$a \in S^{n-1}$ be one of them; then $r^{-1}(a)$ is a submanifold of dimension 1 of D^n, with boundary

$$\partial r^{-1}(a) = r^{-1}(a) \cap \partial D^n = \{a\}\,.$$

But a manifold of dimension 1 with boundary is diffeomorphic to a union of circles and closed intervals, so that its boundary consists of an even number of points. This gives a contradiction, and therefore the existence of a fixed point. □

Exercises

Exercise 10. Show that the vector fields whose flows are drawn in Figure 2.30 are not pseudo-gradient fields.

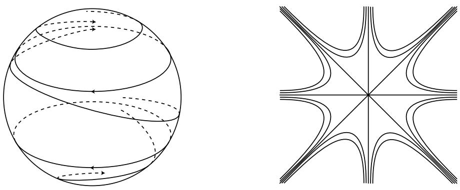

Fig. 2.30

Exercise 11. Let V be a manifold of dimension 2 endowed with a Morse function with a unique critical point of index 1. Show that every pseudo-gradient field adapted to this function satisfies the Smale condition.

Exercise 12. We fix an integer $m \geq 2$. Find all critical points of the function $f : \mathbf{P}^1(\mathbf{C}) \to \mathbf{R}$ defined by

$$f([z_0, z_1]) = \frac{\left|z_0^m + z_1^m\right|^2}{(\left|z_0\right|^2 + \left|z_1\right|^2)^m} = \frac{\left|z^m + 1\right|^2}{(\left|z\right|^2 + 1)^m}$$

(in homogeneous coordinates or in the affine chart $z_1 \neq 0$). Verify that for $m = 2$, the function f is not a Morse function.[3]

We suppose that $m \geq 3$. Show that f is a Morse function and has two local maxima: the points 0 and ∞; m local minima: the m-th roots of -1; and m critical points of index 1: the m-th roots of 1.

[3] It is a Mores–Bott function (see [14]): its critical points form submanifolds (here $\mathbf{P}^1(\mathbf{R})$ for the maximum) and the second-order derivative is transversally nondegenerate.

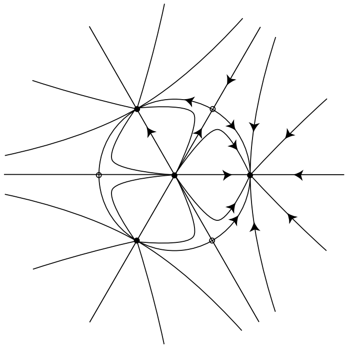

Fig. 2.31

Hint: We can determine the critical points using the derivatives with respect to z and $\overline{z}$, and then use a second-order Taylor expansion of $f(u)$ with respect to u in the neighborhood of 0 (to study the critical points at 0 and ∞) or the analogous expansion of $f(\zeta(1+u))$ (to study the critical points at ζ with $\zeta^m = \pm 1$).

Show that there exists a pseudo-gradient field such as that shown (in an affine chart) in Figure 2.31 (for $m = 3$). More generally, see the article [9] in which an analogous function (defined on $\mathbf{P}^n(\mathbf{C})$) plays an important role.