Homework 12: The Fubini-Study Structure

The purpose of the following exercises is to describe the natural Kähler structure on complex projective space, $\mathbb{CP}^n$.

1. Show that the function on $\mathbb{C}^n$

$$z \longmapsto \log(|z|^2 + 1)$$

is strictly plurisubharmonic. Conclude that the 2-form

$$\omega_{_{\mathrm{FS}}} = \tfrac{i}{2}\partial\bar{\partial}\log(|z|^2 + 1)$$

is a Kähler form. (It is usually called the **Fubini-Study form** on $\mathbb{C}^n$.)

Hint: A hermitian $n \times n$ matrix H is positive definite if and only if $v^* H v > 0$ for any $v \in \mathbb{C}^n \setminus \{0\}$, where v^* is the transpose of the vector $\bar{v}$. To prove positive-definiteness, either apply the Cauchy-Schwarz inequality, or use the following symmetry observation: $\mathrm{U}(n)$ acts transitively on S^{2n-1} and $\omega_{_{\mathrm{FS}}}$ is $\mathrm{U}(n)$-invariant, thus it suffices to show positive-definiteness along *one* direction.

2. Let $\mathcal{U}$ be the open subset of $\mathbb{C}^n$ defined by the inequality $z_1 \neq 0$, and let $\varphi : \mathcal{U} \to \mathcal{U}$ be the map

$$\varphi(z_1, \ldots, z_n) = \tfrac{1}{z_1}(1, z_2, \ldots, z_n) \ .$$

Show that φ maps $\mathcal{U}$ biholomorphically onto $\mathcal{U}$ and that

$$\varphi^* \log(|z|^2 + 1) = \log(|z|^2 + 1) + \log \tfrac{1}{|z_1|^2} \ . \qquad (\star)$$

3. Notice that, for every point $p \in \mathcal{U}$, we can write the second term in $(\star)$ as the sum of a holomorphic and an anti-holomorphic function:

$$-\log z_1 - \log \overline{z_1}$$

on a neighborhood of p. Conclude that

$$\partial\bar{\partial}\varphi^* \log(|z|^2 + 1) = \partial\bar{\partial}\log(|z|^2 + 1)$$

and hence that $\varphi^* \omega_{_{\mathrm{FS}}} = \omega_{_{\mathrm{FS}}}$.

Hint: You need to use the fact that the pullback by a holomorphic map φ^* commutes with the ∂ and $\bar{\partial}$ operators. This is a consequence of φ^* preserving form type, $\varphi^*(\Omega^{p,q}) \subseteq \Omega^{p,q}$, which in turn is implied by $\varphi^* dz_j = \partial\varphi_j \subseteq \Omega^{1,0}$ and $\varphi^* d\overline{z_j} = \bar{\partial}\overline{\varphi_j} \subseteq \Omega^{0,1}$, where φ_j is the jth component of φ with respect to local complex coordinates $(z_1, \ldots, z_n)$.

4. Recall that $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$; $[z_0, \ldots, z_n]$ is the equivalence class of $(z_0, \ldots, z_n)$.

For $i = 0, 1, \ldots, n$, let

$$\begin{aligned} &\mathcal{U}_i = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n \,|\, z_i \neq 0\} \\ &\varphi_i : \mathcal{U}_i \to \mathbb{C}^n \qquad \varphi_i([z_0, \ldots, z_n]) = \left(\tfrac{z_0}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i}\right) . \end{aligned}$$

Homework 11 showed that the collection $\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \ldots, n\}$ is a complex atlas (i.e., the transition maps are biholomorphic). In particular, it was shown that the transition diagram associated with $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ and $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$ has the form

$$\begin{tikzcd} & \mathcal{U}_0 \cap \mathcal{U}_1 \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\ \mathcal{V}_{0,1} \arrow[rr, "\varphi_{0,1}"] & & \mathcal{V}_{1,0} \end{tikzcd}$$

where $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_1 \neq 0\}$ and $\varphi_{0,1}(z_1, \ldots, z_n) = (\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1})$. Now the set $\mathcal{U}$ in exercise 2 is equal to the sets $\mathcal{V}_{0,1}$ and $\mathcal{V}_{1,0}$, and the map φ coincides with $\varphi_{0,1}$.

Show that $\varphi_0^* \omega_{\text{FS}}$ and $\varphi_1^* \omega_{\text{FS}}$ are identical on the overlap $\mathcal{U}_0 \cap \mathcal{U}_1$.

More generally, show that the Kähler forms $\varphi_i^* \omega_{\text{FS}}$ "glue together" to define a Kähler structure on $\mathbb{CP}^n$. This is called the **Fubini-Study form** on complex projective space.

5. Prove that for $\mathbb{CP}^1$ the Fubini-Study form on the chart $\mathcal{U}_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \,|\, z_0 \neq 0\}$ is given by the formula

$$\omega_{\text{FS}} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}$$

where $\frac{z_1}{z_0} = z = x + iy$ is the usual coordinate on $\mathbb{C}$.

6. Compute the total area of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with respect to ω_{FS}:

$$\int_{\mathbb{CP}^1} \omega_{\text{FS}} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2} .$$

7. Recall that $\mathbb{CP}^1 \simeq S^2$ as real 2-dimensional manifolds (Homework 11). On S^2 there is the standard area form ω_{std} induced by regarding it as the unit sphere in $\mathbb{R}^3$ (Homework 6): in cylindrical polar coordinates (θ, h) on S^2 away from its poles ($0 \leq \theta < 2\pi$ and $-1 \leq h \leq 1$), we have

$$\omega_{\text{std}} = d\theta \wedge dh .$$

Using stereographic projection, show that

$$\omega_{\text{FS}} = \frac{1}{4} \omega_{\text{std}} .$$