Homework 13: Simple Pendulum

This problem is adapted from [53].

The **simple pendulum** is a mechanical system consisting of a massless rigid rod of length l, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate in the vertical plane. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

(a) Let θ be the oriented angle between the rod (regarded as a line segment) and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. Show that the function $H : T^*S^1 \to \mathbb{R}$ given by

$$H(\theta, \xi) = \underbrace{\frac{\xi^2}{2ml^2}}_{K} + \underbrace{ml(1 - \cos\theta)}_{V} ,$$

is an appropriate hamiltonian function to describe the simple pendulum. More precisely, check that gravity corresponds to the potential energy $V(\theta) = ml(1 - \cos\theta)$ (we omit universal constants), and that the kinetic energy is given by $K(\theta, \xi) = \frac{1}{2ml^2}\xi^2$.

(b) For simplicity assume that $m = l = 1$.
Plot the level curves of H in the (θ, ξ) plane.

Show that there exists a number c such that for $0 < h < c$ the level curve $H = h$ is a disjoint union of closed curves. Show that the projection of each of these curves onto the θ-axis is an interval of length less than π.

Show that neither of these assertions is true if $h > c$.

What types of motion are described by these two types of curves?
What about the case $H = c$?

(c) Compute the critical points of the function H. Show that, modulo 2π in θ, there are exactly two critical points: a critical point s where H vanishes, and a critical point u where H equals c. These points are called the **stable** and **unstable** points of H, respectively. Justify this terminology, i.e., show that a trajectory of the hamiltonian vector field of H whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?