Homework 11: Complex Projective Space

The complex projective space $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$:

$\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. One denotes by $[z_0, \ldots, z_n]$ the equivalence class of $(z_0, \ldots, z_n)$, and calls $z_0, \ldots, z_n$ the homogeneous coordinates of the point $p = [z_0, \ldots, z_n]$. (The homogeneous coordinates are, of course, only determined up to multiplication by a non-zero complex number λ.)

Let $\mathcal{U}_i$ be the subset of $\mathbb{CP}^n$ consisting of all points $p = [z_0, \ldots, z_n]$ for which $z_i \neq 0$. Let $\varphi_i : \mathcal{U}_i \to \mathbb{C}^n$ be the map

$$\varphi_i([z_0, \ldots, z_n]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right) .$$

1. Show that the collection

$$\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \ldots, n\}$$

is an atlas in the *complex* sense, i.e., the transition maps are biholomorphic. Conclude that $\mathbb{CP}^n$ is a complex manifold.

Hint: Work out the transition maps associated with $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ and $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$. Show that the transition diagram has the form

$$\begin{tikzcd} & \mathcal{U}_0 \cap \mathcal{U}_1 \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\ \mathcal{V}_{0,1} \arrow[rr, "\varphi_{0,1}"] & & \mathcal{V}_{1,0} \end{tikzcd}$$

where $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1 \neq 0\}$ and

$$\varphi_{0,1}(z_1, \ldots, z_n) = \left(\tfrac{1}{z_1}, \tfrac{z_2}{z_1}, \ldots, \tfrac{z_n}{z_1}\right) .$$

2. Show that the 1-dimensional complex manifold $\mathbb{CP}^1$ is diffeomorphic, as a real 2-dimensional manifold, to S^2.

Hint: Stereographic projection.