Assignment 7

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Atmospheric Science Experiment

We will first construct a network graph G(V,E) and then we will use maximum flow in this network to solve the given problem.
Construction of G(V,E)-
The set of vertices in G consists of m+n+2 element:
i) the source s.
ii) the set of vertices $a_1, a_2, ..., a_n$, where vertex a_i represents the ith condition in the atmosphere.
iii) the set of vertices $b_1, b_2, ..., b_m$, where vertex b_j represents the jth balloon.
iv) the sink t.
The edges in G are the following:
i) There is an edge $s \to a_i \; \forall \; i \in [1, n]$. The capacity of each of these edges is k.
ii) There is an edge $a_i \to b_j$, if balloon b_j can measure the condition a_i, i.e. $a_i \in S_j$.The capacity of each of these edges is 1.
iii) There is an edge $b_j \to t \; \forall \; j \in [1, m]$. The capacity of each of these edges is 2.

Proof of correctness

Observe that all the edge capacities in G are integers. Therefore, using integral flow theorem, we can assume that flow through each edge in G is integral.
Lemma 1 - For every possible way of measuring n conditions using m balloons such that each balloon measures at most 2 conditions and each condition is measured by at most k balloons, there exist a valid s-t flow in G.

$\to$ Consider a possible way of measuring conditions with given constraint-
Let, C = $\{(a_i, b_j) \,|1 \le i \le n, 1 \le j \le m, \text{ balloon } b_j \text{ measures condition } a_i\}$
Initially, for all edges in G, flow through the edge is 0. We assign flow(f) in G as follows-
i) For all edges (a_i, b_j), such that $(a_i, b_j) \in C$, let $f(a_i, b_j) = 1$. Since, capacity of all such edges is 1, this flow satisfies capacity constraint.
ii) For all edges (s, a_i), $(1 \le i \le n)$, $f(s, a_i)$ equals to the number of balloons which measures condition i. Since, atmost k balloon can measure condition i, $f(s, a_i) \le k$. Therefore, this also satisfies capacity constraint.
iii) For all edges (b_j, t), $(1 \le j \le m)$, $f(b_j, t)$ equals the number of conditions measured by balloon j. Since, each balloon can measure atmost 2 conditions, capacity constraint is satisfied for these edges.

Consider the node a_i $(1 \le i \le n)$, total outgoing flow $(f_{out}(a_i))$ from this node is -
$f_{out}(a_i) = \sum_{j:(a_i,b_j)\in C} f(a_i, b_j)$
Since, $\forall \; (a_i, b_j) \in C, \; f(a_i, b_j) = 1$. Therefore,
$\implies f_{out}(a_i) =$ number of balloons that measure condition i
Since, the only incoming flow in a_i $(f_{in}(a_i))$ is through edge (s, a_i) and $f(s, a_i)$ equals to the number of balloons which measures condition i. Therefore,
$\implies f_{out}(a_i) = f_{in}(a_i)$

Consider the node b_j ($1 \leq j \leq m$), total incoming flow ($f_{in}(b_j)$) to this node is -
$f_{in}(b_j) = \sum_{i:(a_i,b_j)\in C} f(a_i, b_j)$
Since, $\forall\ (a_i, b_j) \in C,\ f(a_i, b_j) = 1$. Therefore,
$\implies f_{in}(b_j) =$ number of conditions measured by balloon j
Since, the only outgoing flow from b_j ($f_{out}(b_j)$) is through edge (b_j, t) and $f(b_j, t)$ equals to the number of conditions measured by balloon j. Therefore,
$\implies f_{in}(b_j) = f_{out}(b_j)$

Hence, it is proved that all the capacity constraints as well as conservation constraints are satisfied by the flow f. Therefore, f is a valid flow in G.

Lemma 2 - For any valid s-t flow in G, there exist a way to measure the n conditions using m balloons such that any balloon b_j measure conditions only from set S_j and does not measure more than 2 conditions.Also, each condition can be measured by atmost k balloons.

