(1) An appropriate model can be developed by sampling the phenomenon of interest.
- Select a known distribution through educated guesses.
- Make estimate of the parameters
- Test for goodness of fit.

Probability distribution model

Discrete Random Variable

→ X is a discrete random variable if the number of possible values of X is finite, or countable infinite

→ R_X = possible values of X (Range space of X) = {0, 1, 2, ...}

$P(x_i)$ = probability the random variable X is x_i, $p(x_i) = P(X = x_i)$

variable associated with an event

a variable to define a set of event.

→ $p(x_i)$, $i = 1, 2, \ldots$ must satisfy:

1. $p(x_i) \geq 0$, for all i
2. $\sum_{i=1}^{\infty} p(x_i) = 1$.

→ The collection of pairs $(x_i, p(x_i))$, $i = 1, 2, \ldots$, is called the probability distribution of X, and

$p(x_i)$ is called the probability mass function (PMF) of X.

MATHEMATICS CONCEPTS

Continuous Random Variable

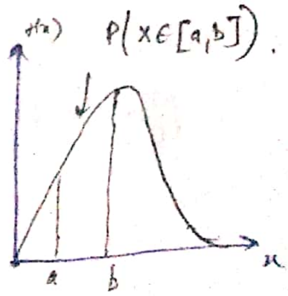
→ X is a continuous random variable if its range space R_X is an interval or a collection of intervals.

→ The probability that X lies in the interval $[a, b]$ is given by:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

probability density function.

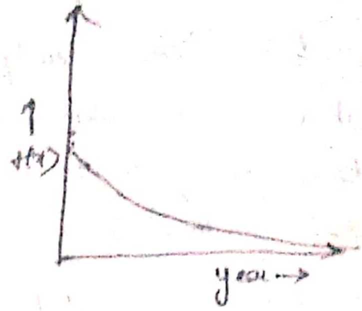
→ ~~f(x) is called the~~
- $f(x) \geq 0$, for all x in R_X
- $\int_{R_X} f(x)\,dx = 1$
- $f(x) = 0$, if x is not in R_X



→ Properties: (i) $P(X = x_0) = 0$,
(ii) $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$

(i) PDF : (life on an inspection device)

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



Ans : $= \frac{1}{2}\int_0^{\infty} e^{-x/2}\,dx$

$$= \frac{\frac{1}{2}}{-\frac{1}{2}}\left[e^{-\infty/2} - e^{-0/2}\right] = 1$$

$-x/2 = t$

$-\frac{dx}{2} = dt$

$\frac{1}{2}\int_0^{-\infty} e^{t}\, (-2)\,dt$

$-\int_0^{-\infty} e^{t}\,dt$

$-\left[e^{-\infty} - e^{0}\right]$

$-\left[0 - 1\right]$

$= 1$

$$= \frac{1}{2}\left[\left[x e^{-x/2}[-2]\right]_0^{\infty} + \int_0^{\infty} e^{-x/2}(+2)\,dx\right]$$

Mean $= \frac{1}{2}\int_0^{\infty} x e^{-x/2}\,dx$

$$= \frac{1}{2}\left[-2(-2)\left[+2\right]\right] = 2$$

$\frac{1}{2}(-2)\left[e^{-x/2}\right.$

$\int_0^{\infty} e^{-x/2}\,dx$

$= -2\left[e^{-\infty} - e^{0}\right]$

$= 2$

$\therefore$ for distribution

$$\therefore f(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Mean $= \lambda$.

CUMULATIVE DISTRIBUTION FUNCTION ($F(x)$)

$$F(x) = P(X \leq x).$$

- If X is discrete: $F(x) = \sum_{x_i \leq x} p(x_i) \rightarrow$ probability distribution of x
- If X is continuous: $F(x) = \int_{-\infty}^{x} f(t)\,dt$ $\rightarrow$ probability distribution func.
- $P(a \leq X \leq b) = F(b) - F(a)$

EXPECTED VALUE (m or μ)

- if X is discrete $\quad E(X) = \sum_{\text{all } i} x_i \, p(x_i)$
- if X is continuous $\quad E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx.$
- also known as mean, m or μ.
- Measure of central tendency

VARIANCE ($V(X)$ or $Var(X)$ or σ^2)

- Definition : $V(X) = E\left((X - E[X])^2\right)$

also $\quad V(X) = E(X^2) - (E(X))^2$

STANDARD DEVIATION (SD or σ)

- Interpret σ always together with the mean.
- Standard deviation of two diff. data sets may be difficult to compare

Eg: for inspection device :

$E(X) = 2, \quad V(X) = E(X^2) - (2)^2.$

$\therefore \quad E(X^2) = \int_0^{\infty} x^2 e^{-x/2} \, dx :$

$= \left[x^2(-2)e^{-x/2}\right]_0^{\infty} + \int_0^{\infty} 2x(+2)e^{-x/2} \, dx$

$= 4\int_0^{\infty} x e^{-x/2} dx = \underline{\underline{8}}.$

$\therefore \quad V(X) = 8 - 4 = 4.$

$\sigma = 2$

COEFF. OF VARIATION

→ Ratio of standard deviation to the mean.

- Normalized measure of dispersion

$$C.O.V = \frac{\text{standard deviation}}{\text{mean}} = \frac{\sigma}{\mu}, \quad \mu > 0.$$

→ Used to compare different datasets, instead of standard deviation.

