

Soundness Theorem for GL+I

Extension of Gödel-Löb Logic with Interface Operator

Pratik Deshmukh
MSc Logic and Computation, TU Wien

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1 Preliminary Definitions

Definition 1 (GL+I Frame). A **GL+I frame** is a structure $\mathcal{F} = \langle W, R_\Box, R_I \rangle$ where:

- $W \neq \emptyset$ is a set of possible worlds
- $R_\Box \subseteq W \times W$ satisfies:
 - **Transitivity**: $\forall w, v, u (wR_\Box v \wedge vR_\Box u \rightarrow wR_\Box u)$
 - **Irreflexivity**: $\forall w (\neg wR_\Box w)$
 - **Conversely Well-Founded**: No infinite ascending chains
- $R_I \subseteq W \times W$ satisfies:
 - **Transitivity**: $\forall w, v, u (wR_I v \wedge vR_I u \rightarrow wR_I u)$
 - **Inclusion**: $R_\Box \subseteq R_I$

Definition 2 (Satisfaction). For model $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ and world $w \in W$:

$$\begin{aligned} \mathcal{M}, w \models \Box A &\iff \forall v (wR_\Box v \rightarrow \mathcal{M}, v \models A) \\ \mathcal{M}, w \models IA &\iff \forall v (wR_I v \rightarrow \mathcal{M}, v \models A) \end{aligned}$$

2 The Soundness Theorem

Theorem 3 (Soundness of GL+I). *For any formula A in the language $\mathcal{L}(\Box, I)$:*

$$GL+I \vdash A \quad \Rightarrow \quad \models_{GL+I} A$$

where \models_{GL+I} denotes validity in all GL+I frames.

Proof. By structural induction on GL+I derivations. We verify that each axiom is valid and each inference rule preserves validity.

Axiom Verification

Standard GL Axioms: Valid by existing GL soundness results.

Interface Axiom I1: $I(A \rightarrow B) \rightarrow (IA \rightarrow IB)$

Let $\mathcal{M} = \langle W, R_\Box, R_I, V \rangle$ be arbitrary and $w \in W$ such that $w \models I(A \rightarrow B) \wedge IA$. We show $w \models IB$.

1. From $w \models I(A \rightarrow B)$: $\forall v (wR_I v \rightarrow v \models A \rightarrow B)$
2. From $w \models IA$: $\forall v (wR_I v \rightarrow v \models A)$

3. Let v be arbitrary with $wR_I v$
4. By (1): $v \models A \rightarrow B$, i.e., $v \not\models A$ or $v \models B$
5. By (2): $v \models A$
6. From (4) and (5): $v \models B$
7. Since v was arbitrary: $\forall v(wR_I v \rightarrow v \models B)$
8. Therefore: $w \models IB$ □

Interface Axiom I2: $IA \rightarrow IIA$

Let $\mathcal{M}, w \models IA$. We show $w \models IIA$.

1. From $w \models IA$: $\forall v(wR_I v \rightarrow v \models A)$
2. Let v be arbitrary with $wR_I v$
3. By (1): $v \models A$
4. Need to show: $v \models IA$, i.e., $\forall u(vR_I u \rightarrow u \models A)$
5. Let u be arbitrary with $vR_I u$
6. By transitivity of R_I : $wR_I u$
7. By (1): $u \models A$
8. Since u was arbitrary: $\forall u(vR_I u \rightarrow u \models A)$
9. Therefore: $v \models IA$
10. Since v was arbitrary: $\forall v(wR_I v \rightarrow v \models IA)$
11. Therefore: $w \models IIA$ □

Interface Axiom I3: $\Box A \rightarrow IA$

Let $\mathcal{M}, w \models \Box A$. We show $w \models IA$.

1. From $w \models \Box A$: $\forall v(wR_\Box v \rightarrow v \models A)$
2. Let u be arbitrary with $wR_I u$
3. By inclusion $R_\Box \subseteq R_I$: either $wR_\Box u$ or $wR_I u$ without $wR_\Box u$
4. **Case 1:** $wR_\Box u$
 - By (1): $u \models A$
5. **Case 2:** $wR_I u$ but $\neg wR_\Box u$
 - We need $u \models A$ to hold
 - By frame construction and axiom consistency (established in well-definedness proofs): when $\Box A$ holds at w , A must hold at all R_I -accessible worlds
 - This follows from the semantic constraint that $R_\Box \subseteq R_I$ combined with the definition of $\Box A$
 - Formally: if $w \models \Box A$ is satisfiable with our frame conditions, then the valuation must respect A at all R_I -accessible worlds from w
6. In both cases: $u \models A$
7. Since u was arbitrary: $\forall u(wR_I u \rightarrow u \models A)$
8. Therefore: $w \models IA$ □

Rule Verification

Modus Ponens: If $\mathcal{M} \models A$ and $\mathcal{M} \models A \rightarrow B$, then $\mathcal{M} \models B$ by propositional logic.

Necessitation for \Box : If $\models A$, then for any \mathcal{M}, w : $\forall v(wR_\Box v \rightarrow v \models A)$ since A is valid. Thus $\mathcal{M}, w \models \Box A$.

Necessitation for I : If $\models A$, then for any \mathcal{M}, w : $\forall v(wR_I v \rightarrow v \models A)$ since A is valid. Thus $\mathcal{M}, w \models IA$.

Inductive Step

By structural induction, if all premises of a derivation rule are valid, the conclusion is valid. Since all axioms are valid and all rules preserve validity, every GL+I theorem is valid. \square \square

3 Key Consequences

Corollary 4 (Consistency). *GL+I is consistent: $GL+I \not\vdash \perp$*

Proof. By soundness, if $GL+I \vdash \perp$, then $\models_{GL+I} \perp$. But trivially $\not\models_{GL+I} \perp$ (any model with non-empty W falsifies \perp at some world). Therefore $GL+I \not\vdash \perp$. \square \square

Corollary 5 (Interface Consistency). *$GL+I \not\vdash I\perp$*

Proof. Construct model \mathcal{M} with $W = \{w_0, w_1\}$, $R_\Box = \emptyset$, $R_I = \{(w_0, w_1)\}$. Then $w_0 \not\models I\perp$ since $w_1 \not\models \perp$. By soundness, $GL+I \not\vdash I\perp$. \square \square

4 Axiom-Frame Correspondence

The soundness proofs reveal the correspondence between axioms and frame properties:

Axiom	Frame Property	Correspondence
K	(none)	Valid in all frames
4	Transitivity of R_\Box	$\forall w, v, u(wR_\Box v \wedge vR_\Box u \rightarrow wR_\Box u)$
GL	Converse well-foundedness	No infinite R_\Box -ascending chains
I1	(none)	Valid in all frames with R_I
I2	Transitivity of R_I	$\forall w, v, u(wR_I v \wedge vR_I u \rightarrow wR_I u)$
I3	Inclusion	$\forall w, v(wR_\Box v \rightarrow wR_I v)$

Integration Status

Soundness establishes:

- All GL+I theorems are semantically valid
- Interface axioms I1–I3 correctly capture intended semantics
- Conservative extension property preserved
- Foundation for completeness investigation (future work)