Linear Algebra Review

Machine Learning

Dinesh K. Vishwakarma, Ph.D.

Overview

- Scalars, Vectors, Matrices and Tensors
- Basic matrix operations (+, -, *)
- Cross and dot products
- Determinants and inverses
- Orthonormal basis
- Tensors

Introduction

- Linear algebra is a branch of mathematics that is widely used throughout science and engineering.
- However, because linear algebra is a form of continuous rather than discrete mathematics, many computer scientists have little experience with it.
- A good understanding of linear algebra is essential for understanding and working with many machine learning algorithms, especially deep learning algorithms.
- Linear Algebra is a branch of mathematics that lets you concisely describe coordinates and interactions of planes in higher dimensions and perform operations on them.

Scaler, Vector, Matrix

Scalar

24

Vector

$\begin{bmatrix} 2 & -8 & 7 \end{bmatrix}$

row

or column $\begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$

Matrix

$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$

row(s) × column(s)

Vector and Matrix

- We want to buy "Apple" and "Banana" at two occasions.
- First occasion two Apple and 5 banana and cost is ₹25. $2A + 5B = 25$
- Second occasion 5 Apple and 7 Banana and Cost is 30. $5A + 7B = 30$

$$\begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \end{bmatrix}$$

Matrix

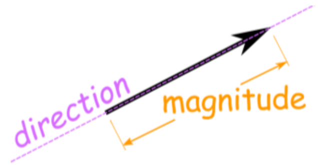
Vector

Scalars

- **Scalars:** A scalar (*a*) is just a single number, in contrast to most of the other objects studied in linear algebra, which are usually arrays of multiple numbers.
- It is written in italics and usually scalars are given in lower-case variable names.
- It can be "integer or real number".
- Scalars are real numbers used in linear algebra.
- Scalar: variable described by a single number (magnitude)
  - **Temperature = 20 °C**
  - **Density = 1 g.cm^{-3}**
  - **Image intensity (pixel value) = 2546 a. u.**

Vectors

- A vector has magnitude (size) and direction.
- A vector is an array of numbers.

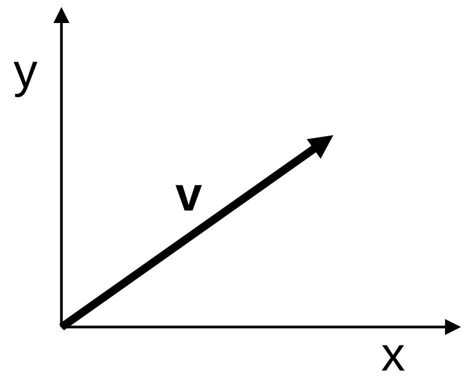


- $x = [x_1\, x_2\ x_3\ \ \dots\dots..x_n]^{\,T}$
- x_{-1} vector containing all element except x_1 .
- Similarly, if $S = \{x_1\, x_2\ x_5\}$
- x_{-S} vector containing all elements except $x_1\, x_2$ and x_5.

Vector Operations

- Vector: 1 x N matrix
- Interpretation: a line in N dimensional space.
- Dot Product, Cross Product, and Magnitude defined on vectors only.

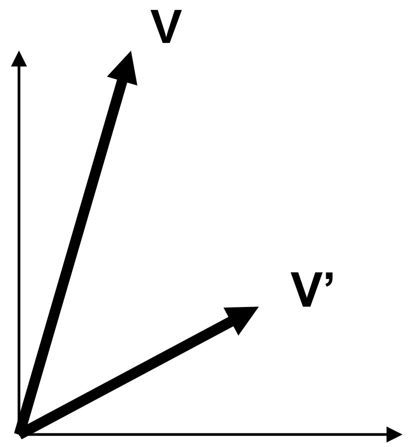
$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Interpretation

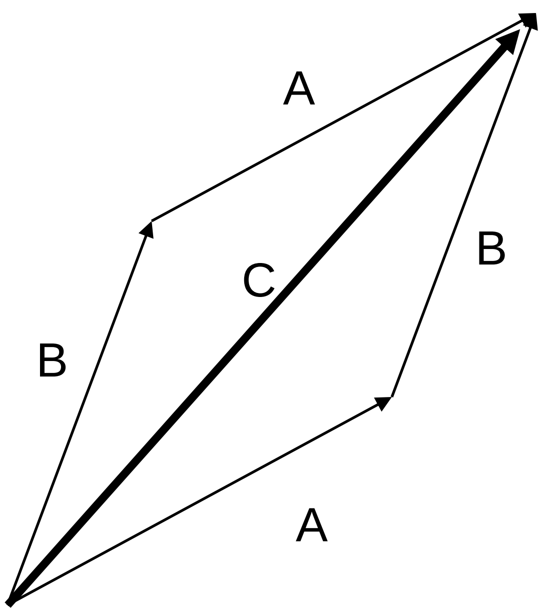
- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Vectors: Dot Product

- Interpretation: the dot product measures to what degree two vectors are aligned



A+B = C
(use the head-to-tail method to combine vectors)

Vectors: Dot Product

$$a \cdot b = ab^T = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

$$\|a\|^2 = aa^T = \sqrt{aa + bb + cc}$$

The magnitude is the dot product of a vector with itself

$$a \cdot b = \|a\| \|b\| \cos(\theta)$$

The dot product is also related to the angle between the two vectors – but it doesn't tell us the angle

Vectors: Cross Product

- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the cosine of the angle between A and B
- The direction of C follows the **right hand rule** – this why we call it a "right-handed coordinate system"

$$\|a \times b\| = \|a\| \|b\| \sin(\theta)$$

What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

rows

columns

(a) 3×2 matrix (3 rows, 2 columns) $\quad \mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

(b) $\mathrm{r} \times \mathrm{c}$ matrix $\quad \mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1c} \\ b_{21} & b_{22} & \cdots & b_{2c} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rc} \end{bmatrix}$ $\quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$

Types of Matrix

Square

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

Strictly Upper Triangular

$$\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Lower Triangular

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Diagonal

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Tridiagonal

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Types of Matrix...

Unit

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Null

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Symmetry $a_{ij} = a_{ji}$

$$\begin{bmatrix} 1.00 & -0.23 & -0.27 \\ -0.23 & 1.00 & 0.64 \\ -0.27 & 0.64 & 1.00 \end{bmatrix}$$

Skew-symmetric matrix: a square matrix in which $a_{ij} = -a_{ji}$ for all *i* and *j*

Transpose Matrix: $A = (A^T)^T$ $\quad a_{ij}^T = a_{ji}$

$$\mathbf{A} = \begin{bmatrix} 140 & 35 & 7.7 \\ 195 & 12 & 7.1 \\ 283 & 53 & 6.4 \\ 132 & 188 & 8.3 \\ 60 & 55 & 6.5 \end{bmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 140 & 195 & 283 & 132 & 60 \\ 35 & 12 & 53 & 188 & 55 \\ 7.7 & 7.1 & 6.4 & 8.3 & 6.5 \end{bmatrix}$$

Basic Operations

- Addition, Subtraction, Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \ldots \\ \ldots & \ldots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & \ldots \\ \ldots & \ldots \end{bmatrix}$$

Is AB = BA? Maybe, but maybe not! Multiplication is not commutative

Basic Operations…

- ***trace*** of a square matrix = sum of diagonal elements.
- ***matrix augmentation***: addition of a column or columns to the initial matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad \mathbf{A}_a = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Rank of Matrix E.g.1

- A matrix of ***r*** rows and ***c*** columns is said to be of order ***r*** by ***c***. If it is a square matrix, ***r*** by ***r***, then the matrix is of order ***r***.
- The rank of a matrix equals the order of highest-order nonsingular submatrix.

$$2 \times 3 \text{ order matrix,} \qquad \mathbf{R} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

$$3 - Sub \text{ Matrices } \mathbf{R}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Each of these has a determinant of 0, so the rank is less than 2. Thus the rank of **R** is 1.

Rank of Matrix E.g.2

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 9 & 10 & 11 \end{bmatrix}$$

Since $|\mathbf{A}|=0$, the rank is not 3. The following submatrix has a nonzero determinant:

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2(3) - 4(1) = 2$$

Thus, the rank of **A** is 2.

Inverse of a Matrix

- Identity matrix: **AI = A**
- Some matrices have an inverse, such that: **AA^{-1} = I**
- Inversion is tricky: **(ABC)$^{-1}$ = C^{-1}B^{-1}A^{-1}**
- Derived from non-commutativity property.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the inverse of a matrix **A** exists, then **A** is said to be <u>*nonsingular*</u> <u>and if d</u>oes not exist, then **A** is said to be <u>*singular*</u>.

<u>Properties</u>

A^{-1} only exists if **A** is **square** (n x n)

If **A^{-1}** exists then **A** is **non-singular** (invertible)

(A B) $^{-1}$ = B^{-1} A^{-1}; B^{-1} A^{-1} A B = B^{-1} B = I

(A^{T}) $^{-1}$ = (A^{-1})^{T}; (A^{-1})^{T} A^{T} = (A A^{-1})^{T} = I

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = 1$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-1} = \left(a_{ij}^{(-1)}\right), \quad a_{ij}^{(-1)} = \frac{C_{ji}}{\det \mathbf{A}} = \frac{\tilde{C}_{ij}}{\det \mathbf{A}}$$

Determinant of a Matrix

- If det(A) = 0, then A has no inverse.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{bmatrix}$$

$$\det(\mathbf{A}) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{1j}(-1)^{(1+j)} M_{1j}$$

Properties

Determinants are defined only for square matrices.

If det(**A**) = 0, **A** is singular, $\mathbf{A}^{-1}$ does not exist

If det(**A**) $\neq$ 0, **A** is non-singular, $\mathbf{A}^{-1}$ exists

Orthonormal Basis

- Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis
- Ortho-Normal: orthogonal + normal
- Orthogonal: dot product is zero
- Normal: magnitude is one
- **Orthogonal** $AA^T = I, \Longrightarrow A^{-1}AA^T = A^{-1}I, \Longrightarrow IA^T = A^{-1}I$
- $A^T = A^{-1}$

Orthonormal Basis

$$x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad x \cdot y = 0$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad x \cdot z = 0$$

$$z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \qquad y \cdot z = 0$$

X, Y, Z is an orthonormal basis. We can describe any 3D point as a linear combination of these vectors.

How do we express any point as a combination of a new basis **U, V, N**, given **X, Y, Z**?

Orthonormal Basis

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} u_1 & v_1 & n_1 \\ u_2 & v_2 & n_2 \\ u_3 & v_3 & n_3 \end{bmatrix} = \begin{bmatrix} a \cdot u + b \cdot u + c \cdot u \\ a \cdot v + b \cdot v + c \cdot v \\ a \cdot n + b \cdot n + c \cdot n \end{bmatrix}$$

(not an actual formula – just a way of thinking about it)

To change a point from one coordinate system to another, compute the dot product of each coordinate row with each of the basis vectors.

Eigen Value and Vector

- Eigenvalue problem (one of the most important problems in the linear algebra):
  - If A is an $n \times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?
  - (The term eigenvalue is from the German word *Eigenwert*, meaning "proper value")

- Eigen Value and Eigen Vector

A: an $n \times n$ matrix

λ: a scalar (could be **zero**)

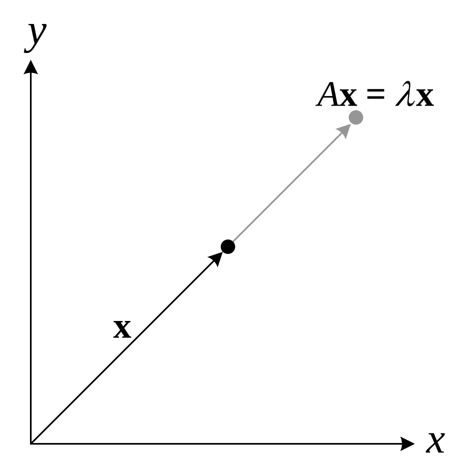
$\mathbf{x}$: a **nonzero** vector in R^n

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvector

Geometric Interpretation



Eigen Values

- To find the eigenvalues of: $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
- Solve the characteristic equation: $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$
- That is: $(\lambda - 1)(\lambda - 5) = 0$
- Eigenvalues are: $\lambda_1 = 1;\ \lambda_2 = 5$

Let x be a ***Eigen Vector*** $\begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

for each λ, *eigen vector can be computed*

$\boldsymbol{\lambda = 1},\ \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$3x_1$+x_2=0 or $x_2 = -3x_1$,

Hence Eigen Vector : $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\boldsymbol{\lambda = 5},\ \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-x_1$+x_2=0 or $x_2 = x_1$,

Hence Eigen Vector : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen Value…

- Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$
  - ✓ Let A be an $n\times n$ matrix.
  - ✓ An eigenvalue of A is a scalar λ such that $(\lambda I - A)\mathbf{x} = \mathbf{0}$
  - ✓ The eigenvectors of A corresponding to λ are the nonzero solutions of $\det(\lambda I - A) = 0$

- **Note: following the definition of the eigenvalue problem**

$$A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; A\mathbf{x} = \lambda I\mathbf{x} \;\Rightarrow\; (\lambda I - A)\mathbf{x} = \mathbf{0} \quad \text{(homogeneous system)}$$

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nonzero solutions for $\mathbf{x}$ iff $\det(\lambda I - A) = 0$

- **Characteristic equation of A:** $\det(\lambda I - A) = 0$

- **Characteristic polynomial of $A \in M_{n\times n}$:**

$$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

Eigen Value E.x.

