Chapter 10

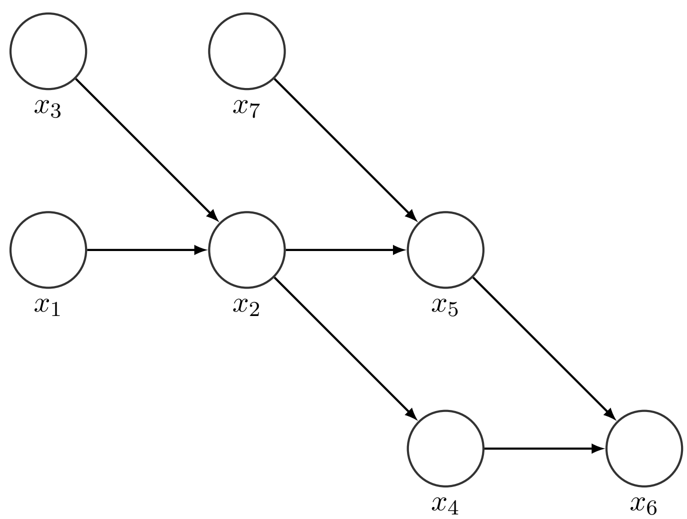
Problems

10.1

The joint probability model between variables $\{x_i\}_{i=1}^{7}$ *factorizes as*

$$Pr(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \\ Pr(x_1)Pr(x_3)Pr(x_7)Pr(x_2|x_1, x_3)Pr(x_5|x_7, x_2)Pr(x_4|x_2)Pr(x_6|x_5, x_4)$$

Draw a directed graphical model relating these variables. Which variables form the Markov blanket of variable x_2*?*



The Markov blanket of x_2 is $\{x_1, x_3, x_4, x_5, x_7\}$ (the parents, children and co-parents of the children of x_2).

10.2

Write out the factorization corresponding to the directed graphical model in Figure 10.14a

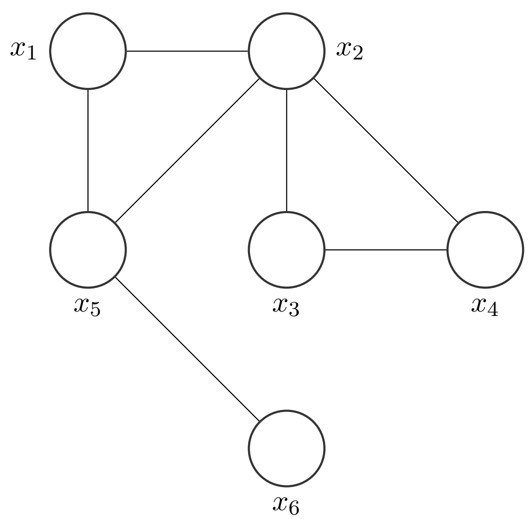
$$\begin{aligned} Pr(x_1, ..., x_{15}) =& Pr(x_1)Pr(x_2)Pr(x_3)Pr(x_4|x_1, x_2)Pr(x_5|x_2, x_3)Pr(x_6)Pr(x_7)Pr(x_8|x_4, x_5) \\ & Pr(x_9|x_3, x_5, x_6)Pr(x_{10}|x_7)Pr(x_{11}|x_8, x_{10})Pr(x_{12}|x_8, x_9)Pr(x_{13}|x_9) \\ & Pr(x_{14}|x_{11})Pr(x_{15}|x_{12}) \end{aligned}$$

10.3

An undirected graphical model has the form

$$Pr(x_1, ..., x_6) = \frac{1}{Z}\phi_1[x_1, x_2, x_5]\phi_2[x_2, x_3, x_4]\phi_3[x_1, x_5]\phi_4[x_5, x_6]$$

Draw the undirected graphical model that corresponds to this factorization.



10.4

Write out the factorization corresponding to the undirected graphical model in Figure 10.14b.

$$
\begin{aligned}
Pr(x_1, ..., x_{15}) = \frac{1}{Z}&\phi_1[x_3]\phi_2[x_1, x_4]\phi_3[x_2, x_4, x_8]\phi_4[x_2, x_5, x_8]\phi_5[x_5, x_9] \\
&\phi_6[x_6, x_9]\phi_7[x_8, x_{12}]\phi_8[x_9, x_{12}, x_{13}, x_{15}]\phi_9[x_8, x_{11}]\phi_{10}[x_7, x_{10}] \\
&\phi_{11}[x_7, x_11]\phi_{12}[x_{11}, x_{14}]
\end{aligned}
$$

