SEEM 5380 HW2

Zhizhong Li
Mar 2017
Information Engineering, CUHK
`1155070507, lz015@ie.cuhk.edu.hk`

1 Problem 1

1.1

Since the solid cone $\mathcal{Q}^3$ is closed, convex, and rotational symmetric along the first axis (t axis), we only need consider a small set S of candidate points when trying to find the projection of a point y into this set $\mathcal{Q}^3$. This is because the point projected to must be unique and thus we can eliminate points that have 'brother/sister' points which shares the same distance to y due to symmetry. Specifically, we can choose S be the (or a) plane that contains both the point y and the first axis. The problem then reduces to a simple two-dimensional geometry problem of middle school level (Figure 1). We can directly write down the solution as specified in the question.

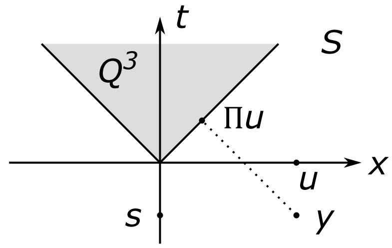

Figure 1: The problem of projecting point $y = (t, u)$ to $\mathcal{Q}^3$ is reduced to this two dimensional problem.

1.2

Let point $z = \Pi_H(y) = (a, b, a) \in H$ be the projected point. Then $z - y = (a - s, b - u_1, a - u_2)$ must parallel to the normal direction of plane H, namely $n = (1, 0, -1)$. Solve this constraint, we get $b = u_1$ and $a = (s + u_2)/2$.

1.3

Let $p = (t, u_1, u_2) \in \mathcal{Q}^3 \cap H$ be a point in the intersection. Since $p \in H$, we have $t = u_2$. Since $p \in \mathcal{Q}$, we have $u_1^2 + u_2^2 = u_1^2 + t^2 \le t^2$, and thus $u_1 \equiv 0$. Together with $t \ge \|x\|_2 \ge 0$, we know the intersection is contained in the ray $R := \{t(1, 0, 1) : t \ge 0\}$. It's easy to see that this ray also contained in the intersection. So $\mathcal{Q}^3 \cap H = R$. Then the projection of point y to this ray R is a simple optimization problem

$$t^* = \operatorname*{argmin}_{t \ge 0} \|(s, u_1, u_2) - t(1, 0, 1)\|_2^2 = \operatorname*{argmin}_{t \ge 0} (s - t)^2 + (u_2 - t)^2. \quad (1)$$

The solution is $t^* = (s + u_2)_+/2$. So comes the desired answer.

SEEM 5380 Homeworks [2].

1.4

Use previous results, we have

$$\begin{aligned}\operatorname{dist}(x^k, \mathcal{Q}^3) &= \left\| (k,1,k) - \frac{k+\sqrt{1+k^2}}{2\sqrt{1+k^2}}(\sqrt{1+k^2},1,k) \right\|_2 \\ &= \left\| \left(\frac{\sqrt{1+k^2}-k}{2}, \frac{k}{2\sqrt{1+k^2}} - \frac{1}{2}, \frac{k}{2\sqrt{1+k^2}}(k-\sqrt{1+k^2}) \right) \right\|_2 \\ &\to 0 \quad (k \to \infty),\end{aligned} \tag{2}$$

$$\operatorname{dist}(x^k, H) = 0, \tag{3}$$

and

$$\operatorname{dist}(x^k, \mathcal{Q}^3 \cap H) = 1. \tag{4}$$

It is evident that the claim in the question holds.

