

# Joint Distribution Functions

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# Partial Derivatives

For a function  $f(x, y)$ , the partial derivative with respect to  $x$  is taken by differentiating  $f(x, y)$  with respect to  $x$  while treating  $y$  as a constant. For example:

$$f(x, y) = x^2 + \ln(y)$$

# Multiple Integration

Multiple integration is when you integrate a multivariate function by multiple variables. This is done by integrating the function by an individual variable at a time. For example:

$f(x, y) = x^2 + y^2$  which can be integrated as:

# Joint Distributions

A joint distribution is a process where more than one random variable is generated; for example, collecting biomedical data, such as multiple biomarkers, are considered to follow a joint distribution. In mathematical terms, instead of dealing with a random variable, we are dealing with a random vector. Observing a particular random vector will have a probability attached to it.

# Bivariate Discrete Distributions

Let  $X$  and  $Y$  be 2 discrete random variables, the joint distribution function of  $(X, Y)$  is defined as

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The properties of a bivariate discrete distribution are

- $p_{X,Y}(x, y) \geq 0$  for all  $x, y$
- $\sum_x \sum_y p_{X,Y}(x, y) = 1$

# Bivariate Continuous Distribution

Let  $X$  and  $Y$  be 2 continuous random variables, the joint distribution function of  $(X, Y)$  is defined as

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

The properties of a bivariate continuous distribution are

- $f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}$
- $f_{X,Y}(x, y) \geq 0$
- $\int_x \int_y f_{X,Y}(x, y) dy dx = 1$

## Example

$$f(x, y) \begin{cases} 3x & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find  $P(0 \leq X \leq 0.5, 0.25 \leq Y)$

# Bivariate Normal Distribution

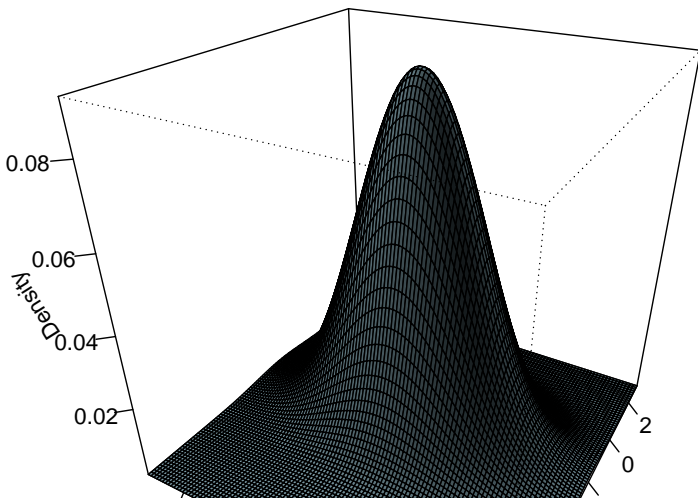
$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{pmatrix} \right]$$



# Bivariate Normal Distribution

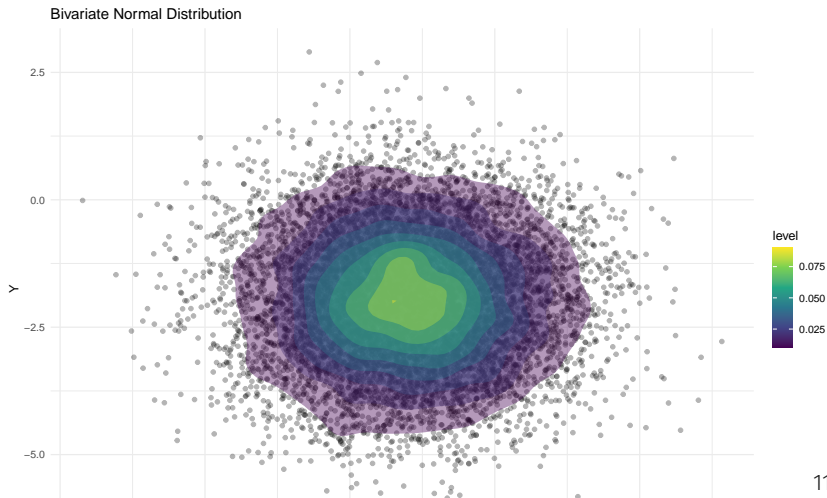
# Bivariate Normal Distribution

$$N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \times \sqrt{2 \times 1.5} \\ 0 \times \sqrt{2 \times 1.5} & 1.5 \end{pmatrix}\right]$$



