Homework 8: Compatible Linear Structures

1. Let $\Omega(V)$ and $J(V)$ be the spaces of symplectic forms and complex structures on the vector space V, respectively. Take $\Omega \in \Omega(V)$ and $J \in J(V)$. Let $\mathrm{GL}(V)$ be the group of all isomorphisms of V, let $\mathrm{Sp}(V, \Omega)$ be the group of symplectomorphisms of (V, Ω), and let $\mathrm{GL}(V, J)$ be the group of complex isomorphisms of (V, J).

   Show that

   $$\Omega(V) \simeq \mathrm{GL}(V)/\mathrm{Sp}(V,\Omega) \quad \text{and} \quad J(V) \simeq \mathrm{GL}(V)/\mathrm{GL}(V,J) \ .$$

   Hint: The group $\mathrm{GL}(V)$ acts on $\Omega(V)$ by pullback. What is the stabilizer of a given Ω?

2. Let $(\mathbb{R}^{2n}, \Omega_0)$ be the standard $2n$-dimensional symplectic euclidean space.

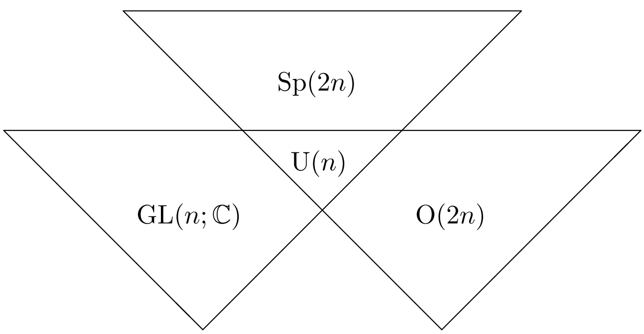
   The **symplectic linear group** is the group of all linear transformations of $\mathbb{R}^{2n}$ which preserve the symplectic structure:

   $$\mathrm{Sp}(2n) := \{A \in \mathrm{GL}(2n;\mathbb{R}) \mid \Omega_0(Au, Av) = \Omega_0(u,v) \text{ for all } u, v \in \mathbb{R}^{2n}\} \ .$$

   Identifying the complex $n \times n$ matrix $X + iY$ with the real $2n \times 2n$ matrix $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, consider the following subgroups of $\mathrm{GL}(2n;\mathbb{R})$:

   $$\mathrm{Sp}(2n) \ , \ \mathrm{O}(2n) \ , \ \mathrm{GL}(n;\mathbb{C}) \ \text{ and } \ \mathrm{U}(n) \ .$$

   Show that the intersection of any two of them is $\mathrm{U}(n)$. (From [83, p.41].)



3. Let (V, Ω) be a symplectic vector space of dimension $2n$, and let $J : V \to V$, $J^2 = -\mathrm{Id}$, be a complex structure on V.

   (a) Prove that, if J is Ω-compatible and L is a lagrangian subspace of (V, Ω), then JL is also lagrangian and $JL = L^\perp$, where $\perp$ denotes orthogonality with respect to the positive inner product $G_{_J}(u, v) = \Omega(u, Jv)$.

   (b) Deduce that J is Ω-compatible if and only if there exists a symplectic basis for V of the form

   $$e_1, e_2, \ldots, e_n, f_1 = Je_1, f_2 = Je_2, \ldots, f_n = Je_n$$

   where $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$.