Chapter 14
Exercises for the Second Part

As its title may suggest, this chapter contains exercises on the second part (of this book).

14.1 Exercises on Chapter 5

Exercise 1. Let E be a symplectic vector space of dimension $2n$. If F is a subset of E, then we let F° denote the orthogonal vector subspace for the symplectic form, which consists of all vectors x such that $\omega(x,y) = 0$ for every $y \in F$.

For a subspace F of E, show that

$$\dim F + \dim F^{\circ} = \dim E.$$

We call F *isotropic* if the form ω is zero on F, that is, if $F \subset F^{\circ}$. What can we say about the dimension of F? We call F *Lagrangian* if F is isotropic and $\dim F = n$.

Let F be a "co-isotropic" subspace (that is, one whose orthogonal is isotropic). Show that ω induces a symplectic form on the quotient vector space F/F°. Let L be a Lagrangian subspace of E such that $L + F = E$. Show that the composition

$$L \cap F \subset F \longrightarrow F/F^{\circ}$$

is the injection of a Lagrangian subspace.

Exercise 2. Let V be a manifold and let η be a 1-form on V. We can view η as a section

$$\eta : V \longrightarrow T^{\star}V$$

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of the cotangent bundle. If λ denotes the Liouville form on $T^\star V$, then what is the form $\eta^\star\lambda$ on V?

Exercise 3 (Lagrangian submanifolds). If (W,ω) is a symplectic manifold of dimension $2n$, then we call a submanifold $j : L \subset W$ *Lagrangian* if $\dim L = n$ and $j^\star\omega = 0$.

(1) Prove that in a symplectic surface, all curves are Lagrangian.
(2) Prove that if L_1 is Lagrangian in W_1 and L_2 is Lagrangian in W_2, then $L_1 \times L_2$ is Lagrangian in $W_1 \times W_2$. Construct a Lagrangian torus T^n in $\mathbf{R}^{2n}$. Prove that every symplectic manifold contains Lagrangian tori.
(3) Let η be a 1-form on a manifold V. We view η as a section of $T^\star V$ (as in Exercise 2 on p. 515). Prove that the image of η is a Lagrangian submanifold if and only if the form η is closed.
(4) Let f be a $\mathcal{C}^\infty$ function on a compact manifold V. By the above, the 1-form df defines a Lagrangian submanifold of $T^\star V$. Prove that it meets the zero section.
(5) Consider the product manifold $W \times W$ endowed with the symplectic form $\omega \oplus (-\omega)$. Prove that the diagonal is a Lagrangian submanifold. Let φ be a diffeomorphism from W into itself. Under what condition on φ is the graph of φ a Lagrangian submanifold of $W \times W$?

Exercise 4. Let w be a $\mathcal{C}^\infty$ map from the sphere S^2 to the cotangent space $T^\star V$ of a manifold V. Show that

$$\int_{S^2} w^\star\omega = 0.$$

Exercise 5. Let ω be a symplectic form on a compact surface $\varSigma$. Prove that $\varSigma$ is oriented, and then that

$$\int_{\varSigma} \omega \neq 0.$$

Consider the inclusion $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C})$ induced by the inclusion $\mathbf{C}^2 \subset \mathbf{C}^{n+1}$. Prove that it is a symplectic submanifold. Deduce that there exists a $\mathcal{C}^\infty$ map

$$w : S^2 \longrightarrow \mathbf{P}^n(\mathbf{C})$$

such that if ω now denotes the symplectic form of $\mathbf{P}^n(\mathbf{C})$, then

$$\int_{S^2} w^\star\omega \neq 0.$$

Exercise 6. Let φ^1 be a Hamiltonian diffeomorphism of W. Show that the graph of φ^1 is transversal to the diagonal of $W \times W$ at (x, x) if and only if the trajectory of x is nondegenerate.

Exercise 7 (The Group of Hamiltonian Diffeomorphisms). In this exercise we consider a compact manifold W endowed with a symplectic form ω. Let $\operatorname{Ham}(W)$ denote the set of diffeomorphisms of W that are Hamiltonian flows at time 1.

(1) Suppose that the Hamiltonian H_t generates the isotopy φ_t and that the Hamiltonian K_t generates ψ_t. Let

$$G_t = H_t + K_t \circ \varphi_t^{-1}.$$

Show that

$$X_{G_t}(x) = X_{H_t}(x) + (T_{\varphi_t^{-1}(x)}\varphi_t)(X_{K_t} \circ \varphi_t^{-1}(x)).$$

Deduce from this that $\varphi_t \circ \psi_t$ is the Hamiltonian isotopy generated by the Hamiltonian[1] $H_t + K_t \circ \varphi_t^{-1}$.

