Homework 3:
Tautological Form and Symplectomorphisms

This set of problems is from [53].

1. Let (M, ω) be a symplectic manifold, and let α be a 1-form such that

$$\omega = -d\alpha \ .$$

   Show that there exists a unique vector field v such that its interior product with ω is α, i.e., $\imath_v \omega = -\alpha$.

   Prove that, if g is a symplectomorphism which preserves α (that is, $g^*\alpha = \alpha$), then g commutes with the one-parameter group of diffeomorphisms generated by v, i.e.,

$$(\exp tv) \circ g = g \circ (\exp tv) \ .$$

   Hint: Recall that, for $p \in M$, $(\exp tv)(p)$ is the *unique* curve in M solving the ordinary differential equation

$$\begin{cases} \frac{d}{dt}(\exp tv(p)) = v(\exp tv(p)) \\ (\exp tv)(p)|_{t=0} = p \end{cases}$$

   for t in some neighborhood of 0. Show that $g \circ (\exp tv) \circ g^{-1}$ is the one-parameter group of diffeomorphisms generated by $g_* v$. (The push-forward of v by g is defined by $(g_* v)_{g(p)} = dg_p(v_p)$.) Finally check that g preserves v (that is, $g_* v = v$).

2. Let X be an arbitrary n-dimensional manifold, and let $M = T^*X$. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X, and let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be the corresponding coordinates on $T^*\mathcal{U}$.

   Show that, when α is the tautological 1-form on M (which, in these coordinates, is $\sum \xi_i \, dx_i$), the vector field v in the previous exercise is just the vector field $\sum \xi_i \frac{\partial}{\partial \xi_i}$.

   Let $\exp tv$, $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v.

   Show that, for every point $p = (x, \xi)$ in M,

$$(\exp tv)(p) = p_t \quad \text{where} \quad p_t = (x, e^t \xi) \ .$$

3. Let M be as in exercise 2.

   Show that, if g is a symplectomorphism of M which preserves α, then

$$g(x,\xi) = (y,\eta) \quad \Longrightarrow \quad g(x,\lambda\xi) = (y,\lambda\eta)$$

   for all $(x,\xi) \in M$ and $\lambda \in \mathbb{R}$.

   Conclude that g has to preserve the cotangent fibration, i.e., show that there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x,\xi) = x$.

   Finally prove that $g = f_\#$, the map $f_\#$ being the symplectomorphism of M lifting f.

   Hint: Suppose that $g(p) = q$ where $p = (x,\xi)$ and $q = (y,\eta)$.
   Combine the identity

$$(dg_p)^* \alpha_q = \alpha_p$$

   with the identity

$$d\pi_q \circ dg_p = df_x \circ d\pi_p \ .$$

   (The first identity expresses the fact that $g^*\alpha = \alpha$, and the second identity is obtained by differentiating both sides of the equation $\pi \circ g = f \circ \pi$ at p.)

4. Let M be as in exercise 2, and let h be a smooth function on X. Define $\tau_h : M \to M$ by setting

$$\tau_h(x,\xi) = (x, \xi + dh_x) \ .$$

   Prove that

$$\tau_h^* \alpha = \alpha + \pi^* dh$$

   where π is the projection map

$$\begin{array}{ccc} M & & (x,\xi) \\ \downarrow \pi & & \downarrow \\ X & & x \end{array}$$

   Deduce that

$$\tau_h^* \omega = \omega \ ,$$

   i.e., that τ_h is a symplectomorphism.