$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ so $A - \lambda I = \begin{bmatrix} 1 - \lambda & -2 \\ 3 & -4 - \lambda \end{bmatrix}$

$\det(A - \lambda I) = (1 - \lambda)(-4 - \lambda) - (3)(-2)$
$= \lambda^2 + 3\lambda + 2$

Set $\lambda^2 + 3\lambda + 2$ to 0

Then = λ = (-3 +/- sqrt(9-8))/2

So the two values of λ are -1 and -2.

Eigen Vector E.x.

Once you have the eigenvalues, you can plug them into the equation A**x** = λ**x** to find the corresponding sets of eigenvectors **x**.

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \begin{aligned} x_1 - 2x_2 &= -x_1 \\ 3x_1 - 4x_2 &= -x_2 \end{aligned}$$

$$\boxed{\begin{aligned} &(1) \quad 2x_1 - 2x_2 = 0 \\ &(2) \quad 3x_1 - 3x_2 = 0 \end{aligned}}$$

These equations are not independent. If you multiply (2) by 2/3, you get (1).

The simplest form of (1) and (2) is $x_1 - x_2 = 0$, or just $x_1 = x_2$.

Eigen Vector E.x...

- Since $\boldsymbol{x_1} = \boldsymbol{x_2}$, we can represent all eigenvectors for eigenvalue -1 as multiples of a simple basis vector: $E = t\begin{bmatrix}1\\1\end{bmatrix}$, where t is a parameter. So $[1\ 1]^T$, $[4\ 4]^T$, $[3000\ 3000]^T$ are all possible eigenvectors for eigenvalue -1.

- For the second eigenvalue (-2) we get

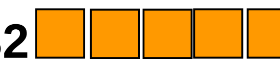
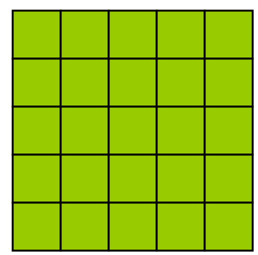
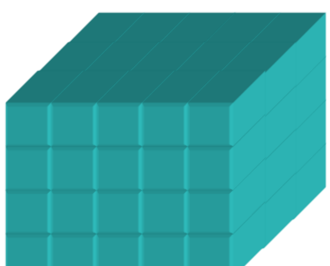
$$x_1 - 2x_2 = -2x_1$$
$$3x_1 - 4x_2 = -2x_2$$

- $\begin{bmatrix}1 & -2\\3 & -4\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix} = -2\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}$

$$\boxed{\begin{array}{ll}(1) & 3x_1 - 2x_2 = 0\\(2) & 3x_1 - 2x_2 = 0\end{array}}$$

so eigenvectors are of the form $t\begin{bmatrix}2\\3\end{bmatrix}$

Tensors (A)

- Generalization of an n-dimensional array.
- Array of number arranged on a regular grid with variable number of axes is called tensor. $(\mathbf{A}_{i,j,k})$
- A multidimensional array.
  - **Scalar:** **11** 
  - **Vector : (order-1 tensor) 10, 11, 12, 32** 
  - **Matrix : (order-2 tensor)** $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{6} & \mathbf{7} & \mathbf{8} \end{pmatrix}$ 
  - **Tensor:** 

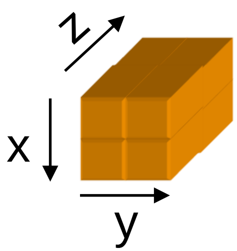
Why tensor?

- Tensors can be used when matrices are not enough.
- A matrix can represent a binary relation.
  - A tensor can represent an n-ary relation
    - E.g. subject–predicate–object data
  - A tensor can represent a set of binary relations
    - Or other matrices