(2) Describe the Hamiltonian isotopy generated by $-H_t \circ \varphi_t$.

(3) Show that the set $\operatorname{Ham}(W)$ is a subgroup of the group of symplectic diffeomorphisms of W.

(4) Let Φ be a symplectic diffeomorphism of W. Describe the Hamiltonian isotopy generated by the Hamiltonian $H_t \circ \Phi$. Prove that the subgroup $\operatorname{Ham}(W)$ is a normal subgroup of the group of all symplectic diffeomorphisms of W.

Exercise 8. Let x be a nondegenerate critical point of an autonomous Hamiltonian H on $\mathbf{R}^{2n}$. We let φ^t denote the flow of X_H. Express X_H in the coordinates (p_i, q_i). Show that the Jacobian matrix of φ^t satisfies

$$\frac{d}{dt}\left(\operatorname{Jac}_x \varphi^t\right) = (J_0 \operatorname{Hess}_x(H)) \cdot \operatorname{Jac}_x \varphi^t$$

and that consequently

$$\operatorname{Jac}_x \varphi^t = e^{tJ_0 \operatorname{Hess}_x(H)}.$$

Let x be a critical point of an autonomous Hamiltonian H on a symplectic manifold W. Prove that if x is nondegenerate as a periodic trajectory, then it is nondegenerate as a critical point of H.

[1] Even if H and K are autonomous, the composed Hamiltonian isotopy does not (in general) come from an autonomous Hamiltonian. Bonus question: When is it the case?

Exercise 9 (Harmonic Oscillator). On the symplectic manifold $\mathbf{R}^{2n}$, we consider the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum \alpha_i(p_i^2 + q_i^2),$$

where the α_i are positive real numbers, say $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

(1) Write down the corresponding Hamiltonian system and solve it.
(2) We suppose that all α_i/α_j are irrational. Show that the system has exactly n families of periodic solutions, each contained in a plane. We fix a solution contained in the plane with equation $(q_j = p_j = 0)_{j\neq i}$. What are the Floquet multipliers of this solution? Is it nondegenerate?
(3) We suppose that all α_i are equal. What can you say about the periodic solutions?

Exercise 10. Let E be a vector space endowed with a symplectic form ω and an almost complex structure J calibrated by ω; we let $\perp$ denote orthogonality for the inner product $\omega(\cdot, J\cdot)$. Show that a subspace L is Lagrangian if and only if $L^\perp = JL$.

Exercise 11. Let (W, ω) be a symplectic manifold and let J be an almost complex structure calibrated by ω. Let $V \subset W$ be a complex submanifold, that is, one that is stable under J:

$$\forall\, x \in V, \quad J_x(T_xV) \subset T_xV.$$

Verify that ω defines a nondegenerate form on V.

Exercise 12 (Complex, but not Symplectic). Consider the map

$$\begin{aligned} f : \mathbf{C}^2 &\longrightarrow \mathbf{C}^2 \\ (z_1, z_2) &\longmapsto (2z_1, 2z_2) \end{aligned}$$

and the quotient $\mathcal{H}$ ("Hopf surface") of $\mathbf{C}^2 - \{0\}$ for the action of $\mathbf{Z}$ given by

$$n \cdot (z_1, z_2) = f^n(z_1, z_2).$$

Show that the quotient is a complex manifold (that is, with complex analytic transition maps) that is diffeomorphic to $S^3 \times S^1$. Consequently, its second de Rham cohomology group is zero. Deduce that $\mathcal{H}$ does not have any symplectic structure.[2]

[2] In [18], you can find examples of manifolds that are symplectic but not complex.

Exercise 13 (Cayley Numbers and Almost Complex Structures). Recall that the algebra of *Cayley numbers* or *octaves* is a real vector space $\mathbf{O}$ of dimension 8, endowed with a basis that we denote by

$$(1, i, j, k, \ell, \ell i, \ell j, \ell k)$$

and with multiplication (that is right and left distributive over addition) defined by

$$i^2 = j^2 = k^2 = \ell^2 = ijk = jki = kij = -1, \quad i\ell = -\ell i, \text{ etc.}$$

Recall that this multiplication is neither commutative nor associative (because $(ij)\ell = -i(j\ell)$), but that it does satisfy $(ab)b = a(bb)$ for all a and b.

