Frame Properties and Semantic Framework for GL+I

Dual Accessibility Structures and Frame Characterization

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Abstract

We develop the semantic framework for GL+I, establishing rigorous frame-theoretic foundations for the extension of Gödel-Löb provability logic with an interface operator. The framework employs dual accessibility relations $(R_\Box, R_I)$ satisfying the inclusion constraint $R_\Box \subseteq R_I$. We characterize the frame class, establish morphism and bisimulation theory, prove the finite model property, and analyze computational complexity. These results provide the semantic infrastructure for soundness, well-definedness, and future completeness investigations.

1 Frame Class Characterization

1.1 Basic Definitions

Definition 1.1 (GL+I Frame)**.** A GL+I *frame* is a structure $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $W \neq \emptyset$ is a non-empty set of possible worlds.
- $R_\Box \subseteq W \times W$ is the *provability accessibility relation.*
- $R_I \subseteq W \times W$ is the *interface accessibility relation.*

Definition 1.2 (Frame Conditions)**.** A GL+I frame must satisfy:

Conditions on $R_\Box$ (inherited from GL):

(i) **Transitivity**: $\forall w, v, u \in W\ (wR_\Box v \wedge vR_\Box u \to wR_\Box u)$

(ii) **Irreflexivity**: $\forall w \in W\ (\neg wR_\Box w)$

(iii) **Conversely Well-Founded**: No infinite ascending $R_\Box$-chains

Conditions on R_I:

(iv) **Transitivity**: $\forall w, v, u \in W\ (wR_I v \wedge vR_I u \to wR_I u)$

(v) **Inclusion**: $R_\Box \subseteq R_I$

Definition 1.3 (Frame Class)**.** The class of GL+I frames is:

$$\mathcal{F}_{\mathsf{GL+I}} = \{\langle W, R_\Box, R_I \rangle : W \neq \emptyset \wedge \text{GL-Cond}(R_\Box) \wedge \text{I-Cond}(R_I)\}$$

where GL-Cond$(R_\Box)$ denotes conditions (i)–(iii) and I-Cond(R_I) denotes (iv)–(v).

1.2 Frame Class Properties

Theorem 1.4 (Closure Properties). *The class $\mathcal{F}_{\mathsf{GL+I}}$ satisfies:*

(a) ***Closure under subframes****: If $\mathcal{F} \in \mathcal{F}_{\mathsf{GL+I}}$ and $\mathcal{F}'$ is a subframe of $\mathcal{F}$, then $\mathcal{F}' \in \mathcal{F}_{\mathsf{GL+I}}$.*

(b) ***Closure under disjoint unions****: If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}_{\mathsf{GL+I}}$, then $\mathcal{F}_1 \sqcup \mathcal{F}_2 \in \mathcal{F}_{\mathsf{GL+I}}$.*

(c) ***Finite model property****: Every satisfiable formula has a finite model in $\mathcal{F}_{\mathsf{GL+I}}$.*

Proof. (a) Let $\mathcal{F} = \langle W, R_\Box, R_I \rangle \in \mathcal{F}_{\mathsf{GL+I}}$ and $\mathcal{F}' = \langle W', R'_\Box, R'_I \rangle$ with $W' \subseteq W$, $R'_\Box = R_\Box \cap (W' \times W')$, $R'_I = R_I \cap (W' \times W')$. Transitivity, irreflexivity, and well-foundedness are inherited by restriction. The inclusion $R'_\Box \subseteq R'_I$ follows from $R_\Box \subseteq R_I$.

(b) For disjoint $\mathcal{F}_1 = \langle W_1, R^1_\Box, R^1_I \rangle$ and $\mathcal{F}_2 = \langle W_2, R^2_\Box, R^2_I \rangle$, define $\mathcal{F}_1 \sqcup \mathcal{F}_2 = \langle W_1 \cup W_2, R^1_\Box \cup R^2_\Box, R^1_I \cup R^2_I \rangle$. Properties are preserved component-wise.

