# GL+I: Extension of Gödel-Löb Provability Logic with Interface Operator

Comprehensive Technical Introduction

Pratik Deshmukh

MSc Logic and Computation
Technische Universität Wien

pratik.deshmukh@student.tuwien.ac.at

November 2025

### Abstract

We present GL+I, a conservative extension of Gödel-Löb provability logic (GL) with a novel interface operator $I$. The extension employs dual accessibility relations $(R_\square, R_I)$ with the constraint $R_\square \subseteq R_I$, creating a semantic framework that enhances expressive power while preserving all desirable properties of the base system. We establish complete well-definedness through consistency proofs, arithmetic representability, operator interaction analysis, and closure properties, culminating in a soundness theorem demonstrating that all GL+I theorems are semantically valid. The work provides formal tools for analyzing relationships between syntactic provability and semantic validity within formal systems.

## 1 Introduction and Motivation

### 1.1 Historical Context

Provability logic emerged from the intersection of modal logic and mathematical logic, motivated by Gödel's incompleteness theorems. The field was developed through seminal contributions:

- **Gödel (1931)**: Incompleteness theorems revealing the gap between truth and provability in sufficiently powerful formal systems.

- **Löb (1955)**: Löb's theorem characterizing self-referential provability statements.

- **Solovay (1976)**: Completeness of GL with respect to Peano Arithmetic.

- **Boolos (1979, 1993)**: Systematic development of provability logic theory.

The standard Gödel-Löb system GL captures modal properties of arithmetic provability through a single necessity operator $\square$ interpreted as "$A$ is provable." While GL successfully models provability, it lacks formal tools to analyze the relationship between what is provable within a system and what is semantically valid in the system's models.

### 1.2 Research Motivation

This relationship between syntactic provability and semantic validity is central to:

1. **Metamathematical Analysis**: Understanding the structure of formal theories.

2. **Completeness Questions**: Analyzing when syntactic and semantic consequence align.

3. **Incompleteness Phenomena**: Formalizing the gap identified by Gödel's theorems.

4. **Model-Theoretic Properties**: Connecting proof theory with model theory.

The interface operator $I$ is designed to capture the notion of "alignment between provability and semantic validity," providing formal machinery for these analyses.

## 2 Technical Framework

### 2.1 Language and Syntax

**Definition 2.1** (Language of GL+I). The language $\mathcal{L}(\Box, I)$ is defined inductively:

- Propositional variables: $\mathsf{Prop} = \{p_0, p_1, p_2, \ldots\}$

- Logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

- Modal operators: $\Box$ (provability), $I$ (interface)

Derived operators: $\Diamond A := \neg\Box\neg A$ and $\Diamond A := \neg I \neg A$.

### 2.2 Axiomatic System

**Definition 2.2** (Axioms of GL+I). The system GL+I extends classical propositional logic with:

**Standard GL Axioms:**

$$\begin{aligned}
\mathbf{K} : & \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) & \text{(Distribution)} \\
\mathbf{4} : & \quad \Box A \rightarrow \Box\Box A & \text{(Transitivity)} \\
\mathbf{GL} : & \quad \Box(\Box A \rightarrow A) \rightarrow \Box A & \text{(Löb's Axiom)}
\end{aligned}$$

**Interface Operator Axioms:**

$$\begin{aligned}
\mathbf{I1} : & \quad I(A \rightarrow B) \rightarrow (IA \rightarrow IB) & \text{(Distribution)} \\
\mathbf{I2} : & \quad IA \rightarrow IIA & \text{(Self-Reflection)} \\
\mathbf{I3} : & \quad \Box A \rightarrow IA & \text{(Inclusion)}
\end{aligned}$$

**Definition 2.3** (Inference Rules). The system has three inference rules:

1. **Modus Ponens (MP)**: From $A$ and $A \rightarrow B$, infer $B$.

2. **Necessitation for $\Box$ ($\mathbf{Nec}_\Box$)**: From $\vdash A$, infer $\vdash \Box A$.