Consider the Euclidean space $\mathbf{R}^7$ of imaginary octaves, that is, the subspace generated by $(i, j, k, \ell i, \ell j, \ell k)$. Let $V \subset \mathbf{R}^7$ be an oriented submanifold of dimension 6. For any point x of V, we let $n(x)$ denote the unitary normal vector (defined by the orientations of V and $\mathbf{R}^7$) and we define

$$J_x : T_x V \longrightarrow \mathbf{R}^7$$

by setting $J_x(u) = n(x) \cdot u$ (multiplication in the sense of octaves). Show that J_x has values in $T_x V$ and that the endomorphisms J_x define an almost complex structure J on V. In this way, all hypersurfaces of $\mathbf{R}^7$ are almost complex.[3]

Exercise 14. We write the matrices with $2n$ lines and $2n$ columns as block matrices

$$A = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}, \quad X, Y, Z, T \in M_n(\mathbf{R}).$$

Under which conditions on the matrices X, Y, Z and T is the matrix A an element of the group $\mathrm{Sp}(2n; \mathbf{R})$?

Exercise 15. Let S be a real symmetric matrix. Show that

$$\exp(J_0 S) \in \mathrm{Sp}(2n; \mathbf{R}).$$

Conversely, suppose that the matrix S is such that

$$\forall\, t, \quad \exp(t J_0 S) \in \mathrm{Sp}(2n; \mathbf{R});$$

what can you say about S?

[3] Replacing $\mathbf{O}$ by $\mathbf{H}$ and 7 by 3 would be an analogous (but more complicated) way in which to show that every oriented surface embedded in $\mathbf{R}^3$ admits an almost complex structure.

Exercise 16. Use Corollary 5.6.7 to prove that the determinant of a symplectic matrix is 1.

Exercise 17 (Symplectization of a Contact Manifold). Let V be a manifold endowed with a 1-form α that is nonsingular (that is, such that $\alpha_x \neq 0$ for every $x \in V$) and such that

$$\forall\, x \in V, \quad (d\alpha)_x|_{\operatorname{Ker} \alpha_x} \text{ is a nondegenerate bilinear form.}$$

We call α a *contact form* on V. The dimension of V must be odd; we write it as $2n+1$.

(1) Show that $\alpha \wedge (d\alpha)^{\wedge n}$ is a volume form on V.
(2) Show that there exists a unique vector field X on V such that

$$i_X \alpha \equiv 1 \quad \text{and} \quad i_X(d\alpha) = 0$$

(we call X the *Reeb vector field* of α).
(3) Consider $W = V \times \mathbf{R}$ endowed with the 2-form ω defined by

$$\omega_{(x,\sigma)} = d(e^\sigma \alpha).$$

Show that (W, ω) is a symplectic manifold (called the *symplectization* of the contact manifold (V, α)).
(4) Determine the Hamiltonian vector field of the function $H(x, \sigma) = \sigma$.

Example.

Let $V = S^{2n+1} \subset \mathbf{C}^{n+1}$ be the unit sphere ($\mathbf{C}^{n+1}$ is endowed with coordinates $(q_1 + ip_1, \dots, q_{n+1} + ip_{n+1})$) and consider the 1-form

$$\alpha = \frac{1}{2} \sum (p_i dq_i - q_i dp_i).$$

Prove that α is a contact form. Determine $\operatorname{Ker} \alpha$ and the Reeb vector field X of α. Prove that the symplectization of (S^{2n+1}, α) is symplectomorphic to $\mathbf{C}^{n+1} - \{0\}$ endowed with its standard symplectic form.

Exercise 18. We call a function H on a symplectic manifold a *periodic* Hamiltonian[4] if there exists a circle action

$$S^1 \times W \longrightarrow W$$

[4] For this notion, basic results, and more, see for example [5].

such that
$$\frac{d}{dt}\left(\exp(2i\pi t)\cdot x\right)|_{t=0} = X_H(x)$$
for every $x \in W$.

(1) Show that the fixed points of the circle action are the critical points of H.
(2) Show that all (periodic) orbits of H are degenerate.

Exercise 19. We let the circle $\mathbf{S}^1$ act on the complex projective space $\mathbf{P}^n(\mathbf{C})$ by
$$u \cdot [z_0, \ldots, z_n] = [u^{m_0} z_0, \ldots, u^{m_n} z_n], \quad m_i \in \mathbf{Z}.$$
Show that the function
$$H([z_0, \ldots, z_n]) = \frac{1}{2}\frac{\sum m_i \left|z_i^2\right|}{\sum |z_i|^2}$$
is a periodic Hamiltonian associated with this action.

