Review:

More Probability Theory

Learning Outcomes

- Define Moment Generating Functions
- Discuss Properties

- **Continuous Random Variables**
- Uniform Distribution
- Normal Distribution
- Moment Generating Functions
- Characteristic Functions
- Poisson Distribution
- Binomial Distribution
- Uniform Distribution
- Normal Distribution
- MGF Properties

Continuous Random Variables

A random variable X is considered continuous if the $P(X = x)$ does not exist.

CDF

The cumulative distribution function of X provides the $P(X \leq x)$, denoted by $F(x)$, for the domain of X.

Properties of the CDF of X:

1. $F(-\infty) \equiv \lim_{y \to -\infty} F(y) = 0$
2. $F(\infty) \equiv \lim_{y \to \infty} F(y) = 1$
3. $F(x)$ is a nondecreaseing function

PDF

The probability density function of the random variable X is given by

$$f(x) = \frac{dF(x)}{d(x)} = F'(x)$$

wherever the derivative exists.

Properties of pdfs:

Expected Value

The expected value for a continuous distribution is defined as

$$E(X) = \int x f(x) dx$$

The expectation of a function $g(X)$ is defined as

$$E\{g(X)\} = \int g(x) f(x) dx$$

Expected Value Properties

1. $E(c) = c$, where c is constant
2. $E\{cg(X)\} = cE\{g(X)\}$
3. $E\{g_1(X) + g_2(X) + \cdots + g_n(X)\} = E\{g_1(X)\} + E\{g_2($

Variance

The variance of continuous variable is defined as

$$Var(X) = E[\{X - E(X)\}^2] = \int \{X - E(X)\}^2 f(x)dx$$

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Uniform Distribution

A random variable is said to follow uniform distribution if the density function is constant between two parameters.

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

Expected Value

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Normal Distribution

A random variable is said to follow a normal distribution if the the frequency of occurrence follow a Gaussian function.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

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Moments

The kth moment is defined as the expectation of the random variable, raised to the kth power, defined as $E(X^k)$.

Moment Generating Functions

The moment generating functions is used to obtain the kth moment. The mgf is defined as

$$m(t) = E(e^{tX})$$

The kth moment can be obtained by taking the kth derivative of the mgf, with respect to t, and setting t equal to 0:

$$E(X^k) = \left.\frac{d^k m(t)}{dt}\right|_{t=0}$$

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Characteristic Functions

$$\phi(t) = E\left(e^{itX}\right) = E\left\{\cos(tX)\right\} + iE\left\{\sin(tX)\right\}$$

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MGF

Expected Value

Variance

Variance

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MGF

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Linearity

Let X follow a distribution f, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

Derivation

Linearity

Let X and Y be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

Derivation

Uniqueness

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Uniqueness

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.