Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

CrossMark

Quadratic transformation inequalities for Gaussian hypergeometric function

Tie-Hong Zhao[1], Miao-Kun Wang[2], Wen Zhang[3] and Yu-Ming Chu[2*]

*Correspondence:
chuyuming2005@126.com
[2]Department of Mathematics,
Huzhou University, Huzhou, China
Full list of author information is
available at the end of the article

Abstract

In the article, we present several quadratic transformation inequalities for Gaussian hypergeometric function and find the analogs of duplication inequalities for the generalized Grötzsch ring function.

MSC: 33C05; 26D20

Keywords: Hypergeometric function; Quadratic transformation; Generalized Grötzsch ring function; Duplication inequality

1 Introduction

The Gaussian hypergeometric function $_2F_1(a, b; c; x)$ with real parameters a, b, and c ($c \neq 0, -1, -2, \dots$) is defined by [1, 4, 24, 41]

$$F(a, b; c; x) =_2 F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}$$

for $x \in (-1, 1)$, where $(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$ for $n = 1, 2, \dots$, and $(a, 0) = 1$ for $a \neq 0$. The function $F(a, b; c; x)$ is called zero-balanced if $c = a + b$. The asymptotical behavior for $F(a, b; c; x)$ as $x \to 1$ is as follows (see [4, Theorems 1.19 and 1.48])

$$\begin{cases} F(a, b; c; 1) = \Gamma(c)\Gamma(c - a - b)/[\Gamma(c - a)\Gamma(c - b)], & a + b < c, \\ B(a, b)F(a, b; c; z) + \log(1 - z) = R(a, b) + O((1 - z)\log(1 - z)), & a + b = c, \\ F(a, b; c; z) = (1 - z)^{c-a-b}F(c - a, c - b; c; z), & a + b > c, \end{cases} \quad (1.1)$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\, dt$ [10, 25, 43, 44, 47] and $B(p, q) = [\Gamma(p)\Gamma(q)]/[\Gamma(p + q)]$ are the classical gamma and beta functions, respectively, and

$$R(a, b) = -\psi(a) - \psi(b) - 2\gamma, \qquad R\left(\frac{1}{4}, \frac{3}{4}\right) = \log 64, \quad (1.2)$$

$\psi(z) = \Gamma'(z)/\Gamma(z))$, and $\gamma = \lim_{n \to \infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.577\dots$ is the Euler–Mascheroni constant [21, 50].

As is well known, making use of the hypergeometric function, Branges proved the famous Bieberbach conjecture in 1984. Since then, $F(a, b; c; x)$ and its special cases and gen-

Springer

eralizations have attracted attention of many researchers, and was studied deeply in various fields [2, 5, 9, 11−18, 20, 22, 23, 26, 30, 31, 35−37, 40, 45, 46, 48]. A lot of geometrical and analytic properties, and inequalities of the Gaussian hypergeometric function have been obtained [3, 6−8, 19, 29, 32, 34, 38, 49].

Recently, in order to investigate the Ramanujan's generalized modular equation in number theory, Landen inequalities, Ramanujan cubic transformation inequalities, and several other quadratic transformation inequalities for zero-balanced hypergeometric function have been proved in [27, 28, 32, 39, 42]. For instance, using the quadratic transformation formula [24, (15.8.15), (15.8.21)]

$$F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{8r(1+r)}{(1+3r)^2}\right) = \sqrt{1+3r}F\left(\frac{1}{4}, \frac{3}{4}; 1; r^2\right), \tag{1.3}$$

Wang and Chu [32] found the maximal regions of the (a, b)-plane in the first quadrant such that inequality

$$F\left(a, b; a+b; \frac{8r(1+r)}{(1+3r)^2}\right) \le \sqrt{1+3r}F\left(a, b; a+b; r^2\right) \tag{1.4}$$

or its reversed inequality

$$F\left(a, b; a+b; \frac{8r(1+r)}{(1+3r)^2}\right) \ge \sqrt{1+3r}F\left(a, b; a+b; r^2\right) \tag{1.5}$$

holds for each $r \in (0, 1)$. Moreover, very recently in [33], some Landen-type inequalities for a class of Gaussian hypergeometric function $_2F_1(a, b; (a+b+1)/2; x)$ $(a, b > 0)$, which can be viewed as a generalization of Landen identities of the complete elliptic integrals of the first kind

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4r}{(1+r)^2}\right) = (1+r)F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right),$$

have also been proved. As an application, the analogs of duplication inequalities for the generalized Grötzsch ring function with two parameters [33]

$$\mu_{a,b}(r) = \frac{B(a, b)}{2} \frac{F(a, b; (a+b+1)/2; 1-r^2)}{F(a, b; (a+b+1)/2; r^2)}, \quad r \in (0, 1), \tag{1.6}$$

have been derived. In fact, the authors have proved

Theorem 1.1 *For* $(a, b) \in \{(a, b)|a, b > 0, ab \ge a+b-10/9, a+b \ge 2\}$, *let* $x = x(r) = 2\sqrt{r}/(1+r)$, *then the Landen-type inequality*

$$(xx')^{(a+b-1)/2}F\left(a, b; \frac{a+b+1}{2}; x^2\right) > (1+r)(rr')^{(a+b-1)/2}F\left(a, b; \frac{a+b+1}{2}; r^2\right) \tag{1.7}$$

holds for all $r \in (0, 1)$.

Theorem 1.2 *For* $(a,b) \in \{(a,b)|a,b > 0, ab \geq a + b - 10/9, a + b \geq 2\}$, *define the function* g *on* $(0,1)$ *by*

$$g(r) = 2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) - \mu_{a,b}(r).$$

Then g *is strictly increasing from* $(0,1)$ *onto* $(-\infty, 0)$. *In particular, the inequality*

$$2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) < \mu_{a,b}(r)$$

holds for each $r \in (0,1)$ *with* $(a,b) \in \{(a,b)|a,b > 0, ab \geq a + b - 10/9, a + b \geq 2\}$.

The purpose of this paper is to establish several quadratic transformation inequalities for Gaussian hypergeometric function $_2F_1(a,b;(a+b+1)/2;x)$ $(a,b > 0)$, such as inequalities (1.4), (1.5) and (1.7), and thereby prove the analogs of Theorem 1.2.

