alpha}$. If $\pi_t = 1$, $\hat{\pi}_t$ cannot be larger than $2\alpha\rho - 1 \in (0,1)$

since $\frac{1}{2} < \alpha\rho < 1$. As soon as $\hat{\pi}_t > 2\alpha\rho - 1$ while $\pi_t = 1$, $\pi_t \to 0$ quickly which pushes $\hat{\pi}_t$ toward 0. Since $\alpha\rho < \frac{2+\sqrt{3}}{4}$, the "ridge line" of $\mathcal{R}_-$ stays above H along $\beta_t(0) = \frac{\gamma}{1-\alpha}$ for any value of $\hat{\pi}_t \in [0,1]$. Thus, $\hat{\pi}_t \to 0$ while $\pi_t = 0$ in the long run. Thus,

$$(\beta(0), \hat{\pi}, \pi) = \left(\frac{\gamma}{1-\alpha}, 0, 0\right)$$

is the only stable point of the mean dynamics of $(\beta_t(0), \hat{\pi}_t, \pi_t)$.

Suppose $\alpha\rho > \frac{2+\sqrt{3}}{4}$. We know that if a stationary point exists, it should be along the intersection of they hyperplane where $\beta(0) = \frac{\gamma}{1-\alpha}$, the boundary of the intersection between H and $\mathcal{R}_-$ and $(\frac{\gamma}{1-\alpha}, 0, 0)$. If $\alpha\rho > \frac{2+\sqrt{3}}{4}$, then $\bar{\pi} > \underline{\pi}$. Thus, $(\frac{\gamma}{1-\alpha}, \underline{\pi}, \underline{\pi})$ and $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ are two stationary points. Note that the line segment connecting $(\frac{\gamma}{1-\alpha}, \underline{\pi}, \underline{\pi})$ and $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ is in $\mathcal{R}_-$, implying that $\pi_t \to 1$ instantaneously from the perspective of $(\beta_t(0), \hat{\pi}_t)$. As a result, along the line segment

$$\dot{\hat{\pi}} = 1 - \hat{\pi} > 0.$$

Similarly, the line segments connecting $(\frac{\gamma}{1-\alpha}, \underline{\pi}, \underline{\pi})$ and $(\frac{\gamma}{1-\alpha}, 0, 0)$ and and connecting $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ and $(\frac{\gamma}{1-\alpha}, 1, 1)$ belong to $\mathcal{R}_+$ so that $\pi_t \to 0$ instantaneously from the perspective of $(\beta_t(0), \hat{\pi}_t)$, implying

$$\dot{\hat{\pi}} = -\hat{\pi} < 0.$$

Thus, $(\frac{\gamma}{1-\alpha}, \underline{\pi}, \underline{\pi})$ is unstable, while $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ is stable. The stability of $(\frac{\gamma}{1-\alpha}, 0, 0)$ follows from the same logic as the proof of the stability of the same stationary point when $\alpha\rho < \frac{2+\sqrt{3}}{4}$. The only exception is that the domain of attraction changes as $\alpha\rho > \frac{2+\sqrt{3}}{4}$.

Recall that if $\alpha\rho > \frac{2+\sqrt{3}}{4}$, the entire state space is the domain of attraction of $(\frac{\gamma}{1-\alpha}, 0, 0)$, which is the only stable stationary point. If $\alpha\rho < \frac{2+\sqrt{3}}{4}$, a simple check of the gradient vector field shows that

$$\{(\beta(0), \hat{\pi}, \pi) \in \mathcal{R}_+ \cap \mathsf{H}_- \mid \hat{\pi} < \underline{\pi}\} \tag{A.53}$$

is the domain of attraction for $(\frac{\gamma}{1-\alpha}, 0, 0)$. Following the same reasoning, we can show that the domain of attraction for $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ is the complement of (A.53). $\square$

$\forall \epsilon > 0$, define

$$T^\epsilon\left(T, \frac{\gamma}{1-\alpha}, \bar{\pi}\right) = \#\left\{t \le T \mid (\beta_t(0), \hat{\pi}_t) \in \mathcal{N}_\epsilon\left(\left(\frac{\gamma}{1-\alpha}, \bar{\pi}\right)\right)\right\}$$

as the number of periods before T rounds when $(\beta_t(0), \hat{\pi}_t)$ is in ϵ neighborhood of $\left(\frac{\gamma}{1-\alpha}, \bar{\pi}\right)$. Note that $\left(\frac{\gamma}{1-\alpha}, \bar{\pi}\right)$ is not a locally stable point, but a projection of the locally stable point $\left(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi}\right)$ to the first two components space.

