



Entanglement and thermodynamics after a quantum quench in integrable systems

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Entanglement and entropy are key concepts standing at the foundations of quantum and statistical mechanics. Recently, the study of quantum quenches revealed that these concepts are intricately intertwined. Although the unitary time evolution ensuing from a pure state maintains the system at zero entropy, local properties at long times are captured by a statistical ensemble with nonzero thermodynamic entropy, which is the entanglement accumulated during the dynamics. Therefore, understanding the entanglement evolution unveils how thermodynamics emerges in isolated systems. Alas, an exact computation of the entanglement dynamics was available so far only for noninteracting systems, whereas it was deemed unfeasible for interacting ones. Here, we show that the standard quasiparticle picture of the entanglement evolution, complemented with integrability-based knowledge of the steady state and its excitations, leads to a complete understanding of the entanglement dynamics in the space–time scaling limit. We thoroughly check our result for the paradigmatic Heisenberg chain.

entanglement | quantum quench | integrability | thermodynamics

Since the early days of quantum mechanics, understanding how statistical ensembles arise from the unitary time evolution of an isolated quantum system has been a fascinating question (1–7). A widely accepted mechanism is that, although the entire system remains in a pure state, the reduced density matrix of an arbitrary finite compact subsystem attains a long time limit that can be described by a statistical ensemble (8). In the last decade, groundbreaking experiments with cold atoms (9–19) simulated with incredible precision the unitary time evolution of many-body quantum systems, reviving the interest in this topic. The simplest out-of-equilibrium protocol in which these ideas can be theoretically and experimentally tested is the quantum quench (20, 21). A system is prepared in an initial state $|\Psi_0\rangle$, typically the ground state of a local Hamiltonian H_0 , and it evolves with a many-body Hamiltonian H . At asymptotically long times, physical observables relax to stationary values, which for generic systems, are described by the Gibbs (thermal) ensemble (3–7), whereas for integrable systems, a generalized Gibbs ensemble has to be used (8, 22–45).

Although these results suggest a spectacular compression of the amount of information needed to describe steady states, state of the art numerical methods, such as the time-dependent density matrix renormalization group (46–49) (tDMRG), can only access the short-time dynamics. Physically, the origin of this conundrum is the fast growth of the entanglement entropy $S \equiv -\text{Tr}_{A\rho_A} \ln \rho_A$, with ρ_A being the reduced density matrix of an interval A of length ℓ embedded in an infinite system. It is well-understood that S grows linearly with the time after the quench (50). This linear behavior implies an exponentially increasing amount of information manipulated during typical tDMRG simulations. Remarkably, the entanglement dynamics has been successfully observed in a very recent cold-atom experiment (19).

In this work, using a standard quasiparticle picture (50), we show that, in integrable models, the steady state and its low-lying excitations encode sufficient information to reconstruct the entanglement dynamics up to short times. According to the quasiparticle picture (50), the prequench initial state acts as a

source of pairs of quasiparticle excitations. Let us first assume that there is only one type of quasiparticles identified by their quasimomentum λ and moving with velocity $v(\lambda)$. Although quasiparticles created far apart from each other are incoherent, those emitted at the same point in space are entangled. Because these propagate ballistically throughout the system, larger regions get entangled. At time t , $S(t)$ is proportional to the total number of quasiparticle pairs that, emitted at the same point in space, are shared between A and its complement (Fig. 1A). Specifically, one obtains

$$S(t) \propto 2t \int_{|v|t < \ell} d\lambda v(\lambda) f(\lambda) + \ell \int_{2|v|t > \ell} d\lambda f(\lambda), \quad [1]$$

