Homework 9: Contractibility

The following proof illustrates in a geometric way the relation between lagrangian subspaces, complex structures and inner products; from [11, p.45].

Let (V, Ω) be a symplectic vector space, and let $\mathcal{J}(V, \Omega)$ be the set of all complex structures on (V, Ω) which are Ω-compatible; i.e., given a complex structure J on V we have

$$J \in \mathcal{J}(V, \Omega) \iff G_{J}(\cdot, \cdot) := \Omega(\cdot, J\cdot) \text{ is a positive inner product on } V \ .$$

Fix a lagrangian subspace L_0 of (V, Ω). Let $\mathcal{L}(V, \Omega, L_0)$ be the space of all lagrangian subspaces of (V, Ω) which intersect L_0 transversally. Let $\mathcal{G}(L_0)$ be the space of all positive inner products on L_0.

Consider the map

$$\begin{array}{rccc} \Psi : & \mathcal{J}(V, \Omega) & \to & \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0) \\ & J & \mapsto & (JL_0, G_{J}|_{L_0}) \end{array}$$

Show that:

1. Ψ is well-defined.

2. Ψ is a bijection.

   Hint: Given $(L, G) \in \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0)$, define J in the following manner: For $v \in L_0$, $v^{\perp} = \{u \in L_0 \mid G(u, v) = 0\}$ is a $(n-1)$-dimensional space of L_0; its symplectic orthogonal $(v^{\perp})^{\Omega}$ is $(n+1)$-dimensional. Check that $(v^{\perp})^{\Omega} \cap L$ is 1-dimensional. Let Jv be the unique vector in this line such that $\Omega(v, Jv) = 1$. Check that, if we take v's in some G-orthonormal basis of L_0, this defines the required element of $\mathcal{J}(V, \Omega)$.

3. $\mathcal{L}(V, \Omega, L_0)$ is contractible.

   Hint: Prove that $\mathcal{L}(V, \Omega, L_0)$ can be identified with the vector space of all symmetric $n \times n$ matrices. Notice that any n-dimensional subspace L of V which is transversal to L_0 is the graph of a linear map $S : JL_0 \to L_0$, i.e.,

$$\begin{array}{rrcl} & L & = & \text{span of } \{Je_1 + SJe_1, \ldots, Je_n + SJe_n\} \\ \text{when} & L_0 & = & \text{span of } \{e_1, \ldots, e_n\} \ . \end{array}$$

4. $\mathcal{G}(L_0)$ is contractible.

   Hint: $\mathcal{G}(L_0)$ is even convex.

Conclude that $\mathcal{J}(V, \Omega)$ is contractible.