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MEAN AND VARIANCE OF SUMS

- If $x_1, x_2, \ldots x_k$ are k random variable and if $a_1, a_2, \ldots a_k$ are k constants, then

$$E(a_1x_1 + a_2x_2 + \ldots + a_kx_k) = a_1E(x_1) + a_2E(x_2) + \ldots$$

- for independent variables

$$Var(a_1x_1 + a_2x_2 + \ldots + a_kx_k) = a_1^2\, Var(x_1) + a_2^2\, Var(x_2) + \ldots$$

COVARIANCE

(i) Given two random variables x and y with μ_x and μ_y, their covariance is defined as.

$$\underline{Cov(x,y)} = \sigma_{xy}^2 = E[(x-\mu_x)(y-\mu_y)] = E(xy) - E(x)E(y)$$

measures dependency of x and y.

(ii) for Independent Variables, Covariance is zero since

$$E(xy) = E(x)E(y)$$

CORRELATION COEFFICIENT

(i) The normalized value of covariance is called the correlation or simply correlation,

$$Correlation(x,y) = \rho_{x,y} = \frac{\sigma_{xy}^2}{\sigma_x\, \sigma_y}.$$

Range $[-1,1]$

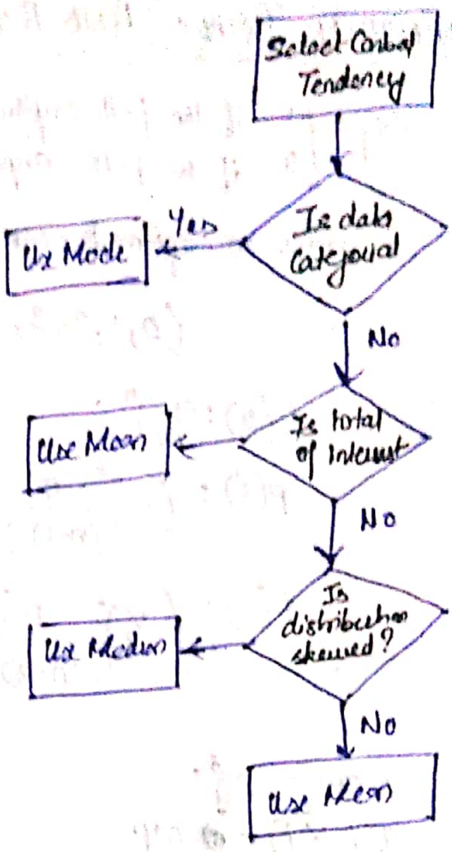
QUANTILE

The x value at which the CDF takes a value α is called α-quantile or 100α-percentile. Denoted by x_α.

$$P(X \le x_\alpha) = F(x_\alpha) = \alpha \quad , \alpha \in [0,1]$$

- Median $x_{0.5}$, is the 50 percentile or 0.5-quantile.
- Mode: for which $p_i(x)$ is highest or $f(x)$ is heighest.

SELECTING CENTRAL TENDENCY



RELATIONSHIP BETWEEN SIMULATION AND PROBABILITY THEORY

CENTRAL LIMIT THEOREM

- Let Z_n be the random variable

$$Z_n = \frac{X(n) - \mu}{\sqrt{\sigma^2/n}}$$

- and $F_n(z)$ be the distribution of Z_n for a sample of size of n, i.e. $F_n(z) = P(Z_n \le z)$ then

$$F_n(z) \xrightarrow[n\to\infty]{} \Phi(z)$$

Normal distribution.

- where $\Phi(z)$ is normal distribution with $\mu = 0$ and $\sigma^2 = 1$.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy \quad \text{for } -\infty < z < \infty .$$

STRONG LAW OF LARGE NUMBERS

- Let $X_1, X_2, \ldots, X_n$ be IID random variable with mean μ.

$$\bar{X}(n) \xrightarrow[n\to\infty]{} \mu \text{ with probability } 1.$$

Sample mean.

DISCRETE DISTRIBUTIONS

→ Bernoulli trials and Bernoulli distribution
→ Binomial distribution
→ Geometric and negative binomial distribution
→ Poisson distribution.

Bernoulli Trials and Bernoulli Distribution.

→ $X_j = \begin{cases} 1 & \text{if the } j\text{-th experiment is a success} \\ 0 & \text{if the } j\text{-th experiment is a failure.} \end{cases}$

consider an experiment consisting of n trials, Range space

$$\{0,1,2,3,\ldots\ldots n\}$$

$P(0) : (1/2)^n$

$P(1) : (1/2)^n \frac{n!}{(n-1)!}$

$P(2) : (1/2)^n \frac{n!}{(n-2)!\,2!}$

$\boxed{\begin{matrix} P(0) = 1-p \\ P(1) = 1. \end{matrix}}$ for $n=1$.

$P(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x=0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$

Mean $= p$.

$E(X_j) = np$

$V(X_j) = p(1-p) = npq$

→ n Bernoulli trials when trials are independent

$$P(x_1, x_2, \ldots, x_n) = P_1(x_1)\,P_2(x_2) \ldots P_n(x_n).$$

Binomial Distribution

Geometric Distribution.