where $f(\lambda)$ depends on the production rate of quasiparticles. Eq. 1 holds in the space–time scaling limit $t, \ell \rightarrow \infty$ at t/ℓ fixed. When a maximum quasiparticle velocity v_M exists [e.g., because of the Lieb–Robinson bound (51)], for $t \leq \ell/(2v_M)$, S grows linearly in time, because the second term in Eq. 1 vanishes. In contrast, for $t \gg \ell/(2v_M)$, the entanglement is extensive (i.e., $S \propto \ell$). This light-cone spreading of entanglement has been confirmed analytically only in free models (52–57), numerically in several studies (58–60), in the holographic framework (61–68), and in a recent experiment (19). The validity of the quasiparticle picture Eq. 1 for the entanglement dynamics has been proven for free models in ref. 52. In the presence of interactions, few results are known. For instance, the validity of Eq. 1 has been proven for rational CFTs (Conformal Field Theories) in ref. 69. In interacting integrable models à la Yang–Baxter, the quasiparticle picture has been used to describe the out-of-equilibrium dynamics after

Significance

Understanding how statistical ensembles arise from the out-of-equilibrium dynamics of isolated pure systems has been a fascinating question since the early days of quantum mechanics. Recently, it has been proposed that the thermodynamic entropy of the long-time statistical ensemble is the stationary entanglement of a large subsystem in an infinite system. Here, we combine this concept with the quasiparticle picture of the entanglement evolution and integrability-based knowledge of the steady state to obtain exact analytical predictions for the time evolution of the entanglement in arbitrary 1D integrable models. These results explicitly show the transformation between the entanglement and thermodynamic entropy during the time evolution. Thus, entanglement is the natural witness for the generalized microcanonical principle underlying relaxation in integrable models.

PHYSICS

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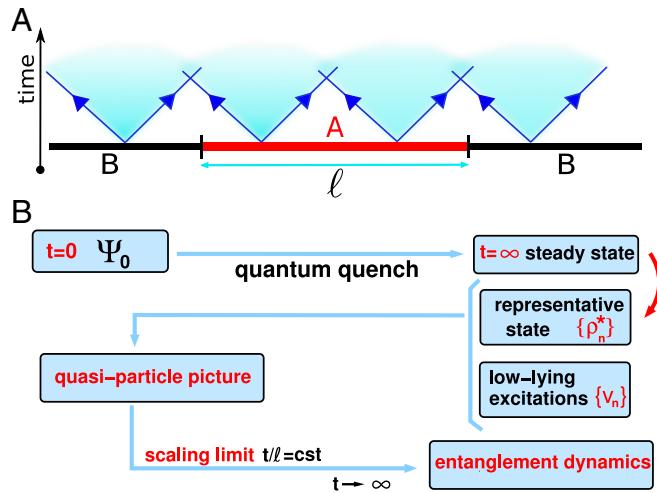


Fig. 1. Entanglement dynamics after a quantum quench: theoretical scheme. (A) Quasiparticle picture. Full lines denote quasiparticles with maximum velocity emitted in the initial state. Shaded cones show the halo of slower quasiparticles. (B) Main steps to calculate the entanglement dynamics using Bethe ansatz and the quasiparticle picture.

an inhomogeneous quench, and it is at the foundation of the integrable hydrodynamics approach for transport in integrable models (70, 71).

Results

In a generic interacting integrable model, there are different species of stable quasiparticles corresponding to bound states of an arbitrary number of elementary excitations. Integrability implies that different types of quasiparticles must be treated independently. It is then natural to conjecture that

$$S(t) = \sum_n \left[2t \int_{2|v_n|t < \ell} d\lambda v_n(\lambda) s_n(\lambda) + \ell \int_{2|v_n|t > \ell} d\lambda s_n(\lambda) \right], \quad [2]$$

where the sum is over the types of particles n , $v_n(\lambda)$ is their velocity, and $s_n(\lambda)$ is their entropy. To give predictive power to Eq. 2, in the following, we show how to determine $v_n(\lambda)$ and $s_n(\lambda)$ in the Bethe ansatz framework for integrable models. Eq. 2 is straightforwardly generalized to the mutual information between two intervals (see *SI Materials and Methods* and Fig. S1).

