Homework 4: Geodesic Flow

This set of problems is adapted from [53].

Let (X, g) be a riemannian manifold. The arc-length of a smooth curve $\gamma : [a, b] \to X$ is

$$\text{arc-length of } \gamma \ := \ \int_a^b \left|\frac{d\gamma}{dt}\right| dt \ , \quad \text{where} \quad \left|\frac{d\gamma}{dt}\right| \ := \ \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)} \ .$$

1. Show that the arc-length of γ is independent of the parametrization of γ, i.e., show that, if we reparametrize γ by $\tau : [a', b'] \to [a, b]$, the new curve $\gamma' = \gamma \circ \tau : [a', b'] \to X$ has the same arc-length. A curve γ is called a curve of *constant velocity* when $\left|\frac{d\gamma}{dt}\right|$ is independent of t. Show that, given any curve $\gamma : [a, b] \to X$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a, b] \to [a, b]$ such that $\gamma \circ \tau : [a, b] \to X$ is of constant velocity.

2. Given a smooth curve $\gamma : [a, b] \to X$, the *action* of γ is $\mathcal{A}(\gamma) \ := \ \displaystyle\int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt$. Show that, among all curves joining x to y, γ minimizes the action if and only if γ is of constant velocity and γ minimizes arc-length.

   Hint: Suppose that γ is of constant velocity, and let $\tau : [a, b] \to [a, b]$ be a reparametrization. Show that $\mathcal{A}(\gamma \circ \tau) \geq \mathcal{A}(\gamma)$, with equality only when $\tau = \text{identity}$.

3. Assume that (X, g) is geodesically convex, that is, any two points $x, y \in X$ are joined by a unique (up to reparametrization) minimizing geodesic; its arc-length $d(x, y)$ is called the riemannian distance between x and y.

   Assume also that (X, g) is *geodesically complete*, that is, every geodesic can be extended indefinitely. Given $(x, v) \in TX$, let $\exp(x, v) : \mathbb{R} \to X$ be the unique minimizing geodesic of constant velocity with initial conditions $\exp(x, v)(0) = x$ and $\frac{d \exp(x,v)}{dt}(0) = v$.

   Consider the function $f : X \times X \to \mathbb{R}$ given by $f(x, y) = -\frac{1}{2} \cdot d(x, y)^2$. Let $d_x f$ and $d_y f$ be the components of $df_{(x,y)}$ with respect to $T^*_{(x,y)}(X \times X) \simeq T^*_x X \times T^*_y X$. Recall that, if

   $$\Gamma^\sigma_f = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X \times X\}$$

   is the graph of a diffeomorphism $f : T^*X \to T^*X$, then f is the symplectomorphism generated by f. In this case, $f(x, \xi) = (y, \eta)$ if and only if $\xi = d_x f$ and $\eta = -d_y f$.

   Show that, under the identification of TX with T^*X by g, the symplectomorphism generated by f coincides with the map $TX \to TX$, $(x, v) \mapsto \exp(x, v)(1)$.

Hint: The metric g provides the identifications $T_xXv \simeq \xi(\cdot) = g_x(v,\cdot) \in T_x^*X$. We need to show that, given $(x,v) \in TX$, the unique solution of $(\star) \begin{cases} g_x(v,\cdot) = d_xf(\cdot) \\ g_y(w,\cdot) = -d_yf(\cdot) \end{cases}$ is $(y,w) = (\exp(x,v)(1), d\frac{\exp(x,v)}{dt}(1))$.
Look up the Gauss lemma in a book on riemannian geometry. It asserts that geodesics are orthogonal to the level sets of the distance function.
To solve the <u>first</u> line in $(\star)$ for y, evaluate both sides at $v = \frac{d\exp(x,v)}{dt}(0)$. Conclude that $y = \exp(x,v)(1)$. Check that $d_xf(v') = 0$ for vectors $v' \in T_xX$ orthogonal to v (that is, $g_x(v,v') = 0$); this is a consequence of $f(x,y)$ being the arc-length of a *minimizing* geodesic, and it suffices to check locally.
The vector w is obtained from the <u>second</u> line of $(\star)$. Compute $-d_yf(\frac{d\exp(x,v)}{dt}(1))$. Then evaluate $-d_yf$ at vectors $w' \in T_yX$ orthogonal to $\frac{d\exp(x,v)}{dt}(1)$; this pairing is again 0 because $f(x,y)$ is the arc-length of a minimizing geodesic. Conclude, using the nondegeneracy of g, that $w = \frac{d\exp(x,v)}{dt}(1)$.
For both steps, it might be useful to recall that, given a function $\varphi : X \to \mathbb{R}$ and a tangent vector $v \in T_xX$, we have $d\varphi_x(v) = \frac{d}{du}\left[\varphi(\exp(x,v)(u))\right]_{u=0}$.