Tensors (A) Reshaping

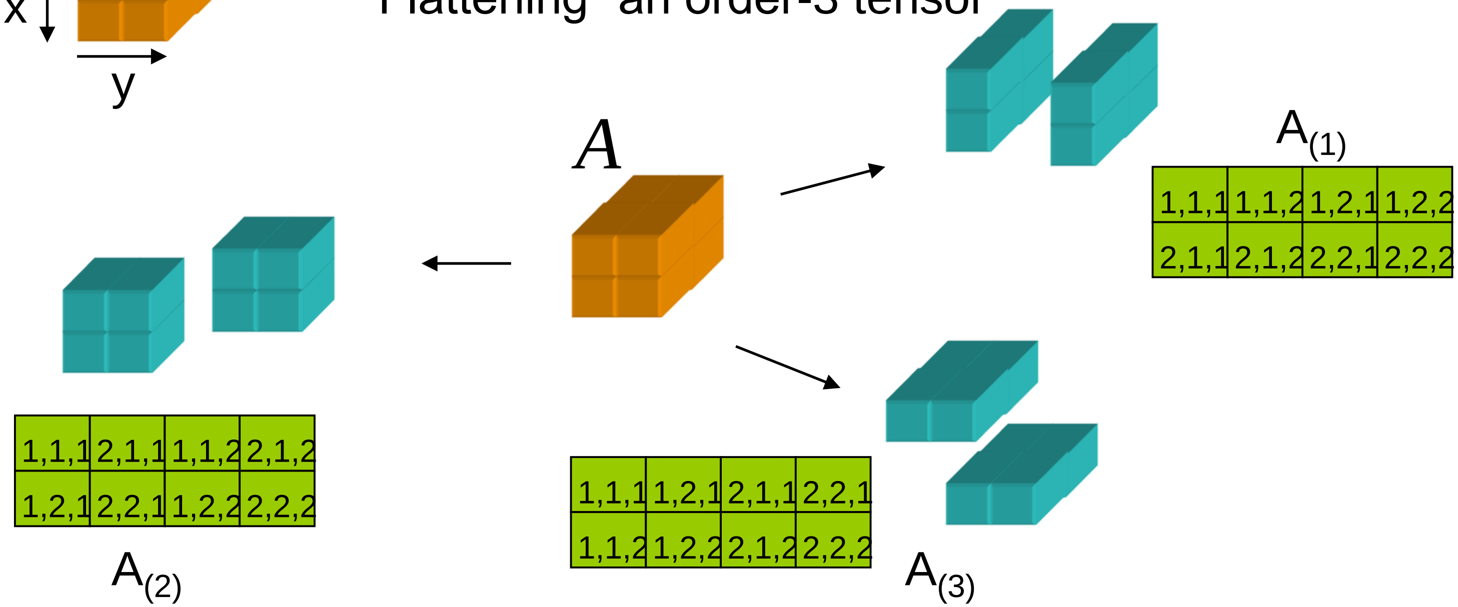
Matrix to vector

"Vectorizing" a matrix



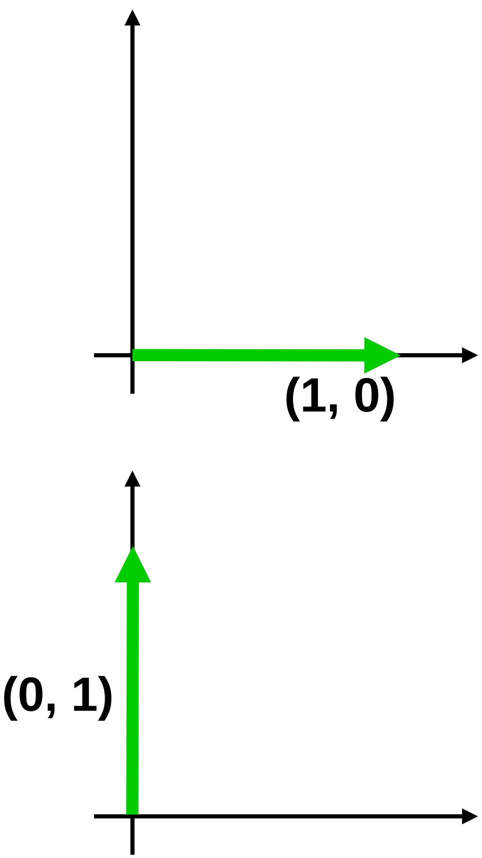
Order-3 tensor to matrix

"Flattening" an order-3 tensor

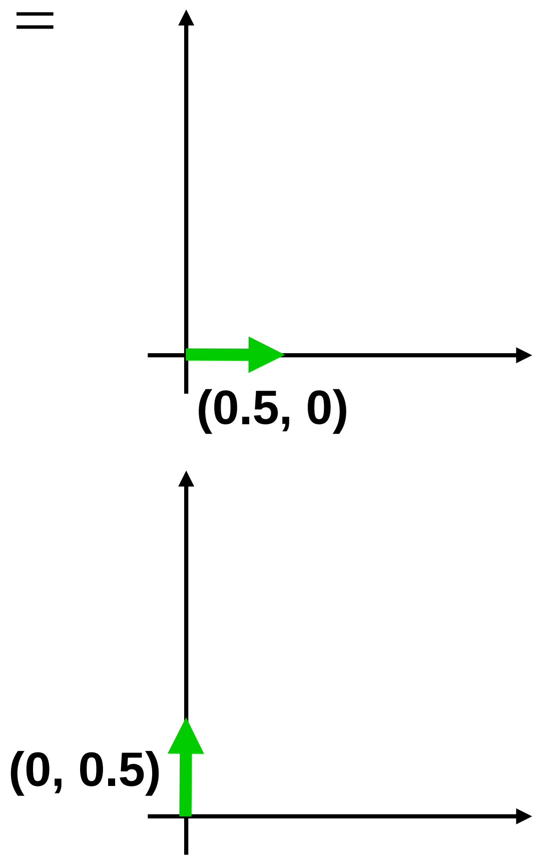


Application of Matrices

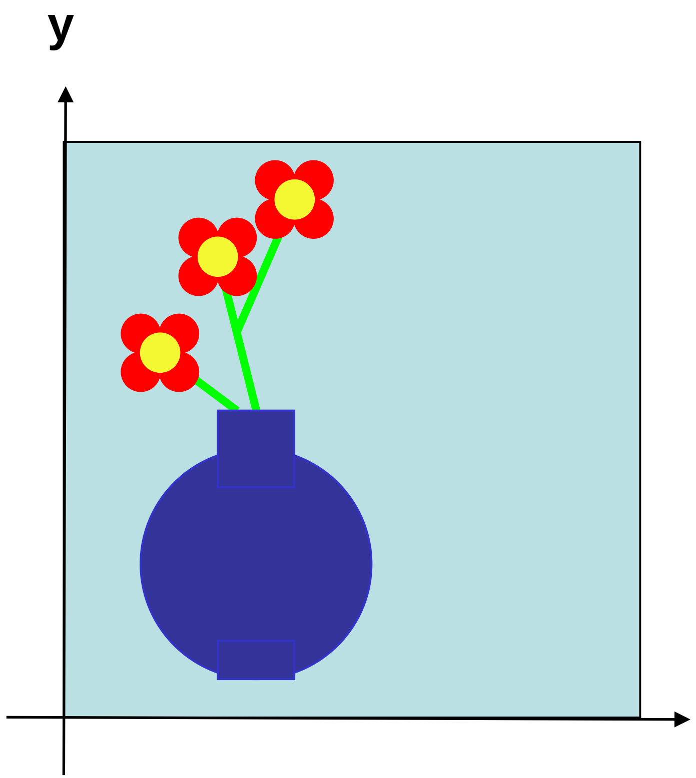
Scaling of a vector by .5



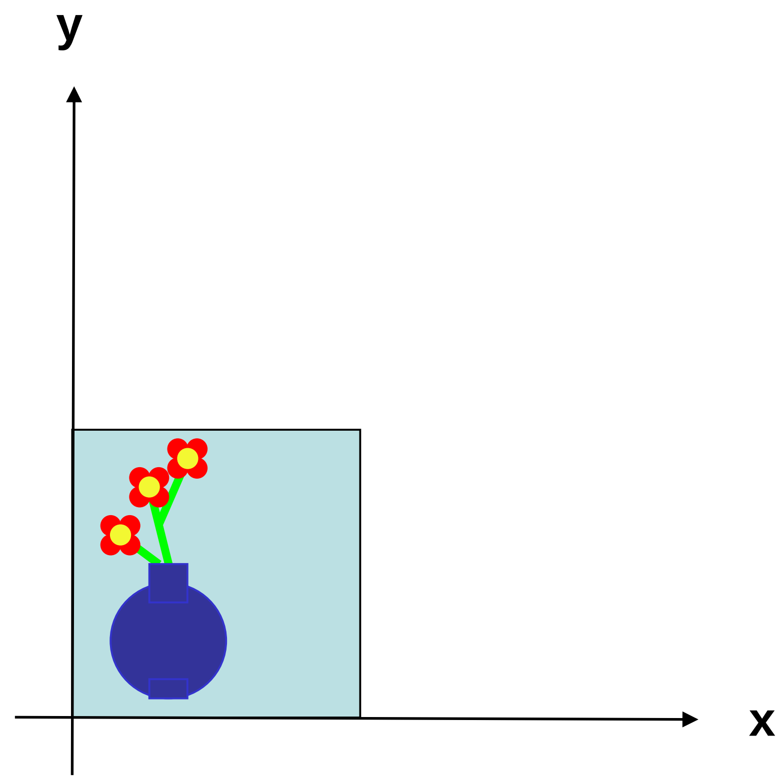
$$\text{scale}(.5, .5) =$$



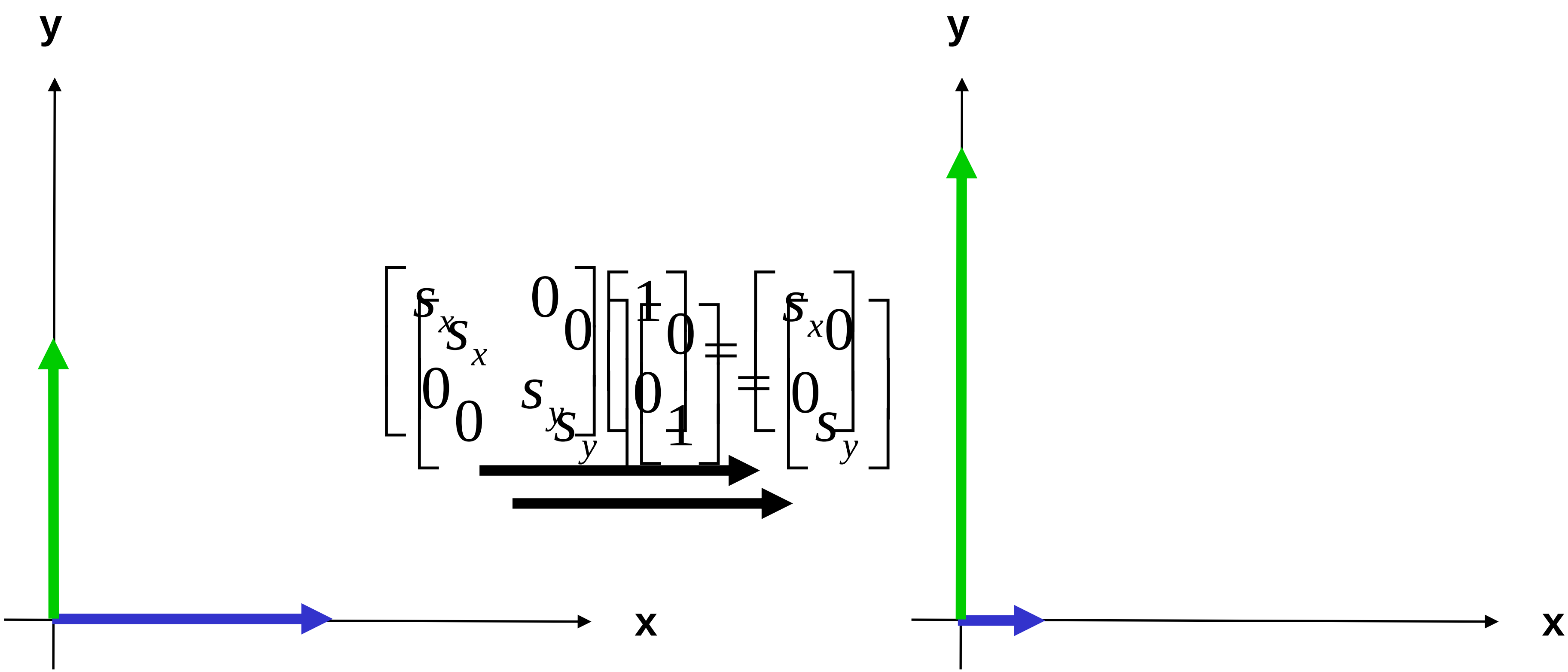
Scaling by .5



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$



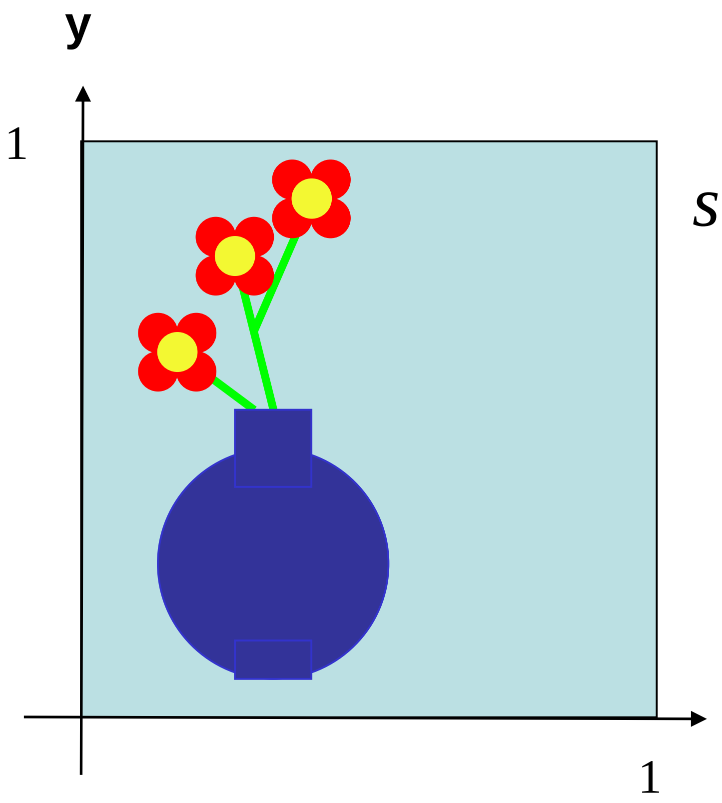
General Scaling



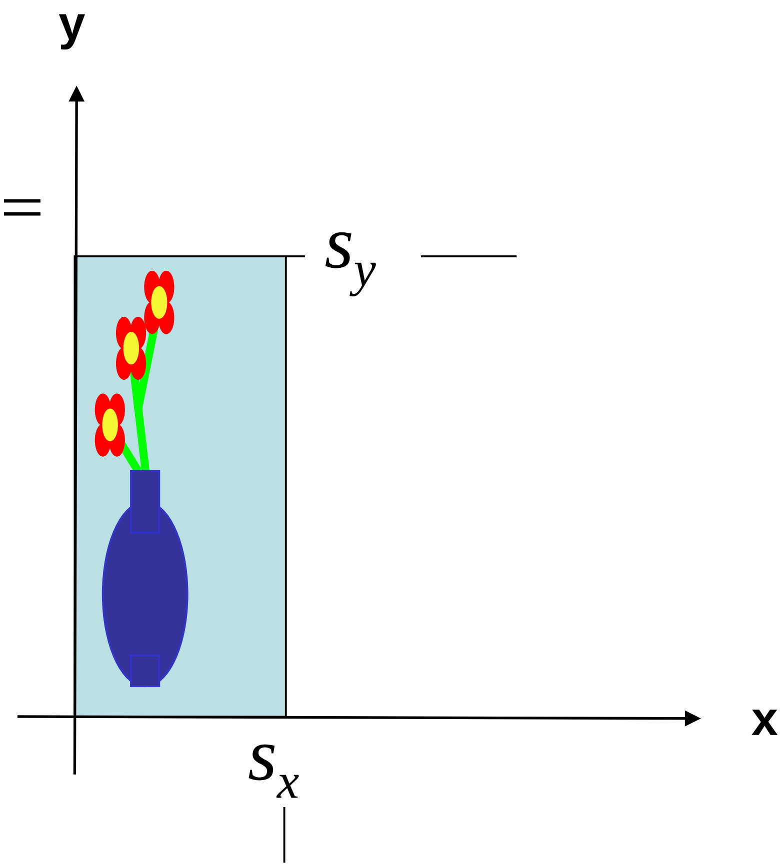
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s_x \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ s_y \end{bmatrix}$$