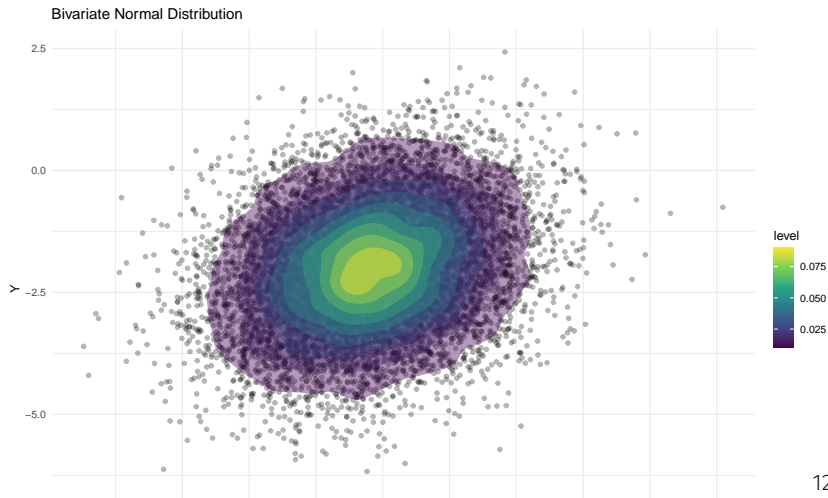
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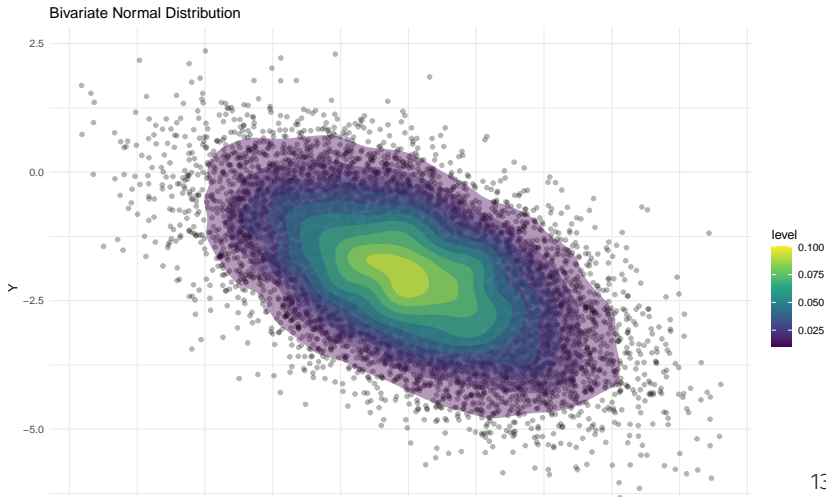
# Bivariate Normal Distribution

$$N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & 0.25 \times \sqrt{2 \times 1.5} \\ 0.25 \times \sqrt{2 \times 1.5} & 1.5 \end{pmatrix}\right]$$



# Bivariate Normal Distribution

$$N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & -0.55 \times \sqrt{2 \times 1.5} \\ -0.55 \times \sqrt{2 \times 1.5} & 1.5 \end{pmatrix}\right]$$



# Marginal Density Functions

A Marginal Density Function is density function of one random variable from a random vector.

# Marginal Discrete Probability Mass Function

Let  $X$  and  $Y$  be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The marginal distribution of  $X$  is defined as

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

# Marginal Continuous Density Function

Let  $X$  and  $Y$  be 2 continuous random variables, with a joint density function of  $f_{X,Y}(x, y)$ . The marginal distribution of  $X$  is defined as

$$f_X(x) = \int_y f_{X,Y}(x, y) dy$$



## Example

$$f_{X,Y}(x,y) \begin{cases} 2x & 0 \leq y \leq 1; 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find  $f_X(x)$

# Bivariate Marginal Density

# Conditional Distributions

A conditional distribution provides the probability of a random variable, given that it was conditioned on the value of a second random variable.

