Homework 2: Symplectic Volume

1. Given a vector space V, the exterior algebra of its dual space is

$$\wedge^*(V^*) = \bigoplus_{k=0}^{\dim V} \wedge^k(V^*) \ ,$$

where $\wedge^k(V^*)$ is the set of maps $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ which are linear in each entry, and for any permutation π, $\alpha(v_{\pi_1}, \ldots, v_{\pi_k}) = (\operatorname{sign} \pi) \cdot \alpha(v_1, \ldots, v_k)$. The elements of $\wedge^k(V^*)$ are known as **skew-symmetric k-linear maps** or **k-forms** on V.

(a) Show that any $\Omega \in \wedge^2(V^*)$ is of the form $\Omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$, where $u_1^*, \ldots, u_k^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ is a basis of V^* dual to the standard basis ($k + 2n = \dim V$).

(b) In this language, a symplectic map $\Omega : V \times V \to \mathbb{R}$ is just a nondegenerate 2-form $\Omega \in \wedge^2(V^*)$, called a **symplectic form** on V.

Show that, if Ω is any symplectic form on a vector space V of dimension $2n$, then the nth exterior power $\Omega^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n}$ does not vanish.

(c) Deduce that the nth exterior power ω^n of any symplectic form ω on a $2n$-dimensional manifold M is a volume form.[2]

Hence, any symplectic manifold (M, ω) is canonically oriented by the symplectic structure. The form $\frac{\omega^n}{n!}$ is called the **symplectic volume** or the **Liouville volume** of (M, ω).

Does the Möbius strip support a symplectic structure?

(d) Conversely, given a 2-form $\Omega \in \wedge^2(V^*)$, show that, if $\Omega^n \neq 0$, then Ω is symplectic.

Hint: Standard form.

2. Let (M, ω) be a $2n$-dimensional symplectic manifold, and let ω^n be the volume form obtained by wedging ω with itself n times.

(a) Show that, if M is compact, the de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ is non-zero.

Hint: Stokes' theorem.

(b) Conclude that $[\omega]$ itself is non-zero (in other words, that ω is not exact).

(c) Show that if $n > 1$ there are no symplectic structures on the sphere S^{2n}.

[2] A **volume form** is a nonvanishing form of top degree.