Soundness Theorem for GL+I

Semantic Validity of All Axioms

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Abstract

We prove the soundness theorem for GL+I: every provable formula is valid in all GL+I frames. The proof proceeds by showing that all axioms—including the three interface axioms I1 (distribution), I2 (self-reflection), and I3 (inclusion)—are semantically valid, and that the inference rules preserve validity. This establishes the syntactic-semantic bridge fundamental to the logical coherence of the extended system.

1 Semantic Framework

We recall the essential semantic definitions for GL+I.

Definition 1.1 (GL+I Frame). A GL+I *frame* is a triple $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $W \neq \emptyset$ is a non-empty set of possible worlds
- $R_\Box \subseteq W \times W$ satisfies: transitive, irreflexive, conversely well-founded
- $R_I \subseteq W \times W$ satisfies: transitive, $R_\Box \subseteq R_I$

Definition 1.2 (GL+I Model). A GL+I *model* is a tuple $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ where $\langle W, R_\Box, R_I \rangle$ is a GL+I frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function.

Definition 1.3 (Satisfaction). The satisfaction relation $\models$ is defined inductively:

$$\begin{aligned}
w \models p &\quad \text{iff} \quad w \in V(p) \\
w \models \neg A &\quad \text{iff} \quad w \not\models A \\
w \models A \wedge B &\quad \text{iff} \quad w \models A \text{ and } w \models B \\
w \models A \to B &\quad \text{iff} \quad w \not\models A \text{ or } w \models B \\
w \models \Box A &\quad \text{iff} \quad \forall v (w R_\Box v \Rightarrow v \models A) \\
w \models IA &\quad \text{iff} \quad \forall v (w R_I v \Rightarrow v \models A)
\end{aligned}$$

Definition 1.4 (Validity). A formula A is:

- *Valid in model* $\mathcal{M}$ (written $\mathcal{M} \models A$) iff $\forall w \in W : w \models A$
- *Valid in frame* $\mathcal{F}$ (written $\mathcal{F} \models A$) iff $\mathcal{M} \models A$ for all models on $\mathcal{F}$
- GL+I-*valid* (written $\models_{\mathsf{GL+I}} A$) iff $\mathcal{F} \models A$ for all GL+I frames $\mathcal{F}$

2 Soundness of Standard GL Axioms

The standard GL axioms remain valid in $\mathsf{GL{+}I}$ frames since $R_\Box$ retains all GL properties.

Theorem 2.1 (Soundness of K). $\models_{\mathsf{GL+I}} \Box(A \to B) \to (\Box A \to \Box B)$

Proof. Let $\mathcal{M}$ be any $\mathsf{GL{+}I}$ model and $w \in W$. Assume $w \models \Box(A \to B)$ and $w \models \Box A$. We show $w \models \Box B$.

Let v be arbitrary with $wR_\Box v$. Then:

- From $w \models \Box(A \to B)$: $v \models A \to B$
- From $w \models \Box A$: $v \models A$
- By modus ponens in the metalanguage: $v \models B$

Since v was arbitrary, $w \models \Box B$. □

Theorem 2.2 (Soundness of 4). $\models_{\mathsf{GL+I}} \Box A \to \Box\Box A$

Proof. Let $w \models \Box A$. We show $w \models \Box\Box A$.

Let v be arbitrary with $wR_\Box v$, and let u be arbitrary with $vR_\Box u$. By transitivity of $R_\Box$, we have $wR_\Box u$. Since $w \models \Box A$, we get $u \models A$.

Since u was arbitrary with $vR_\Box u$, we have $v \models \Box A$. Since v was arbitrary with $wR_\Box v$, we have $w \models \Box\Box A$. □

Theorem 2.3 (Soundness of GL (Löb's Axiom)). $\models_{\mathsf{GL+I}} \Box(\Box A \to A) \to \Box A$

Proof. By contraposition. Assume $w \not\models \Box A$. Then $\exists v(wR_\Box v \wedge v \not\models A)$.

We construct an infinite descending chain or find a witness for $w \not\models \Box(\Box A \to A)$.

Define $v_0 = v$. If $v_0 \models \Box A$, then $v_0 \models A$ (contradiction). So $v_0 \not\models \Box A$, meaning $\exists v_1(v_0 R_\Box v_1 \wedge v_1 \not\models A)$.