# Discrete Conditional Distributions

Let  $X$  and  $Y$  be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The conditional distribution of  $X|Y = y$  is defined as

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

# Continuous Conditional Distributions

Let  $X$  and  $Y$  be 2 continuous random variables, with a joint density function of  $f_{X,Y}(x, y)$ . The conditional distribution of  $X|Y = y$  is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

# Bivariate Normal Conditional Distribution

# Independent Random Variables

Random variables are considered independent of each other if the probability of one variable does not affect the probability of another variable.

# Discrete Independent Random Variables

Let  $X$  and  $Y$  be 2 discrete random variables, with a joint density function of  $p_{X,Y}(x, y)$ .  $X$  is independent of  $Y$  if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$



# Continuous Independent Random Variables

Let  $X$  and  $Y$  be 2 continuous random variables, with a joint density function of  $f_{X,Y}(x, y)$ .  $X$  is independent of  $Y$  if and only if

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# Matrix Algebra

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\det(A) = a_1 a_2$$

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}$$

## Example

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left\{ \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right\}$$

Show that  $X \perp Y$ .

$$f_{X,Y}(x,y) = \det(2\pi\Sigma)^{-1/2} \exp \left\{ -\frac{1}{2}(w - \mu)^T \Sigma^{-1} (w - \mu) \right\}$$

where  $\Sigma = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix}$ ,  $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$ , and

$$w = \begin{pmatrix} x \\ y \end{pmatrix}$$

# Covariance

Let  $X$  and  $Y$  be 2 random variables with mean  $E(X) = \mu_x$  and  $E(Y) = \mu_y$ , respectively. The covariance of  $X$  and  $Y$  is defined as

$$Cov(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\}$$

$$Cov(X, Y) = E(XY) - \mu_x\mu_y$$

If  $X$  and  $Y$  are independent random variables, then

$$\text{Cov}(X, Y) = 0$$

# Correlation

The correlation of  $X$  and  $Y$  is defined as

$$\rho = \text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

# Expectations

Let  $X_1, X_2, \dots, X_n$  be a set of random variables, the expectation of a function  $g(X_1, \dots, X_n)$  is defined as

$$E\{g(X_1, \dots, X_n)\} = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} g(X_1, \dots, X_n) p(x_1, \dots, x_n)$$

or

$$E\{g(X)\} = \int_{x_1 \in X_1} \cdots \int_{x_n \in X_n} g(X) f(X) dx_n \cdots dx_1$$

# Expected Value and Variance of Linear Functions

Let  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_m$  be random variables with  $E(X_i) = \mu_i$  and  $E(Y_j) = \tau_j$ . Furthermore, let  $U = \sum_{i=1}^n a_i X_i$  and  $V = \sum_{j=1}^m b_j Y_j$  where  $\{a_i\}_{i=1}^n$  and  $\{b_j\}_{j=1}^m$  are constants. We have the following properties:

- $E(U) = \sum_{i=1}^n a_i \mu_i$
- $Var(U) = \sum_{i=1}^n a_i^2 Var(X_i) + 2 \sum_{i < j} a_i a_j Cov(X_i, X_j)$
- $Cov(U, V) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j Cov(X_i, Y_j)$



# Conditional Expectations

Let  $X_1$  and  $X_2$  be two random variables, the conditional expectation of  $g(X_1)$ , given  $X_2 = x_2$ , is defined as

$$E\{g(X_1)|X_2 = x_2\} = \sum_{x_1} g(x_1)p(x_1|x_2)$$

or

$$E\{g(X_1)|X_2 = x_2\} = \int_{x_1} g(x_1)f(x_1|x_2)dx_1.$$

# Conditional Expectations

Furthermore,

$$E(X_1) = E_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$

and

$$Var(X_1) = E_{X_2}\{Var_{X_1|X_2}(X_1|X_2)\} + Var_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$