

# INTERSECTION MULTIPLICITIES OF NOETHERIAN FUNCTIONS

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**ABSTRACT.** We provide a partial answer to the following problem: *give an effective upper bound on the multiplicity of non-isolated common zero of a tuple of Noetherian functions*. More precisely, consider a foliation defined by two commuting polynomial vector fields  $V_1, V_2$  in  $\mathbb{C}^n$ , and  $p$  a nonsingular point of the foliation. Denote by  $\mathcal{L}$  the leaf passing through  $p$ , and let  $F, G \in \mathbb{C}[X]$  be two polynomials. Assume that  $F|_{\mathcal{L}} = 0, G|_{\mathcal{L}} = 0$  have several common branches. We provide an effective procedure which allows to bound from above multiplicity of intersection of remaining branches of  $F|_{\mathcal{L}} = 0$  with  $G|_{\mathcal{L}} = 0$  in terms of the degrees and dimensions only.

## 1. INTRODUCTION

Let  $\mathcal{F}$  be the foliation generated in  $X = \mathbb{C}^n$  by several commuting polynomial vector fields  $V_1, \dots, V_k$ . A restriction of a polynomial  $F \in \mathbb{C}[X]$  to a leaf of  $\mathcal{F}$  is called a *Noetherian function*. The main result of this paper is motivated by the following question: *is it possible to effectively bound the topological complexity of objects defined by Noetherian functions solely in terms of discrete parameters of the defining functions (i.e. dimension of spaces, number and degrees of vector fields and of polynomials)?*

An important subclass of the class of Noetherian functions is that of Pfaffian functions. The theory of Fewnomials developed by Khovanskii provides effective upper bounds for global topological invariants (e.g. Betti numbers) of *real* varieties defined by Pfaffian equations (see [6]). Evidently, for complex varieties such bounds are impossible: the only holomorphic functions which admit “finite complexity” (e.g., finitely many zeros) in the entire complex domain are the polynomials. One may trace the dichotomy between the real and complex settings to the absence of a complex analogue of the Rolle theorem (which is a cornerstone of the real Fewnomial theory) — the derivative of a holomorphic function with many zeros may have no zeros at all.

While the global Rolle theorem fails in the complex setting, certain local analogs still hold. Perhaps the simplest of these analogs is the statement that if a derivative  $f'$  admits a zero of multiplicity  $n$  at some point, then  $f$  may admit a zero of multiplicity at most  $n+1$  at the same point. This trivial claim, and ramifications thereof, can be used to build local theory of complex Pfaffian sets, and to provide effective estimates on the *local* complexity of complex Pfaffian sets [3].

The topology of global real Noetherian sets is usually infinite, as demonstrated by the simple Noetherian (but non-Pfaffian) function  $\sin(x)$ , which admits infinitely many zeros. However, a long standing conjecture due to Khovanskii claims that the complexity of the local topology of such functions can be estimated through the discrete parameters of the set. This conjecture is motivated as follows: in view of

Morse theory, to estimate local Betti numbers it is essentially sufficient to bound the number of critical points of Noetherian functions on germs of Noetherian sets. The latter can be bounded through a suitably defined multiplicity of a common zero of a suitable tuple of Noetherian functions. In [5] the multiplicity of an *isolated* common zero is bounded from above. To build the general theory one has to generalize this result to non-isolated intersections.

For non-isolated intersections even the notion of multiplicity becomes non-trivial. For a point  $p$  lying on a leaf  $\mathcal{L}_0$  of the foliation  $\mathcal{F}$  one can define the multiplicity of the common zero  $p$  of the functions  $F_i|_{\mathcal{L}_0}$  as the number of common isolated zeros of  $F_i|_{\mathcal{L}}$  on neighboring leaves  $\mathcal{L}$  converging to  $p$  as  $\mathcal{L} \rightarrow \mathcal{L}_0$ . The multiplicity defined in this manner is not intrinsic to the leaf  $\mathcal{L}$ . It depends on the foliation  $\mathcal{F}$  in which  $\mathcal{L}$  is embedded.

In this paper we restrict attention to the case of foliations with two-dimensional leaves. In this context, we consider another notion of non-isolated multiplicity, suggested by Gabrielov, which is defined intrinsically on the leaf being considered (see subsection 4.2). We prove that this multiplicity can be explicitly bounded in terms of the dimension  $n$  and the degrees of the vector fields defining the foliation.

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## 2. SETUP AND NOTATION

Let  $\mathcal{F}$  be the foliation generated in  $X = \mathbb{C}^n$  by two commuting polynomial vector fields  $V_1, V_2$ . Given two polynomial functions  $F, G$ , we denote by  $\{F, G\}$  the Poisson bracket of  $F$  and  $G$  with respect to the leafs of the foliation,

$$(1) \quad \{f, g\} = V_1(f)V_2(g) - V_2(f)V_1(g)$$

In the entire paper  $\mathcal{L}$  denotes some particular fixed leaf of the foliation, and  $p \in \mathcal{L}$  a particular fixed smooth point of  $\mathcal{L}$ . We will denote functions defined on  $X$  using capital letters, and functions defined on  $\mathcal{L}$  by small letters. We will also denote ideals of functions on  $X$  by capital letter, and ideals of functions on  $\mathcal{L}$  by calligraphic letters.