10.5

Consider the undirected graphical model defined over binary values $\{x_i\}_{i=1}^4 \in \{0, 1\}$ *defined by*

$$Pr(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1),$$

where the function ϕ *is defined by*

$$
\begin{aligned}
\phi(0, 0) &= 1 & \phi(1, 1) &= 2 \\
\phi(0, 1) &= 0.1 & \phi(1, 0) &= 0.1
\end{aligned}
$$

Compute the probability of each of the 16 possible states of this system.

x_1	x_2	x_3	x_4	$\phi(x_1, x_2)$	$\phi(x_2, x_3)$	$\phi(x_3, x_4)$	$\phi(x_4, x_1)$	$ZPr(...)$	$Pr(...)$
0	0	0	0	1.	1.	1.	1.	1.0000	.057870
0	0	0	1	1.	1.	0.1	0.1	0.0100	.000579
0	0	1	0	1.	0.1	0.1	1.	0.0100	.000579
0	0	1	1	1.	0.1	2.	0.1	0.0200	.001157
0	1	0	0	0.1	0.1	1.	1.	0.0100	.000579
0	1	0	1	0.1	0.1	0.1	0.1	0.0001	.000006
0	1	1	0	0.1	2.	0.1	1.	0.0200	.001157
0	1	1	1	0.1	2.	2.	0.1	0.0400	.002315
1	0	0	0	0.1	1.	1.	0.1	0.0100	.000579
1	0	0	1	0.1	1.	0.1	2.	0.0200	.001157
1	0	1	0	0.1	0.1	0.1	0.1	0.0001	.000006
1	0	1	1	0.1	0.1	2.	2.	0.0400	.002315
1	1	0	0	2.	0.1	1.	0.1	0.0200	.001157
1	1	0	1	2.	0.1	0.1	2.	0.0400	.002315
1	1	1	0	2.	2.	0.1	0.1	0.0400	.002315
1	1	1	1	2.	2.	2.	2.	16.0000	.925914
								17.2802	1.000000

We can attempt to check this by sampling from the distribution e.g. using Gibbs sampling. To do this we need the conditional probability distribution for each x_i. By symmetry we just need to calculate the conditional distribution for x_1.

$$
\begin{aligned}
Pr(x_1|x_{\setminus 1}) &= \frac{Pr(x_1, x_2, x_3, x_4)}{\sum_{x_1} Pr(x_1, x_2, x_3, x_4)} \\
&= \frac{\frac{1}{Z}\phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1)}{\sum_{x_1} \frac{1}{Z}\phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1)} \\
&= \frac{\phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1)}{\phi(x_2, x_3)\phi(x_3, x_4)\sum_{x_1} \phi(x_1, x_2)\phi(x_4, x_1)} \\
&= \frac{\phi(x_1, x_2)\phi(x_4, x_1)}{\sum_{x_1} \phi(x_1, x_2)\phi(x_4, x_1)}
\end{aligned}
$$

$$Pr(x_1 = 0|x_2 = 0, x_4 = 0) = \frac{1. \times 1.}{1. \times 1. + 0.1 \times 0.1} = \frac{1.}{1.01} = 0.9901$$
$$Pr(x_1 = 1|x_2 = 0, x_4 = 0) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.9901 = 0.0099$$
$$Pr(x_1 = 0|x_2 = 0, x_4 = 1) = \frac{1. \times 0.1}{1. \times 0.1 + 0.1 \times 2.} = \frac{0.1}{0.3} = 0.3333$$
$$Pr(x_1 = 1|x_2 = 0, x_4 = 1) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.3333 = 0.6667$$
$$Pr(x_1 = 0|x_2 = 1, x_4 = 0) = \frac{0.1 \times 1.}{0.1 \times 1. + 2. \times 0.1} = \frac{0.1}{0.3} = 0.3333$$
$$Pr(x_1 = 1|x_2 = 1, x_4 = 0) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.3333 = 0.6667$$
$$Pr(x_1 = 0|x_2 = 1, x_4 = 1) = \frac{0.1 \times 0.1}{0.1 \times 0.1 + 2. \times 2.} = \frac{0.01}{4.01} = 0.0025$$
$$Pr(x_1 = 1|x_2 = 1, x_4 = 1) = 1 - Pr(x_1 = 0|x_2 = 0, x_4 = 0) = 1 - 0.0025 = 0.9975$$

See problem_10_5.py for an implementation.

