Chapter 10

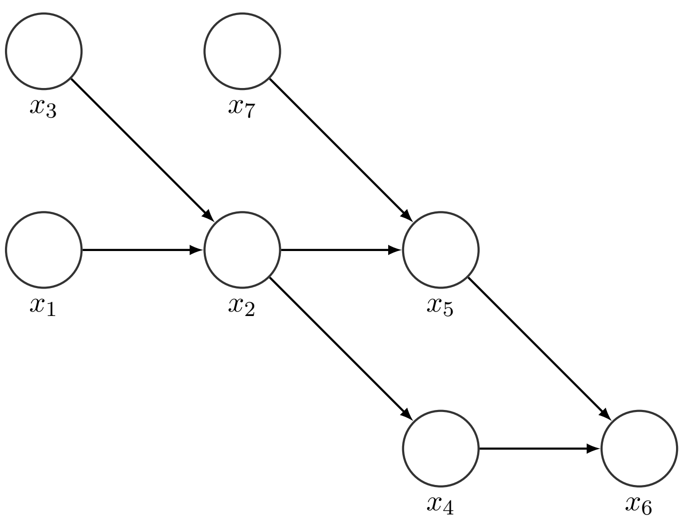
Problems

10.1

The joint probability model between variables $\{x_i\}_{i=1}^{7}$ *factorizes as*

$$Pr(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$$
$$Pr(x_1)Pr(x_3)Pr(x_7)Pr(x_2|x_1, x_3)Pr(x_5|x_7, x_2)Pr(x_4|x_2)Pr(x_6|x_5, x_4)$$

Draw a directed graphical model relating these variables. Which variables form the Markov blanket of variable x_2*?*



The Markov blanket of x_2 is $\{x_1, x_3, x_5, x_7\}$ (the parents, children and co-parents of the children of x_2).

10.2

Write out the factorization corresponding to the directed graphical model in Figure 10.14a

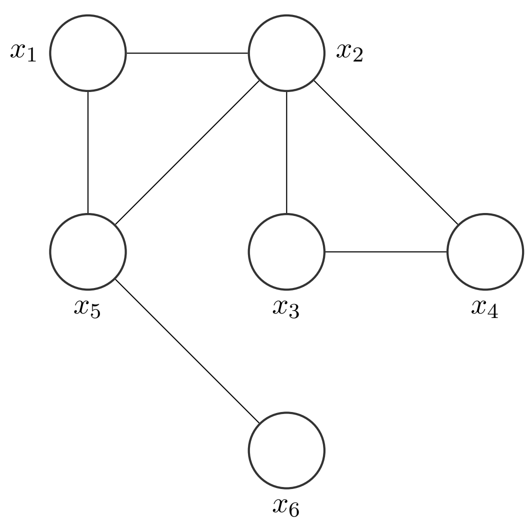
$$\begin{aligned} Pr(x_1, ..., x_{15}) =& Pr(x_1)Pr(x_2)Pr(x_3)Pr(x_4|x_1, x_2)Pr(x_5|x_2, x_3)Pr(x_6)Pr(x_7)Pr(x_8|x_4, x_5) \\ & Pr(x_9|x_3, x_5, x_6)Pr(x_{10}|x_7)Pr(x_{11}|x_8, x_{10})Pr(x_{12}|x_8, x_9)Pr(x_{13}|x_9) \\ & Pr(x_{14}|x_{11})Pr(x_{15}|x_{12}) \end{aligned}$$

10.3

An undirected graphical model has the form

$$Pr(x_1, ..., x_6) = \frac{1}{Z}\phi_1[x_1, x_2, x_5]\phi_2[x_2, x_3, x_4]\phi_3[x_1, x_5]\phi_4[x_5, x_6]$$

Draw the undirected graphical model that corresponds to this factorization.



10.4

Write out the factorization corresponding to the undirected graphical model in Figure 10.14b.

$$Pr(x_1, ..., x_{15}) = \frac{1}{Z}\phi_1[x_3]\phi_2[x_1, x_4]\phi_3[x_2, x_4, x_8]\phi_4[x_2, x_5, x_8]\phi_5[x_5, x_9]$$
$$\phi_6[x_6, x_9]\phi_7[x_8, x_{12}]\phi_8[x_9, x_{12}, x_{13}, x_{15}]\phi_9[x_8, x_{11}]\phi_{10}[x_7, x_{10}]$$
$$\phi_{11}[x_7, x_11]\phi_{12}[x_{11}, x_{14}]$$