2 Problem 2

First we claim that the nuclear norm function $\|X\|_*$ is convex, and treat it as a known fact. Suppose $X = USV^T$ is an SVD of X, then $U \in \mathcal{O}(m), V \in \mathcal{O}(n)$ are two orthogonal matrices.

$$\begin{aligned}\partial\|X\|_* &= \{D : \|X+\Delta\|_* - \|X\|_* \ge D \cdot \Delta, \forall \Delta \in \mathbb{R}^{m\times n}\} \\ &= \{D : \|U^T(X+\Delta)V\|_* - \|U^TXV\|_* \ge (U^TDV)\cdot(U^T\Delta V), \forall \Delta \in \mathbb{R}^{m\times n}\} \\ &= \{D : \|S + U^T\Delta V\|_* - \|S\|_* \ge (U^TDV)\cdot(U^T\Delta V), \forall \Delta \in \mathbb{R}^{m\times n}\} \\ &= \{D : \|S+\Delta\|_* - \|S\|_* \ge (U^TDV)\cdot \Delta, \forall \Delta \in \mathbb{R}^{m\times n}\} \\ &= \{UDV^T : D \in \partial\|S\|_*\}.\end{aligned} \tag{5}$$

So for this problem, we only need compute $\partial\|S\|_*$, where

$$S = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

Claim 1. *Suppose $f(x,y)$ is a convex function where $x \in \mathbb{R}^s, y \in \mathbb{R}^t$. Given point (x_0,y_0), let $g(x) := f(x,y_0)$ and $h(y) := f(x_0,y)$. If the partial gradient $v = \nabla_x g(x_0) \in \mathbb{R}^s$ exists, then*

$$\partial f(x_0,y_0) = \{(v,w) : w \in \partial h(y_0)\}. \tag{7}$$

Proof. (Left $\subset$ Right) $\forall (v,w) \in \partial f(x_0,y_0)$, we have

$$f(x_0+\delta_1, y_0+\delta_2) \ge f(x_0,y_0) + v^T\delta_1 + w^T\delta_2, \quad \forall \delta_1 \in \mathbb{R}^s, \delta_2 \in \mathbb{R}^t. \tag{8}$$

Let $\delta_1 = 0$ and $\delta_2 = 0$ respectively, we will get $w \in \partial h(y_0)$ and $v = \nabla_x g(x_0)$.

(Left $\supset$ Right) Without loss of generality, we can assume $(x_0,y_0) = 0$, $f(x_0,y_0) = 0$ and $(v,w) = 0$. So we have $\nabla_x f(0,0) = 0$ and inequalities

$$f(\delta_1, 0) \ge 0, f(0,\delta_2) \ge 0, \quad \forall \delta_1 \in \mathbb{R}^s, \delta_2 \in \mathbb{R}^t. \tag{9}$$

We need to show that $(0,0) \in \partial f(0,0)$, *i.e.*

$$f(\delta_1,\delta_2) \ge 0, \quad \forall \delta_1 \in \mathbb{R}^s, \delta_2 \in \mathbb{R}^t. \tag{10}$$

If not, then $\exists (a,b) \in \mathbb{R}^{s+t}$, *s.t.* $f(a,b) < 0$. For $t > 0$, we have

$$\begin{aligned} f\left(0, \frac{tb}{1+t}\right) &\le \frac{1}{1+t} f(-ta, 0) + \frac{t}{1+t} f(a,b) \\ &= t\left(\frac{1}{1+t}\frac{f(-ta,0)}{t} + \frac{1}{1+t} f(a,b) \right) \\ &< 0 \quad (\text{for small enough } t).\end{aligned} \tag{11}$$

This contradicts $f(0,\delta_2) \ge 0$. ☐

Next we compute the partial derivatives (if exists) by definition.

$$\left.\frac{\partial\|X\|_*}{\partial x_{ij}}\right|_{X=S} = \lim_{t\to 0}\frac{\|S+tE_{ij}\|_* - \|S\|_*}{t}, \tag{12}$$

where E_{ij} is a all zero matrix except for the (i,j)-th element being 1.

Case $(i \le r, j \le r, i = j)$. For small enough t, we have $\|S + tE_{ij}\|_* = \|S\|_* + t$ (trivial case).