10.6

For directed graphical models Markov blanket of a node is the parents, children and co-parents of children of the node. For undirected graphical models Markov blanket of a node is the set of neighbours of that node. Markov blanket, $blanket[]$ for figures 10.7 and 10.8:

Figure 10.7a

$$blanket[x_1] \in \{x_3\}$$
$$blanket[x_2] \in \{x_3\}$$
$$blanket[x_3] \in \{x_1, x_2\}$$

Figure 10.7b

$$blanket[x_1] \in \{x_3\}$$
$$blanket[x_2] \in \{x_3\}$$
$$blanket[x_3] \in \{x_1, x_2\}$$

Figure 10.7c

$$blanket[x_1] \in \{x_2, x_3\}$$
$$blanket[x_2] \in \{x_1, x_3\}$$
$$blanket[x_3] \in \{x_1, x_2\}$$

Figure 10.8a

$$blanket[x_1] \in \{x_2, x_3\}$$
$$blanket[x_2] \in \{x_1, x_4\}$$
$$blanket[x_3] \in \{x_1, x_4\}$$
$$blanket[x_4] \in \{x_2, x_3\}$$

Figure 10.8b

$$blanket[x_1] \in \{x_2, x_3\}$$
$$blanket[x_2] \in \{x_1, x_3, x_4\}$$
$$blanket[x_3] \in \{x_1, x_2, x_4\}$$
$$blanket[x_4] \in \{x_2, x_3\}$$

10.7

Show that the stated patterns of independence and conditional independence in Figures 10.7 and 10.8 are true.

For Figure 10.7a we need to prove that $x_2 \perp\!\!\!\perp x_1 | x_3$. We use the same strategy as given in the text in "10.2.2 Example 1". We write down the conditional probability statement for x_2 *including* x_1, the variable we are trying to show x_2 is conditionally independent of. Then we show by algebraic manipulation that there is a representation of the conditional probability that does not include x_1

$$\begin{aligned}
Pr(x_2|x_1,x_3) &= \frac{Pr(x_1,x_2,x_3)}{Pr(x_1,x_3)} \\
&= \frac{Pr(x_2|x_3)Pr(x_3|x_1)Pr(x_1)}{\int Pr(x_2|x_3)Pr(x_3|x_1)Pr(x_1)dx_2} \\
&= \frac{Pr(x_2|x_3)Pr(x_3|x_1)Pr(x_1)}{Pr(x_3|x_1)Pr(x_1)\int Pr(x_2|x_3)dx_2} \\
&= \frac{Pr(x_2|x_3)}{\int Pr(x_2|x_3)dx_2}
\end{aligned}$$

The final expression doesn't depend on x_1 so we conclude that the conditional independence statement holds.

For Figure 10.7b this is just the same argument as in "10.3.1. Example 1" of the text but with a relabelling of the graph.

For Figure 10.7c we prove the (unconditional) independence statement $x_2 \perp\!\!\!\perp x_1$ by showing that $Pr(x_2, x_1) = Pr(x_2)Pr(x_1)$

$$\begin{aligned}
Pr(x_2,x_1) &= \int Pr(x_2,x_1,x_3)dx_3 \\
&= \int Pr(x_2)Pr(x_1)Pr(x_3|x_1,x_2)dx_3 \\
&= Pr(x_2)Pr(x_1)\int Pr(x_3|x_1,x_2)dx_3 \\
&= Pr(x_2)Pr(x_1)
\end{aligned}$$

where the last step uses the fact that $\int Pr(x_3|x_1,x_2)dx_3 = 1$.

For Figure 10.8a we again write down the conditional probability statement including the variables we wish to show are (conditionally independent)

$$\begin{aligned}
Pr(x_1|x_4,x_2,x_3) &= \frac{Pr(x_1,x_4,x_2,x_3)}{Pr(x_4,x_2,x_3)} \\
&= \frac{\frac{1}{Z}\phi_1(x_1,x_2)\phi_2(x_2,x_4)\phi_3(x_4,x_3)\phi_4(x_3,x_1)}{\int \frac{1}{Z}\phi_1(x_1,x_2)\phi_2(x_2,x_4)\phi_3(x_4,x_3)\phi_4(x_3,x_1)dx_1} \\
&= \frac{\frac{1}{Z}\phi_1(x_1,x_2)\phi_2(x_2,x_4)\phi_3(x_4,x_3)\phi_4(x_3,x_1)}{\frac{1}{Z}\phi_2(x_2,x_4)\phi_3(x_4,x_3)\int \phi_1(x_1,x_2)\phi_4(x_3,x_1)dx_1} \\
&= \frac{\phi_1(x_1,x_2)\phi_4(x_3,x_1)}{\int \phi_1(x_1,x_2)\phi_4(x_3,x_1)dx_1}
\end{aligned}$$

and the final expression does not rely on x_4

The proof of $x_2 \perp\!\!\!\perp x_3 | x_1, x_4$ is similar:

$$\begin{aligned}
Pr(x_2|x_3, x_1, x_4) &= \frac{Pr(x_1, x_4, x_2, x_3)}{Pr(x_1, x_3, x_4)} \\
&= \frac{\frac{1}{Z}\phi_1(x_1, x_2)\phi_2(x_2, x_4)\phi_3(x_4, x_3)\phi_4(x_3, x_1)}{\int \frac{1}{Z}\phi_1(x_1, x_2)\phi_2(x_2, x_4)\phi_3(x_4, x_3)\phi_4(x_3, x_1)dx_2} \\
&= \frac{\frac{1}{Z}\phi_1(x_1, x_2)\phi_2(x_2, x_4)\phi_3(x_4, x_3)\phi_4(x_3, x_1)}{\frac{1}{Z}\phi_3(x_4, x_3)\phi_4(x_3, x_1)\int \phi_1(x_1, x_2)\phi_2(x_2, x_4)dx_2} \\
&= \frac{\phi_1(x_1, x_2)\phi_2(x_2, x_4)}{\int \phi_1(x_1, x_2)\phi_2(x_2, x_4)dx_2} \\
&\implies x_2 \perp\!\!\!\perp x_3 | x_1, x_4
\end{aligned}$$

For figure 10.8b let's prove the first assertion using the (by now) familiar technique

$$\begin{aligned}
Pr(x_1|x_4, x_2, x_3) &= \frac{Pr(x_1, x_4, x_2, x_3)}{Pr(x_4, x_2, x_3)} \\
&= \frac{Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)Pr(x_4|x_2, x_3)}{\int Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)Pr(x_4|x_2, x_3)dx_1} \\
&= \frac{Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)Pr(x_4|x_2, x_3)}{Pr(x_4|x_2, x_3)\int Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)dx_1} \\
&= \frac{Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)}{\int Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)dx_1} \\
&\implies x_1 \perp\!\!\!\perp x_4 | x_2, x_3
\end{aligned}$$

and then the 2nd assertion

$$
\begin{aligned}
Pr(x_2|x_3,x_1) &= \frac{Pr(x_2,x_3,x_1)}{Pr(x_3,x_1)} \\
&= \frac{\int Pr(x_2,x_3,x_1,x_4)dx_4}{\int\int Pr(x_2,x_3,x_1,x_4)dx_4dx_2} \\
&= \frac{\int Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)Pr(x_4|x_2,x_3)dx_4}{\int\int Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)Pr(x_4|x_2,x_3)dx_4dx_2} \\
&= \frac{Pr(x_1)Pr(x_2|x_1)Pr(x_3|x_1)\int Pr(x_4|x_2,x_3)dx_4}{Pr(x_1)Pr(x_3|x_1)\int Pr(x_2|x_1)\int Pr(x_4|x_2,x_3)dx_4dx_2} \\
&= \frac{Pr(x_2|x_1)\int Pr(x_4|x_2,x_3)dx_4}{\int Pr(x_2|x_1)\int Pr(x_4|x_2,x_3)dx_4dx_2} \\
&= \frac{Pr(x_2|x_1)}{\int Pr(x_2|x_1)dx_2} \\
&= Pr(x_2|x_1)
\end{aligned}
$$

10.8

Draw the factor graphs corresponding to the graphical models in Figures 10.7 and 10.8. You must first establish the factorized joint distribution associated with each graph.

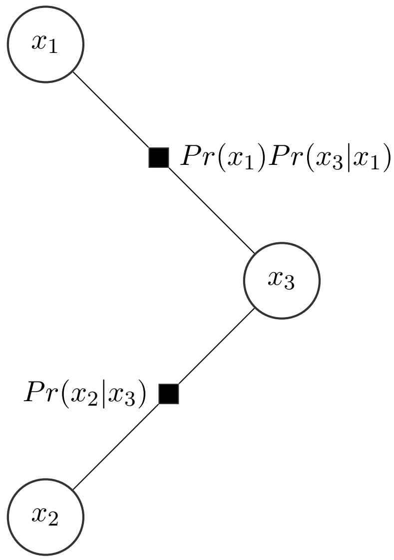


Figure 10.7a

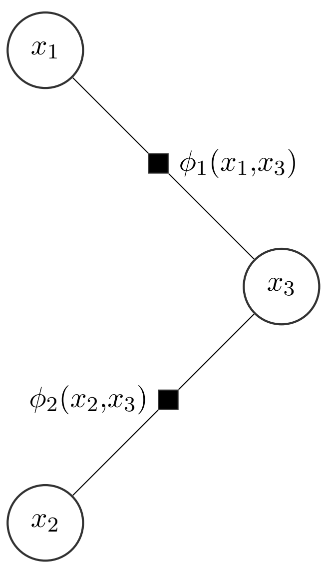


Figure 10.7b

The third graph is a little more tricky. The factorization is $Pr(x_1, x_2, x_3) = Pr(x_1)Pr(x_2)Pr(x_3|x_1, x_2)$. There is no 'nice' way to split the 3rd term so we need to create a 3-way factor.