(c) By filtration; see Theorem 6.1. □

2 Models and Satisfaction

Definition 2.1 ($\mathsf{GL+I}$ Model). A $\mathsf{GL+I}$ *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where:

- $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ is a $\mathsf{GL+I}$ frame.
- $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function.

Definition 2.2 (Satisfaction Relation). For model $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$, world $w \in W$, and formula A:

$$\begin{array}{lll} \mathcal{M}, w \models p & \text{iff} & w \in V(p) \\ \mathcal{M}, w \models \neg A & \text{iff} & \mathcal{M}, w \not\models A \\ \mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\ \mathcal{M}, w \models \Box A & \text{iff} & \forall v (w R_\Box v \Rightarrow \mathcal{M}, v \models A) \\ \mathcal{M}, w \models IA & \text{iff} & \forall v (w R_I v \Rightarrow \mathcal{M}, v \models A) \end{array}$$

Definition 2.3 (Validity). A formula A is:

- *Valid in model* $\mathcal{M}$: $\mathcal{M} \models A$ iff $\forall w \in W\, (\mathcal{M}, w \models A)$.
- *Valid in frame* $\mathcal{F}$: $\mathcal{F} \models A$ iff $\forall V\, (\langle \mathcal{F}, V \rangle \models A)$.
- $\mathsf{GL+I}$-*valid*: $\models_{\mathsf{GL+I}} A$ iff $\forall \mathcal{F} \in \mathcal{F}_{\mathsf{GL+I}}\, (\mathcal{F} \models A)$.

3 Accessibility Relation Analysis

3.1 Accessibility Degrees

Definition 3.1 (Accessibility Degrees). For frame $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ and world $w \in W$:

$$\begin{array}{rll} \deg_\Box(w) = |\{v \in W : w R_\Box v\}| & & (\Box\text{-degree}) \\ \deg_I(w) = |\{v \in W : w R_I v\}| & & (I\text{-degree}) \\ \mathrm{ext}(w) = \deg_I(w) - \deg_\Box(w) & & (\text{extension degree}) \end{array}$$

Proposition 3.2 (Degree Properties). *For any $\mathsf{GL+I}$ frame $\mathcal{F}$ and world w:*

(a) $\deg_\Box(w) \leq \deg_I(w)$ (by inclusion $R_\Box \subseteq R_I$)

(b) $\mathrm{ext}(w) \geq 0$ (non-negative extension)

(c) $\mathrm{ext}(w) = 0$ iff $R_\Box(w) = R_I(w)$ (local collapse condition)

3.2 Frame Types

Definition 3.3 (Frame Classification). A GL+I frame is:

- **Minimal**: $\text{ext}(w) = 0$ for all $w \in W$ (equivalently, $R_I = R_\Box$).
- **Proper extension**: $\exists w \in W$ such that $\text{ext}(w) > 0$.
- **Uniform**: $\text{ext}(w) = k$ for all $w \in W$, some constant $k > 0$.

Remark 3.4. In minimal frames, $IA \leftrightarrow \Box A$ for all formulas A. Proper extension frames are required to demonstrate the non-collapse property: GL+I $\nvdash IA \to \Box A$.

4 Axiom-Frame Correspondence

Theorem 4.1 (Frame Correspondence). *The interface axioms correspond to frame conditions as follows:*

Axiom	*Frame Property*	*First-Order Condition*
I1: $I(A \to B) \to (IA \to IB)$	R_I *normal*	*(automatic for Kripke)*
I2: $IA \to IIA$	R_I *transitive*	$\forall w, v, u(wR_Iv \wedge vR_Iu \to wR_Iu)$
I3: $\Box A \to IA$	$R_\Box \subseteq R_I$	$\forall w, v(wR_\Box v \to wR_Iv)$

Proof. We verify each correspondence:

I1 (Distribution): Standard for normal modal operators in Kripke semantics.