Theorem A.4. *Suppose that* $\alpha\rho > \frac{2+\sqrt{3}}{4}$.

$$\lim_{\sigma_f \to 0} \lim_{\sigma_v \to 0} \lim_{T \to \infty} \mathsf{E} \frac{T^\epsilon\left(T, \frac{\gamma}{1-\alpha}, \bar{\pi}\right)}{T} = 1. \tag{A.54}$$

Proof. We have used the assumption that $T \to \infty$ by focusing on the limit to set $\Sigma(0) = 0$, and then that $\sigma_v \to 0$ to set $\Sigma(1) = 0$. Note that by letting $\sigma_f \to 0$, the domain of attraction of locally stable point $(\frac{\gamma}{1-\alpha}, 0, 0)$ collapses to the hyperplane where $\beta(0) = \frac{\gamma}{1-\alpha}$. Since the variance of ϵ_t remains bounded away from 0, the large deviation rate function of $(\beta_t(0), \hat{\pi}_t, \pi_t)$ converges to 0 as $\sigma_f \to 0$.

On the other hand, the domain of attraction of $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ expands as $\sigma_f \to 0$, which ensures that the large deviation rate function of $(\beta_t(0), \hat{\pi}_t, \pi_t)$ in the neighborhood of $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ remains bounded away from 0 as $\sigma_f \to 0$.

For a sufficiently small $\sigma_f > 0$, the rate function around $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ becomes larger than the rate function around $(\frac{\gamma}{1-\alpha}, 0, 0)$. Consequently, the duration time of $(\beta_t(0), \hat{\pi}_t, \pi_t)$ staying in the neighborhood of $(\frac{\gamma}{1-\alpha}, \bar{\pi}, \bar{\pi})$ becomes exponentially longer than the duration time in the neighborhood of $(\frac{\gamma}{1-\alpha}, 0, 0)$, from which the conclusion follows. $\square$

References

Berk, R. H. (1966): "Limiting Behavior of Posterior Distributions When the Model is Incorrect," *Annals of Mathematical Statistics*, 37, 51–58.

Blinder, A. S. (1998): *Central Banking in Theory and Practice*. MIT Press.

Borkar, V. S. (2008): *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press.

Cho, I.-K., and K. Kasa (2017): "Gresham's Law of Model Averaging," *American Economic Review*, pp. 3589–3616.

Doan, T., R. Litterman, and C. Sims (1984): "Forecasting and Conditional Projection Using Realistic Prior Distributions," *Econometric Reviews*, 3, 1–100.

Esponda, I., and D. Pouzo (2016): "Berk-Nash Equilibrium: A Framework for Modeling Agents With Misspecified Models," *Econometrica*, 84, 1093–1130.

——— (2021): "Equilibrium in Misspecified Markov Decision Processes," *Theoretical Economics*, 16, 717–57.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press.

Evans, G. W., S. Honkapohja, T. J. Sargent, and N. Williams (2013): "Bayesian Model Averaging, Learning, and Model Selection," in *Macroeconomics at the Service of Public Policy*, ed. by T. J. Sargent, and J. Vilmunen. Oxford University Press.

Fudenberg, D., G. Lanzani, and P. Strack (2021): "Limit Points of Endogenous Misspecified Learning," *Econometrica*, 89, 1065–98.

Kushner, H. J., and G. G. Yin (1997): *Stochastic Approximation Algorithms and Applications*. Springer-Verlag.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least Squares Learning Mechanisms in Self Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

Nelson, C. R. (1972): "The Predictive Performance of the FRB-MIT-PENN Model of the U.S. Economy," *American Economic Review*, 62, 902–17.

Olea, J., P. Ortoleva, M. Pai, and A. Prat (2022): "Competing Models," *Quarterly Journal of Economics*, 137, 2419–57.

Department of Economics, Emory University, 1602 Fishburne Dr., Atlanta GA 30322 USA
Email address: `In-koo.cho@emory.edu`
URL: `https://sites.google.com/site/inkoocho`

Department of Economics, Simon Fraser University, 8888 University Drive, Burnaby, B.C. Canada. V5A 1S6
Email address: `kkasa@sfu.ca`
URL: `http://www.sfu.ca/~kkasa`