SEEM 5380: Optim. Methods for High–Dim. Statistics	**2016–17 Second Term**
Homework Set 2	
Instructor: Anthony Man–Cho So	Due: March 20, 2017

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (50pts). Consider the sets $\mathcal{Q}^3 = \{(t,x) \in \mathbb{R} \times \mathbb{R}^2 : \|x\|_2 \le t\}$ and $H = \{x \in \mathbb{R}^3 : x_1 = x_3\}$. Let $y = (s,u) \in \mathbb{R} \times \mathbb{R}^2$ be arbitrary. Given a non–empty closed convex set $C \subset \mathbb{R}^3$, let $\Pi_C(y)$ denote the projection of y onto C.

(a) **(20pts).** Show that

$$\Pi_{\mathcal{Q}^3}(y) = \begin{cases} y & \text{if } \|u\|_2^2 \le s^2 \text{ and } s \ge 0, \\ \mathbf{0} & \text{if } \|u\|_2^2 \le s^2 \text{ and } s \le 0, \\ \dfrac{s + \|u\|_2}{2\|u\|_2}(\|u\|_2, u) & \text{otherwise.} \end{cases}$$

(b) **(15pts).** Show that

$$\Pi_H(y) = \left(\frac{s+u_2}{2}, u_1, \frac{s+u_2}{2} \right).$$

(c) **(10pts).** Show that

$$\Pi_{\mathcal{Q}^3 \cap H}(y) = \frac{(s+u_2)_+}{2}(1,0,1).$$

(d) **(5pts).** Consider the sequence $x^k = (k,1,k) \in \mathbb{R}^3$ for $k \ge 1$. Using the results in (a)–(c), show that there does not exist a constant $\alpha > 0$ such that the inequality

$$\operatorname{dist}(x, \mathcal{Q}^3 \cap H) \le \alpha \left(\operatorname{dist}(x, \mathcal{Q}^3) + \operatorname{dist}(x, H) \right)$$

holds for all $x \in \mathbb{R}^3$.

Problem 2 (20pts). Let $X \in \mathbb{R}^{m \times n}$ be a rank–r $m \times n$ matrix with $1 \le r \le m \le n$. Suppose that

$$X = U \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^T$$

is an SVD of X, where $\Sigma = \operatorname{Diag}(\sigma_1(X), \dots, \sigma_r(X)) \in \mathbb{R}^{r \times r}$ is a diagonal matrix containing the non–zero singular values of X. Let $\|X\|_* = \sum_{i=1}^r \sigma_i(X)$ denote the nuclear norm of X. Show that

$$\partial \|X\|_* = \left\{ U \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & W \end{bmatrix} V^T : \|W\| \le 1 \right\},$$

where I_r is the $r \times r$ identity matrix and $\|W\|$ is the spectral norm (largest singular value) of W.

Problem 3 (30pts). Consider the problem

$$\min_{X \in \mathbb{R}^{2 \times 2}} \{ h(\mathcal{A}(X)) + \|X\|_* \}, \qquad (P)$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is the function given by

$$h(y) = \frac{1}{2} \left\| B^{1/2} y - B^{-1/2} d \right\|_2^2 \quad \text{with} \quad B = \begin{bmatrix} 3/2 & -2 \\ -2 & 3 \end{bmatrix} \succ \mathbf{0} \text{ and } d = \begin{bmatrix} 5/2 \\ -1 \end{bmatrix},$$

and $\mathcal{A} : \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ is the linear operator given by

$$\mathcal{A}(X) = (X_{11}, X_{22}).$$

Using the optimality condition of Problem (P) and the result in Problem 2, show that

$$\bar{X}^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is the unique optimal solution to (P).