We recall some basic facts about $\mu_{a,b}(r)$ (see [33]). The limiting values of $\mu_{a,b}(r)$ at 0 and 1 are

$$\mu_{a,b}(0^+) = \lim_{r \to 0^+} \frac{B(a,b)}{2} F\left(a,b;\frac{a+b+1}{2};1-r^2\right)$$

$$= \begin{cases} \frac{B(a,b)}{2}H(a,b), & a + b < 1, \\ +\infty, & a + b \geq 1, \end{cases} \tag{1.8}$$

$$\mu_{a,b}(1^-) = \lim_{r \to 1^-} \frac{B(a,b)}{2F(a,b;\frac{a+b+1}{2};r^2)} = \begin{cases} \frac{B(a,b)}{2H(a,b)}, & a + b < 1, \\ 0, & a + b \geq 1, \end{cases} \tag{1.9}$$

and the derivative formula of $\mu_{a,b}(r)$ is

$$\frac{d\mu_{a,b}(r)}{dr} = -\frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)} \frac{1}{r^{a+b}r'^{a+b+1}F(a,b;(a+b+1)/2;r^2)^2}. \tag{1.10}$$

Here and in what follows,

$$H(a,b) = \frac{B(\frac{a+b+1}{2}, \frac{1-a-b}{2})}{B(\frac{1+b-a}{2}, \frac{1+a-b}{2})}.$$
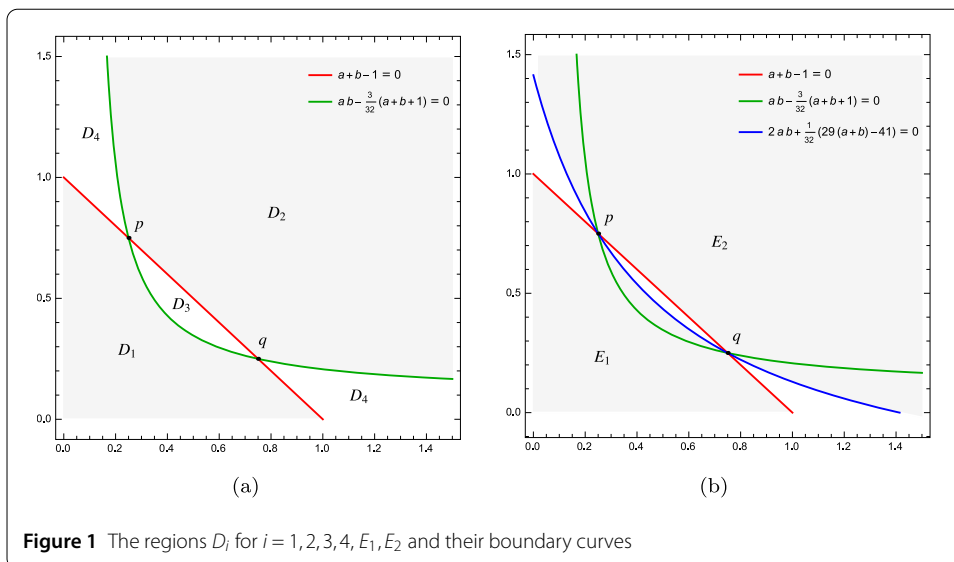
2 Lemmas

In order to prove our main results, we need several lemmas, which we present in this section. Throughout this section, we denote

$$F(x) = F\left(a,b,\frac{a+b+1}{2};x\right), \qquad G(x) = F\left(a+1,b+1;\frac{a+b+3}{2};x\right) \tag{2.1}$$

for $(a,b) \in (0,+\infty) \times (0,+\infty) \setminus \{p,q\}$ with $p = (1/4,3/4)$ and $q = (3/4,1/4)$, and

$$\widehat{F}(x) = \left(\frac{1}{4}, \frac{3}{4}; 1; x\right), \qquad \widehat{G}(x) = F\left(\frac{5}{4}, \frac{7}{4}; 2; x\right). \tag{2.2}$$

Figure 1 The regions D_i for $i = 1, 2, 3, 4, E_1, E_2$ and their boundary curves

For the convenience of readers, we introduce some regions in $\{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}$ and refer to Fig. 1 for illustration:

$$D_1 = \left\{ (a, b) \Big| a, b > 0, a + b \le 1, ab - \frac{3(a + b + 1)}{32} \le 0 \right\},$$

$$D_2 = \left\{ (a, b) \Big| a, b > 0, a + b \ge 1, ab - \frac{3(a + b + 1)}{32} \ge 0 \right\},$$

$$D_3 = \left\{ (a, b) \Big| a, b > 0, a + b < 1, ab - \frac{3(a + b + 1)}{32} > 0 \right\},$$

$$D_4 = \left\{ (a, b) \Big| a, b > 0, a + b > 1, ab - \frac{3(a + b + 1)}{32} < 0 \right\},$$

$$E_1 = \left\{ (a, b) \Big| a, b > 0, a + b \le 1, 2ab + \frac{29(a + b) - 41}{32} \le 0 \right\},$$

$$E_2 = \left\{ (a, b) \Big| a, b > 0, a + b \ge 1, 2ab + \frac{29(a + b) - 41}{32} \ge 0 \right\}.$$

Obviously, $\bigcup_{i=1}^{4} D_i = (0, +\infty) \times (0, +\infty)$ and $D_i \cap D_j = \emptyset$ for $i \ne j \in \{1, 2, 3, 4\}$ except that $D_1 \cap D_2 = \{p, q\}$. Moreover, $D_1 \subset E_1$ and $D_2 \subset E_2$.

Lemma 2.1 ([42, Theorem 2.1]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \{0, 1, 2, \dots\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$, then the following statements hold true:*

1. *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for all $n > 0$, then $h(x)$ is strictly increasing (decreasing) on $(0, r)$;*

2. *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for $0 < n \le n_0$ and decreasing (increasing) for $n > n_0$, then $h(x)$ is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \ge (\le)0$. Moreover, if $H_{f,g}(r^-) < (>)0$, then there exists an $x_0 \in (0, r)$ such that $h(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on (x_0, r).*

Lemma 2.2

1. *The function $\eta(x) = F(x)/\widehat{F}(x)$ is strictly decreasing on $(0,1)$ if $(a,b) \in D_1 \setminus \{p,q\}$ and strictly increasing on $(0,1)$ if $(a,b) \in D_2 \setminus \{p,q\}$. Moreover, if $(a,b) \in D_3$(or D_4), then there exists $\delta_0 \in (0,1)$ such that $\eta(x)$ is strictly increasing (decreasing) on $(0,\delta_0)$ and strictly decreasing (increasing) on $(\delta_0, 1)$.*

2. *The function $\widetilde{\eta}(x) = G(x)/\widehat{G}(x)$ is strictly decreasing on $(0,1)$ if $(a,b) \in E_1 \setminus \{p,q\}$ and strictly increasing on $(0,1)$ if $(a,b) \in E_2 \setminus \{p,q\}$. In the remaining case, namely for $x \in (0,+\infty) \times (0,+\infty) \setminus (E_1 \cup E_2)$, $\widetilde{\eta}(x)$ is piecewise monotone on $(0,1)$.*

