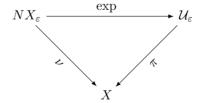
Homework 5: Tubular Neighborhoods in \mathbb{R}^n

1. Let X be a k-dimensional submanifold of an n-dimensional manifold M. Let x be a point in X. The **normal space** to X at x is the quotient space

$$N_x X = T_x M / T_x X$$
,

and the **normal bundle** of X in M is the vector bundle NX over X whose fiber at x is N_xX .

- (a) Prove that NX is indeed a vector bundle.
- (b) If M is \mathbb{R}^n , show that N_xX can be identified with the usual "normal space" to X in \mathbb{R}^n , that is, the orthogonal complement in \mathbb{R}^n of the tangent space to X at x.
- 2. Let X be a k-dimensional compact submanifold of \mathbb{R}^n . Prove the **tubular neighborhood theorem** in the following form.
 - (a) Given $\varepsilon>0$ let $\mathcal{U}_{\varepsilon}$ be the set of all points in \mathbb{R}^n which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p\in\mathcal{U}_{\varepsilon}$ has a *unique* nearest point $\pi(p)\in X$.
 - (b) Let $\pi: \mathcal{U}_{\varepsilon} \to X$ be the map defined in (a) for ε sufficiently small. Show that, if $p \in \mathcal{U}_{\varepsilon}$, then the line segment $(1-t) \cdot p + t \cdot \pi(p)$, $0 \le t \le 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_{\varepsilon}$.
 - (c) Let $NX_{\varepsilon}=\{(x,v)\in NX \text{ such that } |v|<\varepsilon\}$. Let $\exp:NX\to\mathbb{R}^n$ be the map $(x,v)\mapsto x+v$, and let $\nu:NX_{\varepsilon}\to X$ be the map $(x,v)\mapsto x$. Show that, for ε sufficiently small, \exp maps NX_{ε} diffeomorphically onto $\mathcal{U}_{\varepsilon}$, and show also that the following diagram commutes:



3. Suppose that the manifold X in the previous exercise is not compact.

Prove that the assertion about \exp is still true provided we replace ε by a continuous function

$$\varepsilon: X \to \mathbb{R}^+$$

which tends to zero fast enough as x tends to infinity.