Joint Distribution Functions

Partial Derivatives

For a function $f(x, y)$, the partial derivative with respect to x is taken by differentiating $f(x, y)$ with respect to x while treating y as a constant. For example:

$$f(x, y) = x^2 + \ln(y)$$

Multiple Integration

Multiple integration is when you integrate a multivariate function by multiple variables. This is done by integrating the function by an individual variable at a time. For example:

$f(x, y) = x^2 + y^2$ which can be integrated as:

Joint Distributions

A joint distribution is a process where more than one random variable is generated; for example, collecting biomedical data, such as multiple biomarkers, are considered to follow a joint distribution. In mathematical terms, instead of dealing with a random variable, we are dealing with a random vector. Observing a particular random vector will have a probability attached to it.

Bivariate Discrete Distributions

Let X and Y be 2 discrete random variables, the joint distribution function of (X, Y) is defined as

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The properties of a bivariate discrete distribution are

- $p_{X,Y}(x, y) \geq 0$ for all $x,\ y$
- $\sum_x \sum_y p_{X,Y}(x, y) = 1$

Bivariate Continuous Distribution

Let X and Y be 2 continuous random variables, the joint distribution function of (X, Y) is defined as

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

The properties of a bivariate continuous distribution are

- $f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}$
- $f_{X,Y}(x, y) \geq 0$
- $\int_x \int_y f_{X,Y}(x, y) dy dx = 1$

Example

$$f(x,y)\begin{cases} 3x & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0 \le X \le 0.5, 0.25 \le Y)$

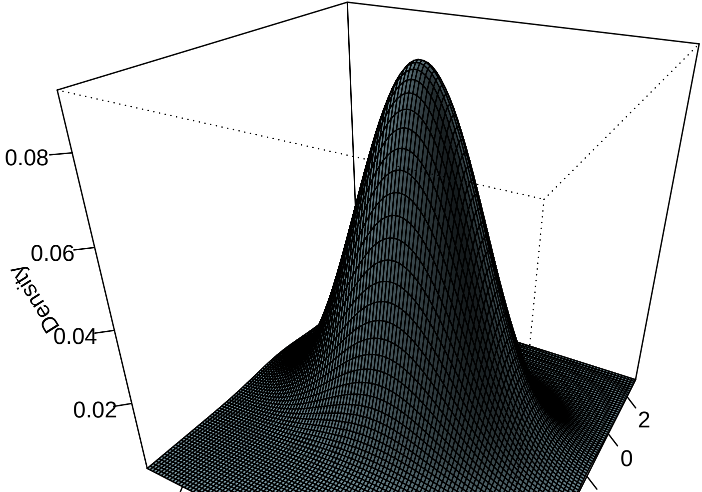
Bivariate Normal Distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{pmatrix}\right]$$

Bivariate Normal Distribution

Bivariate Normal Distribution

$$N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \times \sqrt{2 \times 1.5} \\ 0 \times \sqrt{2 \times 1.5} & 1.5 \end{pmatrix}\right]$$



Bivariate Normal Distribution

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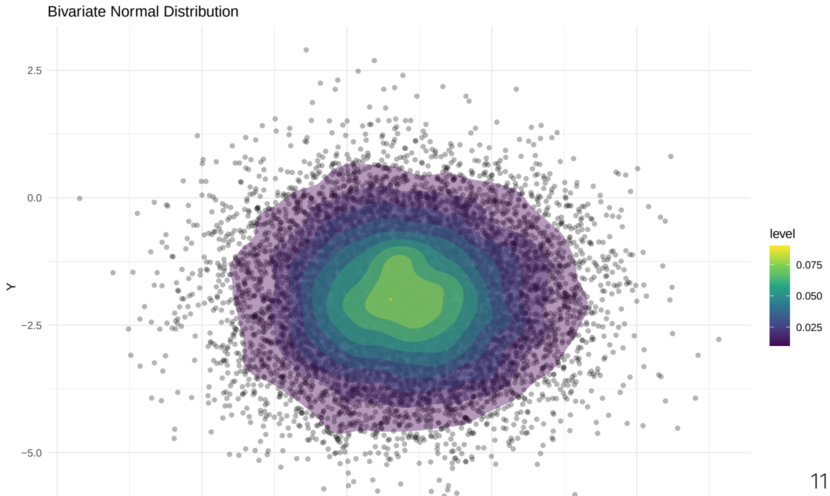