Proof Suppose that

$$A_n = \frac{(a,n)(b,n)}{(\frac{a+b+1}{2},n)n!}, \qquad A_n^* = \frac{(\frac{1}{4},n)(\frac{3}{4},n)}{(1,n)n!},$$

then we have

$$\eta(x) = \frac{F(x)}{\widehat{F}(x)} = \frac{\sum_{n=0}^{\infty} A_n x^n}{\sum_{n=0}^{\infty} A_n^* x^n}. \tag{2.3}$$

It suffices to take into account the monotonicity of $\{A_n/A_n^*\}_{n=0}^{\infty}$. By simple calculations, one has

$$\frac{A_{n+1}}{A_{n+1}^*} - \frac{A_n}{A_n^*} = \frac{A_n \cdot \Delta_n}{A_n^*(\frac{a+b+1}{2})(\frac{1}{4}+n)(\frac{3}{4}+n)}, \tag{2.4}$$

where

$$\Delta_n = \left(\frac{a+b-1}{2}\right)n^2 + \left(ab + \frac{a+b}{2} - \frac{11}{16}\right)n + ab - \frac{3(a+b+1)}{32}. \tag{2.5}$$

We divide the proof into four cases.

CASE 1 $(a,b) \in D_1 \setminus \{p,q\}$. Then it follows easily that $a+b \leq 1$, $ab - \frac{3(a+b+1)}{32} \leq 0$ and $ab + \frac{a+b}{2} - \frac{11}{16} < 0$. This, in conjunction with (2.4) and (2.5), implies that $\{A_n/A_n^*\}_{n=0}^{\infty}$ is strictly decreasing for all $n > 0$. Therefore, (2.3) and Lemma 2.1(1) lead to the conclusion that $\eta(x)$ is strictly decreasing on $(0,1)$.

CASE 2 $(a,b) \in D_2 \setminus \{p,q\}$. Then a similar argument as in CASE 1 yields $\Delta_n > 0$ and this implies that $\eta(x)$ is strictly increasing on $(0,1)$ from (2.3), (2.4) and Lemma 2.1(1).

CASE 3 $(a,b) \in D_3$. It follows from (2.4) and (2.5) that the sequence $\{A_n/A_n^*\}$ is increasing for $0 \leq n \leq n_0$ and decreasing for $n \geq n_0$ for some integer n_0. Furthermore, making use of the derivative formula for Gaussian hypergeometric function

$$\frac{dF(a,b;c;x)}{dx} = \frac{ab}{c}F(a+1,b+1;c+1;x),$$

and in conjunction with (1.1) and $a+b < 1$, we obtain

$$H_{F,\widehat{F}}(x) = \frac{32ab}{3(a+b+1)}\frac{F(a+1,b+1;\frac{a+b+3}{2};x)}{F(\frac{3}{4},\frac{1}{4};2;x)}(1-x)\widehat{F}(x) - F(x)$$

$$\rightarrow -H(a,b) < 0 \tag{2.6}$$

as $x \to 1^-$. Combing with (2.3), (2.6) and Lemma 2.1(2), we conclude that there exists an $x_1 \in (0,1)$ such that $\eta(x)$ is strictly increasing on $(0, x_1)$ and strictly decreasing on $(x_1, 1)$.

CASE 4 $(a, b) \in D_4$. In this case, we follow a similar argument as in CASE 3 and use the fact that

$$
\begin{aligned}
H_{F,\widehat{F}}(x) &= \frac{32ab}{3(a+b+1)}(1-x)\frac{F(a+1, b+1; \frac{a+b+3}{2}; x)}{F(\frac{3}{4}, \frac{1}{4}; 2; x)}\widehat{F}(x) - F(x) \\
&= \frac{32ab}{3(a+b+1)}(1-x)^{\frac{1-a-b}{2}}\left[\frac{F(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+3}{2}; x)}{F(\frac{3}{4}, \frac{1}{4}; 2; x)}F\left(\frac{1}{4}, \frac{3}{4}; 1; x\right) \right. \\
&\quad \left. - F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; x\right)\right] \\
&\to +\infty
\end{aligned}
\tag{2.7}
$$

as $x \to 1^-$ since $a + b > 1$. Therefore, (2.3), (2.7) and Lemma 2.1(2) lead to the conclusion that there exists an $x_2 \in (0,1)$ such that $\eta(x)$ is strictly decreasing on $(0, x_2)$ and strictly increasing on $(x_2, 1)$.

Let

$$
B_n = \frac{(a+1, n)(b+1, n)}{(\frac{a+b+3}{2}, n)n!}, \qquad B_n^* = \frac{(\frac{5}{4}, n)(\frac{7}{4}, n)}{(2, n)n!},
$$

then we can write

$$
\widetilde{\eta}(x) = \frac{G(x)}{\widehat{G}(x)} = \frac{\sum_{n=0}^{\infty} B_n x^n}{\sum_{n=0}^{\infty} B_n^* x^n}.
\tag{2.8}
$$

Easy calculations lead to the conclusion that the monotonicity of $\{B_n/B_n^*\}_{n=0}^{\infty}$ depends on the sign of

$$
\widetilde{\Delta}_n = \left(\frac{a+b-1}{2}\right)n^2 + \left[ab + \frac{3(a+b)}{2} - \frac{27}{16}\right]n + 2ab + \frac{29(a+b) - 41}{32}.
\tag{2.9}
$$

Notice that

$$
\begin{aligned}
H_{G,\widehat{G}}(x) &= \frac{2(a+1)(b+1)}{(a+b+3)} \cdot \frac{32F(a+2, b+2; \frac{a+b+5}{2}; x)}{35F(\frac{9}{4}, \frac{11}{4}; 3; x)}\widehat{G}(x) - G(x) \\
&= (1-x)^{-\frac{1+a+b}{2}}\omega(a, b; x),
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
\omega(a, b; x) &= \frac{64(a+1)(b+1)}{35(a+b+3)}\frac{F(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+5}{2}; x)}{F(\frac{3}{4}, \frac{1}{4}; 3; x)}F\left(\frac{3}{4}, \frac{1}{4}; 2; x\right) \\
&\quad - F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+3}{2}; x\right).
\end{aligned}
\tag{2.11}
$$