Case $(i \le r, j \le r, i < j)$. We transform $S + tE_{ij}$ to a simpler shape. First switch the i-th row and first row, then switch the i-th column and first column, then switch the j-th row and second row, then switch the j-th column and second column. After these operation, the matrix becomes a block diagonal shape without changing the nuclear norm.

$$\tilde{X}(t) = \begin{bmatrix} \sigma_i & t & 0 & 0 \\ 0 & \sigma_j & 0 & 0 \\ 0 & 0 & \tilde{\Sigma} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

where $\tilde{\Sigma}$ is Σ without its original i,j-th rows and columns. So

$$\left.\frac{\partial\|X\|_*}{\partial x_{ij}}\right|_{X=S} = \left.\frac{\partial}{\partial t}\right|_{t=0}\left\| \begin{bmatrix} \sigma_i & t \\ 0 & \sigma_j \end{bmatrix} \right\|_*. \tag{14}$$

From [1], we know that the singular values of matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{15}$$

are

$$\sigma_1 = \sqrt{\frac{p+q}{2}}, \quad \sigma_2 = \sqrt{\frac{p-q}{2}}, \tag{16}$$

where

$$\begin{aligned} p &= a^2 + b^2 + c^2 + d^2, \\ q &= \sqrt{(a^2 + b^2 - c^2 - d^2)^2 + 4(ac+bd)^2}. \end{aligned} \tag{17}$$

Using Equation (16), we can compute (long and tedious) the derivative in Equation (14) and get

$$\left.\frac{\partial\|X\|_*}{\partial x_{ij}}\right|_{X=S} = 0, \quad (i \le r, j \le r, i < j). \tag{18}$$

Similarly, for **case** $(i \le r, j \le r, i > j)$, **case** $(i > r, j \le r)$, and **case** $(i \le r, j > r)$, we also have the partial derivatives being 0.

However, this method fails for **case** $(i > r, j > r)$ because in the corresponding Equation (14)s, $\sigma_i = \sigma_j = 0$. And then Equation (16) will lead to

$$\left\| \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \right\|_* = \left\| \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \right\|_* = |t|, \tag{19}$$

which are not smooth in t.

Let

$$h(Y) := \left\| \begin{bmatrix} \Sigma & 0 \\ 0 & Y \end{bmatrix} \right\|_* = \|\Sigma\|_* + \|Y\|_*. \tag{20}$$

And let

$$g(Y) := \|Y\|_*, \quad Y \in \mathbb{R}^{(m-r)\times(n-r)}. \tag{21}$$

Then $\partial h(0) = \partial g(0)$. Apply Claim 1 and Equation (5), we have

$$\partial\|X\|_* = \left\{ S = U \begin{bmatrix} I_r & 0 \\ 0 & W \end{bmatrix} V^T : W \in \partial h(0) = \partial g(0) \right\}. \tag{22}$$

The remaining part is to prove that

$$\partial g(0) = \{W : \|W\| \le 1\}, \tag{23}$$

where $\|W\|$ is the spectral norm, and $\partial g(0) = \{D : \|\Delta\|_* \geq D \cdot \Delta, \forall \Delta \in \mathbb{R}^{(m-r)\times(n-r)}\}$.

(Left $\subset$ Right) Let $D \in \partial g(0)$, then $\forall \Delta$, one has $D \cdot \Delta \leq \|\Delta\|_*$. Suppose D has an SVD $D = U\Sigma V^T$, then we have

$$(U^T D V) \cdot (U^T \Delta V) \leq \|U^T \Delta V\|_*, \quad \forall \Delta, \tag{24}$$

which is equivalent to

$$\Sigma \cdot \Delta \leq \|\Delta\|_*, \quad \forall \Delta. \tag{25}$$

Let $\Delta = E_{11}$, then we get $\sigma_1 = \|D\| \leq 1$.

