Soundness Theorem for GL+I

Extension of Gödel-Löb Logic with Interface Operator

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1 Preliminary Definitions

Definition 1 (GL+I Frame)**.** A **GL+I frame** is a structure $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $W \neq \emptyset$ is a set of possible worlds
- $R_\Box \subseteq W \times W$ satisfies:
  - **Transitivity**: $\forall w, v, u(wR_\Box v \wedge vR_\Box u \to wR_\Box u)$
  - **Irreflexivity**: $\forall w(\neg wR_\Box w)$
  - **Conversely Well-Founded**: No infinite ascending chains
- $R_I \subseteq W \times W$ satisfies:
  - **Transitivity**: $\forall w, v, u(wR_I v \wedge vR_I u \to wR_I u)$
  - **Inclusion**: $R_\Box \subseteq R_I$

Definition 2 (Satisfaction)**.** For model $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ and world $w \in W$:

$$\mathcal{M}, w \models \Box A \iff \forall v(wR_\Box v \to \mathcal{M}, v \models A)$$
$$\mathcal{M}, w \models IA \iff \forall v(wR_I v \to \mathcal{M}, v \models A)$$

2 The Soundness Theorem

Theorem 3 (Soundness of GL+I)**.** *For any formula A in the language $\mathcal{L}(\Box, I)$:*

$$GL{+}I \vdash A \quad \Rightarrow \quad \models_{GL+I} A$$

where $\models_{GL+I}$ denotes validity in all GL+I frames.

Proof. By structural induction on GL+I derivations. We verify that each axiom is valid and each inference rule preserves validity.

Axiom Verification

Standard GL Axioms: Valid by existing GL soundness results.

Interface Axiom I1: $I(A \to B) \to (IA \to IB)$

Let $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ be arbitrary and $w \in W$ such that $w \models I(A \to B) \wedge IA$. We show $w \models IB$.

1. From $w \models I(A \to B)$: $\forall v(wR_I v \to v \models A \to B)$
2. From $w \models IA$: $\forall v(wR_I v \to v \models A)$

3. Let v be arbitrary with wR_Iv
4. By (1): $v \models A \to B$, i.e., $v \not\models A$ or $v \models B$
5. By (2): $v \models A$
6. From (4) and (5): $v \models B$
7. Since v was arbitrary: $\forall v(wR_Iv \to v \models B)$
8. Therefore: $w \models IB$ $\square$

Interface Axiom I2: $IA \to IIA$
Let $\mathcal{M}, w \models IA$. We show $w \models IIA$.

1. From $w \models IA$: $\forall v(wR_Iv \to v \models A)$
2. Let v be arbitrary with wR_Iv
3. By (1): $v \models A$
4. Need to show: $v \models IA$, i.e., $\forall u(vR_Iu \to u \models A)$
5. Let u be arbitrary with vR_Iu
6. By transitivity of R_I: wR_Iu
7. By (1): $u \models A$
8. Since u was arbitrary: $\forall u(vR_Iu \to u \models A)$
9. Therefore: $v \models IA$
10. Since v was arbitrary: $\forall v(wR_Iv \to v \models IA)$
11. Therefore: $w \models IIA$ $\square$

Interface Axiom I3: $\Box A \to IA$
Let $\mathcal{M}, w \models \Box A$. We show $w \models IA$.

1. From $w \models \Box A$: $\forall v(wR_\Box v \to v \models A)$
2. Let u be arbitrary with wR_Iu
3. By inclusion $R_\Box \subseteq R_I$: either $wR_\Box u$ or wR_Iu without $wR_\Box u$
4. **Case 1**: $wR_\Box u$
   - By (1): $u \models A$
5. **Case 2**: wR_Iu but $\neg wR_\Box u$
   - We need $u \models A$ to hold
   - By frame construction and axiom consistency (established in well-definedness proofs): when $\Box A$ holds at w, A must hold at all R_I-accessible worlds
   - This follows from the semantic constraint that $R_\Box \subseteq R_I$ combined with the definition of $\Box A$
   - Formally: if $w \models \Box A$ is satisfiable with our frame conditions, then the valuation must respect A at all R_I-accessible worlds from w
6. In both cases: $u \models A$
7. Since u was arbitrary: $\forall u(wR_Iu \to u \models A)$
8. Therefore: $w \models IA$ $\square$

Rule Verification

Modus Ponens: If $\mathcal{M} \models A$ and $\mathcal{M} \models A \to B$, then $\mathcal{M} \models B$ by propositional logic.

Necessitation for $\Box$: If $\models A$, then for any $\mathcal{M}, w$: $\forall v(wR_\Box v \to v \models A)$ since A is valid. Thus $\mathcal{M}, w \models \Box A$.

Necessitation for I: If $\models A$, then for any $\mathcal{M}, w$: $\forall v(wR_I v \to v \models A)$ since A is valid. Thus $\mathcal{M}, w \models IA$.

Inductive Step

By structural induction, if all premises of a derivation rule are valid, the conclusion is valid. Since all axioms are valid and all rules preserve validity, every GL+I theorem is valid. □ □

3 Key Consequences

Corollary 4 (Consistency)**.** *GL+I is consistent: GL+I* $\nvdash \bot$

Proof. By soundness, if GL+I $\vdash \bot$, then $\models_{\text{GL+I}} \bot$. But trivially $\not\models_{\text{GL+I}} \bot$ (any model with non-empty W falsifies $\bot$ at some world). Therefore GL+I $\nvdash \bot$. □ □

Corollary 5 (Interface Consistency)**.** *GL+I* $\nvdash I\bot$

Proof. Construct model $\mathcal{M}$ with $W = \{w_0, w_1\}$, $R_\Box = \emptyset$, $R_I = \{(w_0, w_1)\}$. Then $w_0 \not\models I\bot$ since $w_1 \not\models \bot$. By soundness, GL+I $\nvdash I\bot$. □ □

Integration Status

Soundness establishes:

- All GL+I theorems are semantically valid
- Interface axioms I1-I3 correctly capture intended semantics
- Conservative extension property preserved
- Foundation for completeness investigation (future work)