It follows easily from (1.1) and (2.11) that

$$
\lim_{x \to 1^-} \omega(a, b; x) = \frac{64(a+1)(b+1)}{35(a+b+3)}\frac{\Gamma(\frac{a+b+5}{2})\Gamma(\frac{a+b+3}{2})}{\Gamma(a+2)\Gamma(b+2)}\frac{\Gamma(\frac{9}{4})\Gamma(\frac{11}{4})}{\Gamma(3)\Gamma(2)}\frac{\Gamma(2)\Gamma(1)}{\Gamma(\frac{5}{4})\Gamma(\frac{7}{4})}
$$

$$- \frac{\Gamma(\frac{a+b+3}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(a+1)\Gamma(b+1)}$$

$$= \left(\frac{a+b-1}{2}\right)\frac{\Gamma(\frac{a+b+3}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(a+1)\Gamma(b+1)}$$

$$= \begin{cases} <0, & a+b<1, \\ >0, & a+b>1. \end{cases} \tag{2.12}$$

Employing similar arguments mentioned in part (1), we obtain the desired assertions easily from (2.8)–(2.12). □

Lemma 2.3 *Let* $D_0 = \{(a,b)|a,b>0, a+b \geq 7/4, ab \geq a+b-31/28\}$ *and* $x' = \sqrt{1-x^2}$ *for* $0 < x < 1$, *then the function*

$$f(x) = \frac{(xx')^{\frac{a+b-1}{2}} F(a,b;\frac{a+b+1}{2};x^2)}{F(\frac{1}{4},\frac{3}{4};1;x^2)} \tag{2.13}$$

is strictly increasing on $(0,1)$ *if* $(a,b) \in D_0$.

Proof Taking the derivative of $f(x)$ yields

$$f'(x) = \frac{(xx')^{\frac{a+b-3}{2}}}{x'F(\frac{1}{4},\frac{3}{4};1;x^2)^2} f_1(x), \tag{2.14}$$

where

$$f_1(x) = \left[\frac{a+b-1}{2}(1-2x^2)F\left(a,b;\frac{a+b+1}{2};x^2\right) \right.$$
$$\left. + \frac{4ab}{a+b+1}x^2x'^2F\left(a+1,b+1;\frac{a+b+3}{2};x^2\right)\right]F\left(\frac{1}{4},\frac{3}{4};1;x^2\right)$$
$$- \frac{3x^2x'^2}{8}F\left(a,b;\frac{a+b+1}{2};x^2\right)F\left(\frac{5}{4},\frac{7}{4};2;x^2\right). \tag{2.15}$$

We clearly see from (1.1) that

$$x'^2F\left(\frac{5}{4},\frac{7}{4};2;x^2\right) = F\left(\frac{1}{4},\frac{3}{4};2;x^2\right) \leq F\left(\frac{1}{4},\frac{3}{4};1;x^2\right)$$

for $0 < x < 1$. This implies, in conjunction with (2.15), that

$$f_1(x) \geq F\left(\frac{1}{4},\frac{3}{4};1;x^2\right) f_2(x), \tag{2.16}$$

where

$$f_2(x) = \left[\frac{a+b-1}{2} - \left(a+b-\frac{5}{8}\right)x^2\right]F\left(a,b;\frac{a+b+1}{2};x^2\right)$$
$$+ \frac{4ab}{a+b+1}x^2(1-x^2)F\left(a+1,b+1;\frac{a+b+3}{2};x^2\right).$$

It follows from the definition of hypergeometric function that

$$
\begin{aligned}
f_2(x) = {} & \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(\frac{a+b+1}{2},n)} \frac{x^{2n}}{n!} - \left(a+b-\frac{5}{8}\right) \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(\frac{a+b+1}{2},n)} \frac{x^{2n+2}}{n!} \\
& + \frac{4ab}{a+b+1} \left[\sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{(\frac{a+b+3}{2},n)} \frac{x^{2n+2}}{n!} - \sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{(\frac{a+b+3}{2},n)} \frac{x^{2n+4}}{n!} \right] \\
= {} & \frac{a+b-1}{2} + \left[\frac{ab(a+b-1)}{a+b+1} - \left(a+b-\frac{5}{8}\right) + \frac{4ab}{a+b+1} \right] x^2 \\
& + \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{(\frac{a+b+1}{2},n+2)} \frac{x^{2n+4}}{(n+2)!} \\
& - \left(a+b-\frac{5}{8}\right) \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(\frac{a+b+1}{2},n+1)} \frac{x^{2n+4}}{(n+1)!} \\
& + 2 \left[\sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{(\frac{a+b+1}{2},n+2)} \frac{x^{2n+4}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(\frac{a+b+1}{2},n+1)} \frac{x^{2n+4}}{n!} \right] \\
= {} & \frac{a+b-1}{2} \left[1 - \frac{3x^2}{4(a+b+1)} \right] + \frac{4ab(a+b-1) - 4(a-b)^2 + 1}{4(a+b+1)} x^2 \\
& + \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(\frac{a+b+1}{2},n+2)} \frac{C_n}{(n+2)!} x^{2n+4},
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{aligned}
C_n = {} & \frac{a+b-1}{2}(a+n+1)(b+n+1) - \left(a+b-\frac{5}{8}\right)\left(\frac{a+b+1}{2}+n+1\right)(n+2) \\
& + 2(a+n+1)(b+n+1)(n+2) - 2\left(\frac{a+b+1}{2}+n+1\right)(n+1)(n+2) \\
= {} & \left(\frac{4a+4b-7}{8}\right) n^2 + \left[\frac{32ab+5(a+b)-29}{16}\right] n \\
& + \frac{4ab(a+b+3) - 4(a-b)^2 - (3a+3b+5)}{8}.
\end{aligned}
\tag{2.18}
$$

If $(a,b) \in D_0$, namely, $a+b \geq 7/4$ and $ab \geq a+b-31/28$, we can verify

(i)

$$
\begin{aligned}
& 4ab(a+b-1) - 4(a-b)^2 + 1 \\
& \quad \geq 4\left(a+b-\frac{31}{28}\right)(a+b-1) - 4(a-b)^2 + 1 \\
& \quad = \frac{1}{7}\left[112ab - 59(a+b) + 38\right] \geq \frac{53}{7}\left(a+b-\frac{86}{53}\right) \geq \frac{27}{28},
\end{aligned}
$$

(ii)

$$
32ab + 5(a+b) - 29 \geq 32\left(a+b-\frac{31}{28}\right) + 5(a+b) - 29
$$

$$= \frac{37}{7}\left[7(a+b) - \frac{451}{259}\right] \geq \frac{9}{28},$$

(iii)

$$4ab(a+b+3) - 4(a-b)^2 - (3a+3b+5)$$

$$\geq 4\left(a+b - \frac{31}{28}\right)(a+b+3)$$

$$- 4(a-b)^2 - (3a+3b+5) = \frac{16}{7}\left[7ab + 2(a+b) - 8\right]$$

$$\geq \frac{16}{7}\left[7\left(a+b - \frac{31}{28}\right) + 2(a+b) - 8\right] = \frac{36}{7}\left[4(a+b) - 7)\right] \geq 0.$$