(Left $\supset$ Right) Let W satisfies $\|W\| \leq 1$, we want $W \cdot \Delta \leq \|\Delta\|_*$ holds for any Δ. Suppose for a given Δ, it has an SVD $\Delta = U\Sigma V^T$, where the diagonal elements of Σ are non-negative. Then we only need to prove

$$W \cdot \Delta = (U^T W V) \cdot (U^T \Delta V) \leq \|U^T \Delta V\|_* = \|\Delta\|_*, \tag{26}$$

which is equivalent to prove

$$(U^T W V) \cdot \Sigma \leq \|\Sigma\|_*. \tag{27}$$

Let t_i be the diagonal elements of $U^T W V$, then the above equation is equivalent to

$$\sum_i t_i \sigma_i \leq \sum_i \sigma_i, \tag{28}$$

Since $\|U^T W V\| = \|W\| \leq 1$, we have $|t_i| \leq 1$ and thus Equation (28) holds.

Then we finish all the proofs.

3 Problem 3

Since $h(\mathcal{A}(X))$ is smooth, we compute the derivative

$$D := \frac{\partial h(\mathcal{A}(X))}{\partial X} = \begin{bmatrix} \frac{3}{2}X_{11} - 2X_{22} - \frac{5}{2} & 0 \\ 0 & -2X_{11} + 3X_{22} + 1 \end{bmatrix}. \tag{29}$$

The two numbers in the matrix come from

$$\begin{aligned} \frac{\partial h(y)}{\partial y} &= (B^{\frac{1}{2}})^T (B^{\frac{1}{2}} y - B^{-\frac{1}{2}} d) \\ &= By - d. \end{aligned} \tag{30}$$

The optimality condition at point X is

$$0 \in \{D + W : W \in \partial\|X\|_*\}. \tag{31}$$

Or equivalently,

$$-D \in \partial\|X\|_*. \tag{32}$$

For $X = 0$ being rank 0, $\partial\|X\|_* = \{W : \|W\| \leq 1\}$. However, $\|-D\| = 2.5 > 1$. Hence $-D \notin \partial\|X\|_*$ and we know $X = 0$ is not a optimal solution.

If X is rank 1, Then an SVD of X can be written as

$$X = [u\ u^\perp]\text{diag}(\sigma, 0)[v\ v^\perp]^T = \sigma u v^T, \tag{33}$$

where $\sigma > 0, u = (u_1, u_2), u^\perp = (-u_2, u_1), v = (v_1, v_2), v^\perp = (-v_2, v_1)$. So

$$\begin{aligned} \partial\|X\|_* &= \left\{ [u\ u^\perp] \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} [v\ v^\perp]^T : |w| \leq 1 \right\} \\ &= \left\{ \begin{bmatrix} u_1 v_1 + w u_2 v_2 & u_1 v_2 - w u_2 v_1 \\ u_2 v_1 - w u_1 v_2 & u_1 v_1 + w u_2 v_2 \end{bmatrix} : |w| \leq 1 \right\} \end{aligned} \tag{34}$$

So, the optimality condition becomes $\exists |w| \leq 1$, *s.t.*

$$-D = \begin{bmatrix} u_1 v_1 + w u_2 v_2 & u_1 v_2 - w u_2 v_1 \\ u_2 v_1 - w u_1 v_2 & u_1 v_1 + w u_2 v_2 \end{bmatrix}. \tag{35}$$

We have

$$u_1 v_2 = w u_2 v_1, \quad u_2 v_1 = w u_1 v_2. \tag{36}$$

If $u_1 v_2 = u_2 v_1 = 0$, then must have $u = v = (1, 0)$ or $u = v = (0, 1)$.

If $u = v = (1, 0)$, then $-D = I$ and $X = \text{diag}(\sigma, 0)$. We get $X_{11} = 1, X_{22} = 0$. So

$$\hat{X} = \text{diag}(1, 0) \text{ satisfies the optimality condition.} \tag{37}$$

Compute the objective value and we have $h(\mathcal{A}(\hat{X})) + \|\hat{X}\|_* = 9.5$.