The eigenstates of Bethe ansatz solvable models are in correspondence with a set of pseudomomenta (rapidities) λ . In the thermodynamic limit, these rapidities form a continuum. One then introduces the particle densities $\rho_{n,p}(\lambda)$, the hole (i.e., unoccupied rapidities) densities $\rho_{n,h}(\lambda)$, and the total densities $\rho_{n,t}(\lambda) = \rho_{n,p}(\lambda) + \rho_{n,h}(\lambda)$. Every set of densities identifies a thermodynamic “macrostate.” This macrostate corresponds to an exponentially large number of microscopic eigenstates, any of which can be used as a “representative” for the macrostate. The total number of representative microstates is $e^{S_{YY}}$, with S_{YY} as the thermodynamic Yang–Yang entropy of the macrostate

$$\begin{aligned} S_{YY} = s_{YY} L &= L \sum_{n=1}^{\infty} \int d\lambda [\rho_{n,t}(\lambda) \ln \rho_{n,t}(\lambda) \\ &\quad - \rho_{n,p}(\lambda) \ln \rho_{n,p}(\lambda) - \rho_{n,h}(\lambda) \ln \rho_{n,h}(\lambda)] \\ &\equiv L \sum_{n=1}^{\infty} \int d\lambda s_{YY}^{(n)}[\rho_{n,p}, \rho_{n,h}](\lambda). \end{aligned} \quad [3]$$

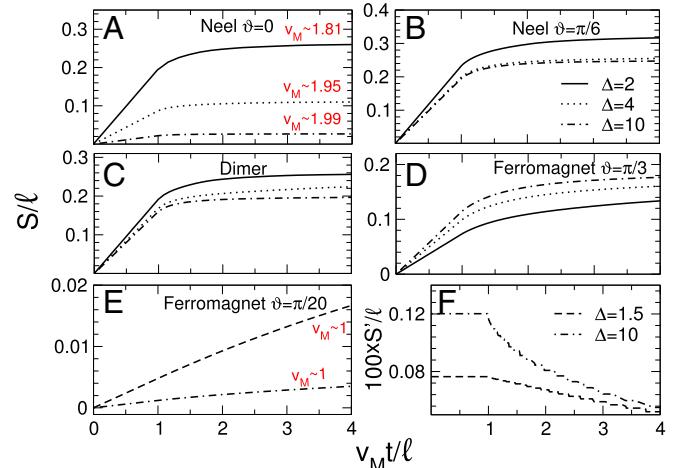


Fig. 2. Analytical predictions for the XXZ chain. Entanglement entropy per site S/ℓ vs. $v_M t/\ell$, with v_M as the maximum velocity. Different panels correspond to the different initial states: Néel (A), tilted Néel (B), Dimer (C), and tilted ferromagnet (D and E). Different lines correspond to different Δ . For $\Delta \rightarrow \infty$, $S \rightarrow 0$ for the Néel quench, whereas it saturates in the other cases. Note in E the substantial entanglement increase for $v_M t/\ell > 1$. (F) The numerical derivative $100 \times S'(v_M t/\ell) \times 100$ for the quench in E.

Physically, S_{YY} corresponds to the total number of ways of assigning the quasimomentum label to the particles, similar to free fermion models.

In the Bethe ansatz treatment of quantum quenches (72–74), local properties of the postquench stationary state are described by a set of densities $\rho_{n,p}^*(\lambda)$ and $\rho_{n,h}^*(\lambda)$. Calculating these densities is a challenging task that has been performed only in a few cases (75–87). From the densities, the thermodynamic entropy of the stationary ensemble Eq. 3 is $s_{YY}[\rho_{n,p}^*, \rho_{n,h}^*](\lambda)$. Physically, $e^{Ls_{YY}}$ is the number of microscopic eigenstates entering in a generalized microcanonical ensemble for quenches, in which all of the microstates corresponding to the macrostate have the same probability.

