Joint Distribution Functions

Conditional Distributions

A conditional distribution provides the probability of a random variable, given that it was conditioned on the value of a second random variable.

Discrete Conditional Distributions

Let X and Y be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The conditional distribution of $X|Y = y$ is defined as

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Continuous Conditional Distributions

Let X and Y be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. The conditional distribution of $X|Y = y$ is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Bivariae Normal Conditional Distribution

Independent Random Variables

Random variables are considered independent of each other if the probability of one variable does not affect the probability of another variable.

Discrete Independent Random Variables

Let X and Y be 2 discrete random variables, with a joint density function of $p_{X,Y}(x,y)$. X is independent of Y if and only if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

Continuous Independent Random Variables

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Matrix Algebra

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\det(A) = a_1 a_2$$

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}$$

Example

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left\{ \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right\}$$

Show that $X \perp Y$.

$$f_{X,Y}(x, y) = \det(2\pi\Sigma)^{-1/2} \exp\left\{ -\frac{1}{2}(w - \mu)^T \Sigma^{-1} (w -$$

where $\Sigma = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$, and

$w = \Big(\; x$

Covariance

Let X and Y be 2 random variables with mean $E(X) = \mu_x$ and $E(Y) = \mu_y$, respectively. The covariance of X and Y is defined as

$$Cov(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\}$$

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

Covariance

If X and Y are independent random variables, then

$$Cov(X, Y) = 0$$

Correlation

The correlation of X and Y is defined as

$$\rho = Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Bivariate Normal Distribution

Expectations

Let $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ be a set of random variables, the expectation of a function $g(X)$ is defined as

$$E\{g(X)\} = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} g(X) p(x, \theta)$$

or

$$E\{g(X)\} = \int_{x_1 \in X_1} \cdots \int_{x_n \in X_n} g(X) f(x, \theta) dx_n \cdots dx_1$$

Expected Value and Variance of Linear Functions

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be random variables with $E(X_i) = \mu_i$ and $E(Y_j) = \tau_j$. Furthermore, let $U = \sum_{i=1}^{n} a_i X_i$ and $V = \sum_{j=1}^{m} b_j Y_j$ where $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ are constants. We have the following properties:

- $E(U) = \sum_{i=1}^{n} a_i \mu_i$
- $Var(U) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum\sum_{i<j} a_i a_j Cov(X_i, X_j)$
- $Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$

Conditional Expectations

Let X_1 and X_2 be two random variables, the conditional expectation of $g(X_1)$, given $X_2 = x_2$, is defined as

$$E\{g(X_1)|X_2 = x_2\} = \sum_{x_1} g(x_1)p(x_1|x_2)$$

or

$$E\{g(X_1)|X_2 = x_2\} = \int_{x_1} g(x_1)f(x_1|x_2)dx_1.$$

Conditional Expectations

Furthermore,

$$E(X_1) = E_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$

and

$$Var(X_1) = E_{X_2}\{Var_{X_1|X_2}(X_1|X_2)\} + Var_{X_2}\{E_{X_1|X_2}$$