

# Homework 11: Complex Projective Space

The complex projective space  $\mathbb{CP}^n$  is the space of complex lines in  $\mathbb{C}^{n+1}$ :

$\mathbb{CP}^n$  is obtained from  $\mathbb{C}^{n+1} \setminus \{0\}$  by making the identifications  $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$  for all  $\lambda \in \mathbb{C} \setminus \{0\}$ . One denotes by  $[z_0, \dots, z_n]$  the equivalence class of  $(z_0, \dots, z_n)$ , and calls  $z_0, \dots, z_n$  the homogeneous coordinates of the point  $p = [z_0, \dots, z_n]$ . (The homogeneous coordinates are, of course, only determined up to multiplication by a non-zero complex number  $\lambda$ .)

Let  $\mathcal{U}_i$  be the subset of  $\mathbb{CP}^n$  consisting of all points  $p = [z_0, \dots, z_n]$  for which  $z_i \neq 0$ . Let  $\varphi_i : \mathcal{U}_i \rightarrow \mathbb{C}^n$  be the map

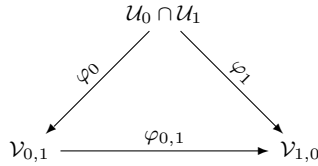
$$\varphi_i([z_0, \dots, z_n]) = \left( \frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i} \right).$$

1. Show that the collection

$$\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \dots, n\}$$

is an atlas in the *complex* sense, i.e., the transition maps are biholomorphic. Conclude that  $\mathbb{CP}^n$  is a complex manifold.

**Hint:** Work out the transition maps associated with  $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$  and  $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$ . Show that the transition diagram has the form



where  $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 \neq 0\}$  and

$$\varphi_{0,1}(z_1, \dots, z_n) = \left( \frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1} \right).$$

2. Show that the 1-dimensional complex manifold  $\mathbb{CP}^1$  is diffeomorphic, as a real 2-dimensional manifold, to  $S^2$ .

**Hint:** Stereographic projection.