We now present our predictions for the entanglement dynamics (Fig. 1B gives a survey of our theoretical scheme). First, in the stationary state, the density of thermodynamic entropy coincides with that of the entanglement entropy in Eq. 2, as it has been shown analytically for free models (52, 88, 89). This identification implies that $s_n(\lambda) = s_{YY}[\rho_{n,p}^*, \rho_{n,h}^*](\lambda)$. Moreover, it is natural to identify the entangling quasiparticles in Eq. 2 with the low-lying excitations around the stationary state ρ^* . Their group

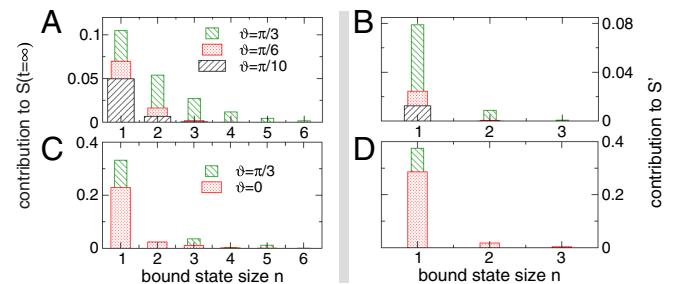


Fig. 3. Bound-state contributions to the entanglement dynamics. On the x axis, n is the bound-state size. (A and B) Quench from the tilted ferromagnet. (A) Bound-state contributions to steady-state entropy density (second term in Eq. 2). (B) Bound-state contributions to the slope of the entanglement growth for $t < \ell/v_M$ (first term in Eq. 2). Different histograms denote different tilting angles ϑ . All of the data are for $\Delta = 2$. (C and D) Same as in A and B for the quench from the tilted Néel state.

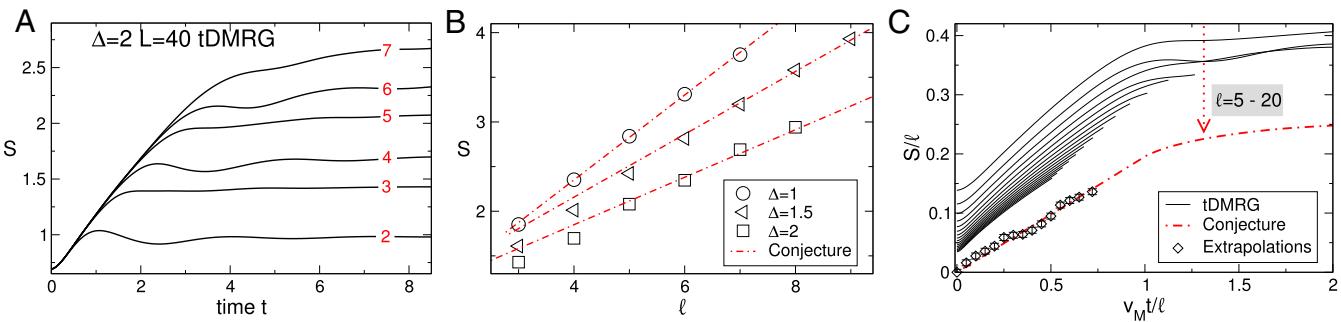


Fig. 4. Comparison with numerical simulations. Entanglement entropy dynamics after the quench from the Néel state in the XXZ chain. (A) tDMRG results for a chain with $L = 40$ sites and $\Delta = 2$. Different curves correspond to different subsystem sizes ℓ (accompanying numbers). (B) The entropy saturation values (tDMRG results at $t \approx 8$) as a function of the block length ℓ for several Δ . The dashed-dotted lines are the conjectured volume-law behavior $S \propto s_{YY}^* \ell$ (compare with Eq. 3). (C) The scaling limit: S/ℓ plotted vs. $v_M t/\ell$. The continuous curves are the tDMRG results for $\ell = 5 - 20$. The diamonds are numerical extrapolations to the thermodynamic limit. The dashed-dotted line is the conjecture Eq. 2.

velocities v_n depend on the stationary state, because the interactions induce a state-dependent dressing of the excitations. These velocities v_n can be calculated by Bethe ansatz techniques (90) (*SI Materials and Methods* and Figs. S2 and S3).