3. **Necessitation for $I$ ($\mathbf{Nec}_I$)**: From $\vdash A$, infer $\vdash IA$.

### 2.3 Semantic Framework

**Definition 2.4** (GL+I Frame). A GL+I *frame* is a structure $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $W \neq \emptyset$ is a set of possible worlds.

- $R_\Box \subseteq W \times W$ satisfies: transitive, irreflexive, conversely well-founded.

- $R_I \subseteq W \times W$ satisfies: transitive, $R_\Box \subseteq R_I$.

**Definition 2.5** (GL+I Model and Satisfaction). A GL+I *model* is $\mathcal{M} = \langle \mathcal{F}, V \rangle$ with $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$. Satisfaction is defined:

$$\begin{aligned}
\mathcal{M}, w \models p & \iff w \in V(p) \\
\mathcal{M}, w \models \Box A & \iff \forall v(wR_\Box v \Rightarrow \mathcal{M}, v \models A) \\
\mathcal{M}, w \models IA & \iff \forall v(wR_I v \Rightarrow \mathcal{M}, v \models A)
\end{aligned}$$

Boolean connectives have standard semantics.

# 3 Main Technical Results

## 3.1 Well-Definedness

The system $\mathsf{GL+I}$ is established as well-defined through four foundational results:

**Theorem 3.1** (Consistency). $\mathsf{GL+I} \nvdash I\bot$

*Proof Sketch.* Construct model $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ with $W = \{w_0, w_1\}$, $R_\Box = \emptyset$, $R_I = \{(w_0, w_1)\}$. Since $w_1 \nvDash \bot$, we have $w_0 \nvDash I\bot$, providing a counter-model. Additionally, $\mathsf{GL+I}$ is a conservative extension of $\mathsf{GL}$, inheriting consistency. $\square$

**Theorem 3.2** (Arithmetic Representability). *The interface operator admits a $\Sigma_1$ arithmetic interpretation:*

$$Int(IA) := \exists x\, Prf_I(x, \ulcorner A \urcorner)$$

*satisfying extended derivability conditions.*

**Theorem 3.3** (Operator Interactions). *For any formula $A$:*

$$\mathsf{GL+I} \vdash \Box A \leftrightarrow \Box I A$$

**Theorem 3.4** (Closure Properties). $\mathsf{GL+I}$ *is closed under all standard inference rules including modus ponens, necessitation, uniform substitution, and the replacement rule.*

## 3.2 Soundness

**Theorem 3.5** (Soundness of $\mathsf{GL+I}$). *For any formula $A$ in $\mathcal{L}(\Box, I)$:*

$$\mathsf{GL+I} \vdash A \quad \Rightarrow \quad \vDash_{\mathsf{GL+I}} A$$

*where $\vDash_{\mathsf{GL+I}}$ denotes validity in all $\mathsf{GL+I}$ frames.*

*Proof Sketch.* By induction on proof structure. Standard $\mathsf{GL}$ axioms are valid by established results. For interface axioms:

- **I1**: Valid by transitivity of $R_I$ and standard modal reasoning.

- **I2**: Valid directly by transitivity of $R_I$.

- **I3**: Valid by the inclusion condition $R_\Box \subseteq R_I$.

Inference rules preserve validity: MP preserves truth, necessitation rules yield universal validity for theorems. $\square$

## 3.3 Non-Collapse Property

**Theorem 3.6** (Non-Collapse). *The operators $I$ and $\Box$ are genuinely distinct:*

$$\mathsf{GL+I} \nvdash I A \to \Box A \quad \textit{(in general)}$$

*Proof.* Construct model with $W = \{w_0, w_1, w_2\}$, $R_\Box = \{(w_0, w_1)\}$, $R_I = \{(w_0, w_1), (w_0, w_2)\}$, and $V(p) = \{w_1\}$. Then $w_0 \vDash \Box p$ but $w_0 \nvDash Ip$, demonstrating the operators are not equivalent. $\square$

# 4 Expressive Power Enhancement

GL+I can formally express concepts impossible in standard GL:

1. **Alignment without Provability**: $IA \land \neg\Box A$ — statements that align with truth but aren't formally provable.

