Homework 15: Legendre Transform

This set of problems is adapted from [54].

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. f is called **strictly convex** if $f''(x) > 0$ for all $x \in \mathbb{R}$. Assuming that f is strictly convex, prove that the following four conditions are equivalent:

   (a) $f'(x) = 0$ for some point x_0,

   (b) f has a local minimum at some point x_0,

   (c) f has a unique (global) minimum at some point x_0,

   (d) $f(x) \to +\infty$ as $x \to \pm\infty$.

   The function f is **stable** if it satisfies one (and hence all) of these conditions.

   For what values of a is the function $e^x + ax$ stable? For those values of a for which it is not stable, what does the graph look like?

2. Let V be an n-dimensional vector space and $F : V \to \mathbb{R}$ a smooth function. The function F is said to be **strictly convex** if for every pair of elements $p, v \in V$, $v \neq 0$, the restriction of F to the line $\{p + xv \,|\, x \in \mathbb{R}\}$ is strictly convex.

   The **hessian** of F at p is the quadratic form

   $$d^2F_p : v \longmapsto \frac{d^2}{dx^2} F(p + xv)|_{x=0} \ .$$

   Show that F is strictly convex if and only if d^2F_p is positive definite for all $p \in V$.

   Prove the n-dimensional analogue of the result you proved in (1). Namely, assuming that F is strictly convex, show that the four following assertions are equivalent:

   (a) $dF_p = 0$ at some point p_0,

   (b) F has a local minimum at some point p_0,

   (c) F has a unique (global) minimum at some point p_0,

   (d) $F(p) \to +\infty$ as $p \to \infty$.

3. As in exercise 2, let V be an n-dimensional vector space and $F : V \to \mathbb{R}$ a smooth function. Since V is a vector space, there is a canonical identification $T_p^*V \simeq V^*$, for every $p \in V$. Therefore, we can define a map

   $$L_F : V \longrightarrow V^* \qquad \textbf{(Legendre transform)}$$

   by setting

   $$L_F(p) = dF_p \in T_p^*V \simeq V^* \ .$$

   Show that, if F is strictly convex, then, for every point $p \in V$, L_F maps a neighborhood of p diffeomorphically onto a neighborhood of $L_F(p)$.

4. A strictly convex function $F : V \to \mathbb{R}$ is **stable** if it satisfies the four equivalent conditions of exercise 2. Given any strictly convex function F, we will denote by S_F the set of $l \in V^*$ for which the function $F_l : V \to \mathbb{R}$, $p \mapsto F(p) - l(p)$, is stable. Prove that:

   (a) The set S_F is open and convex.

   (b) L_F maps V diffeomorphically onto S_F.

   (c) If $\ell \in S_F$ and $p_0 = L_F^{-1}(\ell)$, then p_0 is the unique minimum point of the function F_ℓ.

   Let $F^* : S_F \to \mathbb{R}$ be the function whose value at l is the quantity $-\min_{p \in V} F_l(p)$. Show that F^* is a smooth function.

   The function F^* is called the **dual** of the function F.

5. Let F be a strictly convex function. F is said to have **quadratic growth at infinity** if there exists a positive-definite quadratic form Q on V and a constant K such that $F(p) \geq Q(p) - K$, for all p. Show that, if F has quadratic growth at infinity, then $S_F = V^*$ and hence L_F maps V diffeomorphically onto V^*.

6. Let $F : V \to \mathbb{R}$ be strictly convex and let $F^* : S_F \to \mathbb{R}$ be the dual function. Prove that for all $p \in V$ and all $\ell \in S_F$,

$$F(p) + F^*(\ell) \geq \ell(p) \qquad \textbf{(Young inequality)} \ .$$

7. On one hand we have $V \times V^* \simeq T^*V$, and on the other hand, since $V = V^{**}$, we have $V \times V^* \simeq V^* \times V \simeq T^*V^*$.

   Let α_1 be the canonical 1-form on T^*V and α_2 be the canonical 1-form on T^*V^*. Via the identifications above, we can think of both of these forms as living on $V \times V^*$. Show that $\alpha_1 = d\beta - \alpha_2$, where $\beta : V \times V^* \to \mathbb{R}$ is the function $\beta(p, \ell) = \ell(p)$.

   Conclude that the forms $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ satisfy $\omega_1 = -\omega_2$.

8. Let $F : V \to \mathbb{R}$ be strictly convex. Assume that F has quadratic growth at infinity so that $S_F = V^*$. Let Λ_F be the graph of the Legendre transform L_F. The graph Λ_F is a lagrangian submanifold of $V \times V^*$ with respect to the symplectic form ω_1; why? Hence, Λ_F is also lagrangian for ω_2.

   Let $\mathrm{pr}_1 : \Lambda_F \to V$ and $\mathrm{pr}_2 : \Lambda_F \to V^*$ be the restrictions of the projection maps $V \times V^* \to V$ and $V \times V^* \to V^*$, and let $i : \Lambda_F \hookrightarrow V \times V^*$ be the inclusion map. Show that

$$i^*\alpha_1 = d(\mathrm{pr}_1)^*F \ .$$

   Conclude that

$$i^*\alpha_2 = d(i^*\beta - (\mathrm{pr}_1)^*F) = d(\mathrm{pr}_2)^*F^* \ ,$$

   and from this conclude that the inverse of the Legendre transform associated with F is the Legendre transform associated with F^*.