To substantiate our idea, we focus on the spin-1/2 anisotropic Heisenberg (XXZ) chain, considering quenches from several low-entangled initial states, namely the tilted Néel state, the Majumdar–Ghosh (dimer) state, and the tilted ferromagnetic state (*Materials and Methods*). For these initial states, the densities $\rho_{n(h),p}^*$ are known analytically (77–79).

Fig. 2 summarizes the expected entanglement dynamics in the space–time scaling limit, plotting S/ℓ vs. $v_M t/\ell$. Interestingly, S/ℓ is always smaller than $\ln 2$ (i.e., the entropy of the maximally entangled state). For the Néel quench, because the Néel state becomes the ground state of Eq. 4 in the limit $\Delta \rightarrow \infty$, $S/\ell \approx \ln(\Delta)/\Delta^2$ vanishes, whereas it saturates for all of the other quenches. For the Majumdar–Ghosh state, one obtains $S/\ell = -1/2 + \ln 2$ at $\Delta \rightarrow \infty$. For the tilted ferromagnet with $\vartheta \rightarrow 0$ (Fig. 2E), S/ℓ is small at any Δ , reflecting that the ferromagnet is an eigenstate of the XXZ chain. Surprisingly, the linear growth seems to extend for $v_M t/\ell > 1$. However, dS/dt (Fig. 2F) is flat only for $v_M t/\ell \leq 1$, which signals true linear regime only for $v_M t/\ell \leq 1$. This peculiar behavior is caused by the large entanglement contribution of the slow quasiparticles. In Fig. 3, we report the bound-state resolved contributions to the entanglement dynamics. Fig. 3 A and C focuses on the steady-state entropy (second term in Eq. 2), whereas Fig. 3 B and D shows the bound-state contributions to the slope of the linear growth (first term in Eq. 2). The contribution of the bound states, although never dominant, is crucial to ensure accurate predictions.

Fig. 4A shows tDMRG results for $S(t)$ for the quench from the symmetrized Néel state ($|\uparrow\downarrow\dots\rangle + |\downarrow\uparrow\dots\rangle)/\sqrt{2}$. The data are for the open XXZ chain and subsystems starting from the chain boundary. The qualitative agreement with Eq. 2 is apparent. Fig. 4B reports the steady-state entanglement entropy as a function of ℓ (data at $t \approx 8$ in Fig. 4A). The volume law $S \propto \ell$ is visible. The dashed-dotted lines in Fig. 4B are fits to $S \propto s_{YY}^* \ell$, supporting the equivalence between entanglement and thermodynamic entropy. Fig. 4C focuses on the full-time dependence, plotting S/ℓ vs. $v_M t/\ell$. The dashed-dotted line in Fig. 4C is Eq. 2 with $t \rightarrow t/2$ because of the open boundary conditions (58). Deviations from Eq. 2 because of the finite ℓ are visible. The diamonds in Fig. 4C are numerical extrapolations to the thermodynamic limit. The agreement with Eq. 2 is perfect. Finally, we provide a more stringent check of Eq. 2, focusing on the linear entanglement growth. Fig. 5 shows infinite time-evolving block decimation (iTEBD) results in the thermodynamic limit for

$S'(v_M t)$, with $S'(x) \equiv dS(x)/dx$ taken from ref. 91. For all of the quenches, the agreement with Eq. 2 (horizontal lines in Fig. 5) is spectacular.

