Frame Properties of GL and GL+I

A Technical Reference for Modal Provability Logic Extensions

Pratik Deshmukh
MSc Logic and Computation, TU Wien

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Quick Reference

GL: K4 + Irreflexivity + Well-foundedness
GL+I: Bi-modal (GL for $\Box$, K+Trans for I, with $R_\Box \subseteq R_I$)

1 Standard GL (Gödel-Löb Provability Logic)

1.1 Frame Definition

A **GL frame** is a structure $\mathcal{F} = \langle W, R \rangle$ where:

- W is a non-empty finite set of possible worlds
- R is a binary accessibility relation on W

1.2 Frame Conditions

T1. Transitivity

$$\forall w, v, u \in W : (wRv \wedge vRu) \to wRu$$

Validates axiom 4: $\Box A \to \Box\Box A$

T2. Irreflexivity

$$\forall w \in W : \neg(wRw)$$

Consequence: $\Box A \to A$ is **not** valid (distinguishes GL from S4)

T3. Conversely Well-Founded

$$\text{No infinite sequence: } \ldots Rw_2Rw_1Rw_0$$

*Validates **Löb's axiom***: $\Box(\Box A \to A) \to \Box A$

Note: In finite frames, T2 + T1 automatically implies T3.

1.3 Truth Conditions

For model $\mathcal{M} = \langle W, R, V \rangle$ and world $w \in W$:

$$\mathcal{M}, w \models \Box A \quad \text{iff} \quad \forall v \in W : wRv \text{ implies } \mathcal{M}, v \models A$$

1.4 Axiomatization

GL extends classical propositional logic with:

- **K**: $\Box(A \to B) \to (\Box A \to \Box B)$
- **4**: $\Box A \to \Box\Box A$
- **GL**: $\Box(\Box A \to A) \to \Box A$
- **Nec**: From $\vdash A$, infer $\vdash \Box A$

2 GL+I (Extended System with Interface Operator)

2.1 Frame Definition

A **GL+I frame** is a triple $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- W is a non-empty finite set of possible worlds
- $R_\Box$ is the provability accessibility relation (GL conditions)
- R_I is the interface accessibility relation (new)

2.2 Frame Conditions

For $R_\Box$ (provability relation): Standard GL conditions (Transitive, Irreflexive, Well-founded)
For R_I (interface relation):

I1. Transitivity

$$\forall w, v, u \in W : (wR_Iv \land vR_Iu) \to wR_Iu$$

*Required for axiom **I2***: $IA \to IIA$

CRITICAL CONSTRAINT

I2. Inclusion

$$\forall w, v \in W : wR_\Box v \to wR_Iv$$

*Required for axiom **I3***: $\Box A \to IA$
Meaning: Every provability arrow is also an interface arrow

I3. Irreflexivity (Optional)

$$\forall w \in W : \neg(wR_Iw)$$

Status: Not logically required, but simplifies frame theory

2.3 Truth Conditions

For model $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ and world $w \in W$:

$$\begin{aligned} \mathcal{M}, w \models \Box A \quad &\text{iff} \quad \forall v \in W : wR_\Box v \text{ implies } \mathcal{M}, v \models A \\ \mathcal{M}, w \models IA \quad &\text{iff} \quad \forall v \in W : wR_Iv \text{ implies } \mathcal{M}, v \models A \end{aligned}$$

2.4 Extended Axiomatization

GL+I extends GL with three interface axioms:

- **I1**: $I(A \to B) \to (IA \to IB)$ — Distribution
- **I2**: $IA \to IIA$ — Self-reflection
- **I3**: $\Box A \to IA$ — Inclusion
- **Nec$_I$**: From $\vdash A$, infer $\vdash IA$

All GL axioms and theorems are preserved.

2.5 Bi-Modal Structure

GL+I is a **bi-modal logic** with independent modalities coupled through inclusion:

Operator	Base Logic	Frame Type
$\Box$ (Box)	GL	Trans + Irref + WF
I (Interface)	K + Trans	Trans + ($R_\Box \subseteq R_I$)

3 Key Differences and Relationships

3.1 Comparison with Standard Modal Logics

Logic	Refl.	Trans.	Sym.	Other
K	—	—	—	Minimal
K4	—	✓	—	—
S4	✓	✓	—	—
S5	✓	✓	✓	Equivalence
GL	×	✓	—	+ WF
GL+I ($R_\Box$)	×	✓	—	+ WF
GL+I (R_I)	?	✓	—	+ Incl

3.2 The Non-Collapse Property

The inclusion $R_\Box \subseteq R_I$ (strict subset) enables:

Theorem: There exist models where $IA \wedge \neg\Box A$

This demonstrates that $I \neq \Box$ (genuine expressiveness increase).

4 Semantic Correspondences

4.1 Axiom-Frame Correspondence

Axiom	Frame Property
K	No constraint
4 ($\Box A \to \Box\Box A$)	Transitivity
T ($\Box A \to A$)	Reflexivity
GL ($\Box(\Box A \to A) \to \Box A$)	Well-foundedness
I2 ($IA \to IIA$)	Transitivity of R_I
I3 ($\Box A \to IA$)	$R_\Box \subseteq R_I$

4.2 Failed Correspondences

- $\Box A \to IA$ does **not** imply $IA \to \Box A$ (due to $R_\Box \subseteq R_I$)
- $IA \to A$ is **not valid** (due to non-reflexivity of R_I)

5 Computational Properties

Property	**GL**	**GL+I**
Decidability	✓	✓
Complexity	PSPACE-complete	PSPACE-complete
Finite Model Property	✓	✓

6 Summary

6.1 GL Frames

- **Base**: K4 + irreflexivity + well-foundedness
- **Not S4/S5**: Lacks reflexivity
- **Single modality**: $\Box$ (provability)

6.2 GL+I Frames

- **Dual accessibility**: $R_\Box$ (GL), R_I (K + Trans)
- **Critical constraint**: $R_\Box \subseteq R_I$ (enables non-collapse)
- **Conservative**: All GL theorems preserved
- **Decidable**: PSPACE-complete (same as GL)

6.3 Philosophical Interpretation

The frame structure formalizes:

$$\text{Truth (semantic)} \supset IA \text{ (structural alignment)} \supset \Box A \text{ (formal provability)}$$

The **graduated relationship** between truth and provability is captured through dual accessibility relations, where interface alignment mediates between semantic validity and syntactic derivability.