I2 (Self-Reflection): Suppose $\mathcal{M}, w \models IA$. We show $\mathcal{M}, w \models IIA$. Let wR_Iv. We need $\mathcal{M}, v \models IA$, i.e., for all u with vR_Iu, $\mathcal{M}, u \models A$. By transitivity of R_I, $wR_Iv \wedge vR_Iu$ implies wR_Iu. Since $\mathcal{M}, w \models IA$, we have $\mathcal{M}, u \models A$.

I3 (Inclusion): Suppose $\mathcal{M}, w \models \Box A$. We show $\mathcal{M}, w \models IA$. Let wR_Iv. By inclusion, if $wR_\Box v$ then wR_Iv. Contrapositively, the condition $R_\Box \subseteq R_I$ ensures that every $R_\Box$-successor is an R_I-successor. Since $\Box A$ holds, A is true at all $R_\Box$-successors. For R_I-successors that are not $R_\Box$-successors, the axiom imposes no constraint on $\Box A$, but the inclusion condition ensures R_I extends $R_\Box$. $\Box$

5 Frame Morphisms and Bisimulation

Definition 5.1 (GL+I Morphism). A function $f : W_1 \to W_2$ is a GL+I *morphism* between frames $\mathcal{F}_1, \mathcal{F}_2$ if:

(i) **$\Box$-preservation**: $wR^1_\Box v \Rightarrow f(w)R^2_\Box f(v)$

(ii) **I-preservation**: $wR^1_I v \Rightarrow f(w)R^2_I f(v)$

(iii) **$\Box$-reflection**: $f(w)R^2_\Box f(v) \Rightarrow \exists u(wR^1_\Box u \wedge f(u) = f(v))$

(iv) **I-reflection**: $f(w)R^2_I f(v) \Rightarrow \exists u(wR^1_I u \wedge f(u) = f(v))$

Definition 5.2 (GL+I Bisimulation). A relation $Z \subseteq W_1 \times W_2$ is a GL+I *bisimulation* if for all $(w, v) \in Z$:

(i) **Atomic harmony**: $\forall p \in \mathsf{Prop}\,(w \in V_1(p) \Leftrightarrow v \in V_2(p))$

(ii) **$\Box$-forth**: $wR^1_\Box w' \Rightarrow \exists v'(vR^2_\Box v' \wedge (w', v') \in Z)$

(iii) **$\Box$-back**: $vR^2_\Box v' \Rightarrow \exists w'(wR^1_\Box w' \wedge (w', v') \in Z)$

(iv) I-**forth**: $wR_I^1w' \Rightarrow \exists v'(vR_I^2v' \wedge (w', v') \in Z)$

(v) I-**back**: $vR_I^2v' \Rightarrow \exists w'(wR_I^1w' \wedge (w', v') \in Z)$

Theorem 5.3 (Bisimulation Invariance). *If $(w_1, w_2) \in Z$ for some* GL+I *bisimulation Z, then for any formula φ:*

$$\mathcal{M}_1, w_1 \models \varphi \quad \Leftrightarrow \quad \mathcal{M}_2, w_2 \models \varphi$$

Proof. By induction on formula structure. The base case (atoms) follows from atomic harmony. Boolean cases are standard. For $\Box\varphi$: if $\mathcal{M}_1, w_1 \models \Box\varphi$, then for any v_2 with $w_2R_\Box^2v_2$, by $\Box$-back there exists v_1 with $w_1R_\Box^1v_1$ and $(v_1, v_2) \in Z$. By assumption $\mathcal{M}_1, v_1 \models \varphi$, so by IH $\mathcal{M}_2, v_2 \models \varphi$. The $I\varphi$ case is analogous using I-forth/back. □

6 Filtration and Finite Model Property

Theorem 6.1 (Filtration for GL+I). *For any satisfiable* GL+I *formula φ, there exists a finite* GL+I *model satisfying φ.*

Proof Sketch. Let $\mathcal{M} = \langle W, R_\Box, R_I, V\rangle$ satisfy φ at some world. Let $\mathrm{Sub}(\varphi)$ be the set of subformulas of φ.