Under what condition (on the m_i) is this Hamiltonian a Morse function? What are then the indices of its critical points?

Exercise 20. We return to the quadric Q of Exercise 8 on p. 19. Consider the functions g and h that are restrictions to Q of the functions on $\mathbf{P}^3(\mathbf{C})$ defined by
$$g([z]) = \frac{\operatorname{Im}(z_1\overline{z}_0)}{\sum |z_i|^2} \quad \text{and} \quad h([z]) = \frac{\operatorname{Im}(z_3\overline{z}_2)}{\sum |z_i|^2}.$$
Show that g and h are periodic Hamiltonians on Q.

As in Exercise 8, we fix real numbers λ and μ such that $0 < \lambda < \mu$ and we consider the function f that is the restriction to Q of
$$f([z]) = \frac{\lambda \operatorname{Im}(z_1\overline{z}_0) + \mu \operatorname{Im}(z_3\overline{z}_2)}{\sum |z_i|^2}.$$

Express the Hamiltonian vector field X_f of f using those of g and h. Deduce the critical points of f. What are their indices (this question can be answered without any computations by using Exercises 18 and 21)?

Exercise 21 (Difficult). Consider a periodic Hamiltonian H on a compact symplectic manifold W. Let x be a critical point of H.

(1) Using an equivariant version of Darboux's theorem, show that there exist an almost complex structure and local coordinates in the neighborhood of x such that:

- The linearization of the action of S^1 on the complex vector space T_xW is of the form
$$t \cdot (z_1, \ldots, z_n) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_n} z_n).$$
- The Hamiltonian can be written as
$$H(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{2} \sum_i \alpha_i (x_i^2 + y_i^2).$$

(2) Suppose that the critical point x is *isolated*. Show that it is nondegenerate and of even index.

(3) Show that a connected symplectic manifold endowed with a periodic Hamiltonian whose critical points are isolated is simply connected.

(4) Show that a periodic Hamiltonian on a compact connected symplectic manifold whose critical points are all isolated has a local minimum and a local maximum.

14.2 Exercises on Chapter 6

Exercise 22. Verify that the quotient of the map
$$\begin{aligned} \mathbf{R}^2 &\longrightarrow \mathbf{R}^2 \\ (p, q) &\longmapsto (p, q + 1/2) \end{aligned}$$
on the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ is a diffeomorphism φ that preserves the symplectic form ($\varphi^\star \omega = \omega$) but does not have any fixed points. Show that there exists a vector field X on T^2 such that
$$\varphi = \varphi_X^1$$
but that X is not a Hamiltonian vector field.

Exercise 23. Let W be a symplectic manifold endowed with a circle action that preserves the symplectic form. Suppose that W is simply connected. Prove that the action is associated with a periodic Hamiltonian $H : W \to \mathbf{R}$ (in the sense of Exercise 18 on p. 520). Let $\zeta \in S^1$ and let φ be the diffeomorphism defined by $\varphi(x) = \zeta \cdot x$. Prove that φ is a Hamiltonian diffeomorphism and that it has at least $\sum \dim HM_i(W; \mathbf{Z}/2)$ fixed points.

Exercise 24. Let $x \in \mathcal{L}W$, and let u be a curve in $\mathcal{L}W$ that passes through x, that is, a map

$$u : \mathbf{R} \times S^1 \longrightarrow W$$

with $u(0,t) = x(t)$. Let Y be the vector tangent to $\mathcal{L}W$ defined by u. For a function $f : \mathcal{L}W \to \mathbf{R}$, we set

$$Y(x) \cdot f = \frac{\partial}{\partial s} f \circ u(s,t)|_{s=0}.$$

Show that this formula defines something and that this something is a derivation on the functions.

Exercise 25. On the torus $\mathbf{T}^2 = \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$, we consider the symplectic form $\omega = dy \wedge dx$. Let H and K be the functions from $\mathbf{R}^2$ to $\mathbf{R}$ defined by

$$H(x,y) = \frac{1}{2\pi}\cos(2\pi x), \quad K(x,y) = \frac{1}{2\pi}\sin(2\pi y).$$

Show that H and K define (autonomous) Hamiltonians on $\mathbf{T}^2$ and determine the associated Hamiltonian vector fields X_H and H_K.

Determine the flows φ_t of X_H and ψ_t of X_K as well as the periodic orbits of period 1 of X_H and X_K. Are the Hamiltonians H and K nondegenerate?

