

# Review:

## More Probability Theory

---

# Independent Random Variables

Let  $X$  and  $Y$  be independent random variables with joint density function  $f_{XY}(x, y)$ . Let  $g(X)$  and  $h(Y)$  be functions of  $X$  and  $Y$ , respectively.

Then:

$$E \{g(X)h(Y)\} = E \{g(X)\} E \{h(Y)\}$$

# Linearity

Let  $X$  follow a distribution  $f$ , with the an MGF  $M_X(t)$ ,  
the MGF of  $Y = aX + b$  is given as

$$M_Y(t) = e^{tb} M_X(at)$$

# Linearity

Let  $X$  and  $Y$  be two random variables with MGFs  $M_X(t)$  and  $M_Y(t)$ , respectively, and are independent. The MGF of  $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

# Uniqueness

Let  $X$  and  $Y$  have the following distributions  $F_X(x)$  and  $F_Y(y)$  and MGFs  $M_X(t)$  and  $M_Y(t)$ , respectively.  $X$  and  $Y$  have the same distribution  $F_X(x) = F_Y(y)$  if and only if  $M_X(t) = M_Y(t)$ .

# Uniqueness

Let  $X_1, \dots, X_n$  be independent random variables, where  $X_i \sim N(\mu_i, \sigma_i^2)$ , with  $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2 / 2\}$  for  $i = 1, \dots, n$ . Find the MGF of  $Y = a_1 X_1 + \dots + a_n X_n$ , where  $a_1, \dots, a_n$  are constants.

# Using the Distribution Function

Let there be a random variable  $X$  with a known distribution function  $F_X(x)$ , the density function for the random variable  $Y = g(X)$  can be found with the following steps

1. Find the region of  $Y$  in the space of  $X$ , find  $g^{-1}(y)$
2. Find the region of  $Y \leq y$
3. Find  $F_Y(y) = P(Y \leq y)$  using the probability density function of  $X$  over region  $Y \leq y$
4. Find  $f_Y(y)$  by differentiating  $F_Y(y)$

## Example 1

Let  $X$  have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of  $Y = 3X - 1$ ?



# Using the PDF

Let there be a random variable  $X$  with a known distribution function  $F_X(x)$ , if the random variable  $Y = g(X)$  is either increasing or decreasing, then the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

## Example 2

Let  $X$  have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of  
 $Y = 5 - (X/2)$ ?

# Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable  $X$  with a known distribution function  $F_X(x)$  and random variable  $Y = g(X)$ , the distribution of  $Y$  can be found by:

1. Find the moment generating function of  $Y$ ,  $M_Y(t)$ .
2. Compare  $M_Y(t)$ , with known moment generating functions. If  $M_Y(t) = M_V(t)$ , for all values  $t$ , then  $Y$  and  $V$  have identical distributions.

## Example 3

Let  $X$  follow a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Find the distribution of  $Z = \frac{X - \mu}{\sigma}$ .

## Example 4

Let  $Z$  follow a standard normal distribution with mean 0 and variance 1. Find the distribution of  $Y = Z^2$