10.5

Consider the undirected graphical model defined over binary values $\{x_i\}_{i=1}^4 \in \{0, 1\}$ *defined by*

$$Pr(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1),$$

where the function ϕ *is defined by*

$$\phi(0, 0) = 1 \qquad \phi(1, 1) = 2$$
$$\phi(0, 1) = 0.1 \qquad \phi(1, 0) = 0.1$$

Compute the probability of each of the 16 possible states of this system.

x_1	x_2	x_3	x_4	$\phi(x_1,x_2)$	$\phi(x_2,x_3)$	$\phi(x_3,x_4)$	$\phi(x_4,x_1)$	$ZPr(...)$	$Pr(...)$
0	0	0	0	1.	1.	1.	1.	1.0000	.057870
0	0	0	1	1.	1.	0.1	0.1	0.0100	.000579
0	0	1	0	1.	0.1	0.1	1.	0.0100	.000579
0	0	1	1	1.	0.1	2.	0.1	0.0200	.001157
0	1	0	0	0.1	0.1	1.	1.	0.0100	.000579
0	1	0	1	0.1	0.1	0.1	0.1	0.0001	.000006
0	1	1	0	0.1	2.	0.1	1.	0.0200	.001157
0	1	1	1	0.1	2.	2.	0.1	0.0400	.002315
1	0	0	0	0.1	1.	1.	0.1	0.0100	.000579
1	0	0	1	0.1	1.	0.1	2.	0.0200	.001157
1	0	1	0	0.1	0.1	0.1	0.1	0.0001	.000006
1	0	1	1	0.1	0.1	2.	2.	0.0400	.002315
1	1	0	0	2.	0.1	1.	0.1	0.0200	.001157
1	1	0	1	2.	0.1	0.1	2.	0.0400	.002315
1	1	1	0	2.	2.	0.1	0.1	0.0400	.002315
1	1	1	1	2.	2.	2.	2.	16.0000	.925914
								17.2802	1.000000

We can attempt to check this by sampling from the distribution e.g. using Gibbs sampling. To do this we need the conditional probability distribution for each x_i. By symmetry we just need to calculate the conditional distribution for x_1.

$$
\begin{aligned}
Pr(x_1|x_{\setminus 1}) &= \frac{Pr(x_1,x_2,x_3,x_4)}{\sum_{x_1} Pr(x_1,x_2,x_3,x_4)} \\
&= \frac{\frac{1}{Z}\phi(x_1,x_2)\phi(x_2,x_3)\phi(x_3,x_4)\phi(x_4,x_1)}{\sum_{x_1}\frac{1}{Z}\phi(x_1,x_2)\phi(x_2,x_3)\phi(x_3,x_4)\phi(x_4,x_1)} \\
&= \frac{\phi(x_1,x_2)\phi(x_2,x_3)\phi(x_3,x_4)\phi(x_4,x_1)}{\phi(x_2,x_3)\phi(x_3,x_4)\sum_{x_1}\phi(x_1,x_2)\phi(x_4,x_1)} \\
&= \frac{\phi(x_1,x_2)\phi(x_4,x_1)}{\sum_{x_1}\phi(x_1,x_2)\phi(x_4,x_1)}
\end{aligned}
$$

$$Pr(x_1 = 0|x_2 = 0, x_4 = 0) = \frac{1. \times 1.}{1. \times 1. + 0.1 \times 0.1} = \frac{1.}{1.01} = 0.9901$$
$$Pr(x_1 = 1|x_2 = 0, x_4 = 0) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.9901 = 0.0099$$
$$Pr(x_1 = 0|x_2 = 0, x_4 = 1) = \frac{1. \times 0.1}{1. \times 0.1 + 0.1 \times 2.} = \frac{0.1}{0.3} = 0.3333$$
$$Pr(x_1 = 1|x_2 = 0, x_4 = 1) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.3333 = 0.6667$$
$$Pr(x_1 = 0|x_2 = 1, x_4 = 0) = \frac{0.1 \times 1.}{0.1 \times 1. + 2. \times 0.1} = \frac{0.1}{0.3} = 0.3333$$
$$Pr(x_1 = 1|x_2 = 1, x_4 = 0) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.3333 = 0.6667$$
$$Pr(x_1 = 0|x_2 = 1, x_4 = 1) = \frac{0.1 \times 0.1}{0.1 \times 0.1 + 2. \times 2.} = \frac{0.01}{4.01} = 0.0025$$
$$Pr(x_1 = 1|x_2 = 1, x_4 = 1) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.0025 = 0.9975$$