Extending Gödel-Löb Provability Logic with an Interface Operator

Technical Documentation for Master's Thesis

GL+I: A Conservative Extension with Dual Accessibility Relations

Pratik Deshmukh

MSc Logic and Computation

Technische Universität Wien

Supervisor: Prof. Agata Ciabattoni

pratik.deshmukh@student.tuwien.ac.at

November 2025

Executive Summary

Project Overview

This thesis develops GL+I, a conservative extension of Gödel-Löb provability logic (GL) with a novel interface operator I. The extension employs dual accessibility relations ($R_\Box$ for provability, R_I for interface alignment) with the critical constraint $R_\Box \subseteq R_I$.

Main Contributions:

- Complete axiomatization with three new axioms (I1, I2, I3)
- Rigorous Kripke semantics with dual accessibility
- Full well-definedness proofs (consistency, arithmetic representability, closure)
- Soundness theorem establishing the syntactic-semantic bridge

1 Document Package Contents

This submission package consists of four technical documents providing a complete mathematical foundation for GL+I:

Doc	Title	Pages	Purpose
1	Comprehensive Introduction	5	Motivation, historical context, scope
2	Frame Properties	5	Semantic foundations, frame theory
3	Well-Definedness Proofs	7	Consistency, representability, closure
4	Soundness Theorem	5	Axiom validity, semantic grounding
	Total	**22**	

2 Document Summaries

2.1 Document 1: Comprehensive Introduction

File: `GL_I_Introduction.pdf`
Content Overview:

- Historical context: Gödel (1931), Löb (1955), Solovay (1976), Boolos (1979/1993)
- Technical framework: Language $\mathcal{L}(\Box, I)$, axiom system, inference rules
- Main results summary: Consistency, arithmetic representability, operator interactions
- Expressive power analysis: New concepts expressible in $\mathsf{GL{+}I}$
- Relationship to formal systems and incompleteness phenomena
- Explicit scope limitations (completeness reserved for future work)

Key Contribution: Positions $\mathsf{GL{+}I}$ within the modal logic landscape with appropriate scope claims.

2.2 Document 2: Frame Properties

File: `GL_I_Frame_Properties.pdf`
Content Overview:

- Frame class characterization: $\mathcal{F}_{\mathsf{GL+I}}$ with formal conditions
- Model and satisfaction definitions
- Accessibility degree analysis: $\deg_\Box(w)$, $\deg_I(w)$, extension degree
- Axiom-frame correspondence theorems
- Frame morphisms and bisimulation for $\mathsf{GL{+}I}$
- Filtration and finite model property
- Complexity analysis: Model checking in P, satisfiability PSPACE-complete
- Concrete model examples demonstrating non-collapse

Key Contribution: Establishes rigorous semantic foundations with dual accessibility relations.

2.3 Document 3: Well-Definedness Proofs

File: `GL_I_Well_Definedness.pdf`
Content Overview:

- **Consistency**: $\mathsf{GL{+}I} \nvdash I\bot$ (semantic and syntactic proofs)
- **Arithmetic Representability**: Σ_1 interface provability predicate
- **Operator Interactions**: $\Box A \leftrightarrow \Box I A$ (central equivalence)
- **Closure Properties**: MP, necessitation, substitution, replacement preserved
- Non-triviality results demonstrating $I \neq \Box$
- Extended derivability conditions DI1–DI4

Key Contribution: Demonstrates mathematical coherence and well-behavedness of the extension.

2.4 Document 4: Soundness Theorem

File: `GL_I_Soundness.pdf`

Content Overview:

- Semantic framework recap (frames, models, satisfaction)
- Soundness of standard GL axioms (K, 4, GL/Löb)
- Soundness of interface axioms:
  - I1 (Distribution): $I(A \to B) \to (IA \to IB)$
  - I2 (Self-reflection): $IA \to IIA$
  - I3 (Inclusion): $\Box A \to IA$
- Soundness of inference rules (MP, $\mathrm{Nec}_{\Box}$, Nec_I)
- Main soundness theorem: $\mathsf{GL{+}I} \vdash A \Rightarrow \models_{\mathsf{GL+I}} A$
- Axiom-frame correspondence table

Key Contribution: Establishes the syntactic-semantic bridge, grounding provability in validity.

3 Technical Framework Summary

3.1 The GL+I System

Axiomatization

Language: $\mathcal{L}(\Box, I)$ with propositional variables, Boolean connectives, and modal operators $\Box$ (provability) and I (interface)

Axioms:

- Propositional tautologies
- K: $\Box(A \to B) \to (\Box A \to \Box B)$
- 4: $\Box A \to \Box\Box A$
- GL: $\Box(\Box A \to A) \to \Box A$
- I1: $I(A \to B) \to (IA \to IB)$
- I2: $IA \to IIA$
- I3: $\Box A \to IA$

Rules: Modus Ponens, $\mathrm{Nec}_{\Box}$, Nec_I

3.2 Semantic Framework

Dual Accessibility Semantics

Frame: $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $R_\Box$: transitive, irreflexive, conversely well-founded (standard GL)
- R_I: transitive, with $R_\Box \subseteq R_I$ (critical constraint)

Truth Conditions:

$$\begin{aligned} w \models \Box A \quad &\text{iff} \quad \forall v(wR_\Box v \Rightarrow v \models A) \\ w \models IA \quad &\text{iff} \quad \forall v(wR_I v \Rightarrow v \models A) \end{aligned}$$

4 Main Results Summary

Result	Statement	Doc
Consistency	$\mathsf{GL{+}I} \nvdash I\bot$	3
Conservative Extension	$\mathsf{GL} \vdash \varphi \Leftrightarrow \mathsf{GL{+}I} \vdash \varphi$ (for GL-formulas)	3
Arithmetic Representability	Interface provability is Σ_1	3
Operator Interaction	$\vdash \Box A \leftrightarrow \Box IA$	3
Non-Collapse	$\nvdash IA \to \Box A$ (in general)	2, 3
Finite Model Property	Every satisfiable formula has finite model	2
Complexity	Satisfiability is PSPACE-complete	2
Soundness	$\mathsf{GL{+}I} \vdash A \Rightarrow \models_{\mathsf{GL+I}} A$	4
Axiom Correspondence	I1 $\leftrightarrow$ (none), I2 $\leftrightarrow$ Trans(R_I), I3 $\leftrightarrow$ $R_\Box \subseteq R_I$	2, 4

5 Scope and Future Work

5.1 Current Scope

This work establishes the **foundational theory** of $\mathsf{GL{+}I}$:

- Complete axiomatization with semantic framework
- Full well-definedness proofs
- Soundness theorem
- Expressive power analysis

5.2 Explicit Limitations

The following are **reserved for future work**:

- **Completeness**: Canonical model construction and completeness proof
- **Advanced Fixed Points**: Extended diagonalization techniques
- **Algorithmic Implementation**: Automated reasoning systems
- **Applications**: Concrete applications to formal verification