which defines a morphism of complexes

$$\Phi_{\widetilde{X}} : (C_\star(f_0), \partial_{X_0}) \longrightarrow (C_\star(f_1), \partial_{X_1}),$$

which we can prove, as before, to induce an isomorphism in the homology.

To conclude, in this case, the homology of the complex depends only on V and on the partition of the boundary as $\partial V = \partial_+ V \cup \partial_- V$.

Also note that $\Phi_\star^F$ does not depend on F. To convince yourself of this, for a different "interpolation" F', take $G = \mathrm{Id}$ and $H = F'$ in step (3) of the proof of Theorem 3.4.2.

Exercises

Exercise 13. What is the homology of the complex associated with the function defined in Exercise 12 (p. 51) and the vector field suggested in the same exercise?

Exercise 14. Let E and F be two vector subspaces of a finite-dimensional real vector space. Show that an orientation of E is an equivalence class of bases of E for the equivalence relation

$$\mathcal{B} \sim \mathcal{B}' \Longleftrightarrow \det_{\mathcal{B}} \mathcal{B}' > 0.$$

Likewise, verify that the relation

$$\mathcal{B} \sim \mathcal{B}' \Longleftrightarrow \det_{(\mathcal{B}, \mathcal{B}_0)}(\mathcal{B}', \mathcal{B}_0) > 0$$

defines an equivalence relation on the bases of the complements of F that does not depend on the chosen basis $\mathcal{B}_0$ of F. The equivalence classes are the co-orientations of F.

Verify that if E is oriented, F is co-oriented and E and F are transversal, then $E \cap F$ is co-oriented.

Exercise 15. Determine the homology of the complex $(C_\star(f; \mathbf{Z}), \partial_X)$ for the examples of Morse functions on the manifolds $\mathbf{P}^n(\mathbf{C})$, T^2 and S^n used in this book.