Homework 5: Tubular Neighborhoods in $\mathbb{R}^n$

1. Let X be a k-dimensional submanifold of an n-dimensional manifold M. Let x be a point in X. The **normal space** to X at x is the quotient space

$$N_x X = T_x M / T_x X \ ,$$

and the **normal bundle** of X in M is the vector bundle NX over X whose fiber at x is $N_x X$.

   (a) Prove that NX is indeed a vector bundle.

   (b) If M is $\mathbb{R}^n$, show that $N_x X$ can be identified with the usual "normal space" to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x.

2. Let X be a k-dimensional compact submanifold of $\mathbb{R}^n$. Prove the **tubular neighborhood theorem** in the following form.

   (a) Given $\varepsilon > 0$ let $\mathcal{U}_\varepsilon$ be the set of all points in $\mathbb{R}^n$ which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p \in \mathcal{U}_\varepsilon$ has a *unique* nearest point $\pi(p) \in X$.

   (b) Let $\pi : \mathcal{U}_\varepsilon \to X$ be the map defined in (a) for ε sufficiently small. Show that, if $p \in \mathcal{U}_\varepsilon$, then the line segment $(1-t) \cdot p + t \cdot \pi(p)$, $0 \le t \le 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_\varepsilon$.

   (c) Let $NX_\varepsilon = \{(x, v) \in NX \text{ such that } |v| < \varepsilon\}$. Let $\exp : NX \to \mathbb{R}^n$ be the map $(x, v) \mapsto x + v$, and let $\nu : NX_\varepsilon \to X$ be the map $(x, v) \mapsto x$. Show that, for ε sufficiently small, $\exp$ maps NX_ε diffeomorphically onto $\mathcal{U}_\varepsilon$, and show also that the following diagram commutes:

$$\begin{tikzcd} NX_\varepsilon \arrow[rr, "\exp"] \arrow[dr, "\nu"'] & & \mathcal{U}_\varepsilon \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

3. Suppose that the manifold X in the previous exercise is not compact.

   Prove that the assertion about $\exp$ is still true provided we replace ε by a continuous function

$$\varepsilon : X \to \mathbb{R}^+$$

   which tends to zero fast enough as x tends to infinity.