→ The number of Bernoulli trials, X to achieve the 1st success :

$$P(x) = \begin{cases} q^{x-1}p, & x = 0,1,2\ldots n \\ 0, & \text{otherwise.} \end{cases}$$

$P(1) = p$

$P(2) = qp.$

$E(x) = 1/p$ and $V(x) = q/p^2$

$E(x) = \sum_{i=0} q^{x-1} p\,(x) =$

$= p + 2qp + 3q^2p + \ldots$

$S = p(1 + 2q + 3q^2 + 4q^3 \ldots)$

~~$\int S\,ds = p(q + q^2 + q^3 \ldots q^n)$~~

~~$S^2 = p(pq(1+q+q^2+q^{n-1}) = \frac{pq(q^n-1)}{q-1}$~~

$x = 1 + 2q + 3q^2 + 4q^3$

$\frac{d}{dq}(q + q^2 + q^3 \ldots$

$xq = q + 2q^2 + 3q^3 + 4q$

~~$x(q-1) = 0$~~

$x(p) = 1 + q + q^2 + \ldots$

NEGATIVE BINOMIAL DISTRIBUTION

- The number of Bernoulli trials, X, until the k^{th} success.
- parameters p and k;

$$p(x) = \begin{cases} {}^{x-1}C_{k-1}\, q^{x-k} p^{k} & x = k, k+1, k+2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- $E(X) = k/p$ $\quad V(X) = kq/p^2$

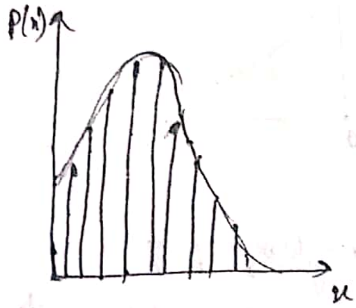
POISSON DISTRIBUTION.

→ Describes many Random processes quite well, and is mathematically quite simple.

- when $\alpha > 0$, PDF and CDF.

$$p(x) = \begin{cases} \frac{\alpha^x}{x!} e^{-\alpha}, & x = 0, 1, \ldots \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \sum_{i=0}^{x} \frac{\alpha^i}{i!} e^{-\alpha}$$

$$E(X) = \alpha = V(X).$$



CONTINUOUS DISTRIBUTIONS

To describe random phenomena in which the variable can take on any value in some interval.

- Uniform
- Exponential
- Weibull
- Normal or Gaussian Distribution function.
- Lognormal

UNIFORM DISTRIBUTION

→ A random variable X is uniformly distributed on the interval (a, b), $U(a, b)$, if its PDF and CDF are

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

$$E(x) = \frac{(a+b)}{2} \qquad V(x) = \frac{(b-a)^2}{12}.$$

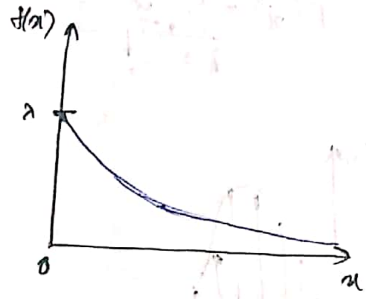
→ $U(0,1)$ provides the means to generate random numbers, from which random variates can be generated.

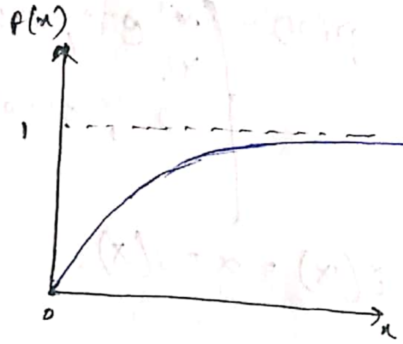
EXPONENTIAL DISTRIBUTION

→ A random variable X is exponentially distributed with parameter $\lambda > 0$ if its PDF and CDF are:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases} \qquad F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t}\,dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

$$E(X) = 1/\lambda \qquad V(X) = 1/\lambda^2.$$





memoryless property

(i) for all s and t greater than or equal to 0:

$$P(X > s+t \mid \underbrace{X > s}_{\text{given } X>s}) = P(X > t) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X > t)$$

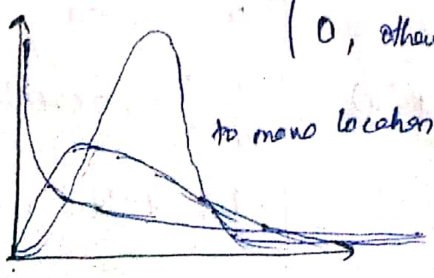
$$= 1 - P(X \leq t)$$

$$= e^{-\lambda t}.$$

WEIBULL DISTRIBUTION.

→ A random variable X has a Weibull distribution if its PDF has the form:

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{\left(-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right)}, & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$



to move location

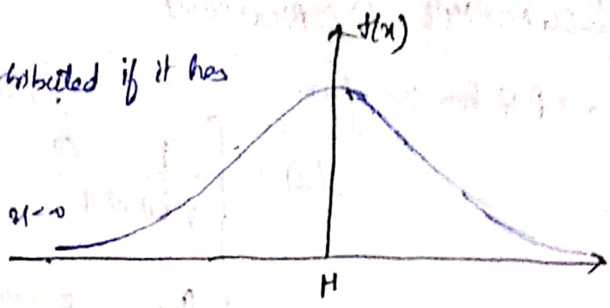
parameters:

(i) location: ν $(-\infty < \nu < \infty)$

(ii) Scale parameter: β, $(\beta > 0)$

(iii) Shape parameter: α, $(\alpha > 0)$

NORMAL DISTRIBUTION



→ A random variable X is normally distributed if it has the PDF:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

Mean: μ.