$\rightarrow$ Consider any valid flow f in G. Also consider a way to measure condition using balloons such that a balloon b_j measures condition a_i if and only in $f(a_i, b_j)$=1.
A balloon b_j measure a_i only if $f(a_i, b_j)$=1. This means balloon b_j measures condition a_i, only if there exist an edge $(a_i \rightarrow b_j)$ and it measures condition a_i exactly once. From the construction of network G, it can be seen that $(a_i \rightarrow b_j)$ is an edge only if $a_i \in S_j$. Therefore, b_j measures condition a_i, only if $a_i \in S_j$.
The number of conditions measured by balloon b_j is equal to number of edges (a_i, b_j) such that $f(a_i, b_j)$=1.
$\implies$ number of conditions measured by balloon $b_j = \sum_{i:a_i\in S_j} f(a_i, b_j)$
Also, incoming flow at b_j, $f_{in}(b_j) = \sum_{i:a_i\in S_j} f(a_i, b_j)$. Therefore,
$\implies$ number of conditions measured by balloon $b_j = f_{in}(b_j)$
Using conservation constraint at node b_j, since f is a valid flow,
$\implies$ number of conditions measured by balloon $b_j = f_{out}(b_j)$
$\implies$ number of conditions measured by balloon $b_j = f(b_j, t) \leq 2$ (since, capacity of edge$(b_j, t) = 2$)
$\implies$ number of conditions measured by balloon $b_j \leq 2$

Condition a_i is measured by balloon b_j if and only if flow through edge (a_i, b_j) is 1. Therefore, number of balloons that measure condition a_i is equal to the number of edges (a_i, b_j) such that $f(a_i, b_j) = 1$.
number of balloons that measure condition $a_i = \sum_{j:f(a_i,b_j)=1} f(a_i, b_j)$
$\implies$ number of balloons that measure condition $a_i = f_{out}(a_i) = f_{in}(a_i) = f(s, a_i)$
$\implies$ number of balloons that measure condition $a_i = f(s, a_i) \leq k$ (using capacity constraint on edge (s, a_i))

In lemma 1 we proved that for all valid ways to measuring n conditions using m balloons there exists a valid s-t flow in G. In lemma 2 we proved that all s-t flow in G represent one way to measure n conditions using m balloons.
Result 1 $\rightarrow$ Using Lemma 1 and Lemma 2, we can say that there exist a way to measure n atmospheric conditions using m balloons, such that each balloon b_j ($1 \leq j \leq m$) measures at most 2 conditions and only from set S_j and also each condition is measured by atmost k balloons if and only if there exist a corresponding valid flow in the above network G.
Therefore, there exist a way to measure atmospheric conditions such that each condition is measured by exactly k different balloons if there exist a flow in network G, such that flow through each of the edge (s, a_i) ($1 \leq i \leq n$) is equal to k. This is because the amount of flow through the edge (s, a_i) is equivalent to the number of balloons measuring the condition i (as shown in above proof). This implies that the total flow in G must be k*n.

Now, consider the cut (S,T) in G , such that S = $\{s\}$ and T = $V - \{s\}$, where V is the set of all vertices in G. The capacity(P) of this cut is-
P = $\sum_{1\leq i\leq n} c(s, a_i)$ where, $c(s, a_i)$ is the capacity of edge (s, a_i)
$\implies P = kn \geq$ min-cut(G) where, min-cut(G) represent the capacity of min-cut in G

Since max-flow = min-cut. Therefore -
Result 2 $\rightarrow$ max-flow in G $\leq$ P = kn

Using Result 1 and Result 2, we can say that there exist a way to measure the atmospheric condition satisfying the given constraints if and only if max-flow in G = kn.

Complexity analysis

Construction of network G takes time which is equal to the number of edges(E) added in G.
$|E| = m + n + \sum_{j=1}^{m} S_j$.
Further, we can compute max-flow in G using edmond-karp algorithm in O($|V||E|^2$), where $|V| = m + n + 2$.
Hence, it is a polynomial time algorithm.