Homework 1: Symplectic Linear Algebra

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in Y\} \ .$$

1. Show that $\dim Y + \dim Y^\Omega = \dim V$.

   Hint: What is the kernel and image of the map

   $$\begin{array}{ccc} V & \longrightarrow & Y^* = \mathrm{Hom}(Y, \mathbb{R}) \quad ? \\ v & \longmapsto & \Omega(v, \cdot)|_Y \end{array}$$

2. Show that $(Y^\Omega)^\Omega = Y$.

3. Show that, if Y and W are subspaces, then

   $$Y \subseteq W \iff W^\Omega \subseteq Y^\Omega \ .$$

4. Show that:

   Y is **symplectic** (i.e., $\Omega|_{Y\times Y}$ is nondegenerate) $\iff$ $Y \cap Y^\Omega = \{0\}$ $\iff$ $V = Y \oplus Y^\Omega$.

5. We call Y **isotropic** when $Y \subseteq Y^\Omega$ (i.e., $\Omega|_{Y\times Y} \equiv 0$).

   Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2} \dim V$.

6. We call Y **coisotropic** when $Y^\Omega \subseteq Y$.

   Check that every codimension 1 subspace Y is coisotropic.

7. An isotropic subspace Y of (V, Ω) is called **lagrangian** when $\dim Y = \frac{1}{2} \dim V$.

   Check that:

   $$Y \text{ is lagrangian} \iff Y \text{ is isotropic and coisotropic} \iff Y = Y^\Omega \ .$$

8. Show that, if Y is a lagrangian subspace of (V, Ω), then any basis $e_1, \ldots, e_n$ of Y can be extended to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of (V, Ω).

   Hint: Choose f_1 in W^Ω, where W is the linear span of $\{e_2, \ldots, e_n\}$.

9. Show that, if Y is a lagrangian subspace, (V, Ω) is symplectomorphic to the space $(Y \oplus Y^*, \Omega_0)$, where Ω_0 is determined by the formula

   $$\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v) \ .$$

   In fact, for any vector space E, the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1, \ldots, e_n$ is a basis of E, and $f_1, \ldots, f_n$ is the dual basis, then $e_1 \oplus 0, \ldots, e_n \oplus 0, 0 \oplus f_1, \ldots, 0 \oplus f_n$ is a symplectic basis for V.