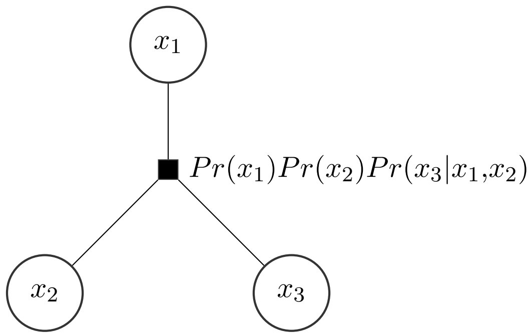


Figure 10.7c

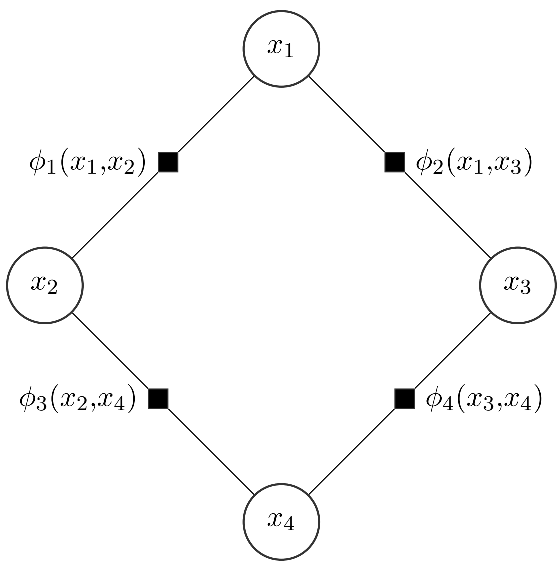


Figure 10.8a

Again it's helpful to write out the full joint probability $Pr(x_1, x_2, x_3, x_4) = Pr(x_1)Pr(x_2|x_1)Pr(x_3, x_1)Pr(x_4|x_2, x_3)$. Once more the last term forces us to have a 3-way factor node. Also we now have options of where to put $Pr(x_1)$. We can put it in the factor node for x_1, x_2 or in the node for x_1, x_3 or we could simply have a a single node for $Pr(x_1)$ hanging off x_1. Here I have decided on the first option.

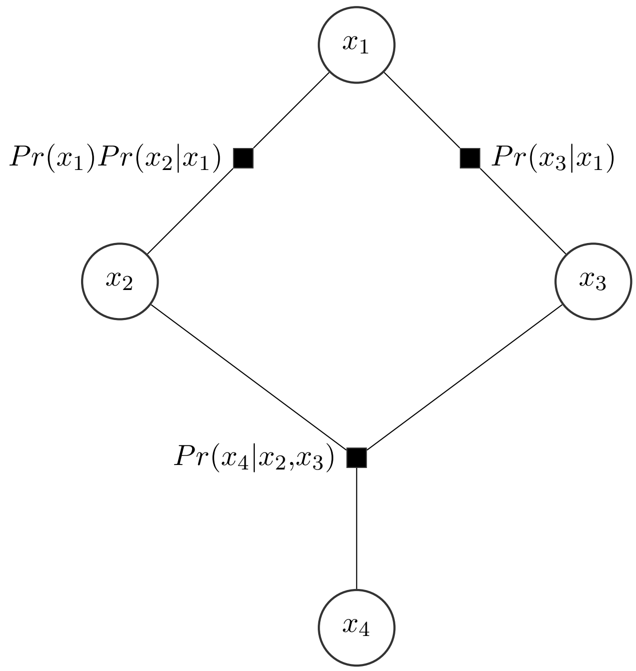


Figure 10.8a

10.9

What is the Markov blanket of variable w_2 in Figure 10.9c?

$$blanket[w_2] = \{w_1, w_5, x_2, w_4\}$$

10.10

What is the Markov blanket of variable w_8 in Figure 10.9e?

We have to combine the definitions of Markov blanket for directed and undirected graphical models. The only directed child, parent or parent-of-child for w_8 is x_8. The undirected neighbours of w_8 are $w_5, w_7, w_9, w_1 1$.

$$blanket[w_8] = \{x_8, w_5, w_7, w_9, w_1 1\}$$