Review:

More Probability Theory

Independent Random Variables

Let X and Y be independent random variables with joint density function $f_{XY}(x, y)$. Let $g(X)$ and $h(Y)$ be functions of X and Y, respectively.

Then:

$$E\{g(X)h(Y)\} = E\{g(X)\}\,E\{h(Y)\}$$

Linearity

Let X follow a distribution f, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

Linearity

Let X and Y be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

Uniqueness

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Uniqueness

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.

Using the Distribution Function

Let there be a random variable X with a known distribution function $F_X(x)$, the density function for the random variable $Y = g(X)$ can be found with the following steps

1. Find the region of Y in the space of X, find $g^{-1}(y)$
2. Find the region of $Y \leq y$
3. Find $F_Y(y) = P(Y \leq y)$ using the probability density function of X over region $Y \leq y$
4. Find $f_Y(y)$ by differentiating $F_Y(y)$

Example 1

Let X have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 3X - 1$?

Using the PDF

Let there be a random variable X with a known distribution function $F_X(x)$, if the random variable $Y = g(X)$ is either increasing or decreasing, than the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

Example 2

Let X have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 5 - (X/2)$?

Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable X with a known distribution function $F_X(x)$ and random variable $Y = g(X)$, the distribution of Y can be found by:

1. Find the moment generating function of Y, $M_Y(t)$.
2. Compare $M_Y(t)$, with known moment generating functions. If $M_Y(t) = M_V(t)$, for all values t, them Y and V have identical distributions.

Example 3

Let X follow a normal distribution with mean μ and variance σ^2. Find the distribution of $Z = \frac{X-\mu}{\sigma}$.

Example 4

Let Z follow a standard normal distribution with mean 0 and variance 1. Find the distribution of $Y = Z^2$