(b) Let us now consider an arbitrary point P. It can be written as (λ, x), where $\lambda \in \overline{\mathcal{L}}(c, d')$ and $x \in \mathcal{M}(d', d)$. Note that for $d' \neq c$, such a point x is not in the domain U on which we have just defined χ: in fact, U is contained in $W^u(c)$. As we have just shown, we can define $\chi = \chi_{d'}$ in a neighborhood $U = U_{d'}$ of $\mathcal{M}(d', d) \cap \{f \geq A + \varepsilon\}$ (d' plays the part of c). We then define χ in the neighborhood of (λ, x) in $\overline{W}^u(c, A+\varepsilon)$ by setting

$$\chi(\lambda', y) = (\lambda', \chi_{d'}(y)).$$

We have now defined χ in an open neighborhood of P. Note that the image of χ contains $\overline{\mathcal{L}}(c, d) \times W^u(d, \varepsilon)$. We extend it to all of $\overline{W}^u(c, A+\varepsilon)$ as in Proposition 4.9.6 and verify without difficulty that it is a homeomorphism.

To Conclude the Proof, the Case Where $\operatorname{Ind}(d) = 0$.

In this case,

$$P = \Big\{(\lambda, x) \in \overline{W}^u(c, A+\varepsilon) \mid x \in W^s(d)\Big\}$$

is a connected component of $\overline{W}^u(c, A+\varepsilon)$.

We define χ on P as follows:

$$\chi(\lambda, x) = \begin{cases} (\lambda, x) & \text{if } f(x) \geq A + \varepsilon + \delta \\ (\lambda, \ell, d) & \text{if } f(x) = A + \varepsilon \\ (\lambda, \varGamma(x)) & \text{if } f(x) \in \left]A + \varepsilon, A + \varepsilon + \delta\right[\end{cases}$$

(ℓ denotes the trajectory of x). In this formula $\varGamma$ is a diffeomorphism between $W^s(d) \cap f^{-1}(]A+\varepsilon, A+\varepsilon+\delta[)$ and $W^s(d) \cap f^{-1}(]f(d), A+\varepsilon+\delta[)$ obtained by pushing along the gradient lines. Elsewhere, we define χ using Proposition 4.9.6 and verify (still without difficulty) that it is a homeomorphism.

The construction shows that χ satisfies the required properties, that is, that its restriction to $\overline{W}^u(c, A + \varepsilon + \delta)$ is the identity and that $\chi(W^u(c, A+\varepsilon)) = W^u(c, A-\varepsilon)$. This concludes the proof of Proposition 4.9.7. □

And that of Theorem 4.9.3.

Exercises

Exercise 16. Does there exist a Morse function on $S^2 \times S^2$ that has a minimum, a maximum, one critical point of index 2, and whose other critical points all have indices 1 or 3?

Exercise 17. Show that the complex projective space $\mathbf{P}^n(\mathbf{C})$ is simply connected.

Exercise 18. Is the quadric Q of Exercise 8 (p. 19) diffeomorphic to $\mathbf{P}^2(\mathbf{C})$?

We send $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ into $\mathbf{P}^3(\mathbf{C})$ using

$$([a,b],[u,v]) \longmapsto [au, bu, av, bv].$$

Show that this is an embedding whose image is described by the equation

$$z_0 z_3 - z_1 z_2 = 0.$$

Prove that Q is diffeomorphic to $S^2 \times S^2$.

Exercise 19. Figure 4.12 shows the graph of a diffeomorphism

$$\varphi :]-1,1[\longrightarrow]-1,1[.$$

We set $a = \varphi(0)$. Show that φ is the flow at time 1 of a vector field on $]-1,1[$, and draw the latter. Let D be the open disk of center 0 with radius 1 in $\mathbf{R}^N$ and let $a \in D$. Construct a vector field on D whose flow satisfies $\varphi^1(0) = a$.

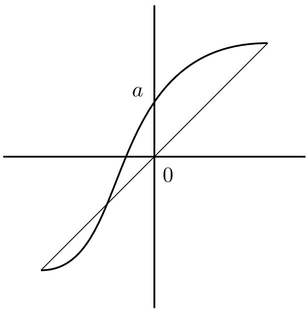

Fig. 4.12

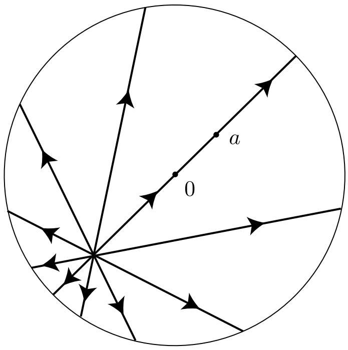

Fig. 4.13

Exercise 20. If $u : V \to W$ is a $\mathcal{C}^\infty$ map and if V and W are compact connected manifolds, then

$$u_\star : HM_0(V; \mathbf{Z}/2) \longrightarrow HM_0(W; \mathbf{Z}/2)$$

is an isomorphism (it is the identity because both spaces are isomorphic to $\mathbf{Z}/2$).

Exercise 21. Let V be a compact connected manifold of dimension n without boundary and let D be a disk of dimension n embedded in V. Show that

$$HM_n(\overline{V - D}; \mathbf{Z}/2) = 0.$$

Deduce that if $u : V^n \to W^n$ is a $\mathcal{C}^\infty$ map that induces a *nonzero* map

$$u_\star : HM_n(V; \mathbf{Z}/2) \longrightarrow HM_n(W; \mathbf{Z}/2),$$

then it is surjective.

Exercise 22. Let W be a compact manifold of dimension $n+1$ with boundary and let V be its boundary, $V = \partial W$. Assume that V is connected:

(1) Prove that the inclusion $V \subset W$ induces the zero map from $HM_n(V; \mathbf{Z}/2)$ to $HM_n(W; \mathbf{Z}/2)$.
(2) Deduce that if $u : V \to Z$ is a $\mathcal{C}^\infty$ map from V to a manifold Z of the same dimension whose induced map $u_\star$ on the homology HM_n is nonzero, then u cannot be extended to a map $\widetilde{u} : W \to Z$.
(3) In particular, there does not exist any map from W to its boundary that restricts to the identity on the boundary (that is, a retraction). The case $W = D^{n+1}$ is that of Brouwer's theorem.

Exercise 23. Let V be a compact manifold. Show that there does not exist any retraction of V onto a proper subset (see Section 4.8.b, if necessary).

Exercise 24. Describe the stable and unstable manifolds, with orientations and co-orientations, for the Morse function f considered in Exercise 6 and a pseudo-gradient field X on $\mathbf{P}^2(\mathbf{R})$. Compute the homology of the complex $(C_\star(f; \mathbf{Z}), \partial_X)$. Show that

$$HM_k(\mathbf{P}_2(\mathbf{R}); \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ \mathbf{Z}/2 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 25. Consider the cellular decomposition of S^n with two cells in any dimension, one being the image of the other under the antipodal map. Determine the differential ∂ of the cellular complex induced on $\mathbf{P}^n(\mathbf{R})$ (as defined in Subsection 4.9.a). Compute $HM_k(\mathbf{P}^n(\mathbf{R}); \mathbf{Z}/2)$. Fix an orientation on each cell of $\mathbf{P}^n(\mathbf{R})$ and use the same method to compute $HM_k(\mathbf{P}^n(\mathbf{R}); \mathbf{Z})$.