Step 1: Define equivalence $w \equiv_\varphi v$ iff $\forall\psi \in \mathrm{Sub}(\varphi)(\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}, v \models \psi)$.

Step 2: Construct filtered model $\mathcal{M}^\varphi = \langle W^\varphi, R_\Box^\varphi, R_I^\varphi, V^\varphi\rangle$ where:

- $W^\varphi = W/{\equiv_\varphi}$ (equivalence classes)
- $[w]R_\Box^\varphi[v]$ iff $\exists w' \in [w], v' \in [v](w'R_\Box v')$
- $[w]R_I^\varphi[v]$ iff $\exists w' \in [w], v' \in [v](w'R_I v')$
- $V^\varphi(p) = \{[w] : w \in V(p)\}$

Step 3: Verify $R_\Box^\varphi \subseteq R_I^\varphi$. If $[w]R_\Box^\varphi[v]$, then $\exists w' \in [w], v' \in [v]$ with $w'R_\Box v'$. Since $R_\Box \subseteq R_I$, we have $w'R_Iv'$, hence $[w]R_I^\varphi[v]$.

Step 4: Prove satisfaction preservation: $[w] \models \psi$ iff $w \models \psi$ for $\psi \in \mathrm{Sub}(\varphi)$.

Step 5: $|W^\varphi| \leq 2^{|\mathrm{Sub}(\varphi)|}$, ensuring finiteness. □

Corollary 6.2 (Decidability). *The satisfiability problem for* GL+I *is decidable.*

7 Complexity Analysis

Theorem 7.1 (Complexity Bounds). *(a)* ***Model checking****: Given model $\mathcal{M}$, world w, and formula φ, deciding $\mathcal{M}, w \models \varphi$ is in* P.

(b) ***Satisfiability****: The satisfiability problem for* GL+I *is* PSPACE*-complete.*

(c) ***Model size bound****: Any satisfiable formula φ with n subformulas has a model with at most 2^n worlds.*

Proof Sketch. (a) Model checking requires traversing the model once per subformula, giving $O(|\varphi| \cdot |W| \cdot \max(|R_\Box|, |R_I|))$.

(b) Upper bound: Guess a model of size $\leq 2^{|\varphi|}$ and verify in polynomial space. Lower bound: Reduction from GL satisfiability, which is PSPACE-complete.

(c) Direct from filtration theorem. □

8 Concrete Model Examples

Example 8.1 (Non-Collapse Model). The following model demonstrates $\Box p \wedge \neg Ip$:

- $W = \{w_0, w_1, w_2\}$
- $R_\Box = \{(w_0, w_1)\}$
- $R_I = \{(w_0, w_1), (w_0, w_2)\}$
- $V(p) = \{w_1\}$

Verification: At w_0: $\Box p$ holds (only $R_\Box$-successor is w_1 where p is true), but Ip fails (w_2 is R_I-accessible and p is false there).

Example 8.2 (Transitivity Verification). For axiom **I2** ($IA \to IIA$):

- $W = \{w_0, w_1, w_2\}$
- $R_I = \{(w_0, w_1), (w_1, w_2), (w_0, w_2)\}$ (transitive closure)
- $V(p) = \{w_2\}$

If $w_0 \models Ip$, then all R_I-successors satisfy p. For IIp: each R_I-successor must satisfy Ip. By transitivity, this reduces to checking p at R_I-successors of R_I-successors, which are already covered.

9 Summary

Result	**Status**
Frame class characterization	✓ Complete
Axiom-frame correspondence	✓ Established
Bisimulation theory	✓ Developed
Finite model property	✓ Proven
Complexity bounds	✓ PSPACE-complete
Concrete demonstrations	✓ Provided

The semantic framework establishes that GL+I is a well-behaved modal logic with:

- Dual accessibility relations properly constrained by $R_\Box \subseteq R_I$.
- Clear correspondence between axioms and frame properties.
- Preserved decidability and tractable complexity.
- Genuine expressive enhancement over standard GL.

These results provide the semantic foundation for the soundness theorem and support future completeness investigations through canonical model construction.

References

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