This, in conjunction with (2.17) and (2.18), implies that $f_2(x) > 0$ for $0 < x < 1$. Therefore, $f(x)$ is strictly increasing on $(0,1)$, which follows from (2.14) and (2.16) if $(a,b) \in D_0$. □

Remark 2.4 The function $f(x)$ defined in Lemma 2.3 is not monotone on $(0,1)$ if two positive numbers a,b satisfy $a+b < 1$, since $\lim_{x\to 0^+} f(x) = \lim_{x\to 1^-} f(x) = +\infty$ and Lemma 2.1(1) shows the monotonicity of $f(x)$ on $(0,1)$ if $a+b = 1$. In the remaining case $a+b > 1$, it follows from (2.15) that $f_1(0^+) = (a+b-1)/2 > 0$. This, in conjunction with (2.14), implies that $f(x)$ is strictly increasing on $(0, x^*)$ for a sufficiently small $x^* > 0$. This enables us to find a sufficient condition for a,b with $a+b > 1$ such that $f(x)$ is strictly increasing on $(0,1)$ in Lemma 2.3.

The following corollary can be derived immediately from the monotonicity of $f(x)$ in Lemma 2.3 and the quadratic transformation equality (1.3).

Corollary 2.5 *Let $x = x(r) = \sqrt{8r(1+r)}/(1+3r)$, if $(a,b) \in D_0$, then the inequality*

$$\left(xx'\right)^{\frac{a+b-1}{2}} F\left(a,b; \frac{a+b+1}{2}; x^2\right) > \sqrt{1+3r}\left(rr'\right)^{\frac{a+b-1}{2}} F\left(a,b; \frac{a+b+1}{2}; r^2\right) \tag{2.19}$$

holds for all $r \in (0,1)$.

3 Main results

Theorem 3.1 *The quadratic transformation inequality*

$$F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right) \leq \sqrt{1+3r}\, F\left(a,b; \frac{a+b+1}{2}; r^2\right) \tag{3.1}$$

holds for all $r \in (0,1)$ with $a,b > 0$ if and only if $(a,b) \in D_1$ and the reversed inequality

$$F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right) \geq \sqrt{1+3r}\, F\left(a,b; \frac{a+b+1}{2}; r^2\right) \tag{3.2}$$

takes place for all $r \in (0,1)$ if and only if $(a,b) \in D_2$, with equality only for $(a,b) = p$ or q.

In the remaining case $(a,b) \in D_3 \cup D_4$, neither of the above inequalities holds for all $r \in (0,1)$.

Proof Suppose that $x(r) = [8r(1+r)]/(1+3r)^2$, then we clearly see that $x(r) > r^2$ for $0 < r < 1$. It follows from Lemma 2.1(1) that $\eta(x(r)) < \eta(r^2)$ for $(a,b) \in D_1 \setminus \{p,q\}$ and $\eta(x(r)) > \eta(r^2)$ for $(a,b) \in D_2 \setminus \{p,q\}$. This, in conjunction with the quadratic transformation formula (1.3), implies

$$F\big(x(r)\big) < \frac{\widehat{F}(x(r))}{\widehat{F}(r^2)} F\big(r^2\big) = \sqrt{1+3r} F\big(r^2\big)$$

for $(a,b) \in D_1 \setminus \{p,q\}$, and it degenerates to the quadratic transformation equality for $(a,b) = p (\text{or} q)$. This completes the proof of (3.1).

Inequality (3.2) can be derived analogously, and the remaining case follows easily from Lemma 2.2(1). □

Theorem 3.2 *We define the function*

$$\varphi(r) = \sqrt{1+3\sqrt{r}} F\left(a,b,; \frac{a+b+1}{2}; r\right) - F\left(a,b; \frac{a+b+1}{2}; \frac{8\sqrt{r}(1+\sqrt{r})}{(1+3\sqrt{r})^2}\right)$$

for $r \in (0,1)$ with $a,b > 0$ and $(a,b) \neq p,q$. Let $L_1 = \{(a,b)|a+b = 1, 0 < a < \frac{1}{4} \text{ or } \frac{3}{4} < a < 1\}$ and $L_2 = \{(a,b)|a+b = 1, \frac{1}{4} < a < \frac{3}{4}\}$. Then the following statements hold true:

1. *If $(a,b) \in L_1 (\text{or } L_2)$, then $\varphi(r)$ is strictly increasing (resp., decreasing) from $(0,1)$ onto $(0, [R(a,b) - \log 64]/B(a,b))$ (resp., $([R(a,b) - \log 64]/B(a,b), 0))$.*
2. *If $(a,b) \in D_1 \setminus L_1$, then $\varphi(r)$ is strictly increasing from $(0,1)$ onto $(0, H(a,b))$;*
3. *If $(a,b) \in D_2 \setminus L_2$, then $\varphi(r)$ is strictly decreasing from $(0,1)$ onto $(-\infty, 0)$.*

As a consequence, the inequality

$$F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right) \leq \sqrt{1+3r} F\left(a,b,; \frac{a+b+1}{2}; r^2\right)$$

$$\leq F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right) + H(a,b) \tag{3.3}$$

holds for all $r \in (0,1)$ if $(a,b) \in D_1 \setminus L_1$, and the following inequality is valid for all $r \in (0,1)$:

$$F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right)$$

$$\leq (\geq) \sqrt{1+3r} F\left(a,b,; \frac{a+b+1}{2}; r^2\right)$$

$$\leq (\geq) F\left(a,b; \frac{a+b+1}{2}; \frac{8r(1+r)}{(1+3r)^2}\right) + \frac{R(a,b) - \log 64}{B(a,b)} \tag{3.4}$$

if $(a,b) \in L_1 (\text{resp., } L_2)$.

