Exercises

Exercise 1. Let U be an open subset of $\mathbf{R}^n$ and let $f: U \to \mathbf{R}$ be a $\mathcal{C}^\infty$ function. Let V be an open subset of $\mathbf{R}^n$ and let $\varphi: V \to U$ be a diffeomorphism. Compute $(d^2(f \circ \varphi))_y$ (for $y \in V$).

Let M be a manifold and let $g: M \to \mathbf{R}$ be a function. Show that the bilinear form $(d^2 g)_x$ is well defined on the vector subspace

$$\operatorname{Ker}(dg)_x \subset T_x M.$$

Exercise 2. The cotangent vector bundle $T^\star V$ of the manifold V is endowed with its natural manifold structure (recalled in Appendix A). Verify that we can see the differential df of a function $f: V \to \mathbf{R}$ as a section

$$df: V \longrightarrow T^\star V$$

of the cotangent vector bundle (which is an embedding of V in $T^\star V$).

What are the critical points of f in this terminology? Show that the point a is a nondegenerate critical point of f if and only if the submanifold $df(V)$ is transverse to the zero section at the point in question.

Prove that a nondegenerate critical point of a function is isolated (without using the Morse lemma!).

Exercise 3 (Monkey Saddle). We consider the function

$$\begin{aligned} f : \mathbf{R}^2 &\longrightarrow \mathbf{R} \\ (x, y) &\longmapsto x^3 - 3xy^2. \end{aligned}$$

Study the critical point $(0, 0)$. Is it nondegenerate? Draw a few regular level sets of the function f as well as its graph.

Exercise 4. If $f : V \to \mathbf{R}$ and $g : W \to \mathbf{R}$ are Morse functions, then $f + g : V \times W \to \mathbf{R}$ is also a Morse function whose critical points are the pairs of critical points of f and g.

Exercise 5 (On the Complex Projective Space). By passing to the quotient, the function on $\mathbf{C}^{n+1} - \{0\}$ defined by

$$f(z_0, \ldots, z_n) = \frac{\sum_{j=0}^n j\,|z_j|^2}{\sum_{j=0}^n |z_j|^2}$$

induces a function on the complex projective space $\mathbf{P}^n(\mathbf{C})$, which we still denote by f.

Verify that this is a Morse function whose critical points are the points $[1, 0, \ldots, 0]$ (of index 0), $[0, 1, 0, \ldots, 0]$ (of index 2) and so on up to $[0, \ldots, 0, 1]$ (of index $2n$).

Readers eager to know where this function may well have come from (from [54], of course, but before that?) can turn to Remark 2.1.12.

Exercise 6 (On the sphere and on the real projective plane). With essentially the same formula, let us now consider the function

$$\begin{aligned} f : S^2 &\longrightarrow \mathbf{R} \\ (x, y, z) &\longmapsto y^2 + 2z^2 \end{aligned}$$

and the resulting function $g : \mathbf{P}^2(\mathbf{R}) \to \mathbf{R}$ on the real projective plane that follows from it by passing to the quotient. Verify that f (and therefore g) is a Morse function and show that it has six critical points (and therefore that g has three):

- Two points of index 0, the points $(\pm 1, 0, 0)$, at level 0
- Two points of index 1, the points $(0, \pm 1, 0)$, at level 1
- Two points of index 2, the points $(0, 0, \pm 1)$, at level 2.

Figure 1.7 shows a few level sets of this function on the sphere. The critical level set containing the two points of index 1 consists of the two circles defined by intersecting the sphere with the planes $z = \pm x$.

Exercise 7. In a square, draw the level sets of the height function on the "inner tube" torus (the answer is somewhere in this book).

Exercise 8 (A projective quadric). We consider the "quadric" Q in $\mathbf{P}^3(\mathbf{C})$ defined by the equation

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0.$$

Show that Q is a compact submanifold of the manifold $\mathbf{P}^3(\mathbf{C})$ of (real) dimension 4.

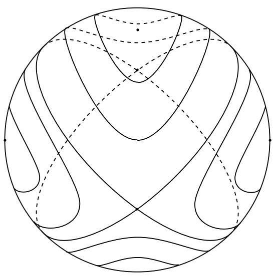

Fig. 1.7 A Morse function on the real projective plane

Let us write $z_j = x_j + iy_j$ (for $0 \le j \le 3$) and $z = x + iy$ for $z \in \mathbf{C}^4$, with $x, y \in \mathbf{R}^4$. We fix two real numbers λ and μ and, for $z \in \mathbf{C}^4$, we set

$$\widetilde{f}(z) = \lambda(x_0y_1 - x_1y_0) + \mu(x_2y_3 - x_3y_2).$$

Verify that for $|u| = 1$, we have

$$\widetilde{f}(uz) = \widetilde{f}(z)$$

and deduce from it that $\widetilde{f}$ defines a function

$$f : \mathbf{P}^3(\mathbf{C}) \longrightarrow \mathbf{R}.$$

Suppose that the real numbers λ and μ occurring in the definition of f satisfy $0 < \lambda < \mu$. Let g be the restriction of f to Q. Show that g is a Morse function that has a local minimum, a local maximum and two critical points of index 2.

Exercise 9. In this exercise, we propose a different proof of the existence of numerous Morse functions on a manifold, and more precisely of Theorem 1.2.5:

(1) Using, for example, Exercise 2, show that the set of Morse functions on the compact manifold V is open for the $\mathcal{C}^2$ topology, and therefore also for the $\mathcal{C}^\infty$ topology.
(2) Let $a \in V$ and let U be the open set of a chart centered at a. We let $\mathcal{F}$ denote the space of differentials of the functions defined on U (or of exact 1-forms), seen as a subset of $\mathcal{C}^\infty(U; (\mathbf{R}^n)^\star)$ (for every $x \in U$, $(df)_x$ is a linear functional on $\mathbf{R}^n$). Note that $\mathcal{F}$ is not an open subset. Let α be a plateau function with support in U and value 1 on a compact neighborhood K of a. For $\varphi = df \in \mathcal{F}$, we let

$$\begin{aligned} F : U \times (\mathbf{R}^n)^\star &\longrightarrow \mathbf{R} \\ (x, A) &\longmapsto f(x) + \alpha(x) A \cdot x. \end{aligned}$$

Compute[3] $(D_1F)_{(x,A)}$ for $x \in K$ and show that F is a submersion

$$K \times (\mathbf{R}^n)^\star \longrightarrow (\mathbf{R}^n)^\star.$$

Show that $\mathcal{F}$ is locally transversal to $\{0\}$ (in the sense of Subsection A.3.c).

[3] If f is a function of two arguments $(u, v) \in U \times V \subset E \times F$, then we let $(D_1f)_{(u,v)}$ denote the linear map

$$(D_1f)_{(u,v)} : E \longrightarrow \mathbf{R}$$

that is the partial differential of f with respect to the first variable. The analogue holds for $(D_2f)_{(u,v)}$.

(3) Show that every point a of V admits a compact neighborhood K such that the set of functions $f : V \to \mathbf{R}$ whose critical points in K are nondegenerate is dense in $\mathcal{C}^\infty(V;\mathbf{R})$.

(4) Deduce a proof of Theorem 1.2.5 from this.