Joint Distribution Functions

Conditional Distributions

A conditional distribution provides the probability of a random variable, given that it was conditioned on the value of a second random variable.

Discrete Conditional Distributions

Let X and Y be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

The conditional distribution of $X|Y = y$ is defined as

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Continuous Conditional Distributions

Let X and Y be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. The conditional distribution of $X|Y = y$ is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Bivariae Normal Conditional Distribution

Multivariate Normal

The normal distribution function can be extended to from univariate (n=1), bivariate (n=2), to a k-dimensional normal distribution function:

$$X \sim N_k(\mu, \Sigma)$$

Multivariate Normal Parameters

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} \quad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{21} & \cdots & \sigma_{k1} \\ \sigma_{21} & \sigma_2^2 & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_k^2 \end{pmatrix}$$

Multivariate Normal PDF

$$f_X(x) = \det(2\pi\Sigma)^{-1/2} \exp\left\{-\frac{1}{2}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right\}$$

Marginal Normal Distribution

$$X = \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} \quad \mathcal{X}_1 = \begin{pmatrix} X_1 \\ \vdots \\ X_j \end{pmatrix} \quad \mathcal{X}_2 = \begin{pmatrix} X_{j+1} \\ \vdots \\ X_k \end{pmatrix}$$

Marginal Normal Distribution

$$\mu = \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix} \ \varpi_1 = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_j \end{pmatrix} \ \varpi_2 = \begin{pmatrix} \mu_{j+1} \\ \vdots \\ \mu_k \end{pmatrix}$$

Marginal Normal Distribution

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{21} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}$$

Marginal Normal Distribution

$$\mathcal{X}_1 \sim N_j(\varpi_1, \varSigma_{11})$$

$$\mathcal{X}_2 \sim N_{k-j}(\varpi_2, \varSigma_{22})$$

Conditional Normal Distribution

$$\mathcal{X}_2 | \mathcal{X}_1 \sim N_j(\varpi_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathcal{X}_1 - \varpi_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

Independent Random Variables

Random variables are considered independent of each other if the probability of one variable does not affect the probability of another variable.

Discrete Independent Random Variables

Let X and Y be 2 discrete random variables, with a joint density function of $p_{X,Y}(x, y)$. X is independent of Y if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Continuous Independent Random Variables

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Matrix Algebra

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\det(A) = a_1 a_2$$

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}$$

Example

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right\}$$

Show that $X \perp Y$ (**this will be on the upcoming test!**).

Covariance

Let X and Y be 2 random variables with mean $E(X) = \mu_x$ and $E(Y) = \mu_y$, respectively. The covariance of X and Y is defined as

$$Cov(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\}$$

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

Covariance

If X and Y are independent random variables, then

$$Cov(X,Y) = 0$$

Correlation

The correlation of X and Y is defined as

$$\rho = Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Expectations

Let $X = (X_1, X_2, \dots, X_n)^{\mathrm{T}}$ be a set of random variables, the expectation of a function $g(X)$ is defined as

$$E\{g(X)\} = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} g(X) p(x, \theta)$$

or

$$E\{g(X)\} = \int_{x_1 \in X_1} \cdots \int_{x_n \in X_n} g(X) f(x, \theta) dx_n \cdots dx_1$$

Expected Value and Variance of Linear Functions

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be random variables with $E(X_i) = \mu_i$ and $E(Y_j) = \tau_j$. Furthermore, let $U = \sum_{i=1}^{n} a_i X_i$ and $V = \sum_{j=1}^{m} b_j Y_j$ where $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ are constants. We have the following properties:

- $E(U) = \sum_{i=1}^{n} a_i \mu_i$
- $Var(U) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum\sum_{i<j} a_i a_j Cov(X_i, X_j)$
- $Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$

Conditional Expectations

Let X_1 and X_2 be two random variables, the conditional expectation of $g(X_1)$, given $X_2 = x_2$, is defined as

$$E\{g(X_1)|X_2 = x_2\} = \sum_{x_1} g(x_1)p(x_1|x_2)$$

or

$$E\{g(X_1)|X_2 = x_2\} = \int_{x_1} g(x_1)f(x_1|x_2)dx_1.$$

Conditional Expectations

Furthermore,

$$E(X_1) = E_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$

and

$$Var(X_1) = E_{X_2}\{Var_{X_1|X_2}(X_1|X_2)\}+Var_{X_2}\{E_{X_1|X_2}$$