General Scaling

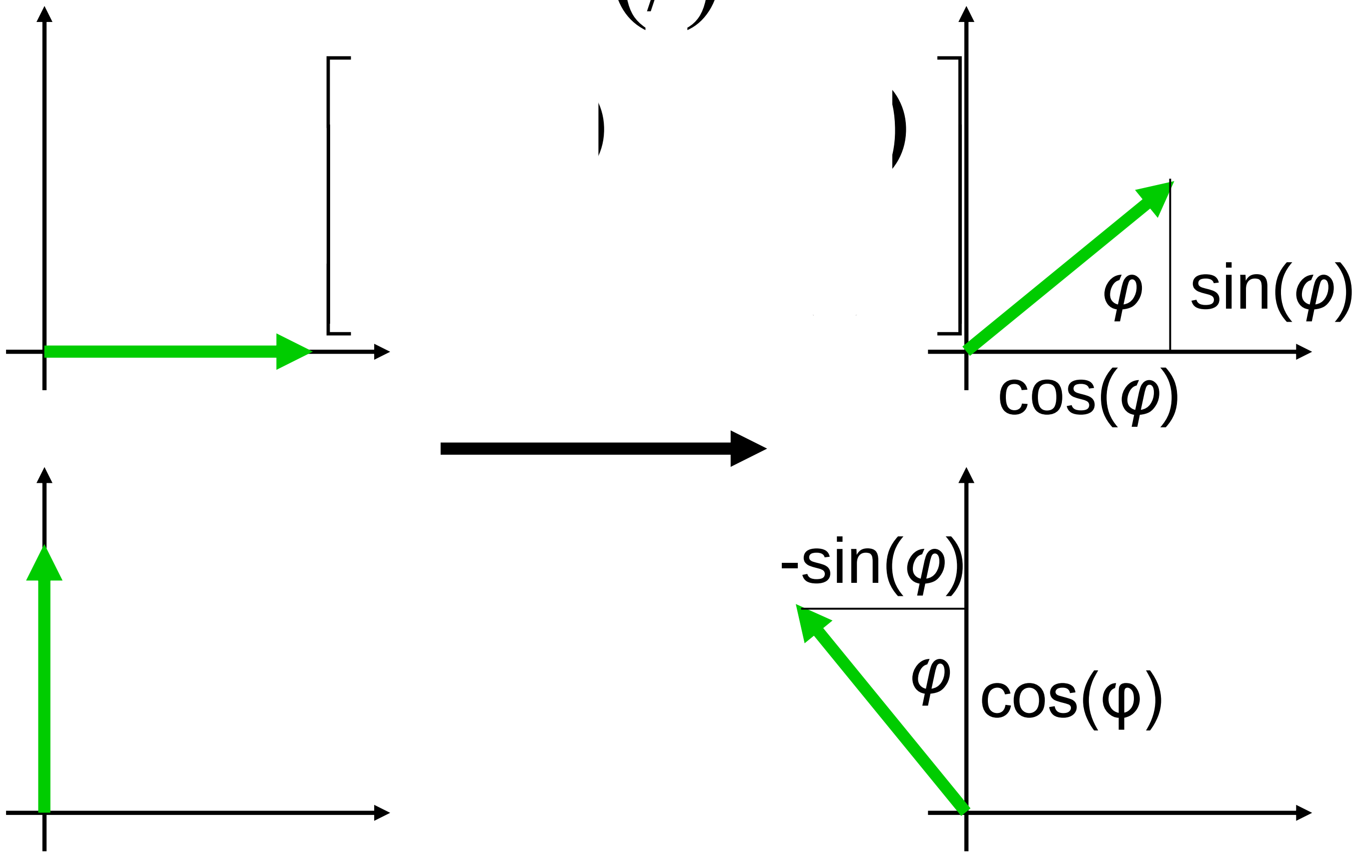


$$scale(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



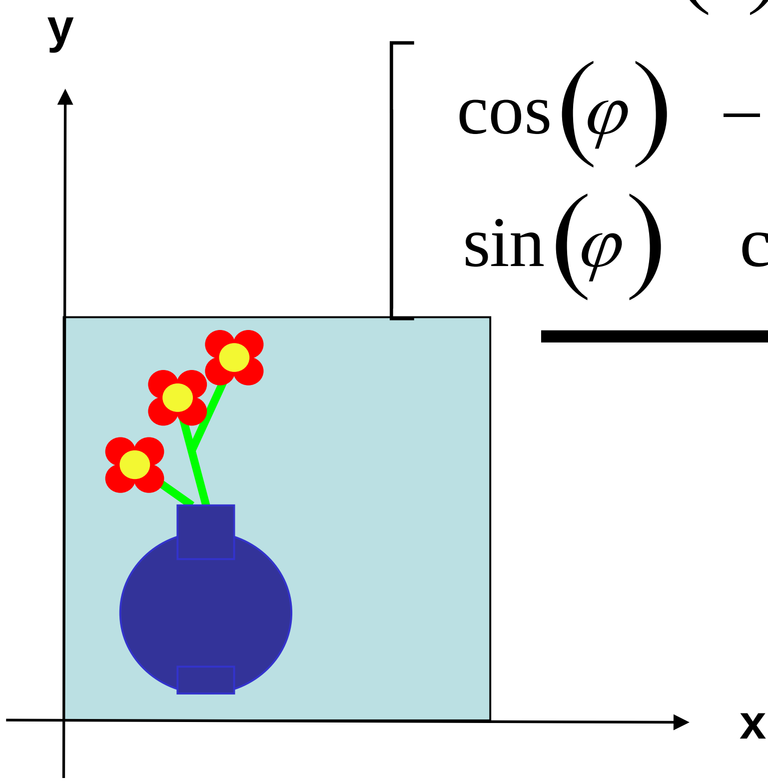
Rotation

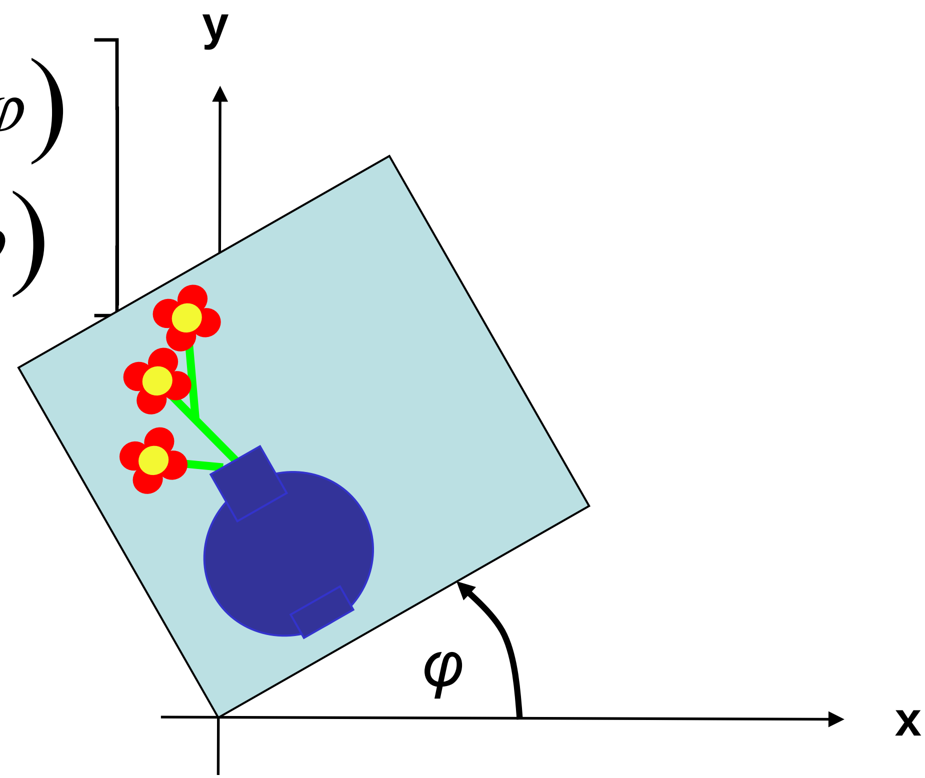
$$\mathrm{rot}(\varphi) =$$



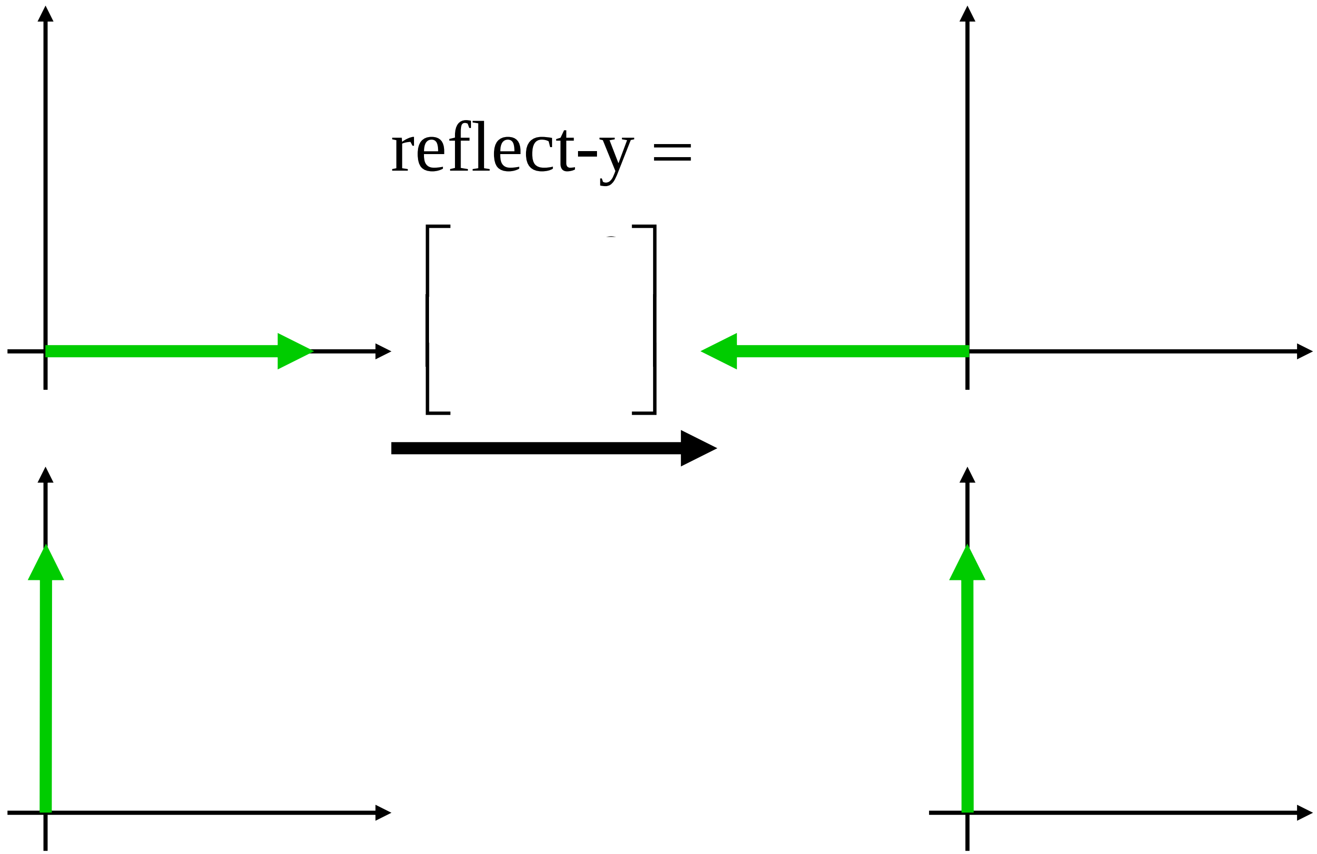
Rotation

$$\mathrm{rot}(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

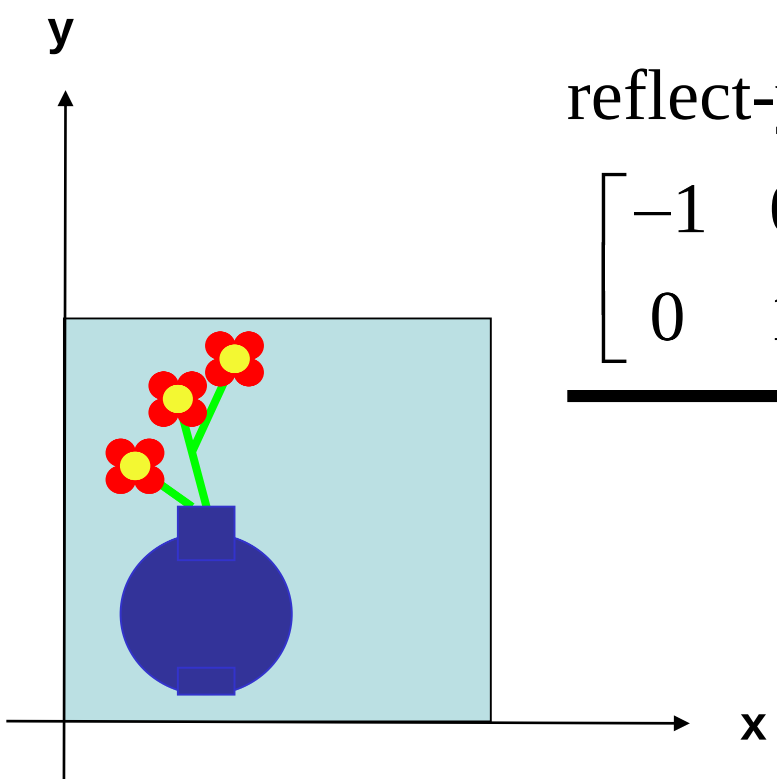




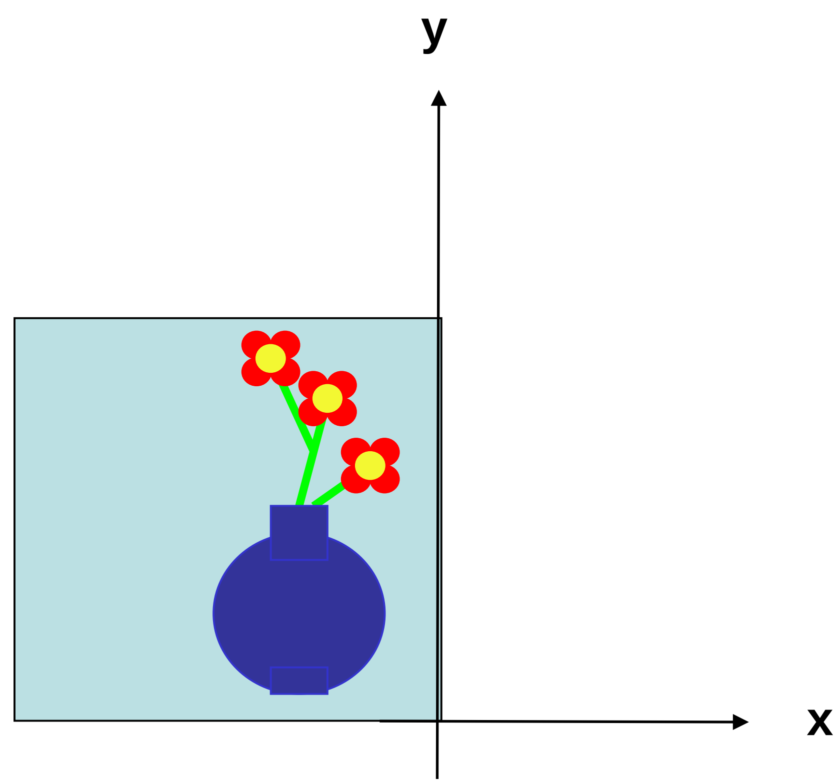
Reflection in y-axis



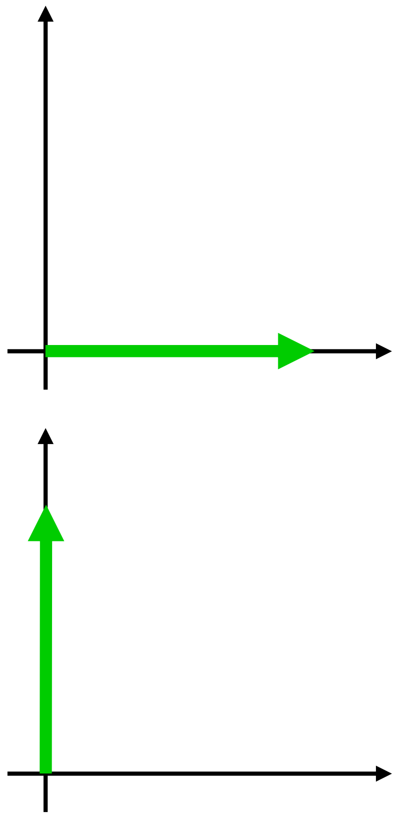
Reflection in y-axis



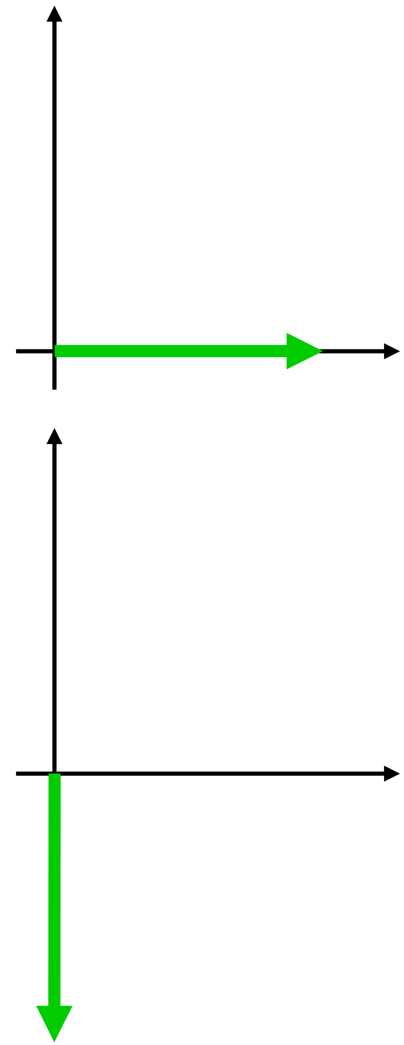
$$\text{reflect-y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