Compute the composition $\sigma_t = \psi_t \circ \varphi_t$. Determine the Hamiltonian F_t that generates the Hamiltonian isotopy σ_t (see Exercise 7 on p. 517). Determine the periodic orbits of period 1 of X_{F_t} and indicate which are contractible. Show that F_t is nondegenerate.

Exercise 26. On the space $\mathcal{L}W$, consider the action form

$$(\alpha_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), Y(t))\,dt.$$

Let $x_0 \in \mathcal{L}W$ be a fixed loop. We fix an extension u_0 of x_0 to the disk D^2. Show that x_0 has a neighborhood U in $\mathcal{L}W$ such that every loop $x \in U$ admits an extension u_x to the disk D^2, so that

$$\mathcal{A}_H(x) = -\int_D u_x^\star \omega + \int_0^1 H_t(x(t))\,dt$$

defines a $\mathcal{C}^\infty$ function on U. Deduce that the form α_H is closed.

Exercise 27. Let α be a closed form on a manifold V and let X be a pseudo-gradient field adapted to α. We define the *energy* of a trajectory γ of the vector field X to be

$$E(\gamma) = \int_{-\infty}^{+\infty} -\alpha(\dot{\gamma}(s))\,ds.$$

Show that if the energy of γ is finite, then γ connects two zeros of α.

Exercise 28 (Naturality of the Floer Equation). Let $u : \mathbf{R} \times S^1$ be a solution of the Floer equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

We are given a time-dependent Hamiltonian K_t with $K_{t+1} = K_t$ and the flow of symplectic diffeomorphisms ψ^t that it generates. Prove that the map $\widetilde{u}$ defined by

$$\widetilde{u}(s,t) = (\psi^t)^{-1}(u(s,t))$$

satisfies

$$\frac{\partial \widetilde{u}}{\partial s} + \widetilde{J}(\widetilde{u})\frac{\partial \widetilde{u}}{\partial t} + \operatorname{grad} \widetilde{H}_t(\widetilde{u}) = 0$$

for almost complex structures $\widetilde{J}$, and a Hamiltonian $\widetilde{H}_t$ that needs to be determined.

Exercise 29. Let V be a compact manifold endowed with a Riemannian metric and let f be a Morse function on V. Consider the functional

$$\gamma \longmapsto E(\gamma) = \frac{1}{2}\int_{\mathbf{R}} \left(\left\|\frac{d\gamma}{ds}\right\|^2 + \left\|\operatorname{grad}_{\gamma(s)} f\right\|^2\right) ds.$$

Let a and b be two critical points of f and let $\gamma : \mathbf{R} \to V$ be such that

$$\lim_{s\to-\infty} \gamma(s) = a, \qquad \lim_{s\to+\infty} \gamma(s) = b.$$

Show that

$$E(\gamma) = \frac{1}{2}\int_{\mathbf{R}} \left\|\frac{d\gamma}{ds} + \operatorname{grad}_{\gamma(s)} f\right\|^2 ds + f(a) - f(b).$$

Determine the extrema of E on the curves γ connecting a to b.

Exercise 30. Prove that the set of critical points of $\mathcal{A}_H$ (without a nondegeneracy assumption) is compact (this is a consequence of the Arzelà–Ascoli theorem).

Exercise 31 ("Removable" Singularities—difficult). Let (W, ω) be a compact symplectic manifold and let J be an almost complex structure calibrated by ω. Let $u : \mathbf{C} \to W$ be a J-holomorphic curve, that is, in coordinates $s + it \in \mathbf{C}$, a curve such that

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0.$$

Show that if u has finite energy, then it extends to a J-holomorphic map

$$\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\} \longrightarrow W.$$

Exercise 32. Consider the map $z \mapsto z^n$ from $\mathbf{C}$ to $\mathbf{C}$. Let $A(r)$ be the area of the image of the disk of center 0 with radius r and let $\ell(r)$ be its circumference. Compute $A(r)$ and $\ell(r)$. Can the inequality

$$\ell(r)^2 \leq 2\pi r A'(r)$$

obtained in the proof of Proposition 6.6.2 be improved?

Exercise 33. Consider the complex curve C_1 with equation

$$y^2 = 4x^3 - x - 1$$

and the map

$$u_\alpha : C_1 \longrightarrow \mathbf{C}^2$$

defined by $(x, y) \mapsto (\alpha^2 x, \alpha^3 y)$. We complete the curve C_1 to obtain a curve in $\mathbf{P}^2(\mathbf{C})$ and extend u_α to a map

$$u_\alpha : C_1 \longrightarrow \mathbf{P}^2(\mathbf{C}).$$

Study the limit of u_α when α tends to 0 (Figure 14.1).

