In-class Exercise

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6 Limit of function

For any $\varepsilon > 0$, there is $\delta = \sqrt{2}\varepsilon$, *s.t.*, if $|x-1| < \delta$, then

$$|\sqrt{1+x} - \sqrt{2}| = \frac{|x-1|}{\sqrt{1+x}+\sqrt{2}} < \epsilon.$$

7 Continuity/uniform continuity of functions

7.1

yes, yes. (reason omitted, you should add by yourself)

7.2

yes, no. (reason omitted, you should add by yourself)

8 Limits of multivariate functions

8.1

no.

Take the limit along the line $y = kx$, then the limit is $\lim_{x\to 0} \frac{kx^2}{x^2+2k^2x^2} = \frac{k}{1+2k^2}$. They are different for different k.

8.2

yes.

For any $\varepsilon > 0$, there is $\delta = 4\varepsilon^2$, *s.t.*, if $\sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{xy}{\sqrt{x^2+2y^2}}\right| \le \frac{x^2+y^2}{2\sqrt{x^2+y^2}} < \epsilon.$$

8.3

no.

Go along the curve $y = kx^3$, you will find they approach different values for different k.

9 Pointwise & uniform convergence of sequence of functions

9.1

Converge pointwise, because for all $x \in [-a, a]$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}.$$

Converge uniformly, because

$$\sup_x \left| f_n(x) - \frac{1}{1-x} \right| = \sup_x \left| \sum_{i=n+1}^{\infty} x^i \right| \le \frac{|a|^{n+1}}{1 - |a|},$$

and when $n \to \infty$ the limit is 0.

9.2

Let $f(x) = 1$ for $x = k\pi, k \in \mathbb{Z}$, and $f(x) = 0$ otherwise. $f_n(x)$ converges to $f(x)$ pointwise (omitted the proof, you should do it by yourself). It is not uniformly convergence, because for all n,

$$\sup_x |f_n(x) - f(x)| = 1.$$

9.3

For any x,

$$|f_n(x) - f(x)| \le \sup_t |f_n(t) - f(t)|,$$

the convergence of the right hand side to 0 implies the convergence of left hand side to 0.

10 Derivatives

10.1

We know that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

That is, for any $\varepsilon > 0$, there exists $0 < \delta < \frac{\varepsilon}{|f'(x_0)|+\varepsilon}$, *s.t.*, for any $|h| < \delta$,

$$\left| \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \right| < \varepsilon,$$

so,

$$|f(x_0 + h) - f(x_0)| < |h|(|f'(x_0)| + \varepsilon) < \varepsilon.$$

This means that $f(x)$ is continuous at x_0.

10.2

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h} = nx^{n-1}.$$

10.3

A hint:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=\lim_{h\to 0}\frac{\big(f(x+h)g(x+h)-f(x)g(x+h)\big)+\big(f(x)g(x+h)-f(x)g(x)\big)}{h}=\cdots.$$

11 Mean value theorem

11.1

Let $F(x):=\int_a^x f(t)dt$, then $F'(x)=f(x)$ on (a,b). By mean value theorem, there exist some $c\in(a,b)$, *s.t.*,

$$f(c)=F'(c)=\frac{F(b)-F(a)}{b-a}=\frac{\int_a^b f(x)dx}{b-a}.$$

11.2

$a=-\pi, b=\pi, f(x)=\sin x, g(x)=\cos x.$

12 Cauchy's mean value theorem

We prove for the case $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$. The $\pm\infty$ case can be obtained from the 0 case. (why? explain it by yourself.)

Let $x\in I$ and $x\neq c$. Without loss of generality, we consider the case $x<c$. Then f and g are continuous on $[x,c]$, and differentiable on (x,c). By Cauchy's mean value theorem, there is a $\xi\in(x,c)$, *s.t.*, when $g(x)\neq 0$,

$$\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(\xi)}{g'(\xi)}.$$

Take limit $x\to c$ on both sides, we get the answer.

13 Remainder term for Taylor series

$$\sup_x R_k\left(\frac{1}{1-x}\right)=\sup_x\sum_{i=k+1}^{\infty}x^i=\sup_x x^{k+1}\sum_{i=0}^{\infty}x^i=\sup_x\frac{x^{k+1}}{1-x}=\frac{a^{k+1}}{1-a}\leq\varepsilon.$$

We have $k\geq\frac{\ln\varepsilon(1-a)}{\ln a}-1$.

14 Riemann integral

Choose a partition $P: 0=\frac{0\pi}{2n},\frac{1\pi}{2n},\dots,\frac{n\pi}{2n}=\frac{\pi}{2}$.

$$R(f,P)=\sum_{k=0}^{n-1}\sin\frac{k\pi}{2n}=Im\sum_{k=0}^{n-1}\exp\frac{k\pi i}{2n}=\cdots$$

(omitted the rest of the computation, do it by yourself.)

15 Remainder term for numerical integration

15.1

(omit)

15.2

By the error analysis, we know there is a number $\xi \in (-1, 1)$,

$$error = -\frac{2^3}{12N^2} f''(\xi).$$

First estimate the range of $f''(x)$, and then estimate the required value of N. (Details omitted.)

15.3

Similar to above.