Variance: $\sigma^2 > 0$.

Denoted as $X \sim N(\mu, \sigma^2)$

→ Properties.

- $\lim_{x \to -\infty} f(x) = 0$, and $\lim_{x \to \infty} f(x) = 0$
- $f(\mu - x) = f(\mu + x)$; the PDF is symmetric about μ.
- Maximum value of PDF occurs at $x = \mu$.
  - ↳ Mean = Mode.

→ Evaluating the distribution

- Independent of μ and σ, using the standard normal distribution:

$$Z \sim N(0,1)$$

- Transformation of variables: let

$$Z = \frac{X - \mu}{\sigma};$$

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z)\, dz = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad \text{where } \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

also $\Phi(-x) = 1 - \Phi(x)$

→
- Most commonly used distribution in data analytics.
- The sum of n independent normal variates is a normal variate.
- The sum of a large number of independent observations from any distribution has a normal distribution.

LOGNORMAL DISTRIBUTION

→ Pdf has the form

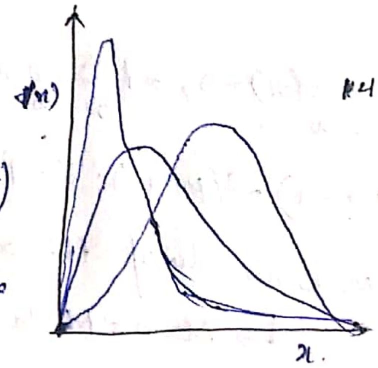
$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma x} e^{\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right]}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Mean $E(X) = e^{\mu + \sigma^2/2}$

Variance $V(X) = e^{2\mu + \sigma^2/2}\left(e^{\sigma^2} - 1\right)$

→ Relationship with normal distribution

- When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim \text{lognormal}(\mu, \sigma^2)$
- Parameters μ and σ^2 are not the random variable X.



POISSON PROCESS

→ Definition: $N(t)$ is a counting function that represents the number of events occurred in $[0, t]$.

→ A counting process $\{N(t), t \geq 0\}$ is a poisson process with mean rate λ if:

- Arrivals occur one at a time.
- $\{N(t), t \geq 0\}$ has stationary increments.
  → No of arrivals in $[t, t+s]$ depends only on s, not on starting point t
  → Arrivals are completely random.
- $\{N(t), t \geq 0\}$ has independent increments
  → Number of arrivals during non-overlapping time intervals are independent
  → future arrivals occur completely random.

→ Properties,

$$P[N(t)=n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \text{ for } t \geq 0 \text{ and } n = 0,1,2..$$

Equal mean and variance: $E[N(t)] = V[N(t)] = \lambda t$.

- Stationary increment:
  The number of arrivals in time s to t, with $s<t$, is also poisson-distributed with mean $\lambda(t-s)$.

POISSON PROCESS : INTERARRIVAL TIMES

- Interarrival times are exponentially distributed and independent with mean $1/\lambda$, if the arrival time follows a poisson distribution process

EMPIRICAL DISTRIBUTIONS

→ A distribution whose parameters are the observed values in a sample of data.

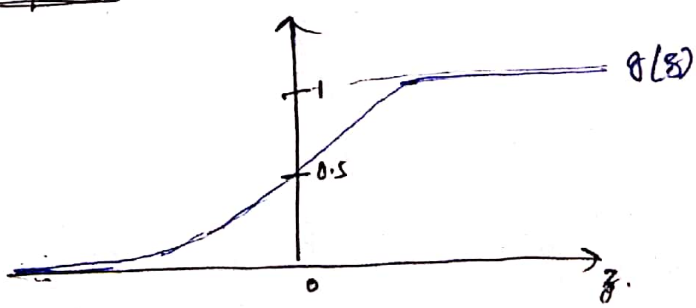
POISSON PROCESS : SPLITTING AND POOLING

Anaconda Ide

Libraries : pscwy, numpy, pandas, nltk, tensorflow, matplotlib, beautiful soup, regex.

Sigmoid function / logistic function

$$g(z) = \frac{1}{1+e^{-z}}$$



Hypothesis test : Goodness of fit

Day :	M	T	W	T	F	S	
Expected % :	10	10	15	20	30	15	→ percentage distribution
Observed :	30	14	34	45	57	20	

Total : 200

H_0 (Null hypothesis) : Owner's distribution is correct

H_1 : Not correct distribution (should not rely on owner's distribution)

α : significance level = 0.05 = 5%

Checking, it has a chi-square distribution, I am checking the probability of getting the result.

Assuming : 20 20 30 40 60 30 total = 200
(What should have been correct)

Chi² statistic (χ^2) $= \left(\frac{(30-20)^2}{20}\right) + \left(\frac{14-20}{20}\right)^2 + \left(\frac{34-30}{30}\right)^2$

$\frac{(\text{error})^2}{\text{expected}}$

$+ \left(\frac{45-40}{45}\right)^2 + \left(\frac{60-57}{60}\right)^2 + \left(\frac{20-30}{30}\right)^2$

$= \frac{100}{20} + \frac{36}{20} + \frac{16}{30} + \frac{25}{45} + \frac{9}{60} + \frac{100}{30}$

$= 11.44$.