Proof Let $z = z(r) = [8\sqrt{r}(1 + \sqrt{r})]/(1 + 3\sqrt{r})^2$, then we clearly see that

$$\frac{dz}{dr} = \frac{4(1-\sqrt{r})}{\sqrt{r}(1+3\sqrt{r})^3} = \frac{4(1-z)}{\sqrt{r}(1-\sqrt{r})(1+3\sqrt{r})}. \tag{3.5}$$

Taking the derivative of $\varphi(r)$ with respect to r and using (3.5) yields

$$\sqrt{r}(1 + 3\sqrt{r})\varphi'(r) = \frac{3\sqrt{1 + 3\sqrt{r}}}{4}F(r) + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3 \frac{2ab}{a + b + 1}G(r)$$
$$- \frac{2ab}{a + b + 1}\frac{4(1 - z)}{1 - \sqrt{r}}G(z). \qquad (3.6)$$

We substitute $\sqrt{r}$ for r in the quadratic transformation equality (1.3), then differentiate it with respect to r to obtain

$$\frac{4(1 - z)}{1 - \sqrt{r}}\widehat{G}(z) = 4\sqrt{1 + 3\sqrt{r}}\widehat{F}(r) + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3\widehat{G}(r),$$

in other words,

$$\frac{4(1 - z)}{1 - \sqrt{r}}\frac{\widehat{G}(z)}{\widehat{G}(r)} = 4\sqrt{1 + 3\sqrt{r}}\frac{\widehat{F}(r)}{\widehat{G}(r)} + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3. \qquad (3.7)$$

If $(a, b) \in D_1 \setminus \{p, q\}$, then it follows from Lemma 2.2(2) that $G(x)/\widehat{G}(x)$ is strictly decreasing on $(0, 1)$. This, in conjunction with $z > r$, implies that $G(z)/\widehat{G}(z) < G(r)/\widehat{G}(r)$, that is,

$$G(z) < \frac{\widehat{G}(z)}{\widehat{G}(r)}G(r). \qquad (3.8)$$

Combing (3.6), (3.7) with the inequality (3.8), we clearly see that

$$\sqrt{r}(1 + 3\sqrt{r})\varphi'(r)$$

$$= \frac{3\sqrt{1 + 3\sqrt{r}}}{4}F(r) + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3\frac{2ab}{a + b + 1}G(r) - \frac{2ab}{a + b + 1}\frac{4(1 - z)}{1 - \sqrt{r}}G(z)$$

$$> \frac{3\sqrt{1 + 3\sqrt{r}}}{4}F(r) + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3\frac{2ab}{a + b + 1}G(r) - \frac{2ab}{a + b + 1}\frac{4(1 - z)}{1 - \sqrt{r}}\frac{\widehat{G}(z)}{\widehat{G}(r)}G(r)$$

$$= \frac{3\sqrt{1 + 3\sqrt{r}}}{4}F(r) + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3\frac{2ab}{a + b + 1}G(r)$$

$$- \frac{2ab}{a + b + 1}\left[4\sqrt{1 + 3\sqrt{r}}\frac{\widehat{F}(r)}{\widehat{G}(r)} + \sqrt{r}(\sqrt{1 + 3\sqrt{r}})^3\right]G(r)$$

$$= 4\sqrt{1 + 3\sqrt{r}}\left[\frac{3}{16}F(r) - \frac{2ab}{a + b + 1}\frac{\widehat{F}(r)}{\widehat{G}(r)}G(r)\right]$$

$$= 4\sqrt{1 + 3\sqrt{r}}\frac{F(r)^2}{\widehat{G}(r)}\left(\frac{\widehat{F}(r)}{F(r)}\right)'. \qquad (3.9)$$

It follows from Lemma 2.2(1) that $\widehat{F}(r)/F(r)$ is strictly increasing on $(0, 1)$ if $(a, b) \in D_1 \setminus \{p, q\}$. This, in conjunction with (3.9), implies that $\varphi(r)$ is strictly increasing on $(0, 1)$ if $(a, b) \in D_1$.

Analogously, if $(a, b) \in D_2 \setminus \{p, q\}$, then we obtain the following inequality:

$$G(z) > \frac{\widehat{G}(z)}{\widehat{G}(r)}G(r).$$

By using a similar argument as above, we have

$$\sqrt{r}\varphi'(r) < \frac{4F^2(r)}{\widehat{G}(r)}\left(\frac{\widehat{F}(r)}{F(r)}\right)' < 0,$$

since $F(r)/\widehat{F}(r)$ is strictly increasing on $(0,1)$ if $(a,b) \in D_2 \setminus \{p,q\}$ by Lemma 2.2(1). Hence, $\varphi(r)$ is strictly decreasing on $(0,1)$ if $(a,b) \in D_2$.

Notice that $\varphi(0^+) = 0$ and

$$\lim_{r \to 1^-} \varphi(r) = \begin{cases} H(a,b), & a+b < 1, \\ \frac{R(a,b)-\log 64}{B(a,b)}, & a+b = 1, \\ -\infty, & a+b > 1. \end{cases} \tag{3.10}$$

Therefore, we obtain the desired assertion from (3.10). □

Theorem 3.3 *If we define the function*

$$\phi(r) = 2\mu_{a,b}\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) - \mu_{a,b}(r),$$

for $(a,b) \in D_0$, then $\phi(r)$ is strictly increasing from $(0,1)$ onto $(-\infty,0)$. As a consequence, the inequality

$$2\mu_{a,b}\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right) < \mu_{a,b}(r)$$

holds for all $r \in (0,1)$ if $(a,b) \in D_0$.

Proof Remark 2.4 enables us to consider the case for $a+b > 1$. Note that $\phi(1^-) = 0$ and

$$\lim_{r \to 0^+} \phi(r)$$
$$= \lim_{r \to 0^+} \frac{B(a,b)}{2}\left[2F\left(a,b;\frac{a+b+1}{2};\left(\frac{1-r}{1+3r}\right)^2\right) - F\left(a,b;\frac{a+b-1}{2};1-r^2\right)\right]$$
$$= B(a,b)\lim_{r \to 0^+}\left[\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right)^{1-a-b}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};\left(\frac{1-r}{1+3r}\right)^2\right)\right.$$
$$\left. - \frac{1}{2}r^{1-a-b}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};1-r^2\right)\right]$$
$$= \frac{1}{2}B\left(\frac{a+b+1}{2},\frac{a+b-1}{2}\right)\lim_{r \to 0^+}\left[2\left(\frac{\sqrt{8r(1+r)}}{1+3r}\right)^{1-a-b} - r^{1-a-b}\right]$$
$$= -\infty. \tag{3.11}$$

Let $x = x(r) = \sqrt{8r(1+r)}/(1+3r)$ and $x' = \sqrt{1-x^2}$. Then

$$\frac{dx}{dr} = \frac{\sqrt{2}(1-r)}{\sqrt{r(1+r)}(1+3r)^2} = \frac{x'(1+3x')^2}{4x}. \tag{3.12}$$

