Homework 16: Hermitian Matrices

Let $\mathcal{H}$ be the vector space of $n \times n$ complex hermitian matrices.

The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation: $A \cdot \xi = A \xi A^{-1}$, for $A \in \mathrm{U}(n)$, $\xi \in \mathcal{H}$.

For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ complex hermitian matrices whose spectrum is λ.

1. Show that the orbits of the $\mathrm{U}(n)$-action are the manifolds $\mathcal{H}_\lambda$.
   For a fixed $\lambda \in \mathbb{R}^n$, what is the stabilizer of a point in $\mathcal{H}_\lambda$?

   Hint: If $\lambda_1, \dots, \lambda_n$ are all distinct, the stabilizer of the diagonal matrix is the torus $\mathbb{T}^n$ of all diagonal unitary matrices.

2. Show that the symmetric bilinear form on $\mathcal{H}$, $(X, Y) \mapsto \mathrm{trace}\ (XY)$, is nondegenerate.
   For $\xi \in \mathcal{H}$, define a skew-symmetric bilinear form ω_ξ on $\mathfrak{u}(n) = T_1\mathrm{U}(n) = i\mathcal{H}$ (space of skew-hermitian matrices) by

   $$\omega_\xi(X, Y) = i\,\mathrm{trace}\ ([X, Y]\xi) \ , \qquad X, Y \in i\mathcal{H} \ .$$

   Check that $\omega_\xi(X, Y) = i\,\mathrm{trace}\ (X(Y\xi - \xi Y))$ and $Y\xi - \xi Y \in \mathcal{H}$.
   Show that the kernel of ω_ξ is $K_\xi := \{Y \in \mathfrak{u}(n) \mid [Y, \xi] = 0\}$.

3. Show that K_ξ is the Lie algebra of the stabilizer of ξ.

   Hint: Differentiate the relation $A\xi A^{-1} = \xi$.

   Show that the ω_ξ's induce nondegenerate 2-forms on the orbits $\mathcal{H}_\lambda$.
   Show that these 2-forms are closed.
   Conclude that all the orbits $\mathcal{H}_\lambda$ are compact symplectic manifolds.

4. Describe the manifolds $\mathcal{H}_\lambda$.
   When all eigenvalues are equal, there is only one point in the orbit.
   Suppose that $\lambda_1 \neq \lambda_2 = \ldots = \lambda_n$. Then the eigenspace associated with λ_1 is a line, and the one associated with λ_2 is the orthogonal hyperplane. Show that there is a diffeomorphism $\mathcal{H}_\lambda \simeq \mathbb{CP}^{n-1}$. We have thus exhibited a lot of symplectic forms on $\mathbb{CP}^{n-1}$, on for each pair of distinct real numbers.
   What about the other cases?

   Hint: When the eigenvalues $\lambda_1 < \ldots < \lambda_n$ are all distinct, any element in $\mathcal{H}_\lambda$ defines a family of pairwise orthogonal lines in $\mathbb{C}^n$: its eigenspaces.

5. Show that, for any skew-hermitian matrix $X \in \mathfrak{u}(n)$, the vector field on $\mathcal{H}$ generated by $X \in \mathfrak{u}(n)$ for the $\mathrm{U}(n)$-action by conjugation is $X^\#_\xi = [X, \xi]$.