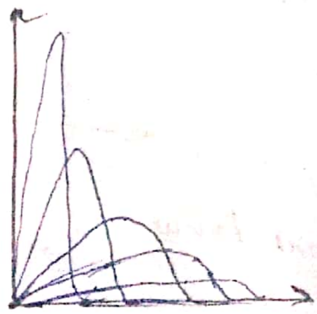
Degree's of freedom : 5 ⟶ Critical χ^2 value ≈ 11.07

Since, 11.44 > 11.07, (the Critical Value) we will reject our null hypothesis

Rayleigh Distribution

$f(x : \sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0$

σ ↳ scale parameter



$F(x ; \sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}$

CDF

Variance : $E[x] = \sigma\sqrt{\pi/2}$

$Var[x] = \sigma^2\left(\frac{4-\pi}{2}\right)$

IID process - Independent & Identically distributed

Suppose two processes (H, T) in a coin

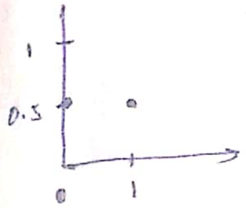
~~Getting Head on coin toss~~

$X = \{1, 0\}$

if Head occur — if tails occur

$P(1)$ or $P(H) = 1/2$

$P(0)$ or $P(E) = 1/2$

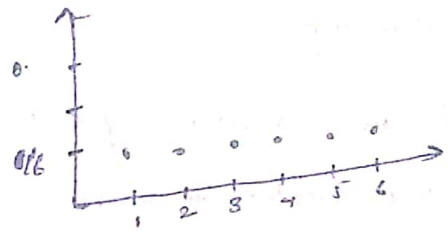


Throwing a dice

$X = \{1, 2, 3, 4, 5, 6\}$

randomvariable — sample space

$P(1) = P(2) = P(3) = P(4) \dots = 1/6$.



Distribution: with what probability are the set of numbers distributed through an event

event provides distribute to a set of numbers, carry a parameter.

$X_{xy} = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(0,1), (0,2), (0,3), (0,4), (0,5), (0,6)\}$

Equal chance in the one — Individual process

expected random variable to occur

$\mu_x = E(X)$

$= 1(0.5) = 0.5$

$\mu_y =$

$E(X) = (1+0) \times 0.5$

$= 0.5$

$V(X) = E(X - \mu_x)^2$

$= 0.5((0.5)^2 + (-0.5)^2)$

$= 0.5(0.5) = 0.25$

$S.D = 0.5$

50% chance of being wrong

$E(X) = (1+2+3+4+5+6) \times 1/6$

$= \frac{6 \times 3 + 3}{6} = \frac{7}{2} = 3.5$

$V(x) = E((X - \mu_x)^2)$

$= 1/6\left((2.5)^2 + (1.5)^2 + (0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2\right)$

$= 2/6(6.25 + 2.25 + 0.25)$

$= 1/3(8.75) = 2.916$

$S.D = 1.7076$ for mean.

Covariance

$\sigma^2_{xy} = E((x - \mu_x)(y - \mu_y))$

$= \frac{1}{12}\left((0.5)[-2.5 - 1.5 - 0.5 + 0.5 + 1.5 + 2.5] + (-0.5)[0]\right)$

$= 0$ (from our independent process)

Two process are independent if

$P(X \cap Y) = P(X)P(Y)$ where X, Y are sample space for random variables.

Q. If X and Y are random variables with probability mass functions P_x and P_y, then if $P_x = P_y$, then X and Y are said to be identically distributed.

Fourier Transform

→ All waveforms can be represented as a sum of sine waves.

→ finding fourier transformation:

$$f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \end{cases} \qquad F(\lambda) = \int_0^\infty f(u).e^{-i\lambda u}.du$$

$\lambda \rightarrow$ frequency, ω.

Given: $e^{-x} \rightarrow e^{-u}$

$$\therefore F(\lambda) = \int_0^\infty e^{-u}.e^{-i\lambda u}.du = \int_0^\infty e^{-u(1+\lambda i)}du$$

$$= \left[\frac{e^{-u(1+\lambda i)}}{-(1+i\lambda)}\right]_0^\infty = \frac{e^{-0}}{1+i\lambda} = \frac{1}{1+\lambda i} \times \frac{1-\lambda i}{1-\lambda i}$$

$$= \frac{1-\lambda i}{1^2-(\lambda i)^2} = \frac{1-\lambda i}{1+\lambda^2}$$

$$= \frac{1-\lambda i}{1+\lambda^2}$$

→ Find fourier transform of

$$f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1. \end{cases}$$ Hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$.