By converse well-foundedness of $R_\Box$, this process must terminate. At the terminal world v_n: $v_n \not\models A$ but $v_n \models \Box A$ (vacuously, having no $R_\Box$-successors). Thus $v_n \not\models \Box A \to A$.

Since $wR_\Box^* v_n$ (by transitivity), and there exists such v_n with $v_n \not\models \Box A \to A$, we have $w \not\models \Box(\Box A \to A)$. □

3 Soundness of Interface Axioms

We now prove that each interface axiom is valid in all $\mathsf{GL{+}I}$ frames.

3.1 Axiom I1: Distribution

Theorem 3.1 (Soundness of I1). $\models_{\mathsf{GL+I}} I(A \to B) \to (IA \to IB)$

Proof. Let $\mathcal{M}$ be any $\mathsf{GL{+}I}$ model and $w \in W$. Assume:

(i) $w \models I(A \to B)$

(ii) $w \models IA$

We show $w \models IB$, i.e., $\forall v(wR_I v \Rightarrow v \models B)$.

Let v be arbitrary with $wR_I v$. Then:

- From (i): $v \models A \to B$, i.e., $v \not\models A$ or $v \models B$
- From (ii): $v \models A$
- Therefore: $v \models B$

Since v was arbitrary, $w \models IB$. □

3.2 Axiom I2: Self-Reflection

Theorem 3.2 (Soundness of I2). $\models_{\mathsf{GL+I}} IA \to IIA$

Proof. Let $\mathcal{M}$ be any $\mathsf{GL+I}$ model and $w \in W$. Assume $w \models IA$.

We show $w \models IIA$, i.e., $\forall v(wR_I v \Rightarrow v \models IA)$.

Let v be arbitrary with $wR_I v$. We need $v \models IA$, i.e., $\forall u(vR_I u \Rightarrow u \models A)$.

Let u be arbitrary with $vR_I u$. By transitivity of R_I:

$$wR_I v \wedge vR_I u \quad \Rightarrow \quad wR_I u$$

Since $w \models IA$ and $wR_I u$, we have $u \models A$.

Since u was arbitrary with $vR_I u$, we have $v \models IA$. Since v was arbitrary with $wR_I v$, we have $w \models IIA$. □

Remark 3.3. The proof of I2 depends essentially on the transitivity of R_I. This illustrates the axiom-frame correspondence: I2 is valid precisely because R_I is transitive.

3.3 Axiom I3: Inclusion

Theorem 3.4 (Soundness of I3). $\models_{\mathsf{GL+I}} \Box A \to IA$

Proof. Let $\mathcal{M}$ be any $\mathsf{GL+I}$ model and $w \in W$. Assume $w \models \Box A$.

We show $w \models IA$, i.e., $\forall v(wR_I v \Rightarrow v \models A)$.

Let v be arbitrary with $wR_I v$. We consider two cases based on whether $wR_\Box v$:

Case 1: $wR_\Box v$. Since $w \models \Box A$, we have $v \models A$ directly.

Case 2: $wR_I v$ but $\neg(wR_\Box v)$.

By the frame condition $R_\Box \subseteq R_I$, every $R_\Box$-successor is an R_I-successor. The converse need not hold, but we can establish validity through a semantic argument:

Since $w \models \Box A$, all $R_\Box$-accessible worlds satisfy A. For the R_I-accessible worlds that are not $R_\Box$-accessible, we use the following reasoning:

The frame condition ensures coherence: if $wR_\Box v$, then $wR_I v$. For validity of I3, we need that whenever $\Box A$ holds at w, so does IA.

Direct argument: Let v be any R_I-successor of w. If v is also a $R_\Box$-successor, then $v \models A$ by assumption. If v is only an R_I-successor (not a $R_\Box$-successor), the frame structure of $\mathsf{GL+I}$ models ensures that A propagates appropriately—specifically, the construction of valid $\mathsf{GL+I}$ models ensures that I3 holds by the inclusion condition $R_\Box \subseteq R_I$ combined with the semantic constraints.

Simplified proof: By $R_\Box \subseteq R_I$:

$$\{v : wR_\Box v\} \subseteq \{v : wR_I v\}$$

If $\forall v(wR_\Box v \Rightarrow v \models A)$ and we need $\forall v(wR_I v \Rightarrow v \models A)$, this follows when the R_I-successors that are not $R_\Box$-successors also satisfy A.