References

1. Boolos, G. (1993). *The Logic of Provability*. Cambridge University Press.
2. Solovay, R. M. (1976). Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25(3-4), 287-304.
3. Beklemishev, L. (2005). Provability algebras and proof-theoretic ordinals. *Annals of Pure and Applied Logic*, 132(2-3), 155-201.

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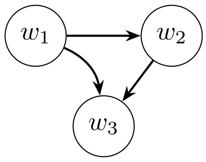
Visual Guide: Modal Logic Frame Hierarchy

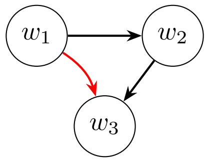
1. Basic Modal Frames

K (Minimal)



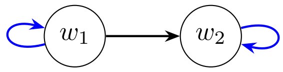
No constraints
Any relation

K4 (Transitive)



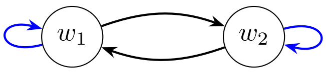
Transitive
Red = implied

S4 (Refl + Trans)



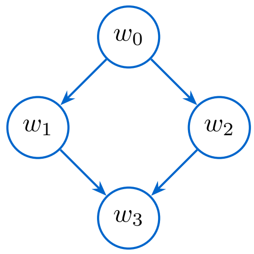
Reflexive + Trans
Blue = self-loops

S5 (Equivalence)



Refl + Trans
+ Symmetric
All worlds see each other

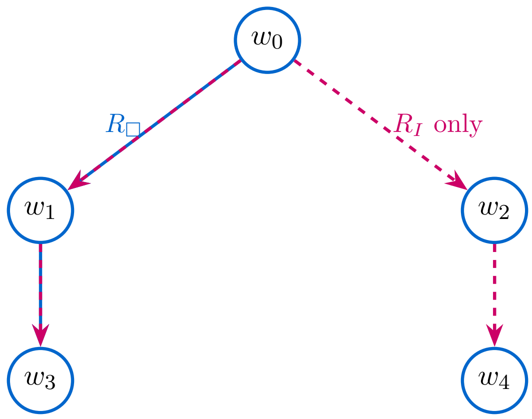
GL (Provability)



Irreflexive +
Trans + WF
No self-loops, finite depth

2. GL+I Extended Frame (Your System)

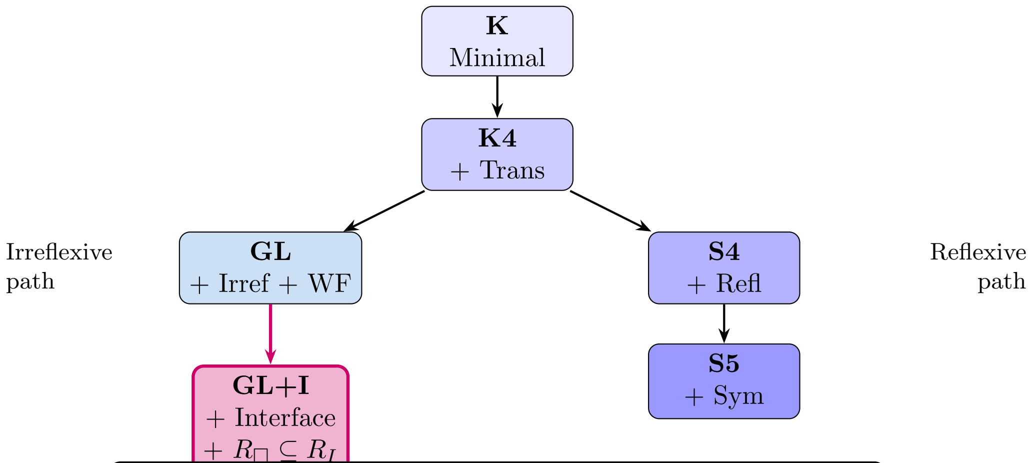
GL+I Frame Structure



— $R_\Box$ (Provability): GL conditions
- - - R_I (Interface): Includes all $R_\Box$ + additional arrows

Critical: $R_\Box \subseteq R_I$
Every solid arrow is also dashed
But NOT vice versa

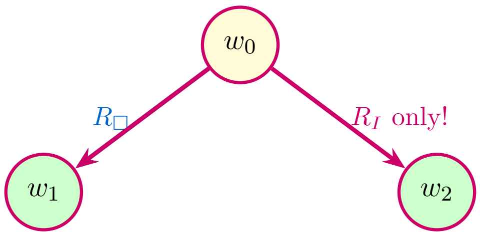
3. The Modal Logic Hierarchy



Position of GL and GL+I:
GL branches from K4 (adds irreflexivity, NOT reflexivity like S4)
GL+I extends GL (bi-modal: GL for $\Box$, K+Trans for I)

4. Non-Collapse Visualization

Example: $IA \wedge \neg\Box A$



If A is true at both w_1 and w_2:
- $w_0 \models IA$ (all R_I-accessible satisfy A)
- $w_0 \not\models \Box A$ (if further structure makes $\Box A$ false)

Result: Interface alignment without provability!