Taking the derivative of $\phi(r)$ and using (3.12) leads to

$$
\begin{aligned}
\phi'(r) = {} & -2\frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)}\frac{1}{x^{a+b}x'^{a+b+1}F(a,b;\frac{a+b+1}{2};x^2)^2}\cdot\frac{x'(1+3x')^2}{4x} \\
& +\frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)}\frac{1}{r^{a+b}r'^{a+b+1}F(a,b;\frac{a+b+1}{2};r^2)^2} \\
= {} & \frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)}\frac{(1+3x')^2}{2(1+3r)x^{a+b+1}x'^{a+b}F(a,b;\frac{a+b+1}{2};x^2)^2} \\
& \times\left[\frac{(xx')^{a+b-1}F(a,b;\frac{a+b+1}{2};x^2)^2}{(rr')^{a+b-1}F(a,b;\frac{a+b+1}{2};r^2)^2}-(1+3r)\right].
\end{aligned}
\tag{3.13}
$$

Therefore, the monotonicity of $\phi(r)$ follows immediately from (2.19) and (3.13). This, in conjunction with (3.11), gives rise to the desired result. □

4 Results and discussion

In the article, we establish several quadratic transformation inequalities for Gaussian hypergeometric function $_2F_1(a,b;(a+b+1)/2;x)$ $(0 < x < 1)$. As applications, we provide the analogs of duplication inequalities for the generalized Grötzsch ring function

$$
\mu_{a,b}(r)=\frac{B(a,b)}{2}\frac{F(a,b;(a+b+1)/2;1-r^2)}{F(a,b;(a+b+1)/2;r^2)}
$$

introduced in [33].

5 Conclusion

We find several quadratic transformation inequalities for the Gaussian hypergeometric function and Grötzsch ring function. Our approach may have further applications in the theory of special functions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
^{1}Department of Mathematics, Hangzhou Normal University, Hangzhou, China. ^{2}Department of Mathematics, Huzhou University, Huzhou, China. ^{3}Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, New York, USA.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Dover, New York (1965). Available online at http://people.math.sfu.ca/~cbm/aands/
2. Anderson, G.D., Qiu, S.-L., Vamanamurthy, M.K., Vuorinen, M.: Generalized elliptic integrals and modular equations. Pac. J. Math. **192**(1), 1–37 (2000). Available online at https://msp.org/pjm/2000/192-1/pjm-v192-n1-p01-s.pdf

3. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Hypergeometric functions and elliptic integrals. In: Srivastava, H.M., Owa, S. (eds.) Current Topics in Analytic Function Theory, pp. 48–85. World Scientific, Singapore (1992). https://doi.org/10.1142/1628.

4. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Conformal Invariants, Inequalities, and Quasiconformal Maps. John Wiley & Sons, New York (1997). Available online at https://www.isbns.net/isbn/9780471594864/

5. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Topics in special functions, in papers on analysis: a volume dedicated to Olli Martio on the occasion of his 60th birthday. Report. Univ. Jyväskylä **83**, 5–26 (2001). Available online at http://cn.arxiv.org/abs/0712.3856

6. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Topics in special functions II. Conform. Geom. Dyn. **11**, 250–270 (2007). https://doi.org/10.1090/S1088-4173-07-00168-3

7. Barnard, R.W., Pearce, K., Richards, K.C.: An inequality involving the generalized hypergeometric function and the arc length of an ellipse. SIAM J. Math. Anal. **31**(3), 693–699 (2000). https://doi.org/10.1137/S0036141098341575

8. Bhayo, B.A., Vuorinen, M.: On generalized complete elliptic integrals and modular functions. Proc. Edinb. Math. Soc. (2) **55**(3), 591–611 (2012). https://doi.org/10.1017/S0013091511000356

9. Borwein, J.M., Borwein, P.B.: Pi and AGM. Wiley, New York (1987). Available online at http://ww1.openisbn.com/download/0471831387.pdf

10. Chen, C.-P., Qi, F., Srivastava, H.M.: Some properties of functions related to the gamma and psi functions. Integral Transforms Spec. Funct. **21**(1–2), 153–164 (2010). https://doi.org/10.1080/10652460903064216

11. Chu, Y.-M., Qiu, S.-L., Wang, M.-K.: Sharp inequalities involving the power mean and complete elliptic integral of the first kind. Rocky Mt. J. Math. **43**(3), 1489–1496 (2013). https://doi.org/10.1216/RMJ-2013-43-5-1489

12. Chu, Y.-M., Qiu, Y.-F., Wang, M.-K.: Hölder mean inequalities for the complete elliptic integrals. Integral Transforms Spec. Funct. **23**(7), 521–527 (2012). https://doi.org/10.1080/10652469.2011.609482

13. Chu, Y.-M., Wang, M.-K.: Optimal Lehmer mean bounds for the Toader mean. Results Math. **61**(3–4), 223–229 (2012). https://doi.org/10.1007/s00025-010-0090-9

14. Chu, Y.-M., Wang, M.-K., Jiang, Y.-P., Qiu, S.-L.: Concavity of the complete elliptic integrals of the second kind with respect to Hölder mean. J. Math. Anal. Appl. **395**(2), 637–642 (2012). https://doi.org/10.1016/j.jmaa.2012.05.083

15. Chu, Y.-M., Wang, M.-K., Ma, X.-Y.: Sharp bounds for Toader mean in terms of contraharmonic mean with applications. J. Math. Inequal. **7**(2), 161–166 (2013). https://doi.org/10.7153/jmi-07-15

16. Chu, Y.-M., Wang, M.-K., Qiu, S.-L.: Optimal combinations bounds of root-square and arithmetic means for Toader mean. Proc. Indian Acad. Sci. Math. Sci. **122**(1), 41–51 (2012). Available online at https://link.springer.com/article/10.1007

17. Chu, Y.-M., Wang, M.-K., Qiu, S.-L., Jiang, Y.-P.: Bounds for complete integrals of the second kind with applications. Comput. Math. Appl. **63**(7), 1177–1184 (2012). https://doi.org/10.1016/j.camwa.2011.12.038

18. Chu, Y.-M., Wang, M.-K., Qiu, S.-L., Qiu, Y.-F.: Sharp generalized Seiffert mean bounds for Toader mean. Abstr. Appl. Anal. **2011**, Article ID 605259 (2011). https://doi.org/10.1155/2011/605259

19. Heikkala, V., Vamanamurthy, M.K., Vuorinen, M.: Generalized elliptic integrals. Comput. Methods Funct. Theory **9**(1), 75–109 (2009). https://doi.org/10.1007/BF0332171

20. Hua, Y., Qi, F.: The best bounds for Toader mean in terms of the centroidal and arithmetic means. Filomat **28**(4), 775–780 (2014). https://doi.org/10.2298/FIL1404775H

21. Huang, T.-R., Han, B.-W., Ma, X.-Y., Chu, Y.-M.: Optimal bounds for the generalized Euler–Mascheroni constant. J. Inequal. Appl. **2018**, Article ID 118 (2018). https://doi.org/10.1186/s13660-018-1711-1