If $u = v = (0, 1)$, then $-D = wI$ and $X = \text{diag}(0, \sigma)$. Solve them we get $\sigma = 0.7, w = 1.1 > 1$, a contradiction with condition $|w| \leq 1$.

If $u_1 v_2 \neq 0$, then we have $u_1 v_2 = w u_2 v_1 = w^2 u_1 v_2$ and thus $w = \pm 1$.

If $w = 1$, we have $u_1 v_2 - u_2 v_1 = 0$, *i.e.* u and v are colinear. We must have $u = \pm v$. If $u = v$, then $X = \sigma u u^T$ and $-D = I$. we get $X_{11} = 1, X_{22} = 0$. Then we have $\sigma u_2^2 = X_{22} = 0$ and thus $u_2 = 0$, a contradiction with $u_2 v_1 = u_1 v_2 \neq 0$. If $u = -v$, then $X = -\sigma u u^T$ and $-D = -I$. we get $X_{11} = 21, X_{22} = 14$. Then we have $\|X\|_* \geq 21$, a value being too large for an optimal.

If $w = -1$, we have $u_1 v_2 + u_2 v_1 = 0$, and thus (u_1, u_2) and $(-v_1, v_2)$ are colinear. We must have $(u_1, u_2) = \pm(-v_1, v_2)$. If $(u_1, u_2) = (-v_1, v_2)$, then $-D = -I$ and we have $X_{11} = 21, X_{22} = 14$. This is too large. If $(u_1, u_2) = (v_1, -v_2)$, then $X = \sigma u v^T$ and $-D = I$. we get $X_{11} = 1, X_{22} = 0$. Then we have $-\sigma u_2^2 = X_{22} = 0$ and thus $u_2 = 0$, a contradiction with $u_2 v_1 = u_1 v_2 \neq 0$.

If X **is rank 2**, suppose one of its SVD is $X = U \Sigma V^T$ and $\sigma_1 \geq \sigma_2 > 0$. Then $\partial \|X\|_* = UV^T$. We get the constraint

$$-D = UV^T. \tag{38}$$

The right hand side can be seen as an SVD of $-D$ with $\sigma_1(D) = \sigma_2(D) = 1$. Since D is a diagonal matrix, we have

$$D = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}. \tag{39}$$

If $D = -I$, then we get $X_{11} = 1, X_{22} = 0$. So $\sigma_1(X) \geq 1$. Since X is rank 2, $\sigma_2(X) > 0$. We have $\|X\|_* > 1$. So the objective function will have value larger than 9.5.

If $D = I$, then we get $X_{11} = 21, X_{22} = 14$. This is too large.

For $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ or $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let $U = [u \;\; k_1 u^\perp], V = [v \;\; k_2 v^\perp]$, where $k_1 = \pm 1, k_2 = \pm 1$. Then

$$-D = UV^T = \begin{bmatrix} u_1 v_1 + k_1 k_2 u_2 v_2 & u_1 v_2 - k_1 k_2 u_2 v_1 \\ u_2 v_1 - k_1 k_2 u_1 v_2 & u_1 v_1 + k_1 k_2 u_2 v_2 \end{bmatrix}. \tag{40}$$

This leads to a contradiction because the diagonal elements of the right hand side have the same sign, while those of the left hand side matrix $-D$ have opposite signs.

As argued above, we found that the unique optimal solution is $\hat{X} = \text{diag}(1, 0)$.

References

[1] P. Lucidarme. Singular value decomposition of a 2x2 matrix, 2017. URL `http://www.lucidarme.me/?p=4624`.

[2] A. M. So. Assignments, 2017. URL `http://www1.se.cuhk.edu.hk/~manchoso/1617/seem5380/`.