SEEM 5380: Optim. Methods for High–Dim. Statistics	**2016–17 Second Term**
Homework Set 1	
Instructor: Anthony Man–Cho So	Due: February 24, 2017

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (25pts). In low–rank matrix recovery problems, one is often interested in the set of $m \times n$ matrices (with $m \le n$) whose rank is less than a given integer $r \ge 1$. For a given $X \in \mathbb{R}^{m\times n}$, let $\mathrm{row}(X) \subseteq \mathbb{R}^n$ and $\mathrm{col}(X) \subseteq \mathbb{R}^m$ denote the row space and column space of X, respectively. For a given pair of r–dimensional subspaces (U, V) such that $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, consider the following subspaces:

$$\begin{aligned} \mathcal{M}(U,V) &= \left\{ X \in \mathbb{R}^{m\times n} : \mathrm{row}(X) \subseteq V,\ \mathrm{col}(X) \subseteq U \right\}, \\ \overline{\mathcal{M}}^{\perp}(U,V) &= \left\{ X \in \mathbb{R}^{m\times n} : \mathrm{row}(X) \subseteq V^{\perp},\ \mathrm{col}(X) \subseteq U^{\perp} \right\}. \end{aligned}$$

(a) **(15pts).** Show that $\mathcal{M} \subsetneq \overline{\mathcal{M}}$.

(b) **(10pts).** Given $X \in \mathbb{R}^{m\times n}$, the *nuclear norm* of X is defined as

$$\|X\|_* = \sum_{i=1}^{m} \sigma_i(X),$$

where $\sigma_1(X), \ldots, \sigma_m(X)$ are the singular values of X. Show that the nuclear norm is decomposable with respect to $(\mathcal{M}, \overline{\mathcal{M}}^{\perp})$.

Problem 2 (40pts). Let $g \sim \mathcal{N}(\mathbf{0}, I_n)$ be an n–dimensional standard Gaussian random vector. We are interested in upper and lower bounds on $\mathbb{E}\left[\|g\|_2\right]$.

(a) **(10pts).** Show that $\mathbb{E}\left[\|g\|_2\right] \le \sqrt{n}$.

To obtain a lower bound on $\mathbb{E}\left[\|g\|_2\right]$, we observe that for any $\alpha \in (0,1)$,

$$\begin{aligned} \mathbb{E}\left[\|g\|_2\right] &= \int_{\mathbb{R}^n} \|x\|_2 \, dG(x) \\ &\ge \int_{\{x\in\mathbb{R}^n : \|x\|_2 \ge \sqrt{\alpha n}\}} \|x\|_2 \, dG(x) \\ &\ge \sqrt{\alpha n} \cdot \Pr(\|g\|_2 \ge \sqrt{\alpha n}), \end{aligned}$$

where

$$dG(x) = \frac{1}{(2\pi)^{n/2}} \exp(-\|x\|_2^2/2)\, dx$$

is the standard n–dimensional Gaussian measure. Hence, it remains to lower bound $\Pr(\|g\|_2 \ge \sqrt{\alpha n})$.

(b) **(20pts).** For any $t > 0$, we have

$$\Pr(\|g\|_2 \le \sqrt{\alpha n}) = \Pr(\|g\|_2^2 \le \alpha n) = \Pr\left(\exp\left(t(\alpha n - \|g\|_2^2)\right) \ge 1\right).$$

Using Markov's inequality and the moment generating function of a standard real Gaussian random variable, show that

$$\Pr(\|g\|_2^2 \le \alpha n) \le \exp(t\alpha n) \cdot (1 + 2t)^{-n/2}. \tag{1}$$

(c) **(10pts).** Using the result in (b), show that

$$\Pr(\|g\|_2^2 \le \alpha n) \le \exp\left[\frac{n}{2}(1 - \alpha + \ln \alpha)\right].$$

(*Hint: Choose t to optimize the bound in* (1).)

Problem 3 (35pts). Let $v \in \mathbb{R}^n$ and $\mu > 0$ be given. Consider the following problem:

$$\min_{x \in \mathbb{R}^n} \left\{ \mu \|x\|_1 + \frac{1}{2}\|x - v\|_2^2 \right\}. \tag{2}$$

(a) **(20pts).** Compute the subdifferential of the function $x \mapsto \|x\|_1$. Hence, write down the optimality condition for problem (2).

(b) **(15pts).** Using the result in (a), give an explicit expression for the optimal solution x^* to problem (2).