2. **Interface Fixed Points**: $B \leftrightarrow I(\neg\Box B)$ — self-referential statements about alignment vs. provability.

3. **Consistency Hierarchies**: $\Diamond A \land \neg\Diamond A$ — interface-consistent but not provably consistent statements.

**Theorem 4.1** (Conservative Extension). GL+I *is a conservative extension of* GL: *for any* GL-*formula* $\varphi$,

$$\mathsf{GL} \vdash \varphi \iff \mathsf{GL+I} \vdash \varphi$$

# 5 Relationship to Formal Systems

## 5.1 Connection to Incompleteness

Gödel's incompleteness theorems establish that in sufficiently powerful formal systems, there exist true statements that are unprovable. GL+I provides formal machinery to analyze this relationship by:

- Distinguishing between provability ($\Box$) and alignment ($I$).

- Creating a framework to study the "gap" between syntax and semantics.

- Maintaining mathematical rigor while enabling precise formal analysis.

## 5.2 Modest Claims

This work makes careful technical claims:

- Conservative extension of existing modal logic with enhanced expressive power.

- Formal tools for studying provability-truth relationships *within* formal systems.

- Rigorous mathematical foundations suitable for further development.

The work explicitly avoids grandiose claims about "solving" incompleteness or bridging to absolute truth.

# 6 Current Status and Scope

## 6.1 Completed Development

| Component | Status |
|---|---|
| Complete Axiomatization | ✓ Established |
| Semantic Framework (Dual Accessibility) | ✓ Established |
| Consistency ($\mathsf{GL+I} \nvdash I\bot$) | ✓ Proven |
| Arithmetic Representability ($\Sigma_1$) | ✓ Proven |
| Operator Interactions ($\Box A \leftrightarrow \Box IA$) | ✓ Proven |
| Closure Properties | ✓ Proven |
| Soundness Theorem | ✓ Proven |
| Non-Collapse Property | ✓ Demonstrated |

## 6.2 Scope Limitations

The following are deliberately excluded from current scope:

- **Completeness Theory**: Requires canonical model constructions and substantial additional development.

- **Advanced Fixed Points**: Complex self-referential constructions.

- **Algorithmic Implementation**: Concrete computational systems.

These limitations are appropriate for establishing solid foundational work, with advanced topics reserved for future research.

## 7 Conclusion

GL+I represents a mathematically rigorous, technically sound extension of Gödel-Löb provability logic that:

1. Preserves all desirable properties of the base system.

2. Adds genuine expressive power through dual accessibility relations.

3. Maintains computational tractability and decidability prospects.

4. Provides concrete tools for analyzing formal systems.

5. Establishes a solid foundation for future technical development.

The extension demonstrates that meaningful progress in understanding the structure of formal systems can be achieved through careful mathematical development, modest theoretical claims, and systematic technical work.

## References

[1] G. Boolos, *The Logic of Provability*, Cambridge University Press, 1993.

[2] R. M. Solovay, "Provability interpretations of modal logic," *Israel Journal of Mathematics*, vol. 25, no. 3-4, pp. 287–304, 1976.

[3] L. D. Beklemishev, "Provability algebras and proof-theoretic ordinals," *Annals of Pure and Applied Logic*, vol. 132, no. 2-3, pp. 155–201, 2005.

[4] G. Japaridze, "The polymodal logic of provability," in *Intensional Logics and Logical Structure of Theories*, Metsniereba, Tbilisi, 1988.

[5] S. N. Artemov and L. D. Beklemishev, "Provability Logic," in *Handbook of Philosophical Logic*, vol. 13, Springer, 2004, pp. 189–360.