22. Jiang, W.-D., Qi, F.: A double inequality for the combination of Toader mean and the arithmetic mean in terms of the contraharmonic mean. Publ. Inst. Math. **99**(113), 237–242 (2016). https://doi.org/10.2298/PIM141026009J

23. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane, 2nd edn. Springer, New York (1973)

24. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge Univ. Press, Cambridge (2010)

25. Qi, F.: Bounds for the ratio of two gamma functions. J. Inequal. Appl. **2010**, Article ID 493058 (2010). https://doi.org/10.1155/2010/493058

26. Qian, W.-M., Chu, Y.-M.: Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters. J. Inequal. Appl. **2017**, Article ID 274 (2017). https://doi.org/10.1186/s13660-017-1550-5

27. Qiu, S.-L., Vuorinen, M.: Landen inequalities for hypergeometric functions. Nagoya Math. J. **154**, 31–56 (1999). https://doi.org/10.1017/S0027763000025290

28. Simić, S., Vuorinen, M.: Landen inequalities for zero-balanced hypergeometric functions. Abstr. Appl. Anal. **2012**, Article ID 932061 (2012). https://doi.org/10.1155/2012/932061

29. Song, Y.-Q., Zhou, P.-G., Chu, Y.-M.: Inequalities for the Gaussian hypergeometric function. Sci. China Math. **57**(11), 2369–2380 (2014). https://doi.org/10.1007/s11425-014-4858-3

30. Sugawa, T., Wang, L.-M.: Spirallikeness of shifted hypergeometric functions. Ann. Acad. Sci. Fenn., Math. **42**(2), 963–977 (2017). https://doi.org/10.5186/aasfm.2017.4257

31. Wang, G.-D., Zhang, X.-H., Chu, Y.-M.: A power mean inequality involving the complete elliptic integrals. Rocky Mt. J. Math. **44**(5), 1661–1667 (2014). https://doi.org/10.1216/RMJ-2014-44-5-1661

32. Wang, M.-K., Chu, Y.-M.: Refinements of transformation inequalities for zero-balanced hypergeometric functions. Acta Math. Sci. **37B**(3), 607–622 (2017). https://doi.org/10.1016/S0252-9602(17)30026-7

33. Wang, M.-K., Chu, Y.-M.: Landen inequalities for a class of hypergeometric functions with applications. Math. Inequal. Appl. **21**(2), 521–537 (2018). https://doi.org/10.7153/mia-2018-21-38

34. Wang, M.-K., Chu, Y.-M., Jiang, Y.-P.: Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions. Rocky Mt. J. Math. **46**(2), 679–691 (2016). https://doi.org/10.1216/RMJ-2016-46-2-679

35. Wang, M.-K., Chu, Y.-M., Qiu, S.-L.: Some monotonicity properties of generalized elliptic integrals with applications. Math. Inequal. Appl. **16**(3), 671–677 (2013). https://doi.org/10.7153/mia-16-50

36. Wang, M.-K., Chu, Y.-M., Qiu, S.-L.: Sharp bounds for generalized elliptic integrals of the first kind. J. Math. Anal. Appl. **429**(2), 744–757 (2015). https://doi.org/10.1016/j.jmaa.2015.04.035

37. Wang, M.-K., Chu, Y.-M., Qiu, Y.-F., Qiu, S.-L.: An optimal power mean inequality for the complete elliptic integrals. Appl. Math. Lett. **24**(6), 887–890 (2011). https://doi.org/10.1016/j.aml.2010.12.044

38. Wang, M.-K., Chu, Y.-M., Song, Y.-Q.: Asymptotical formulas for Gaussian and generalized hypergeometric functions. Appl. Math. Comput. **276**, 44–60 (2016). https://doi.org/10.1016/j.amc.2015.11.088

39. Wang, M.-K., Li, Y.-M., Chu, Y.-M.: Inequalities and infinite product formula for Ramanujan generalized modular equation function. Ramanujan J. **46**(1), 189–200 (2018). https://doi.org/10.1007/s11139-017-9888-3

40. Wang, M.-K., Qiu, S.-L., Chu, Y.-M.: Infinite series formula for Hübner upper bounds function with applications to Hersch–Pfluger distortion function. Math. Inequal. Appl. **21**(3), 629–648 (2018). https://doi.org/10.7153/mia-2018-21-46

41. Yang, Z.-H., Chu, Y.-M.: A monotonicity property involving the generalized elliptic integral of the first kind. Math. Inequal. Appl. **20**(3), 729–735 (2017). https://doi.org/10.7153/mia-20-46

42. Yang, Z.-H., Chu, Y.-M., Wang, M.-K.: Monotonicity criterion for the quotient of power series with applications. J. Math. Anal. Appl. **428**(1), 587–604 (2015) https://doi.org/10.1016/j.jmaa.2015.03.043.

43. Yang, Z.-H., Qian, W.-M., Chu, Y.-M.: On rational bounds for the gamma function. J. Inequal. Appl. **2017**, Article ID 210 (2017). https://doi.org/10.1186/s13660-017-1484-y

44. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On approximating the error function. Math. Inequal. Appl. **21**(2), 469–479 (2018). https://doi.org/10.7153/mia-2018-2

45. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind. J. Math. Anal. Appl. **462**(2), 1714–1726 (2018). https://doi.org/10.1016/j.jmaa.2018.03.005

46. Yang, Z.-H., Tian, J.F.: A comparison theorem for two divided differences and applications to special functions. J. Math. Anal. Appl. **464**(1), 580–595 (2018). https://doi.org/10.1016/j.jmaa.2018.04.024

47. Yang, Z.-H., Zhang, W., Chu, Y.-M.: Sharp Gautschi inequality for parameter $0 < p < 1$ with applications. Math. Inequal. Appl. **20**(4), 1107–1120 (2017) https://doi.org/10.7153/mia-201720-71

48. Yin, L., Mi, L.-F.: Landen type inequality for generalized elliptic integrals. Adv. Stud. Contemp. Math. **26**(4), 717–722 (2016). Available online at http://www.jangjeon.or.kr/etc/view.html?id=1924

49. Zhang, X.-H., Wang, G.-D., Chu, Y.-M.: Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions. J. Math. Anal. Appl. **353**(1), 256–259 (2009). https://doi.org/10.1016/j.jmaa.2008.11.068

50. Zhao, T.-H., Chu, Y.-M., Wang, H.: Logarithmically complete monotonicity properties relating to the gamma function. Abstr. Appl. Anal. **2011**, Article ID 896483 (2011). https://doi.org/10.1155/2011/896483