Homework 7: Manifolds of Contact Elements

Given any manifold X of dimension n, there is a canonical symplectic manifold of dimension $2n$ attached to it, namely its cotangent bundle with the standard symplectic structure. The exercises below show that there is also a canonical *contact* manifold of dimension $2n-1$ attached to X.

The **manifold of contact elements** of an n-dimensional manifold X is

$$\mathcal{C} = \{(x, \chi_x) \,|\, x \in X \text{ and } \chi_x \text{ is a hyperplane in } T_xX\} \ .$$

On the other hand, the projectivization of the cotangent bundle of X is

$$\mathbb{P}^*X = (T^*X \setminus \text{zero section})/\sim$$

where $(x, \xi) \sim (x, \xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ (here $x \in X$ and $\xi, \xi' \in T_x^*X \setminus \{0\}$). We will denote elements of $\mathbb{P}^*X$ by $(x, [\xi])$, $[\xi]$ being the $\sim$ equivalence class of ξ.

1. Show that $\mathcal{C}$ is naturally isomorphic to $\mathbb{P}^*X$ as a bundle over X, i.e., exhibit a diffeomorphism $\varphi : \mathcal{C} \to \mathbb{P}^*X$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathbb{P}^*X \\ \pi \downarrow & & \downarrow \pi \\ X & = & X \end{array}$$

   where the vertical maps are the natural projections $(x, \chi_x) \mapsto x$ and $(x, \xi) \mapsto x$.

   Hint: The kernel of a non-zero $\xi \in T_x^*X$ is a hyperplane $\chi_x \subset T_xX$. What is the relation between ξ and ξ' if $\ker \xi = \ker \xi'$?

2. There is on $\mathcal{C}$ a canonical field of hyperplanes $\mathcal{H}$ (that is, a smooth map attaching to each point in $\mathcal{C}$ a hyperplane in the tangent space to $\mathcal{C}$ at that point): $\mathcal{H}$ at the point $p = (x, \chi_x) \in \mathcal{C}$ is the hyperplane

$$\mathcal{H}_p = (d\pi_p)^{-1}\chi_x \ \subset T_p\mathcal{C} \ ,$$

   where

$$\begin{array}{ccccc} \mathcal{C} & \qquad & p = (x, \chi_x) & \qquad & T_p\mathcal{C} \\ \downarrow \pi & & \downarrow & & \downarrow d\pi_p \\ X & & x & & T_xX \end{array}$$

   are the natural projections, and $(d\pi_p)^{-1}\chi_x$ is the preimage of $\chi_x \subset T_xX$ by $d\pi_p$.

   Under the isomorphism $\mathcal{C} \simeq \mathbb{P}^*X$ from exercise 1, $\mathcal{H}$ induces a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^*X$. Describe $\mathbb{H}$.

   Hint: If $\xi \in T_x^*X \setminus \{0\}$ has kernel χ_x, what is the kernel of the canonical 1-form $\alpha_{(x,\xi)} = (d\pi_{(x,\xi)})^*\xi$?

3. Check that $(\mathbb{P}^*X, \mathbb{H})$ is a contact manifold, and therefore $(\mathcal{C}, \mathcal{H})$ is a contact manifold.

   Hint: Let $(x, [\xi]) \in \mathbb{P}^*X$. For any ξ representing the class $[\xi]$, we have

   $$\mathbb{H}_{(x,[\xi])} = \ker\left((d\pi_{(x,[\xi])})^*\xi\right) .$$

   Let $x_1, \ldots, x_n$ be local coordinates on X, and let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be the associated local coordinates on T^*X. In these coordinates, $(x, [\xi])$ is given by $(x_1, \ldots, x_n, [\xi_1, \ldots, \xi_n])$. Since at least one of the ξ_i's is nonzero, without loss of generality we may assume that $\xi_1 \neq 0$ so that we may divide ξ by ξ_1 to obtain a representative with coordinates $(1, \xi_2, \ldots, \xi_n)$. Hence, by choosing always the representative of $[\xi]$ with $\xi_1 = 1$, the set $x_1, \ldots, x_n, \xi_2, \ldots, \xi_n$ defines coordinates on some neighborhood $\mathcal{U}$ of $(x, [\xi])$ in $\mathbb{P}^*X$. On $\mathcal{U}$, consider the 1-form

   $$\alpha = dx_1 + \sum_{i \geq 2} \xi_i dx_i .$$

   Show that α is a contact form on $\mathcal{U}$, i.e., show that $\ker \alpha_{(x,[\xi])} = \mathbb{H}_{(x,[\xi])}$, and that $d\alpha_{(x,[\xi])}$ is nondegenerate on $\mathbb{H}_{(x,[\xi])}$.

4. What is the symplectization of $\mathcal{C}$?

   What is the manifold $\mathcal{C}$ when $X = \mathbb{R}^3$ and when $X = S^1 \times S^1$?

Remark. Similarly, we could have defined the **manifold of oriented contact elements** of X to be

$$\mathcal{C}^o = \left\{ (x, \chi_x^o) \;\middle|\; x \in X \text{ and } \begin{array}{c} \chi_x^o \text{ is a hyperplane in } T_xX \\ \text{equipped with an orientation} \end{array} \right\} .$$

The manifold $\mathcal{C}^o$ is isomorphic to the cotangent sphere bundle of X

$$S^*X := (T^*X \setminus \text{zero section})/ \approx$$

where $(x, \xi) \approx (x, \xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R}^+$.

A construction analogous to the above produces a canonical contact structure on $\mathcal{C}^o$. See [3, Appendix 4].

$\diamondsuit$