Homework 10: Integrability

This set of problems is from [11, p.46-47].

1. Let (M, J) be an almost complex manifold. Its **Nijenhuis tensor** $\mathcal{N}$ is:

$$\mathcal{N}(v,w) := [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w] \ ,$$

where v and w are vector fields on M, $[\cdot,\cdot]$ is the usual bracket

$$[v,w] \cdot f := v \cdot (w \cdot f) - w \cdot (v \cdot f) \ , \quad \text{for } f \in C^\infty(M) \ ,$$

and $v \cdot f = df(v)$.

   (a) Check that, if the map $v \mapsto [v, w]$ is complex linear (in the sense that it commutes with J), then $\mathcal{N} \equiv 0$.

   (b) Show that $\mathcal{N}$ is actually a tensor, that is: $\mathcal{N}(v, w)$ at $x \in M$ depends only on the values $v_x, w_x \in T_xM$ and not really on the vector fields v and w.

   (c) Compute $\mathcal{N}(v, Jv)$. Deduce that, if M is a surface, then $\mathcal{N} \equiv 0$.

A theorem of Newlander and Nirenberg [89] states that an almost complex manifold (M, J) is a complex (analytic) manifold if and only if $\mathcal{N} \equiv 0$. Combining (c) with the fact that any orientable surface is symplectic, we conclude that any orientable surface is a complex manifold, a result already known to Gauss.

2. Let $\mathcal{N}$ be as above. For any map $f : \mathbb{R}^{2n} \to \mathbb{C}$ and any vector field v on $\mathbb{R}^{2n}$, we have $v \cdot f = v \cdot (f_1 + if_2) = v \cdot f_1 + i\, v \cdot f_2$, so that $f \mapsto v \cdot f$ is a complex linear map.

   (a) Let $\mathbb{R}^{2n}$ be endowed with an almost complex structure J, and suppose that f is a J-**holomorphic function**, that is,

$$df \circ J = i\, df \ .$$

   Show that $df(\mathcal{N}(v, w)) = 0$ for all vector fields v, w.

   (b) Suppose that there exist n J-holomorphic functions, $f_1, \ldots, f_n$, on $\mathbb{R}^{2n}$, which are independent at some point p, i.e., the real and imaginary parts of $(df_1)_p, \ldots, (df_n)_p$ form a basis of $T_p^*\mathbb{R}^{2n}$. Show that $\mathcal{N}$ vanishes identically at p.

   (c) Assume that M is a complex manifold and J is its complex structure. Show that $\mathcal{N}$ vanishes identically everywhere on M.

In general, an almost complex manifold has *no* J-holomorphic functions at all. On the other hand, it has *plenty* of J-**holomorphic curves**: maps $f : \mathbb{C} \to M$ such that $df \circ i = J \circ df$. J-holomorphic curves, also known as **pseudo-holomorphic curves**, provide a main tool in symplectic topology, as first realized by Gromov [49].