

## **Review:**

More Probability Theory

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## Moment Generating Functions

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## Moments

The  $k$ th moment is defined as the expectation of the random variable, raised to the  $k$ th power, defined as  $E(X^k)$ .

## Moment Generating Functions

The moment generating functions is used to obtain the  $k$ th moment. The mgf is defined as

$$m(t) = E(e^{tX})$$

The  $k$ th moment can be obtained by taking the  $k$ th derivative of the mgf, with respect to  $t$ , and setting  $t$  equal to 0:

$$E(X^k) = \frac{d^k m(t)}{dt^k} \Big|_{t=0}$$

## Characteristic Functions

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## Characteristic Functions

$$\phi(t) = E(e^{itX}) = E\{\cos(tX)\} + iE\{\sin(tX)\}$$

## Poisson Distribution

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**MGF**

## Expected Value

# Variance

## Binomial Distribution

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**MGF**

## Uniform Distribution

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**MGF**

## Normal Distribution

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MGF

## MGF Properties

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## Linearity

Let  $X$  follow a distribution  $f$ , with the MGF  $M_X(t)$ , the MGF of  $Y = aX + b$  is given as

$$M_Y(t) = e^{tb} M_X(at)$$

# Derivation

## Linearity

Let  $X$  and  $Y$  be two random variables with MGFs  $M_X(t)$  and  $M_Y(t)$ , respectively, and are independent. The MGF of  $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

# Derivation

## Uniqueness

Let  $X$  and  $Y$  have the following distributions  $F_X(x)$  and  $F_Y(y)$  and MGFs  $M_X(t)$  and  $M_Y(t)$ , respectively.  $X$  and  $Y$  have the same distribution  $F_X(x) = F_Y(y)$  if and only if  $M_X(t) = M_Y(t)$ .

## Uniqueness

Let  $X_1, \dots, X_n$  be independent random variables, where

$X_i \sim N(\mu_i, \sigma_i^2)$ , with  $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$  for  $i = 1, \dots, n$ .

Find the MGF of  $Y = a_1 X_1 + \dots + a_n X_n$ , where  $a_1, \dots, a_n$  are constants.

## Function of Random Variables

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# Function of Random Variables

## Obtaining the PDFs

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## Using the Distribution Function

Let there be a random variable  $X$  with a known distribution function  $F_X(x)$ , the density function for the random variable  $Y = g(X)$  can be found with the following steps

1. Find the region of  $Y$  in the space of  $X$ , find  $g^{-1}(y)$
2. Find the region of  $Y \leq y$
3. Find  $F_Y(y) = P(Y \leq y)$  using the probability density function of  $X$  over region  $Y \leq y$
4. Find  $f_Y(y)$  by differentiating  $F_Y(y)$

## Example 1

Let  $X$  have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of  $Y = 3X - 1$ ?

## Using the PDF

Let there be a random variable  $X$  with a known distribution function  $F_X(x)$ , if the random variable  $Y = g(X)$  is either increasing or decreasing, than the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

## Example 2

Let  $X$  have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of  $Y = 5 - (X/2)$ ?

## Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable  $X$  with a known distribution function  $F_X(x)$  and random variable  $Y = g(X)$ , the distribution of  $Y$  can be found by:

1. Find the moment generating function of  $Y$ ,  $M_Y(t)$ .
2. Compare  $M_Y(t)$ , with known moment generating functions. If  $M_Y(t) = M_V(t)$ , for all values  $t$ , then  $Y$  and  $V$  have identical distributions.

### Example 3

Let  $X$  follow a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Find the distribution of  $Z = \frac{X-\mu}{\sigma}$ .

## Example 4

Let  $Z$  follow a standard normal distribution with mean 0 and variance 1.  
Find the distribution of  $Y = Z^2$