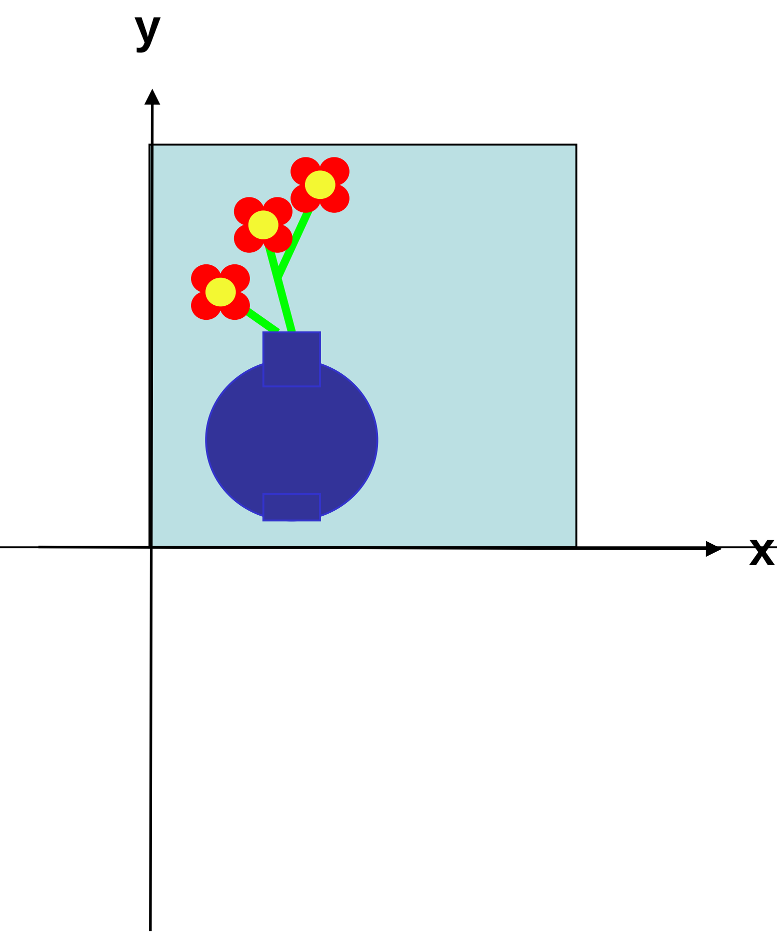
Reflection in x-axis



$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

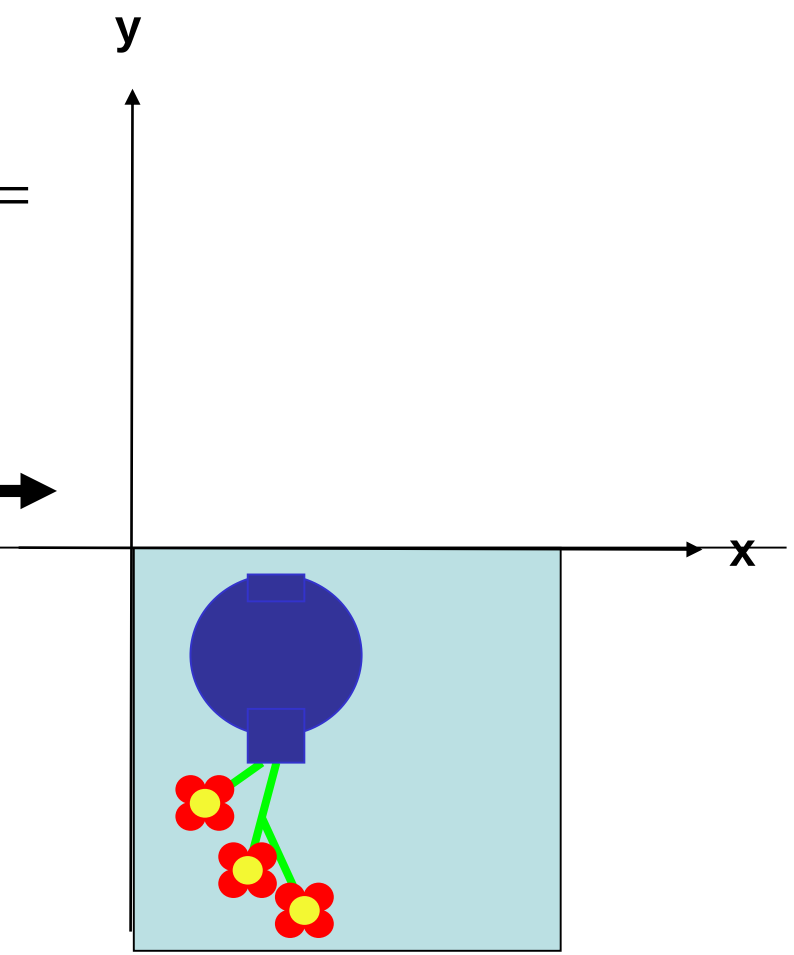


Reflection in x-axis



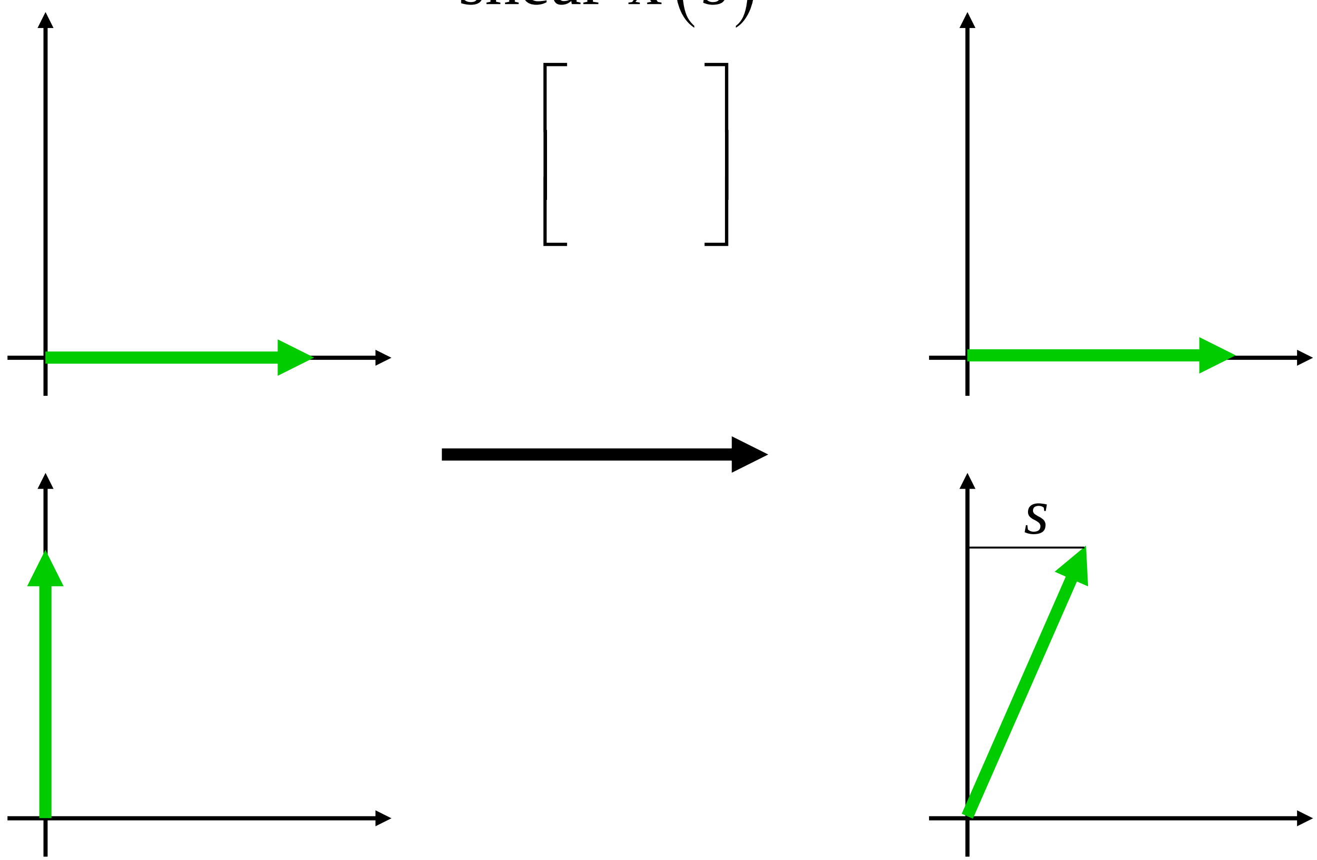
reflect-x =

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

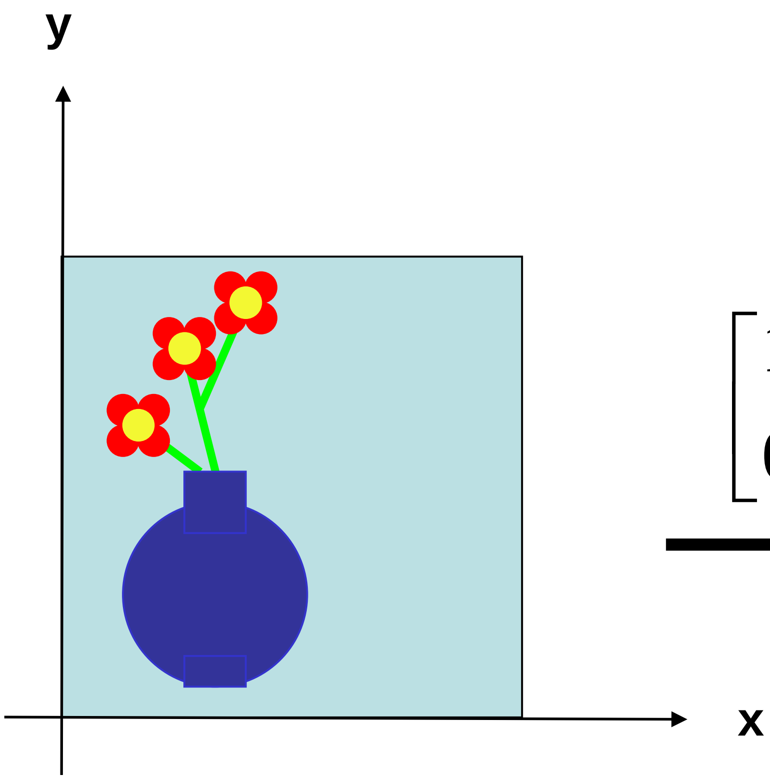


Shear-x

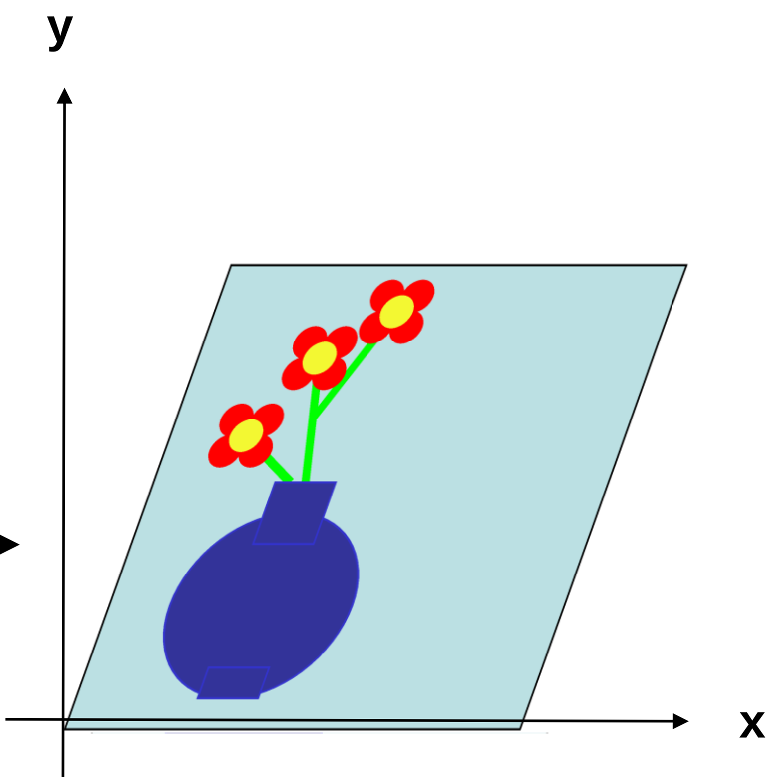
$$\text{shear-x}(s) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$



Shear x

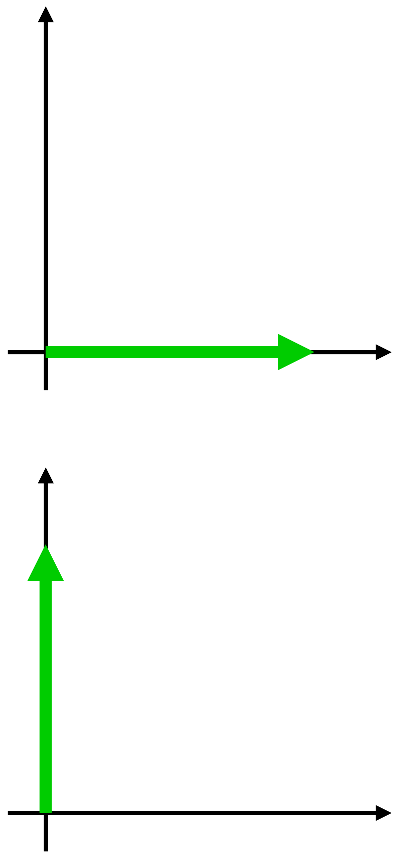


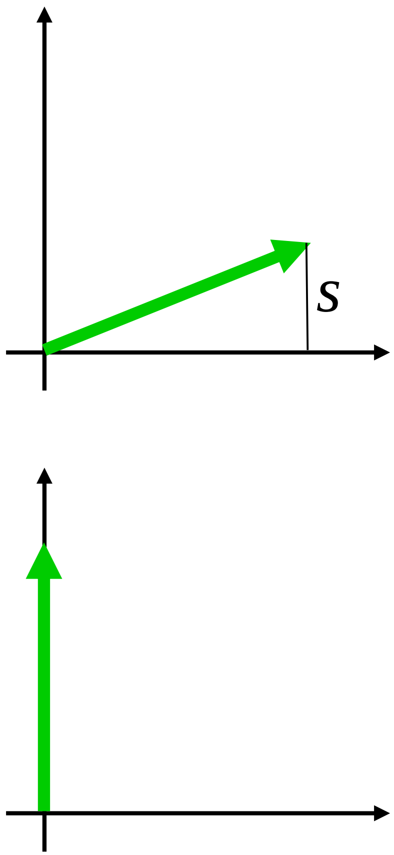
$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$



