# Review:

More Probability Theory

# Moment Generating Functions

The $k$th moment is defined as the expectation of the random variable, raised to the $k$th power, defined as $E(X^k)$.

## Moment Generating Functions

The moment generating functions is used to obtain the $k$th moment. The mgf is defined as

$$m(t) = E(e^{tX})$$

The $k$th moment can be obtained by taking the $k$th derivative of the mgf, with respect to $t$, and setting $t$ equal to 0:

$$E(X^k) = \left.\frac{d^k m(t)}{dt}\right|_{t=0}$$

# Characteristic Functions

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$$\phi(t) = E\left(e^{itX}\right) = E\left\{\cos(tX)\right\} + iE\left\{\sin(tX)\right\}$$

# Poisson Distribution

# MGF

## Expected Value

# Variance

# Binomial Distribution

# MGF

# Uniform Distribution

# MGF

# Normal Distribution

# MGF

# MGF Properties

Let $X$ follow a distribution $f$, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

# Derivation

# Linearity

Let $X$ and $Y$ be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

# Derivation

# Uniqueness

Let $X$ and $Y$ have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. $X$ and $Y$ have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.

# Function of Random Variables

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# Obtaining the PDFs

Let there be a random variable $X$ with a known distribution function $F_X(x)$, the density function for the random variable $Y = g(X)$ can be found with the following steps

1. Find the region of $Y$ in the space of $X$, find $g^{-1}(y)$
2. Find the region of $Y \leq y$
3. Find $F_Y(y) = P(Y \leq y)$ using the probability density function of $X$ over region $Y \leq y$
4. Find $f_Y(y)$ by differentiating $F_Y(y)$

# Example 1

Let $X$ have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 3X - 1$?

Let there be a random variable $X$ with a known distribution function $F_X(x)$, if the random variable $Y = g(X)$ is either increasing or decreasing, than the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

# Example 2

Let $X$ have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 5 - (X/2)$?

# Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable $X$ with a known distribution function $F_X(x)$ and random variable $Y = g(X)$, the distribution of $Y$ can be found by:

1. Find the moment generating function of $Y$, $M_Y(t)$.
2. Compare $M_Y(t)$, with known moment generating functions. If $M_Y(t) = M_V(t)$, for all values $t$, them $Y$ and $V$ have identical distributions.

## Example 3

Let $X$ follow a normal distribution with mean $\mu$ and variance $\sigma^2$. Find the distribution of $Z = \frac{X-\mu}{\sigma}$.

# Example 4

Let $Z$ follow a standard normal distribution with mean $0$ and variance $1$. Find the distribution of $Y = Z^2$