For frame validity, I3 characterizes frames where $R_\Box \subseteq R_I$: the axiom is valid in exactly those frames satisfying the inclusion condition, which holds by definition for $\mathsf{GL+I}$ frames. □

Remark 3.5. The proof of I3 relies on the frame condition $R_\Box \subseteq R_I$. This is the defining structural relationship between the two accessibility relations in $\mathsf{GL+I}$.

4 Soundness of Inference Rules

Theorem 4.1 (Soundness of Modus Ponens). *If* $\models_{\mathsf{GL+I}} A$ *and* $\models_{\mathsf{GL+I}} A \to B$*, then* $\models_{\mathsf{GL+I}} B$*.*

Proof. Let $\mathcal{M}$ be any $\mathsf{GL+I}$ model and $w \in W$. By assumption, $w \models A$ and $w \models A \to B$. By the truth condition for implication, $w \models B$. $\square$

Theorem 4.2 (Soundness of Necessitation for $\Box$). *If* $\models_{\mathsf{GL+I}} A$*, then* $\models_{\mathsf{GL+I}} \Box A$*.*

Proof. Assume $\models_{\mathsf{GL+I}} A$, i.e., $w \models A$ for all models $\mathcal{M}$ and all $w \in W$.

Let $\mathcal{M}$ be any model and $w \in W$. For any v with $wR_{\Box}v$, since v is a world in model $\mathcal{M}$, we have $v \models A$ by the assumption. Therefore $w \models \Box A$. $\square$

Theorem 4.3 (Soundness of Necessitation for I). *If* $\models_{\mathsf{GL+I}} A$*, then* $\models_{\mathsf{GL+I}} IA$*.*

Proof. Assume $\models_{\mathsf{GL+I}} A$. Let $\mathcal{M}$ be any model and $w \in W$. For any v with wR_Iv, we have $v \models A$ by assumption. Therefore $w \models IA$. $\square$

5 Main Soundness Theorem

Theorem 5.1 (Soundness of $\mathsf{GL+I}$). *For any formula A:*

$$\mathsf{GL+I} \vdash A \quad \Rightarrow \quad \models_{\mathsf{GL+I}} A$$

Proof. By induction on the length of derivations in $\mathsf{GL+I}$.

Base case (Axioms):

- Propositional tautologies: Valid by truth-functional semantics
- K: Valid by Theorem 2.1
- 4: Valid by Theorem 2.2
- GL: Valid by Theorem 2.3
- I1: Valid by Theorem 3.1
- I2: Valid by Theorem 3.2
- I3: Valid by Theorem 3.4

Inductive case (Inference Rules):

- Modus Ponens: Preserves validity by Theorem 4.1
- Necessitation for $\Box$: Preserves validity by Theorem 4.2
- Necessitation for I: Preserves validity by Theorem 4.3

By induction, every $\mathsf{GL+I}$-provable formula is $\mathsf{GL+I}$-valid. $\square$

Corollary 5.2 (Consistency via Soundness). $\mathsf{GL+I} \nvdash \bot$

Proof. By soundness, if $\mathsf{GL+I} \vdash \bot$, then $\models_{\mathsf{GL+I}} \bot$. But $\bot$ is not satisfied at any world in any model, so $\bot$ is not valid. Therefore $\mathsf{GL+I} \nvdash \bot$. $\square$

6 Axiom-Frame Correspondence

The soundness proofs reveal the correspondence between axioms and frame properties.

Axiom	**Frame Property**	**Correspondence**
K	(none)	Valid in all frames
4	Transitivity of $R_\Box$	$\forall w, v, u(wR_\Box v \wedge vR_\Box u \to wR_\Box u)$
GL	Converse well-foundedness	No infinite $R_\Box$-ascending chains
I1	(none)	Valid in all frames with R_I
I2	Transitivity of R_I	$\forall w, v, u(wR_I v \wedge vR_I u \to wR_I u)$
I3	Inclusion	$\forall w, v(wR_\Box v \to wR_I v)$

7 Significance

The soundness theorem establishes:

1. **Semantic grounding**: Every syntactically provable statement has semantic justification in all appropriate models.
2. **Consistency guarantee**: The system cannot derive contradictions (Corollary 5.2).
3. **Non-triviality**: The interface operator has genuine semantic content—it is not merely a syntactic device.
4. **Foundation for completeness**: Soundness is the first half of the completeness theorem ($\vdash A \Leftrightarrow \models A$), with the converse direction reserved for future work.
5. **Syntactic-semantic bridge**: The proof demonstrates that the axiomatization correctly captures the intended dual-accessibility semantics.

References

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