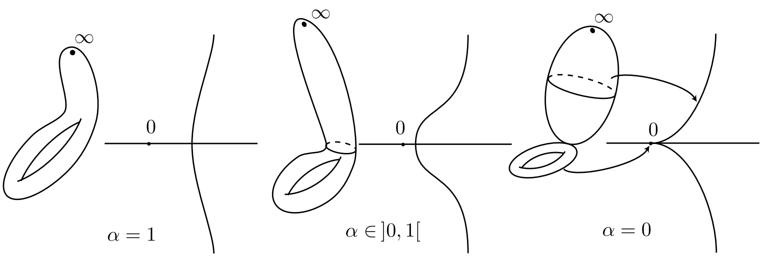

Fig. 14.1

Exercise 34 (Forms with Integral Periods). Let α be a closed 1-form on a manifold V. Suppose that α has "integral periods",[5] that is, with the notation of Section 6.7.a, that the image of φ_α is contained in $\mathbf{Z} \subset \mathbf{R}$. Show that the formula

$$f(x) = \int_{x_0}^{x} \alpha$$

defines a map $f : V \to \mathbf{R}/\mathbf{Z}$ and that

$$f^\star \, d\theta = 2\pi\alpha.$$

[5] This means that the de Rham cohomology class of α is contained in the image of $H^1(V; \mathbf{Z}) \to H^1(V; \mathbf{R})$.

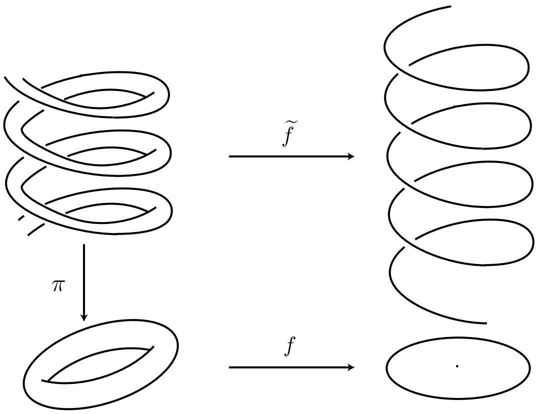

Fig. 14.2

Show that the integration cover $\pi : \widetilde{V} \to V$ is the pullback of the cover in the diagram

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"'] & \mathbf{R} \arrow[d, "\exp"] \\ V \arrow[r, "f"'] & \mathbf{R}/\mathbf{Z} \end{tikzcd}$$

or in Figure 14.2 and that it is cyclic. Next consider the action form α_H (as in Section 6.7.b) on the projective space $\mathbf{P}^n(\mathbf{C})$. Verify that we still have

$$\int_{S^2} w^\star \omega \in \mathbf{Z}.$$

Deduce that there exists an infinite cyclic cover of $\mathcal{L}\mathbf{P}^n(\mathbf{C})$ on which α_H has a primitive $\widetilde{f}_H$.

14.3 Exercises on Chapter 7

Exercise 35. What does the relation ${}^tAJA = J$ imply for the determinant of a symplectic matrix A?

To prove that this determinant is $+1$, we can proceed as in Section 5.6.d. Another method consists in showing that the symplectic group is generated by the maps

$$x \longmapsto x + \lambda\omega(x, a)a$$

(symplectic transvections). This is what we ask you to do in this exercise.

Exercise 36. Let T be a symplectic transvection (Exercise 35). Compute $\rho(T)$. Does the map ρ satisfy

$$\rho(AB) = \rho(A)\rho(B)\,?$$

Exercise 37 (Fundamental Group of U(n)). The group $\mathrm{SU}(n)$ acts on the unit sphere $S^{2n-1} \subset \mathbf{C}^n$ in such a way that the stabilizer the last vector of the canonical basis can be identified with $\mathrm{SU}(n-1)$. Deduce that $\mathrm{SU}(n)$ is simply connected.

Show that the map

$$\mathrm{U}(n) \xrightarrow{\det} S^1$$

induces an isomorphism of the fundamental groups.

Exercise 38. Consider the matrix

$$A(t) = \begin{pmatrix} 1+4\pi^2t^2 & 2\pi t \\ 2\pi t & 1 \end{pmatrix} \quad \text{for } t \in [0,1].$$

Verify that the path $t \mapsto A(t)$ is in $\mathcal{S}$ (defined in Section 7.1.a) and compute its Maslov index.