Bivariate Normal Distribution

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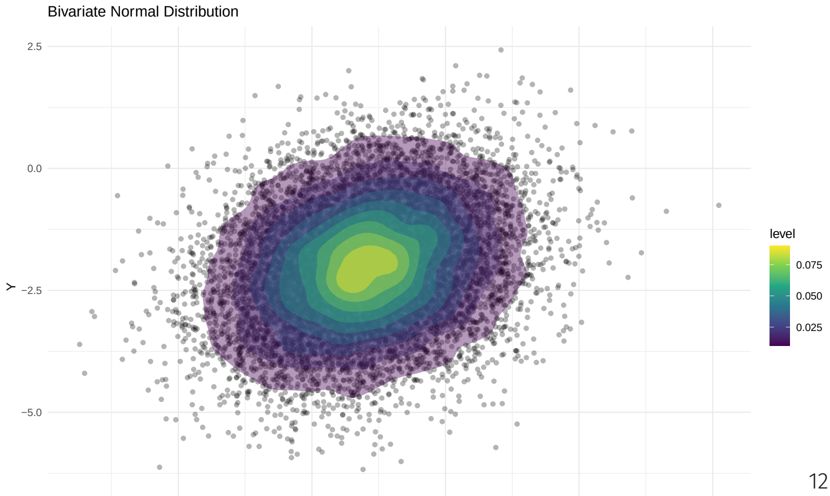
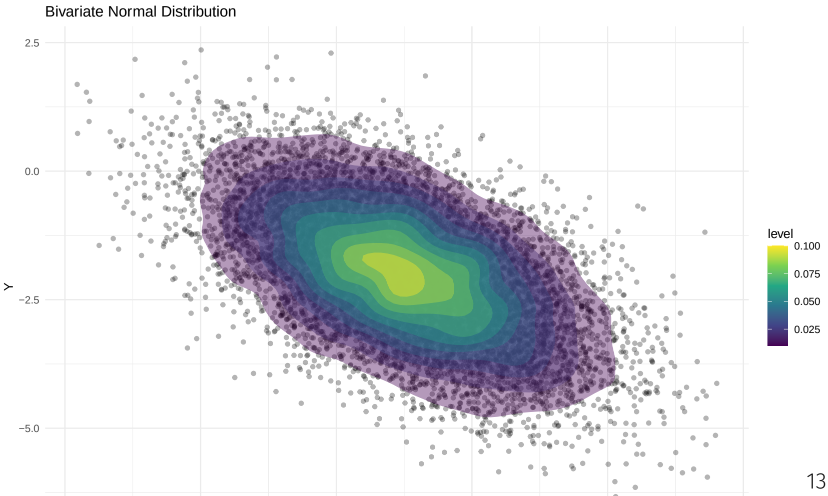


Bivariate Normal Distribution

$$N\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & -0.55 \times \sqrt{2 \times 1.5} \\ -0.55 \times \sqrt{2 \times 1.5} & 1.5 \end{pmatrix}\right]$$



Marginal Density Functions

A Marginal Density Function is density function of one random variable from a random vector.

Marginal Discrete Probability Mass Function

Let X and Y be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

The marginal distribution of X is defined as

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

Marginal Continuous Density Function

Let X and Y be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. The marginal distribution of X is defined as

$$f_X(x) = \int_y f_{X,Y}(x, y)dy$$

Example

$$f_{X,Y}(x,y)\begin{cases} 2x & 0 \leq y \leq 1;\ 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $f_X(x)$

Bivariate Marginal Density

Conditional Distributions

A conditional distribution provides the probability of a random variable, given that it was conditioned on the value of a second random variable.

Discrete Conditional Distributions

Let X and Y be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

The conditional distribution of $X|Y = y$ is defined as

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Continuous Conditional Distributions

Let X and Y be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. The conditional distribution of $X|Y = y$ is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Bivariae Normal Conditional Distribution

Independent Random Variables

Random variables are considered independent of each other if the probability of one variable does not affect the probability of another variable.

Discrete Independent Random Variables

Let X and Y be 2 discrete random variables, with a joint density function of $p_{X,Y}(x, y)$. X is independent of Y if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Continuous Independent Random Variables

Let X and Y be 2 continuous random variables, with a joint density function of $f_{X,Y}(x,y)$. X is independent of Y if and only if

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Matrix Algebra

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\det(A) = a_1 a_2$$

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}$$

Example

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left\{\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}\right\}$$

Show that $X \perp Y$.

$$f_{X,Y}(x, y) = \det(2\pi\Sigma)^{-1/2} \exp\left\{-\frac{1}{2}(w - \mu)^T\Sigma^{-1}(w -$$

where $\Sigma = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$, and

$$w = \begin{pmatrix} x \\ \end{pmatrix}$$

Covariance

Let X and Y be 2 random variables with mean $E(X) = \mu_x$ and $E(Y) = \mu_y$, respectively. The covariance of X and Y is defined as

$$Cov(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\}$$

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

Covariance

If X and Y are independent random variables, then

$$Cov(X, Y) = 0$$

Correlation

The correlation of X and Y is defined as

$$\rho = Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Expectations

Let $X_1, X_2, \dots, X_n$ be a set of random variables, the expectation of a function $g(X_1, \dots, X_n)$ is defined as

$$E\{g(X_1, \dots, X_n)\} = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} g(X_1, \dots, X_n) p(x_1$$

or

$$E\{g(X)\} = \int_{x_1 \in X_1} \cdots \int_{x_n \in X_n} g(X) f(X) dx_n \cdots dx_1$$

- $X = (X_1, \dots, X_n)$

Expected Value and Variance of Linear Functions

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be random variables with $E(X_i) = \mu_i$ and $E(Y_j) = \tau_j$. Furthermore, let $U = \sum_{i=1}^{n} a_i X_i$ and $V = \sum_{j=1}^{m} b_j Y_j$ where $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ are constants. We have the following properties:

- $E(U) = \sum_{i=1}^{n} a_i \mu_i$
- $Var(U) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum\sum_{i<j} a_i a_j Cov(X_i, X_j)$
- $Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$

Conditional Expectations

Let X_1 and X_2 be two random variables, the conditional expectation of $g(X_1)$, given $X_2 = x_2$, is defined as

$$E\{g(X_1)|X_2 = x_2\} = \sum_{x_1} g(x_1)p(x_1|x_2)$$

or

$$E\{g(X_1)|X_2 = x_2\} = \int_{x_1} g(x_1)f(x_1|x_2)dx_1.$$

Conditional Expectations

Furthermore,

$$E(X_1) = E_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$

and

$$Var(X_1) = E_{X_2}\{Var_{X_1|X_2}(X_1|X_2)\}+Var_{X_2}\{E_{X_1|X_2}$$