Ans: $F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x).e^{isx}dx.$

$$= \int_{-\infty}^{-1} f(x)e^{isx}dx + \int_{-1}^{1} f(x)e^{isx}dx + \int_1^\infty f(x)e^{isx}dx$$

$$= \int_{-1}^{1} e^{isx}dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is} = \frac{2\sin s}{s}.$$

→ $F\{f(x)\} = F(s) = \frac{2\sin s}{s}$, since $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Now, by inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}e^{-isx}ds.$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx} ds.$$

Put $x=0$,

$$f(0) = 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} ds.$$

$$\int_{-\infty}^{\infty} \frac{\sin s\, ds}{s} = \pi$$ even ~~odd~~ function. $f(-x) = f(x)$.

$$2\int_{0}^{\infty} \frac{\sin s}{s} = \pi$$

or $$\int_{0}^{\infty} \frac{\sin s}{s} = \pi/2.$$

→ Ubuntu:

MATHEMATICAL CONCEPTS

(i) VECTORS

- A vector is an array of number

$$x = [x_1\ x_2\ x_3 \cdots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$x_{-1}: \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix}$ (A vector containing all except x_1).

- Interpretation.

$r = v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow$ a Line in a N dimensional space (3D space here)

$$\vec{r} = [\hat{i}\ \hat{j}\ \hat{k}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\hat{i} + b\hat{j} + c\hat{k}$$

- Matrices are transformation on a set of vectors, or a set of lines

$$\underset{r}{\begin{bmatrix} x \\ y \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underset{r'}{\begin{bmatrix} x' \\ y' \end{bmatrix}}$$

(ii) • Dot product (Measures what degree are two vectors alligned).

(for non-vector) $\underbrace{\hat{a}.\hat{b}}_{\text{dot product}} = |a||b| \underbrace{\cos\theta}_{\text{angle between } \hat{a} \text{ and } \hat{b}}$.

for vectors

$$\hat{a}.\hat{b} = ab^T = [a\ b\ c] \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf.$$

- Cross product (A vector perpendicular to both $\hat{a}$ and $\hat{b}$)

(for vector) $\underline{\hat{a} \times \hat{b}} = |a||b| \sin\theta$

(follows Right-Hand Rule; ie, we follow a right-handed co-ordinate system")

(ii) MATRIX.

- Upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times3}$$

- Lower triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Tridiagonal

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

- Symmetric, $a_{ij} = a_{ji}$

Skew-symmetric, $a_{ij} = -a_{ji}$

(iii) BASIC OPERATIONS

- Trace = sum of diagonal elements.
- Augmentation = addition of a column or colums to the initial matrix

$C = (A|B)$ → Augmenting operator

(iv) RANK OF MATRIX

- Rank equals the order of highest-order non singular submatrix.

Eg: 2×3 order matrix, $R = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

Transformation, $R \equiv \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$\therefore$ rank = 1

- $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 9 & 10 & 11 \end{bmatrix}$ → Since, $|A| = 0$, rank $\neq 3$
Since, here determinant $\neq 0$,
$\therefore$ rank = 2.

(v) INVERSE OF A MATRIX

- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$

A → non singular if inverse exist.

$A^{-1} = \dfrac{C^T}{|A|}$ → Cofactor matrix.

- ~~Deter~~ if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\quad |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

- if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cofactor = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$\text{Cofactor}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(vi) ORTHONORMAL BASIS

Basis : a space is totally defined by a set of vector.

Ortho-Normal : orthogonal + normal → magnitude is one.
dot product is zero

Orthogonal : $AA^T = I \Rightarrow A^{-1}AA^T = A^{-1} \rightarrow A^T = A^{-1}$
I → magnitude 1.

(~~Orthonormal~~)

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$$x = [1\ 0\ 0]^T = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$y = [0\ 1\ 0]^T = \begin{bmatrix}0\\1\\0\end{bmatrix}$$ — Basis

$$x \cdot y = x \cdot y^T = \underset{3\times1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \underset{1\times3}{[0\ 1\ 0]} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$y \cdot z = x \cdot z$$

$$z = [0\ 0\ 1]^T = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

X, Y, Z is an orthonormal basis, We can describe any 3D point as a linear combination of these vectors.

for any vector.

$$\underset{3\times3}{\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}} \underset{3\times3}{\begin{bmatrix}u_1 & v_1 & n_1\\u_2 & v_2 & n_2\\u_3 & v_3 & n_3\end{bmatrix}} = \underset{3\times3}{\begin{bmatrix}au_1 & av_1 & an_1\\bu_2 & bv_2 & bn_2\\cu_3 & cv_3 & cn_3\end{bmatrix}}$$

$$= \begin{bmatrix}a.u + b.u + cu\\a.v + bv + cv\\a.n + b.n + cn\end{bmatrix}$$

To change a point from one coordinate system to another, compute the dot product of each coordinate row with each of the basis vectors

vi) Eigen Value and Vector

→ Problem : if A is a $n \times n$ matrix, do there exist non zero vectors x in R^n such that

$$Ax = \lambda x$$
↳ non zero vector in $\mathbb{R}^n$.

to find eigen value

$$|A'x| = \lambda|x|$$
$$|x||A-\lambda| = 0.$$

ie for $A = \begin{bmatrix}4 & 1\\3 & 2\end{bmatrix}$

$$\begin{vmatrix}4-\lambda & 1\\3 & 2-\lambda\end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3 = 0$$
$$\Rightarrow 8 - 6\lambda + \lambda^2 - 3 = 0$$
$$\lambda^2 - 6\lambda + 5 = 0$$
$$(\lambda-5)(\lambda-1) = 0$$

$\lambda_1 = 5$ $\lambda_2 = 5$.

