Homework 14: Minimizing Geodesics

This set of problems is adapted from [53].

Let (M, g) be a riemannian manifold. From the riemannian metric, we get a function $F : TM \to \mathbb{R}$, whose restriction to each tangent space T_pM is the quadratic form defined by the metric.

Let p and q be points on M, and let $\gamma : [a, b] \to M$ be a smooth curve joining p to q. Let $\tilde{\gamma} : [a, b] \to TM$, $\tilde{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$ be the lift of γ to TM. The **action** of γ is

$$\mathcal{A}(\gamma) = \int_a^b (\tilde{\gamma}^* F)\ dt = \int_a^b \left| \frac{d\gamma}{dt} \right|^2 dt\ .$$

1. Let $\gamma : [a, b] \to M$ be a smooth curve joining p to q. Show that the arc-length of γ is independent of the parametrization of γ, i.e., show that if we reparametrize γ by $\tau : [a', b'] \to [a, b]$, the new curve $\gamma' = \gamma \circ \tau : [a', b'] \to M$ has the same arc-length.

2. Show that, given any curve $\gamma : [a, b] \to M$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a, b] \to [a, b]$ such that $\gamma \circ \tau : [a, b] \to M$ is of constant velocity, that is, $|\frac{d\gamma}{dt}|$ is independent of t.

3. Let $\tau : [a, b] \to [a, b]$ be a smooth monotone map taking the endpoints of $[a, b]$ to the endpoints of $[a, b]$. Prove that

$$\int_a^b \left(\frac{d\tau}{dt} \right)^2 dt \geq b - a\ ,$$

with equality holding if and only if $\frac{d\tau}{dt} = 1$.

4. Let $\gamma : [a, b] \to M$ be a smooth curve joining p to q. Suppose that, as s goes from a to b, its image $\gamma(s)$ moves at constant velocity, i.e., suppose that $|\frac{d\gamma}{ds}|$ is constant as a function of s. Let $\gamma' = \gamma \circ \tau : [a, b] \to M$ be a reparametrization of γ. Show that $\mathcal{A}(\gamma') \geq \mathcal{A}(\gamma)$, with equality holding if and only if $\tau(t) \equiv t$.

5. Let $\gamma_0 : [a, b] \to M$ be a curve joining p to q. Suppose that γ_0 is **action-minimizing**, i.e., suppose that

$$\mathcal{A}(\gamma_0) \leq \mathcal{A}(\gamma)$$

for any other curve $\gamma : [a, b] \to M$ joining p to q. Prove that γ_0 is also **arc-length-minimizing**, i.e., show that γ_0 is the shortest geodesic joining p to q.

6. Show that, among all curves joining p to q, γ_0 minimizes the action if and only if γ_0 is of constant velocity and γ_0 minimizes arc-length.

7. On a coordinate chart $(\mathcal{U}, x^1, \ldots, x^n)$ on M, we have

$$F(x, v) = \sum g_{ij}(x) v^i v^j \ .$$

Show that the Euler-Lagrange equations associated to the action reduce to the **Christoffel equations** for a geodesic

$$\frac{d^2\gamma^k}{dt^2} + \sum (\Gamma^k_{ij} \circ \gamma) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0 \ ,$$

where the Γ^k_{ij}'s (called the **Christoffel symbols**) are defined in terms of the coefficients of the riemannian metric by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{\ell k} \left(\frac{\partial g_{\ell i}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell} \right) \ ,$$

(g^{ij}) being the matrix inverse to (g_{ij}).

8. Let p and q be two non-antipodal points on S^n. Show that the geodesic joining p to q is an arc of a great circle, the great circle in question being the intersection of S^n with the two-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by p and q.

   Hint: No calculations are needed: Show that an isometry of a riemannian manifold has to carry geodesics into geodesics, and show that there is an isometry of $\mathbb{R}^{n+1}$ whose fixed point set is the plane spanned by p and q, and show that this isometry induces on S^n an isometry whose fixed point set is the great circle containing p and q.