Do the same (this is more delicate) for the matrix

$$A(t) = \begin{pmatrix} 1-4\pi^2t^2 & -2\pi t \\ 2\pi t & 1 \end{pmatrix} \quad \text{for } t \in [0,1].$$

Exercise 39. In $\mathbf{R}^4$ endowed with the coordinates (q_1, q_2, p_1, p_2), the symplectic form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ and the complex structure $J = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$, consider the quadratic form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (q_2p_1 - q_1p_2)$$

and the associated symmetric matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\alpha & \mathrm{Id} \end{pmatrix} \quad \text{with } \alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that

$$A = \exp(tJS) = \begin{pmatrix} \exp(t\alpha) & t\exp(t\alpha) \\ 0 & \exp(t\alpha) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & t\cos t & -t\sin t \\ \sin t & \cos t & t\sin t & t\cos t \\ & 0 & \cos t & -\sin t \\ & & \sin t & \cos t \end{pmatrix},$$

that this is a symplectic matrix with (double) eigenvalues $e^{\pm it}$, that it is not diagonalizable (for $t \neq 0$) and that it is in $\mathrm{Sp}(2n)^+$ (for $t \neq 0$).

Next, compute $\rho(A)$ (for $t \in]0, \pi[$):

(1) Show that $m_0 = 0$ and that $\rho(A) = (e^{it})^\sigma$, where σ is the signature of Q on the characteristic space E corresponding to the eigenvalue e^{it}.
(2) Show that $X = (1, -i, 0, 0) \in E$ and that $\operatorname{Im} \omega(\overline{X}, X) = 0$. Deduce that $\sigma = 0$ and that $\rho(A) = 1$.

Exercise 40 (Grassmannian of the Lagrangians). Consider the space Λ_n of Lagrangian vector subspaces of $\mathbf{R}^{2n} = \mathbf{C}^n$. Prove that the group $\mathrm{U}(n)$ acts transitively on Λ_n and that the stabilizer of

$$\mathbf{R}^n = \{X \in \mathbf{C}^n \mid \operatorname{Im}(X) = 0\}$$

is isomorphic to the orthogonal group $\mathrm{O}(n)$. Deduce that Λ_n is a connected compact manifold of dimension $n(n+1)/2$.

Show that the map

$$\det{}^2 : \mathrm{U}(n) \longrightarrow S^1$$

defines a continuous map from Λ_n to S^1 and that it induces an isomorphism of the fundamental groups.

Exercise 41 (Maslov Class of a Lagrangian Immersion). Let $f : L \to \mathbf{R}^n$ be an immersion of a manifold of dimension n in $\mathbf{R}^{2n}$. Suppose that f is *Lagrangian*, that is, that $f^\star \omega = 0$ or equivalently that for every x in L, $T_x f(T_x L)$ is a Lagrangian subspace of $\mathbf{R}^{2n}$.

So, sending each point to the tangent space at that point defines a "Gauss" map $\gamma(f) : L \to \Lambda_n$. The composition

$$\gamma(f)_\star : \pi_1(L) \longrightarrow \pi_1(\Lambda_n) = \mathbf{Z}$$

therefore sends each loop in L to an integer: its Maslov class. Determine the Maslov classes of the (Lagrangian) immersions of S^1 in $\mathbf{R}^2$ defined by the drawings in Figure 14.3. Show that the Maslov class of an embedded circle is ± 2 (use the "turning tangents" theorem; see [8]).

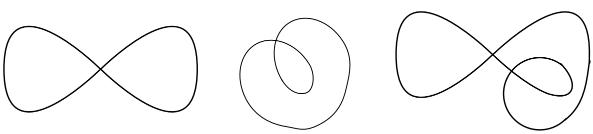

Fig. 14.3

Exercise 42 (Relative Maslov Index). Let W be a manifold endowed with a symplectic form ω and let

$$j : L \hookrightarrow W$$

be a Lagrangian embedding (see Exercise 41). With each disk

$$u : D^2 \longrightarrow W$$

with boundary in L, that is, with $u(\partial D^2) \subset j(L)$, we can associate an integer $\mu_L(u)$ as follows: trivialize $u^\star TW$ using a symplectic trivialization

$$\Phi : u^\star TW \longrightarrow D^2 \times \mathbf{R}^{2n};$$

the class of the loop

$$\begin{aligned} S^1 &\longrightarrow \Lambda_n \\ z &\longmapsto \Phi(T_{u(z)}L) \end{aligned}$$

in $\pi_1(\Lambda_n) \equiv \mathbf{Z}$ is then the integer $\mu_L(u)$ in question. Verify that this integer does not depend on the choice of the trivialization Φ.