∴ eigen vector

$$\begin{bmatrix}4-\lambda & 1\\3 & 2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$$

$$\begin{bmatrix}1\\-3\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix}$$

Note $(\lambda I - A)x = 0$

has non zero solution for x iff $\det(\lambda I - A) = 0$.

Characteristic equation $|A - \lambda I| = 0$.

Note: $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad \left(b=3,\ c=2 \quad x = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} = \frac{-2}{2}, \frac{-4}{2} \right) \quad -1, -2$$

Q1 $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ so $A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$-(1-\lambda)(4+\lambda) + 6 = 0$$

$$-(4 + \underbrace{\lambda - 4\lambda}_{-3\lambda} - \lambda^2) + 6 = 0$$

$$\lambda^2 + 3\lambda \cancel{-4+6} = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$-4+1$

$$\lambda = -1, -2$$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \cancel{3} & -2 \\ 3 & -\cancel{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - 2x_2 \\ 3x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \cancel{\begin{bmatrix} -x_1 - 2x_2 \\ 3x_1 - 6x_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = x_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\cancel{-x - 2x_2}$

$x = -2x_2$

$\cancel{3x_1 = 6x_2}$

$\frac{x_1}{x_2} = 2/3 \qquad 3x_1 = 2x_2$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

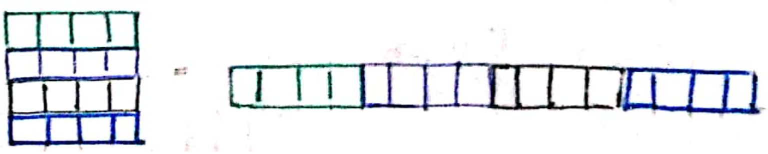
(vi) Tensors (A)

→ Generalisation of an n-dimensional array,

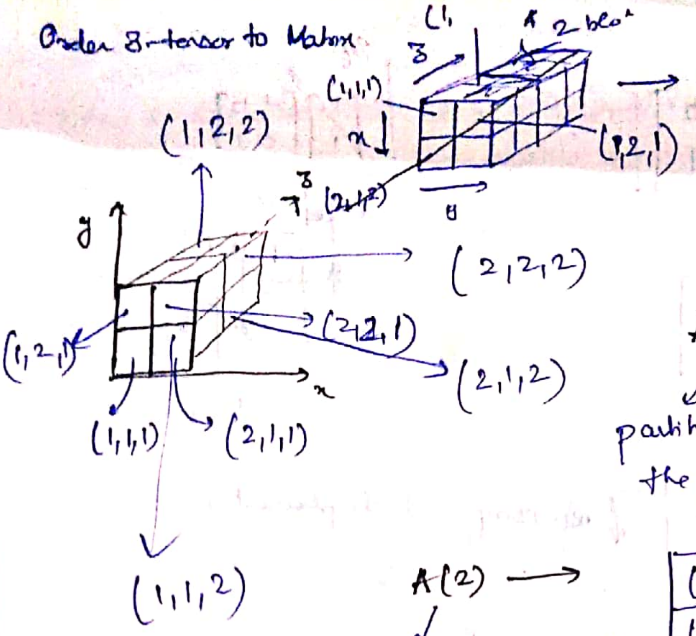
→ A matrix can represent a binary relation

- A tensor can represent an n-ary relation.
- A tensor can represent a set of binary relations.

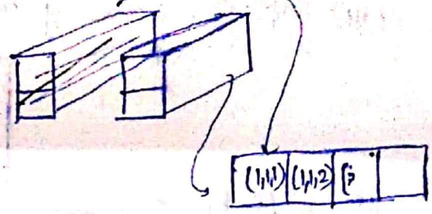
(vii) Tensors (A) Reshaping.



Order 3-tensor to Matrix



$\rightarrow$ A(1)



A(1) $\rightarrow$

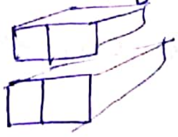
(1,1,1)	(1,1,2)	(1,2,1)	(1,2,2)
(2,1,1)	(2,1,2)	(2,2,1)	(2,2,2)

partition along the y-z axis plane

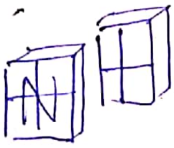
A(2) $\longrightarrow$

(1,1,1)	(2,1,1)	(1,1,2)	(2,1,2)
(1,2,1)	(2,2,1)	(1,2,2)	(2,2,2)

partition across the x-z axis plane



A(3) $\longrightarrow$



(1,1,1)	(1,2,1)	(2,1,1)	(2,2,1)
(1,1,2)	(1,2,2)	(2,1,2)	(2,2,2)

across z ~~x axis~~ x-y plane

(x)

(x) Application of Matrices.