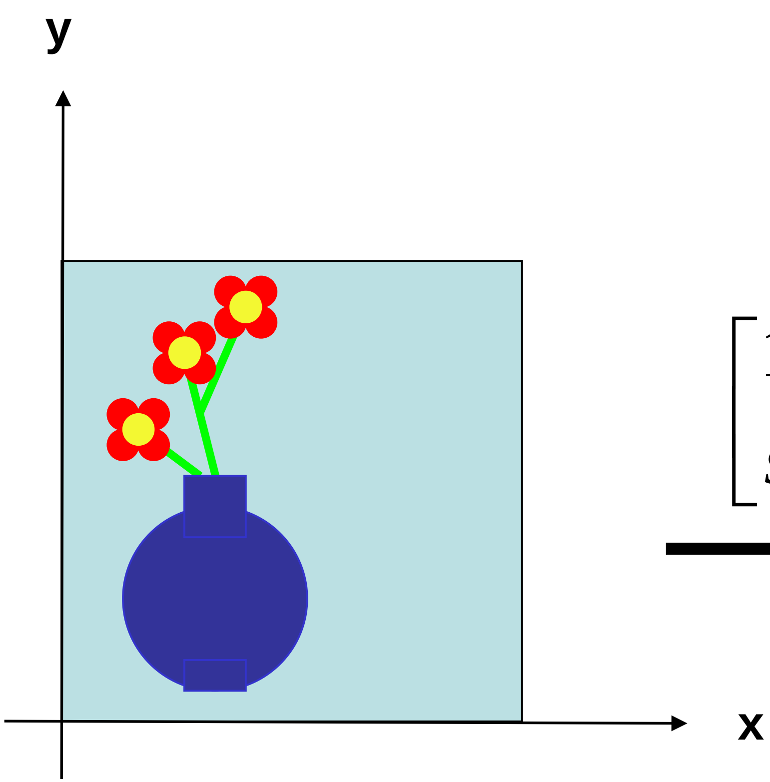
Shear-y

$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

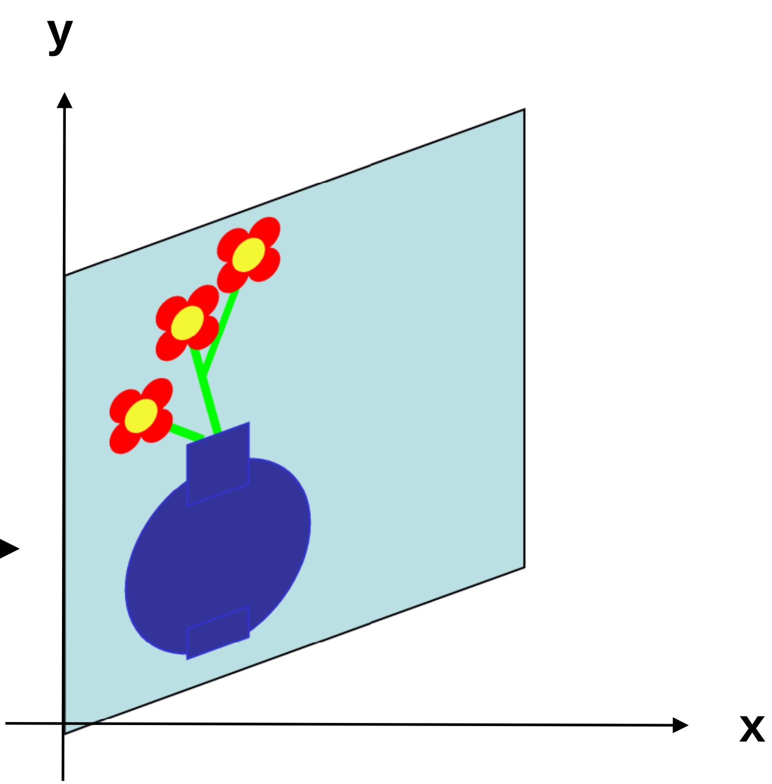




Shear y



$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$



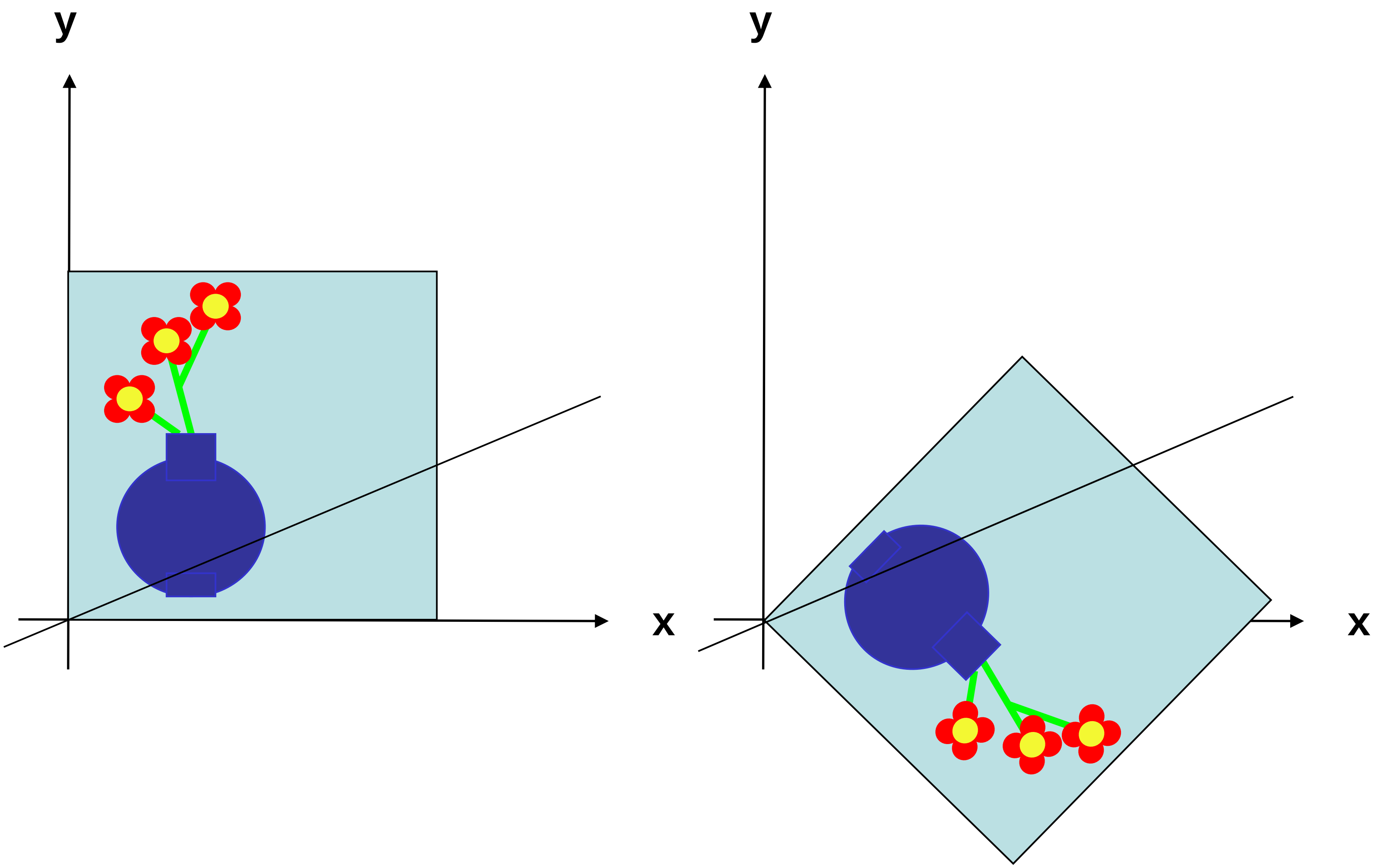
Linear Transformations

- Scale, Reflection, Rotation, and Shear are all linear transformations
- They satisfy: $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$
  - $\mathbf{u}$ and $\mathbf{v}$ are vectors
  - a and b are scalars
- If T is a linear transformation
  - $T((0, 0)) = (0, 0)$

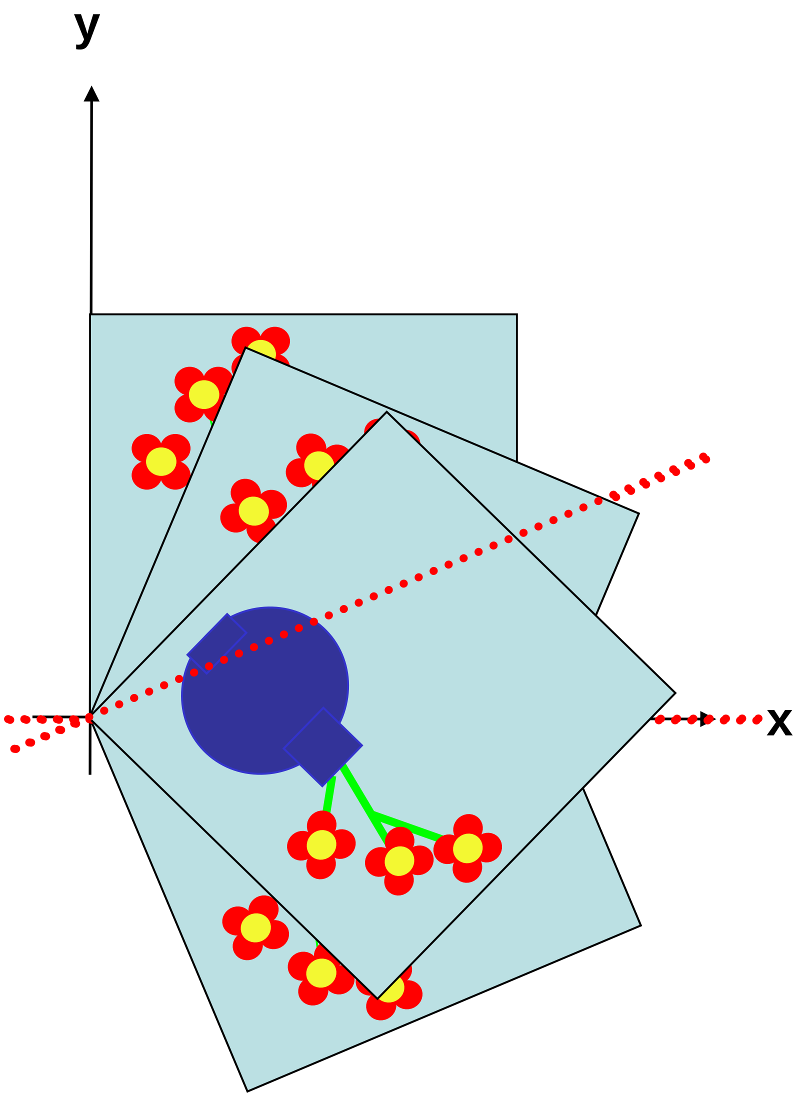
Composing Linear Transformations

- If T_1 and T_2 are transformations
  - $T_2\, T_1(\mathbf{v}) =_{def} T_2(\, T_1(\mathbf{v}))$
- If T_1 and T_2 are linear and are represented by matrices M_1 and M_2
  - $T_2\, T_1$ is represented by $M_2\, M_1$
  - $T_2\, T_1(\mathbf{v}) = T_2(\, T_1(\mathbf{v})) = (M_2\, M_1)(\mathbf{v})$

Reflection About an Arbitrary Line (through the origin)