Let v be another disk in W with boundary in L. Suppose that u and v are homotopic relative to L, that is, that there exists a homotopy

$$h : D^2 \times [0,1] \longrightarrow W$$

such that

$$\begin{cases} h(\cdot, 0) = u \\ h(\cdot, 1) = v \\ h(z,t) \in L \quad \text{if } z \in S^1. \end{cases}$$

Prove that $\mu_L(u) = \mu_L(v)$. So μ_L defines a map from the group $\pi_2(W, L)$ of relative homotopy classes to $\mathbf{Z}$. Prove that this map is a group homomorphism.

From now on, suppose that $\pi_2(W) = 0$. Prove that in this case, $\mu_L(u)$ depends only on the restriction of u to the boundary and therefore defines a group homomorphism $\pi_1(L) \to \mathbf{Z}$.

Exercise 43. Let P be a polynomial with complex coefficients and let $\alpha \in \mathbf{C}$ be a root of P of multiplicity m. We begin by recalling a proof of Rouché's theorem.

(1) Let γ be the circle $\gamma(t) = \alpha + \varepsilon e^{2i\pi t}$ ($t \in [0,1]$). Show that if ε is sufficiently small so that α is the unique root of P in the closed disk B_ε with

boundary γ, then

$$m = \frac{1}{2i\pi}\int_\gamma \frac{P'(z)}{P(z)}\,dz.$$

(2) Let $\delta = \sup_{z\in \operatorname{Im}\gamma} |P(z)|$. Let Q be a polynomial with

$$\sup_{z\in B_\varepsilon} |P(z) - Q(z)| < \delta.$$

Verify that Q does not have any roots in the circle γ. Prove that the image of γ under h is contained in the open disk of center 1 with radius 1. Deduce that

$$\int_\gamma \frac{h'(z)}{h(z)}\,dz = 0.$$

(3) Prove that Q has exactly m roots (counted with multiplicities) in the disk B_ε. This is Rouché's theorem.

Deduce a proof of Proposition 7.3.6.

14.4 Exercises on Chapter 8

Exercise 44. Show that the kernel of the operator Γ considered in Proposition 8.1.4 is not finite-dimensional.

Exercise 45 (Another Proof of Theorem 8.6.11). Let $\Sigma : \mathbf{R} \to \operatorname{End}(\mathbf{R}^{2n})$ be a continuous map such that

$$\Sigma(s) = \pi\,\mathrm{Id} \text{ for } s < -s_0 \quad \text{and} \quad \Sigma(s) = 3\pi\,\mathrm{Id} \text{ for } s > s_0.$$

Show that the operator F defined by

$$F(Y) = \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + \sigma \cdot Y$$

is a Fredholm operator from $W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n})$ to $L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n})$.

Show that

$$\dim \operatorname{Ker} F = 2n\#\left\{\ell \in \mathbf{Z}^\star \mid 1 < 2\ell < 3\right\} = 2n,$$

and then that $\dim \operatorname{Ker} F^\star = 0$ and that F is surjective.

14.5 Exercises on Chapter 10

Exercise 46. Consider the operator

$$L_0 : W^{1,2}(\mathbf{R};\mathbf{R}^N) \longrightarrow L^2(\mathbf{R};\mathbf{R}^N)$$
$$Y \longmapsto \frac{dY}{ds} + A_0(s),$$

where $A_0(s)$ is a diagonal matrix (for every s) that is constant for $|s|$ sufficiently large, and is of the form

$$A = \begin{pmatrix} \mathrm{Id}_{m^+} & 0 \\ 0 & -\mathrm{Id}_{n^+} \end{pmatrix} \text{ for } s \geq M \quad \text{and} \quad \begin{pmatrix} \mathrm{Id}_{m^-} & 0 \\ 0 & -\mathrm{Id}_{n^-} \end{pmatrix} \text{ for } s \leq -M.$$

(1) Verify that L_0 is a Fredholm operator. Determine its index as a function of $m^\pm$ and $n^\pm$.
(2) Taking inspiration from the methods of Section 8.8, deduce another proof of Proposition 10.2.8.

14.6 Exercises on Chapter 11

Exercise 47. We use the notation and results of Exercise 25 on p. 523, where we determined the contractible and periodic solutions of period 1 of a Hamiltonian F_t on the torus $\mathbf{T}^2$. Compute the indices of these trajectories (a computation essentially asked in Exercise 38) and determine the Floer complex for this Hamiltonian (which we assume to be regular).