Conclusions

The main result of this work is the analytical prediction in Eq. 2 for the time-dependent entanglement entropy after a generic quantum quench in an integrable model. We tested our conjecture for several quantum quenches in the XXZ spin chain, although we expect Eq. 2 to be more general. Additional checks of Eq. 2 (e.g., for the Lieb–Liniger gas) are desirable. It would be also interesting to generalize Eq. 2 to quenches from inhomogeneous initial states, exploiting the recent analytical results (70, 71, 92). Although we are not able yet to provide an ab initio derivation of Eq. 2, we find it remarkable that it is possible to characterize analytically the dynamics of the entanglement entropy, whereas its equilibrium behavior is still an open challenge. Finally, we believe that Eq. 2 represents a deep conceptual breakthrough, because it shows in a single compact formula the relation between entanglement and thermodynamic

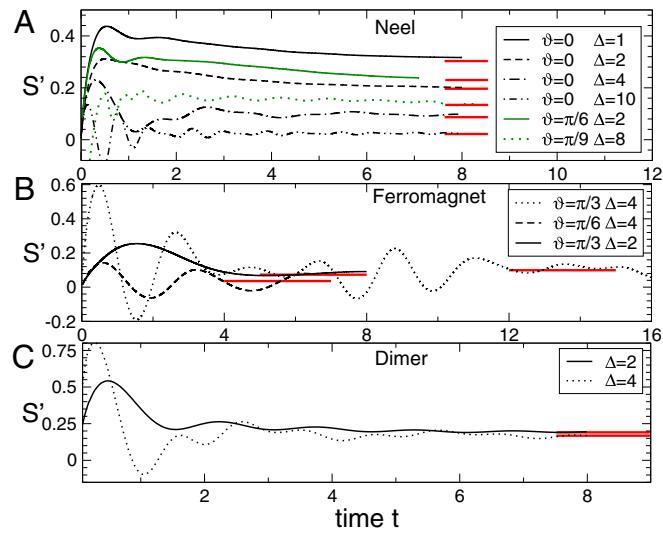


Fig. 5. Comparison with numerical simulations: the short-time regime. Derivative $S'(v_M t)$ as a function of time. Different panels correspond to different initial states: tilted Néel (A), tilted ferromagnet (B), and dimer state (C); ϑ is the tilting angle. In each panel, the different curves are iTEBD results for different Δ . The horizontal lines are the conjecture Eq. 2.

entropy for integrable models. An analogous description for nonintegrable systems, where quasiparticles have finite lifetime or do not exist at all, could lead to a deeper understanding of thermalization (19).

Materials and Methods

The anisotropic spin-1/2 Heisenberg chain is defined by the Hamiltonian

$$H = \sum_{i=1}^L \left[\frac{1}{2} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \Delta \left(S_i^z S_{i+1}^z - \frac{1}{4} \right) \right], \quad [4]$$

where S_i^α are spin-1/2 operators, and Δ is the anisotropy parameter. Here, we considered as prequench initial states the tilted Néel state $|\vartheta, \nearrow \searrow \dots\rangle \equiv e^{i\vartheta \sum_j S_j^y} |\uparrow \downarrow \dots\rangle$, the Majumdar–Ghosh (dimer) state $|MG\rangle \equiv ((|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle)/2)^{\otimes L/2}$, and the tilted ferromagnetic state $|\vartheta, \nearrow \nearrow \rangle \equiv e^{i\vartheta \sum_j S_j^y} |\uparrow \uparrow \dots\rangle$. The Heisenberg spin chain is the prototype of all integrable models. Moreover, for all of the initial states considered in this

work, the postquench steady state can be characterized analytically via the macrostate densities $\rho_{ph}^{\alpha\beta}$. Specifically, a set of recursive relations for these densities can be obtained (*SI Materials and Methods*). The group velocities of the low-lying excitations around the steady state (i.e., the entangling quasiparticles) are obtained by solving numerically an infinite set of second-type Fredholm integral equations (details are in *SI Materials and Methods*).

The numerical data for the postquench dynamics of the entanglement entropy presented in Fig. 4 were obtained using the standard tDMRG (46–49) in the framework of matrix product states. For the implementation, we used the iTENSOR library (itensor.org). The data presented in Fig. 5 are obtained using the iTEBD method (93) and they are a courtesy of Mario Collura.

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