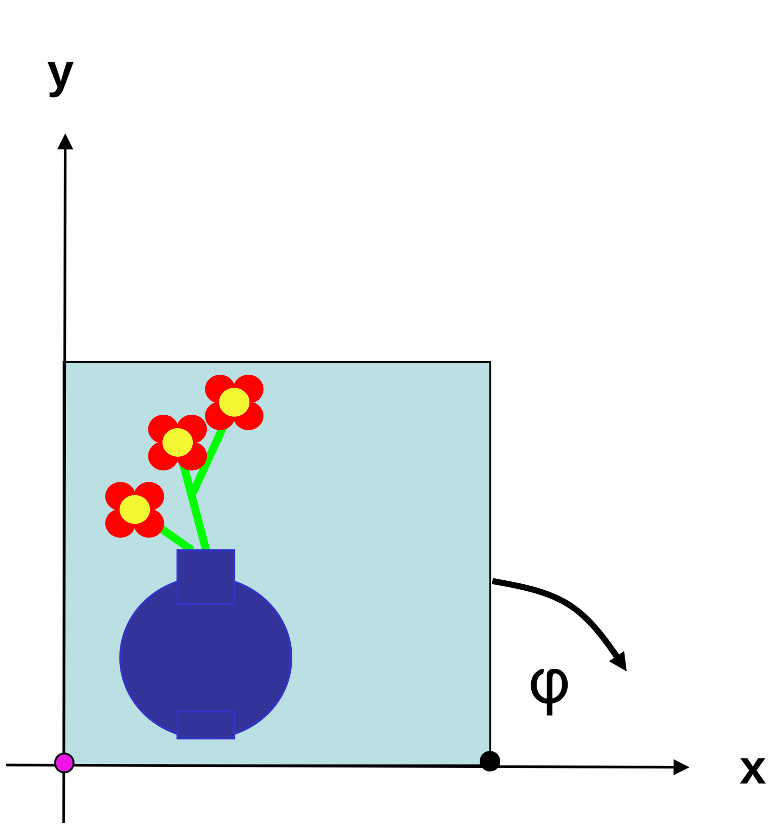
Reflection as a Composition

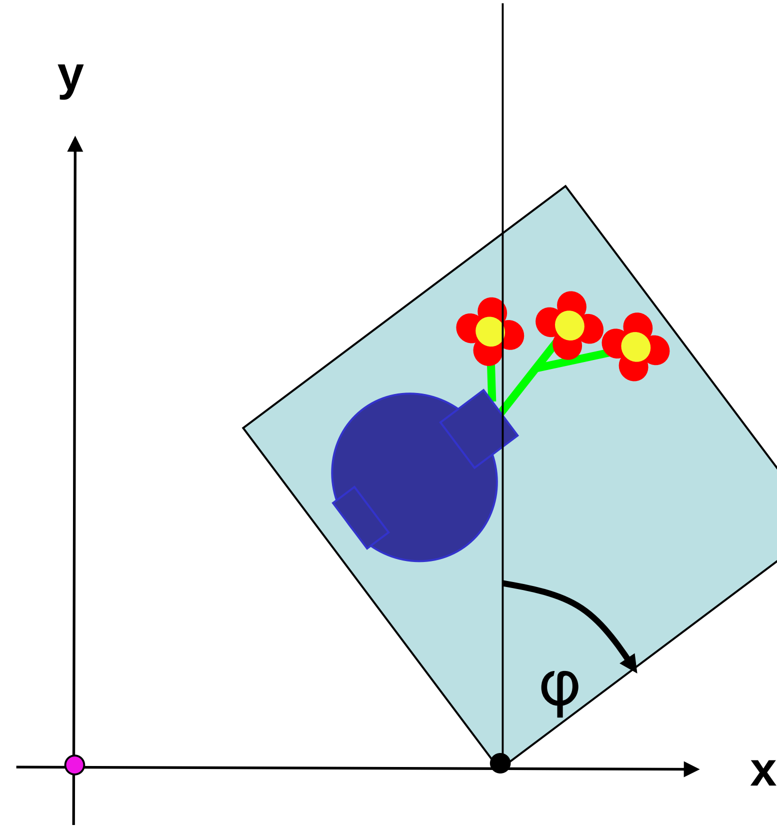


Decomposing Linear Transformations

- Any 2D Linear Transformation can be decomposed into the product of a rotation, a scale, and a rotation if the scale can have negative numbers.
- $M = R_1 S R_2$

Rotation about an Arbitrary Point

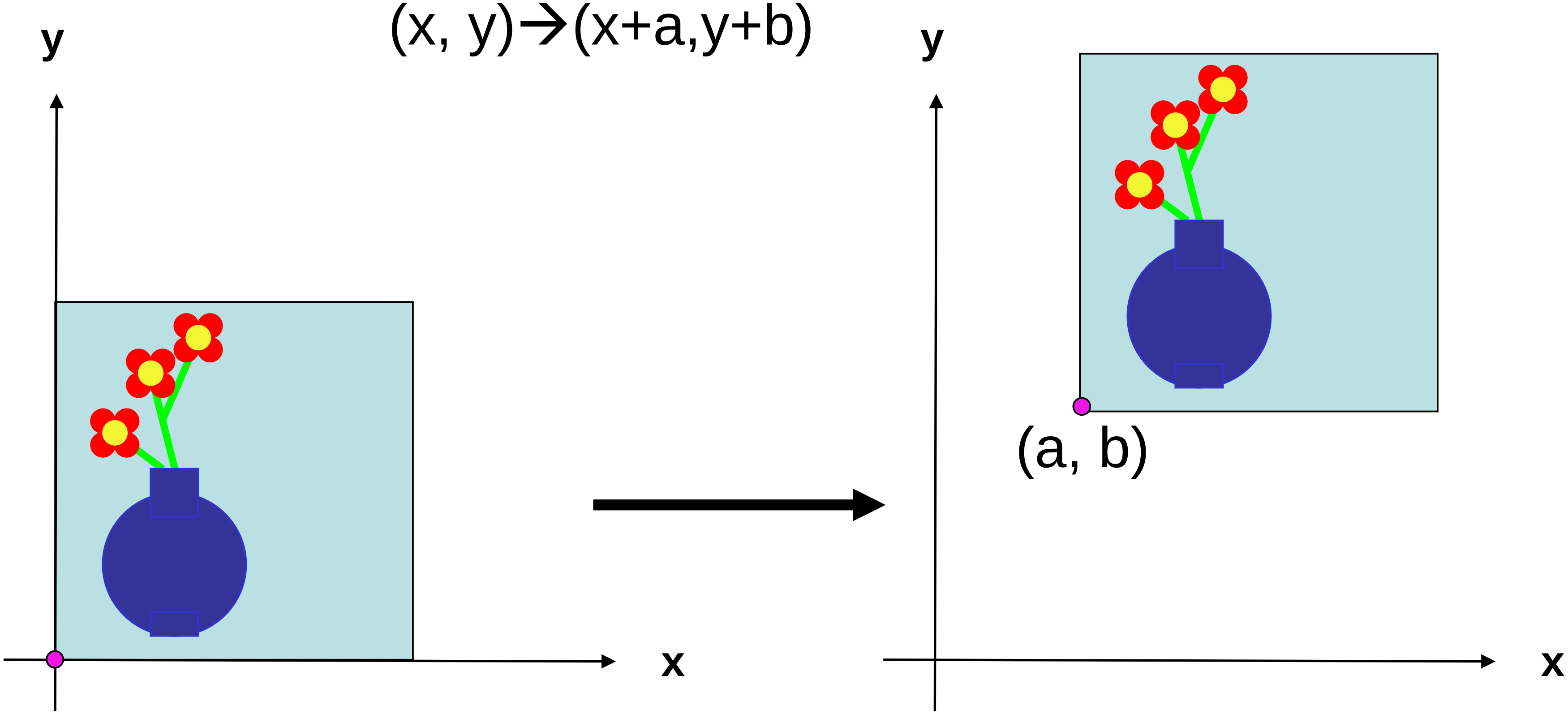




This is not a linear transformation. The origin moves.

Translation

$(x, y) \rightarrow (x+a, y+b)$

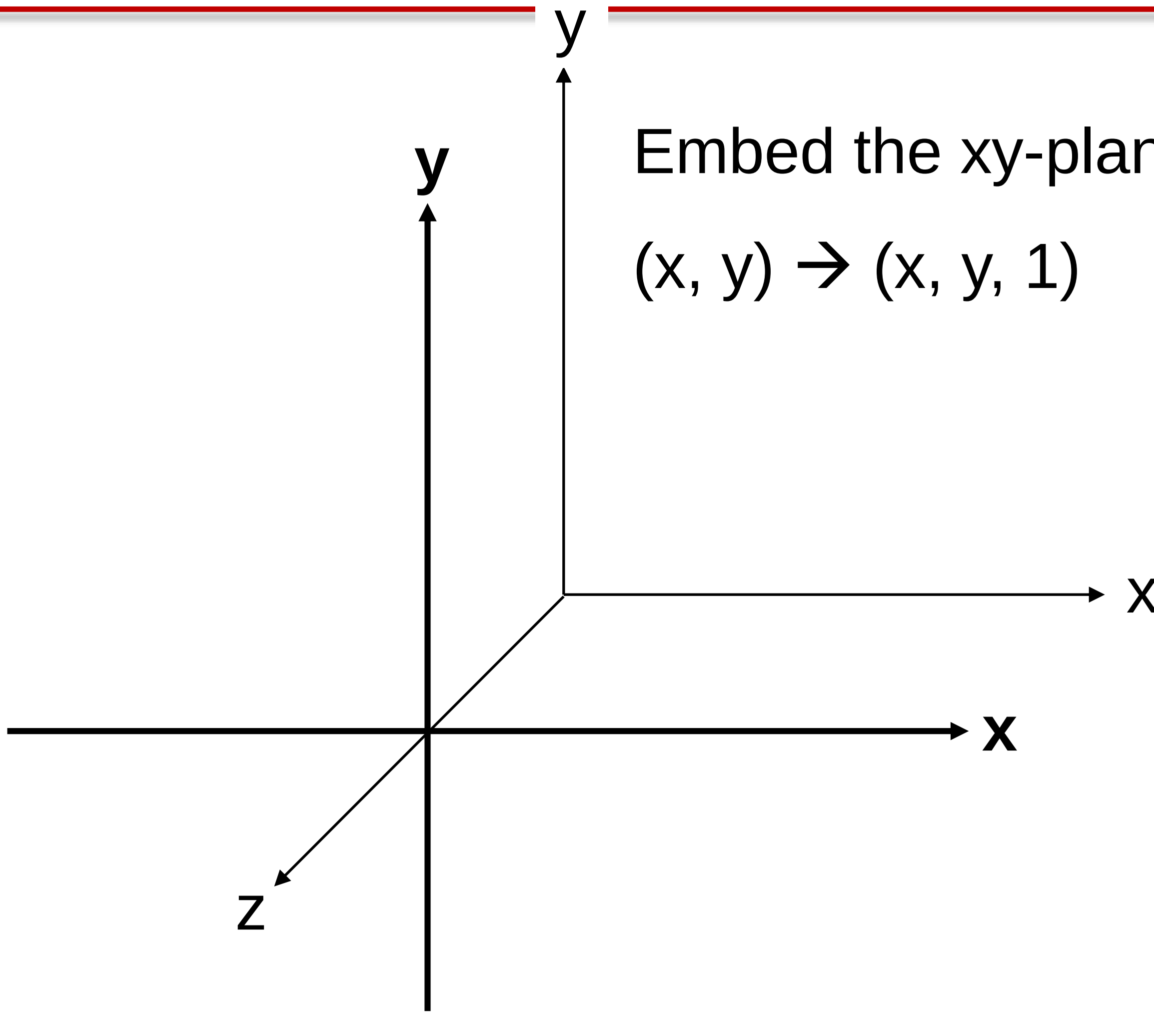


This is not a linear transformation. The origin moves.

Homogeneous Coordinates

Embed the xy-plane in R^3 at $z = 1$.

$(x, y) \rightarrow (x, y, 1)$



2D Linear Transformations as 3D Matrices

Any 2D linear transformation can be represented by a 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

or a 3x3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$

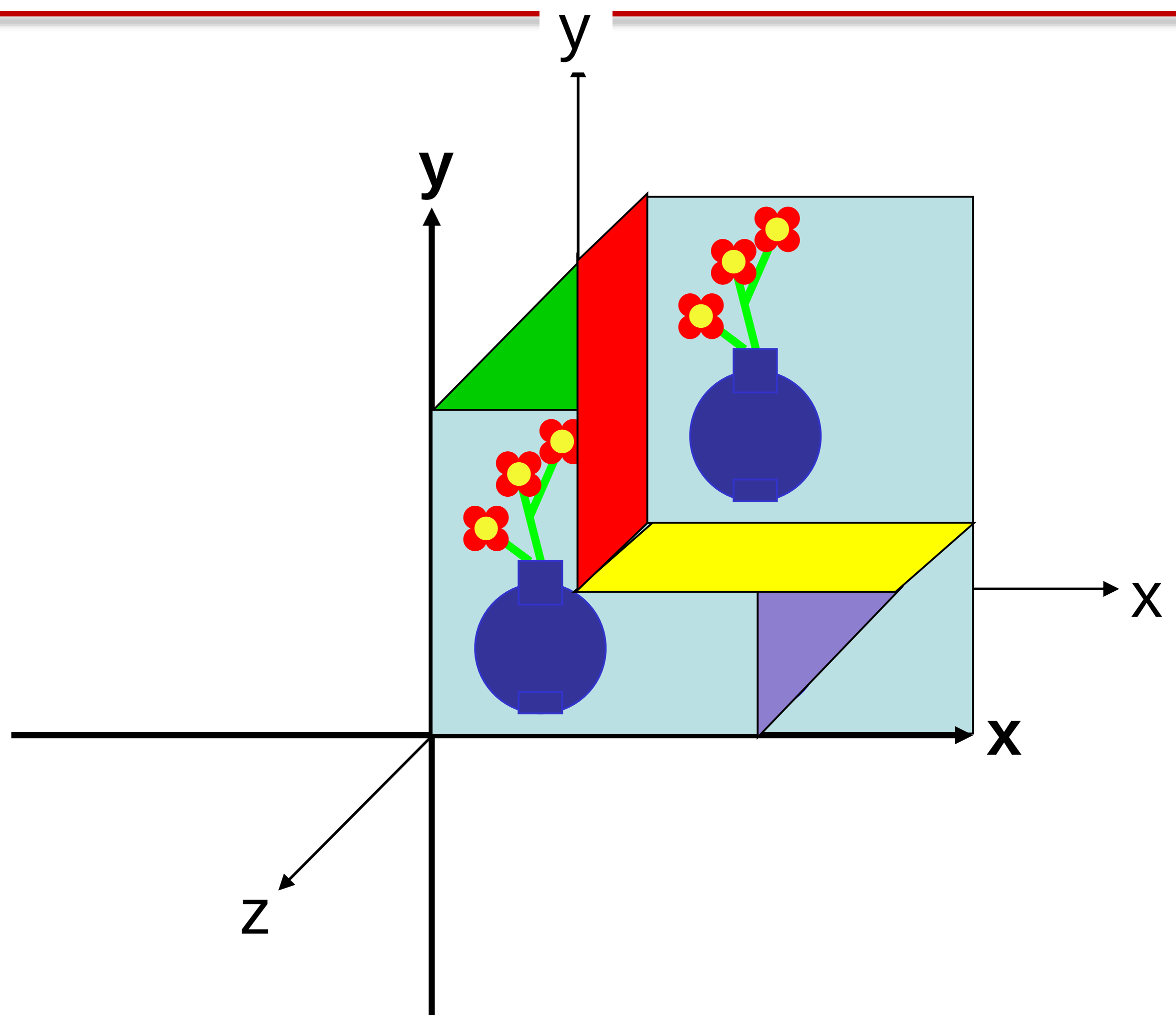
2D Linear Translations as 3D Matrices

Any 2D translation can be represented by a 3x3 matrix.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

This is a 3D shear that acts as a translation on the plane z = 1.