(i) Scaling of a vector. $\rightarrow$ $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ (diagonal matrix with $\lambda = 0.5$

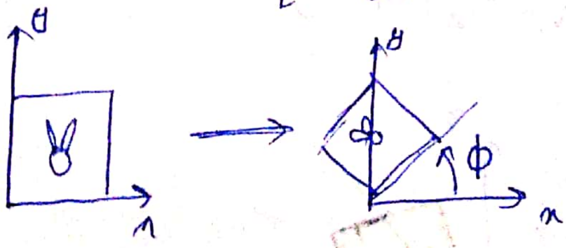
$$\begin{bmatrix} u \\ v \end{bmatrix} [0.5] = \begin{bmatrix} 0.5u \\ 0.5v \end{bmatrix}$$

$\hookrightarrow$ scale $(\underset{x}{0.5}, \underset{y}{0.5}, \underset{z}{0.5})$

$\therefore$ general scaling $\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$

(ii) Rotation (ϕ)

$$\text{rot}(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$



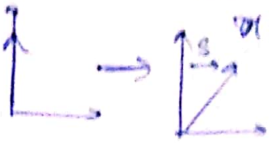
(iii) Reflection across y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(iv) Reflection in x axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(v) Shear-x (s)



x coordinate $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ (by y-component displacement)

vi) Shear-y (s)

$$\begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix}$$ (x-component displacement)

(X) LINEAR TRANSFORMATIONS

→ Scale, Reflection, Rotation, Shear are Linear transformation because they satisfy

$$T(a\underline{U} + b\underline{V}) = aT(U) + bT(V)$$

Linear, Vectors

(XI) COMPOSING LT (Linear transformation)

- $T_2T_1(V) \overset{def}{=} T_2(T_1(V))$

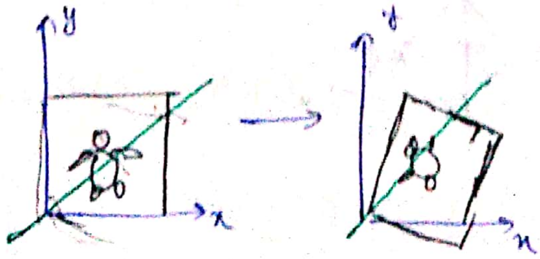
  transformation

- if T_2 and T_1 are Linear and are represented by matrices M_2 & M_1

  T_2T_1 represented M_2M_1

$$T_2T_1(V) = T_2(T_1(V)) = (M_2M_1)(V)$$

(XII)



(xii) DECOMPOSING LINEAR TRANSFORM

→ Any 2D linear transformation can be decomposed into the product of a rotation, a scale, and a rotation if the scale can have negative numbers

$$M = R_1 S R_2$$

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

(XIII) Translation. The origin move

$$(x, y) \longrightarrow (x+a, y+b)$$

(IVX) Homogeneous Coordinates Embed the xy plane in R^3 at $z=1$

$$(x, y) \longrightarrow (x, y, 1)$$

(XV) 2D linear transformations to 3D Matrices

eg
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

V vector

$\Downarrow$ 3D

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$

eg: A 3D shear that acts as a translation on the plane $z=1$

for 2D in x direction
(1) shear by a and y-direction by b →
(2) ~~shear y by b~~
~~in y direction~~

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & ab+1 \end{bmatrix}}$$

$$(x, y, 1) \longrightarrow (x+a, y+b, 1)$$

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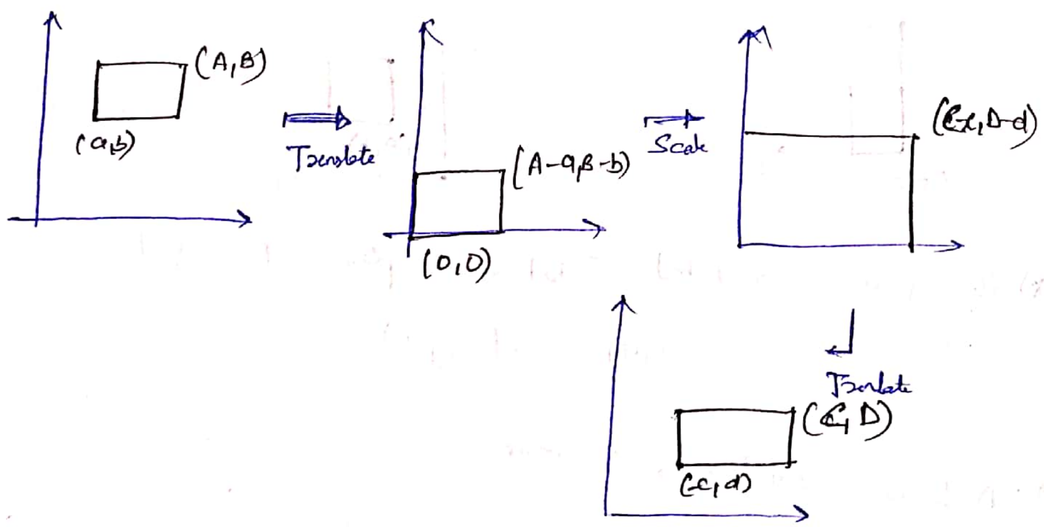
(XVI) 2D-Affine Transformation

→ An affine transformation is any transformation that preserves co-linearity.
(ie all points lying on a line initially still lie on a line after transformation)
and ratio of distance (eg midpoint of a line segment, remains the midpoint after transformation.

→ With homogeneous coordinates, we can represent all 2D affine transformations as 3D linear transformations.

→ We can use matrix multiplication to transform objects

(XVII) Windowing Transforms



(XIX) 3D Affine Transformations

$$Scale(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$translate(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

that ln

$$rotate_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rotate_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$