Homework 6: Oriented Surfaces

1. The standard symplectic form on the 2-sphere is the standard area form: If we think of S^2 as the unit sphere in 3-space

$$S^2 = \{u \in \mathbb{R}^3 \text{ such that } |u| = 1\} \ ,$$

then the induced area form is given by

$$\omega_u(v, w) = \langle u, v \times w \rangle$$

where $u \in S^2$, $v, w \in T_u S^2$ are vectors in $\mathbb{R}^3$, $\times$ is the exterior product, and $\langle \cdot, \cdot \rangle$ is the standard inner product. With this form, the total area of S^2 is 4π.

Consider cylindrical polar coordinates (θ, z) on S^2 away from its poles, where $0 \le \theta < 2\pi$ and $-1 \le z \le 1$.

Show that, in these coordinates,

$$\omega = d\theta \wedge dz \ .$$

2. Prove the Darboux theorem in the 2-dimensional case, using the fact that every nonvanishing 1-form on a surface can be written locally as $f\,dg$ for suitable functions f, g.

   Hint: $\omega = df \wedge dg$ is nondegenerate $\iff$ (f, g) is a local diffeomorphism.

3. Any oriented 2-dimensional manifold with an area form is a symplectic manifold.

   (a) Show that convex combinations of two area forms ω_0, ω_1 that induce the same orientation are symplectic.

   This is wrong in dimension 4: find two symplectic forms on the vector space $\mathbb{R}^4$ that induce the same orientation, yet some convex combination of which is degenerate. Find a path of symplectic forms that connect them.

   (b) Suppose that we have two area forms ω_0, ω_1 on a compact 2-dimensional manifold M representing the same de Rham cohomology class, i.e., $[\omega_0] = [\omega_1] \in H^2_{\mathrm{deRham}}(M)$.

   Prove that there is a 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_1^* \omega_0 = \omega_1$, $\varphi_0 = \mathrm{id}$, and $\varphi_t^* \omega_0$ is symplectic for all $t \in [0, 1]$.

   Hint: Exercise (a) and the Moser trick.

   Such a 1-parameter family φ_t is called a *strong isotopy* between ω_0 and ω_1. In this language, this exercise shows that, up to strong isotopy, there is a unique symplectic representative in each non-zero 2-cohomology class of M.