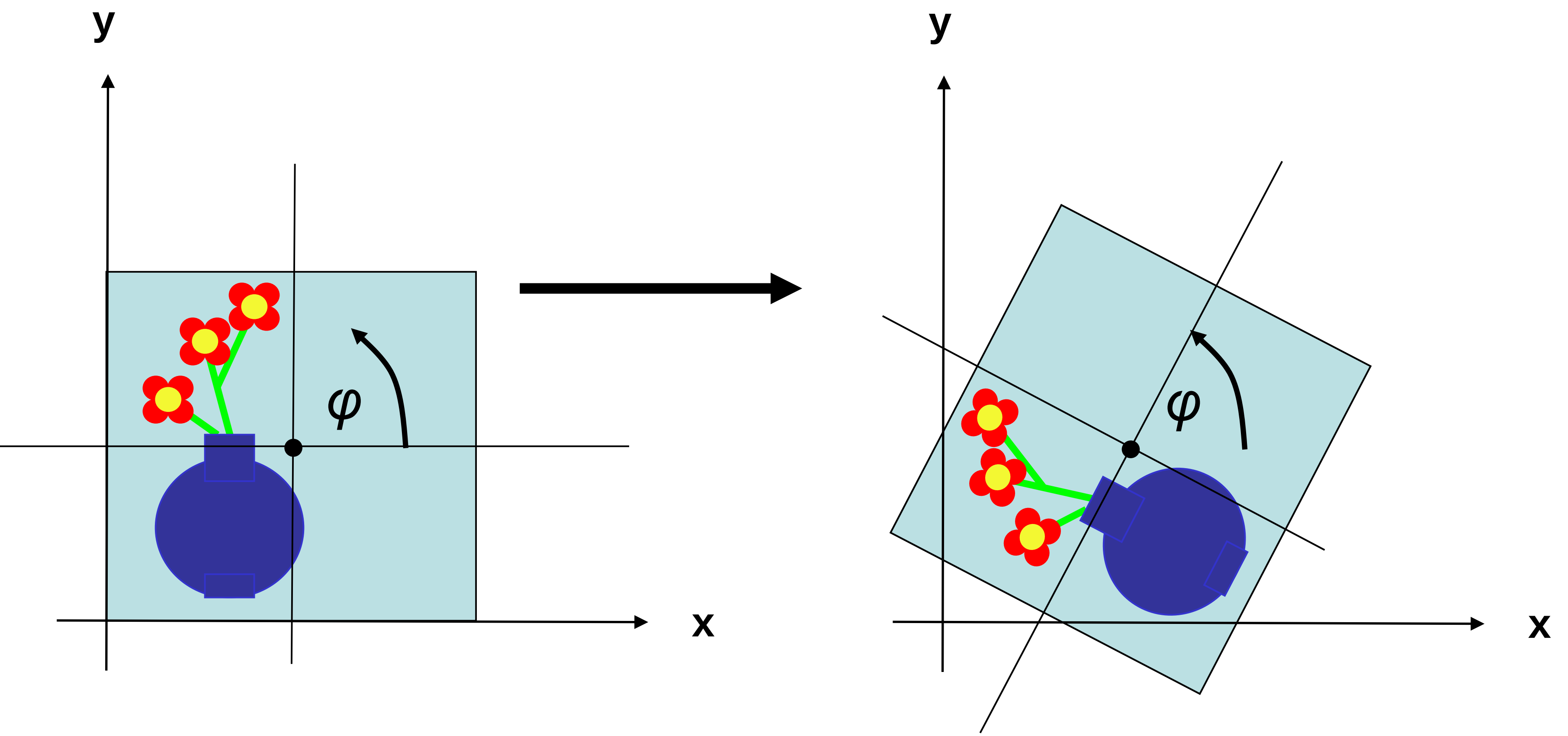
Translation as a Shear



2D Affine Transformations

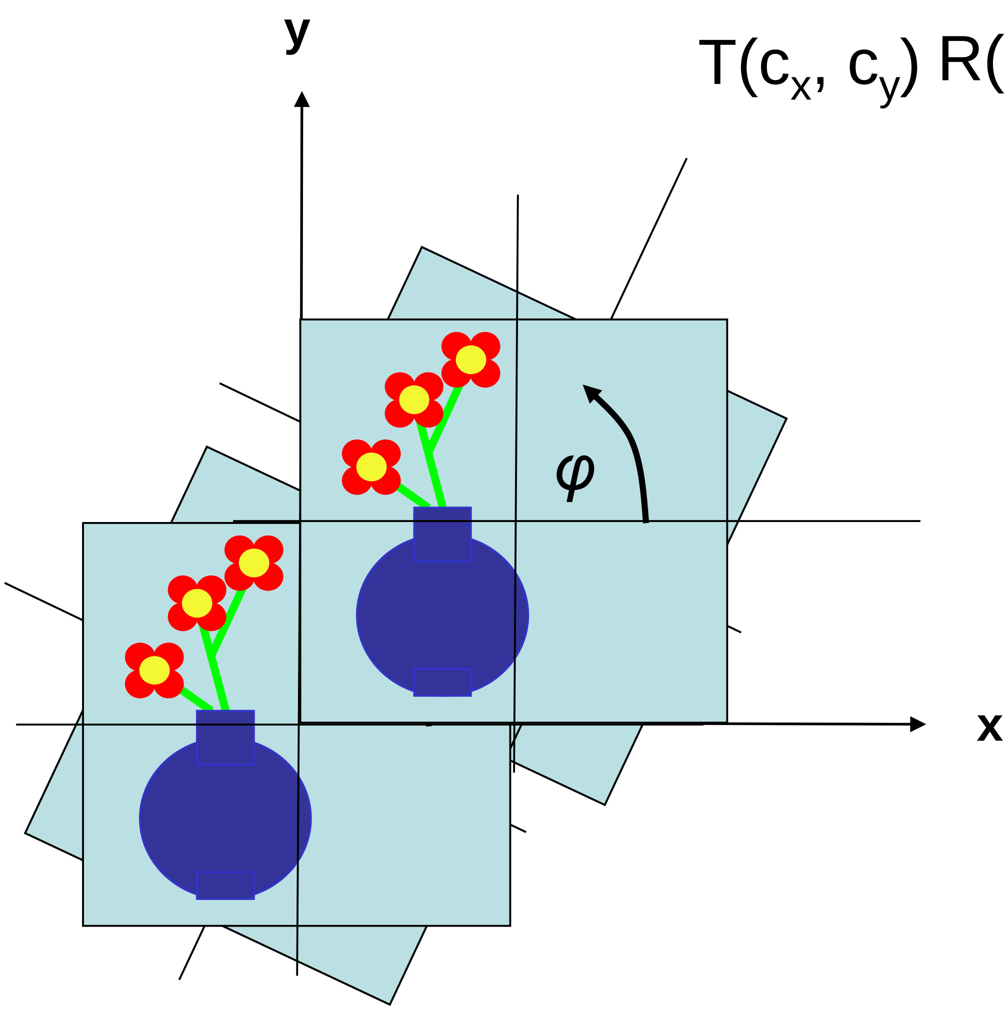
- An *affine transformation* is any transformation that preserves co-linearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation).
- With homogeneous coordinates, we can represent all 2D affine transformations as 3D linear transformations.
- We can then use matrix multiplication to transform objects.

Rotation about an Arbitrary Point

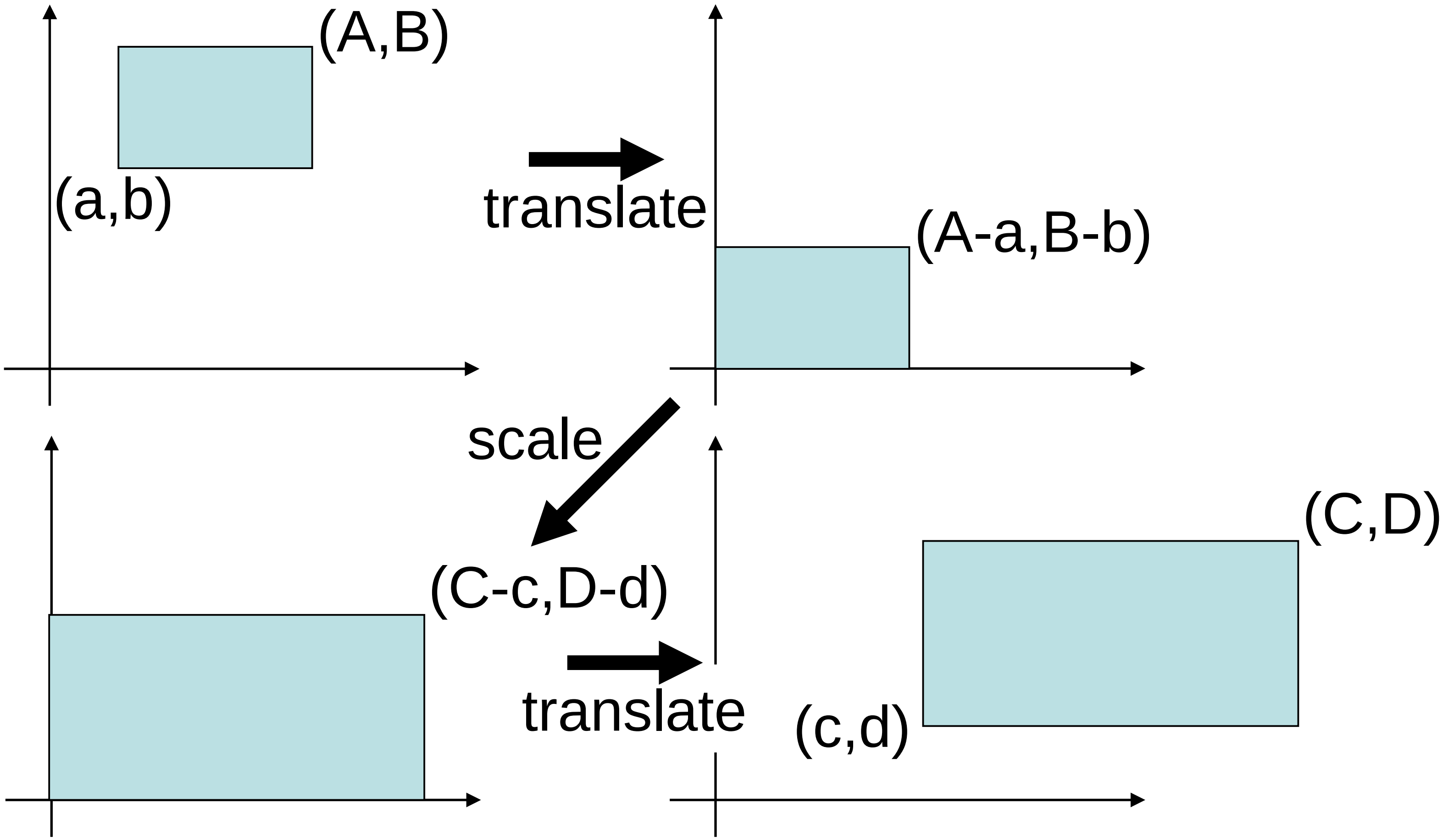


Rotation about an Arbitrary Point

$T(c_x, c_y)\ R(\varphi)\ T(-c_x, -c_y)$



Windowing Transforms



3D Transformations

Remember:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A 3D linear transformation can be represented by a 3x3 matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Affine Transformations

$$\text{scale}\left(s_x, s_y, s_z\right) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{translate}\left(t_x, t_y, t_z\right) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotations

$$\text{rotate}_{\text{x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_{\text{y}}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_{\text{z}}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$