We denote by  $\mathcal{O}(X)$  the ring of polynomial functions on  $X$ , and by  $\mathcal{O}_p(\mathcal{L})$  the ring of germs of analytic functions on  $\mathcal{L}$  at  $p$ . Given an ideal  $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$  we denote by  $\text{mult}_p \mathcal{I}$  the dimension  $\dim_{\mathbb{C}} \mathcal{O}_p(\mathcal{L})/\mathcal{I}$ . We also denote by  $V(I)$  the variety associated to  $I$ .

## 3. NOETHERIAN PAIRS AND CONTROLLABLE INCLUSIONS

In studying the restriction of algebraic functions to the leafs of a foliation we shall often find it necessary to simultaneously keep track of functions defined locally on a particular leaf, and their algebraic counterparts defined globally. In this section we introduce notation and terminology to facilitate the manipulation of such data.

**Definition 1.** *A pair of ideals  $(I, \mathcal{J})$  with  $I \subset \mathcal{O}(X), \mathcal{J} \subset \mathcal{O}_p(\mathcal{L})$  is called a noetherian pair for the leaf  $\mathcal{L}$  if  $I|_{\mathcal{L}} \subset \mathcal{J}$ .*

By an inclusion of pairs  $(I, \mathcal{J}) \subset (J, \mathcal{J})$  we mean simply that  $I \subset J, \mathcal{J} \subset \mathcal{J}$ . The pair  $(J, \mathcal{J})$  is said to *extend* the pair  $(I, \mathcal{J})$ .

We will introduce a number of operations which generate for a given pair  $(I, \mathcal{J})$  an extension  $(J, \mathcal{J})$ . The goal is to form an extension in such a way that the multiplicity of  $\mathcal{J}$  can be estimated from that of  $\mathcal{J}$ . More precisely we introduce the following notion.

**Definition 2.** An inclusion  $(I, \mathcal{J}) \subset (J, \mathcal{J})$  is said to be controllable if:

- The complexity of  $J$  can be bounded in terms of the complexity of  $I$ .
- The multiplicity of  $\mathcal{J}$  can be bounded in terms of the multiplicity of  $\mathcal{J}$ .

The precise estimates on the complexity and the multiplicity in the definition above will vary for the various types of inclusions we form.

We record two immediate consequences of this definition.

**Proposition 3.** If each inclusion in the sequence

$$(2) \quad (I_0, \mathcal{J}_0) \subset \cdots \subset (I_k, \mathcal{J}_k)$$

is controllable, then the inclusion  $(I_0, \mathcal{J}_0) \subset (I_k, \mathcal{J}_k)$  is controllable.

**Proposition 4.** Suppose  $(I, \mathcal{J}) \subset (J, \mathcal{J})$  is an controllable inclusion, and  $p \notin V(J)$ . Then one can give an upper bound for  $\text{mult}_p \mathcal{J}$ .

*Proof.* By definition, one can give an upper bound for  $\text{mult}_p(\mathcal{J})$  in terms of  $\text{mult}_p(\mathcal{J})$ . But  $J \subset \mathcal{J}$  and  $p \notin V(J)$ , so  $\text{mult}_p \mathcal{J} = 0$  and the proposition follows.  $\square$

These two proposition lay out the general philosophy of this paper. We start with a pair  $(I, \mathcal{J})$  and attempt, by forming a sequence of controllable inclusions, to reach a pair  $(J, \mathcal{J})$  with  $V(J)$  as small as possible. If we manage to get a pair with  $p \notin V(J)$  then we obtain an upper bound for  $\text{mult}_p \mathcal{J}$ .

We now introduce the three controllable inclusions which will be used in this paper.

**3.1. Radical extension.** Given a pair  $(I, \mathcal{J})$  we define a new pair  $(J, \mathcal{J})$  by letting  $J = \sqrt{I}$  and  $\mathcal{J} = \langle \mathcal{J}, J|_{\mathcal{L}} \rangle$ . The complexity of the ideal  $J$  can be estimated from that of  $I$  using effective radical extraction algorithms. The multiplicity of  $\mathcal{J}$  can be estimated from that of  $\mathcal{J}$  by combining the effective Nullstellensatz with the following simple lemma.

**Lemma 5.** Let  $\mathcal{K} \subset \mathcal{O}_p(\mathcal{L})$  be an ideal of finite multiplicity, and suppose  $f^n \in \mathcal{K}$ . Then

$$(3) \quad \text{mult } \mathcal{K} \leq n \text{ mult } \langle \mathcal{K}, f \rangle$$

*Proof.* Let  $\{h_1, \dots, h_k\}$  generate the local algebra  $\mathcal{O}_p(\mathcal{L})/\langle \mathcal{K}, f \rangle$ . Then a simple computation shows that  $\{h_i f^j\}_{1 \leq i \leq k, 0 \leq j \leq n-1}$  generate  $\mathcal{O}_p(\mathcal{L})/\mathcal{K}$ .  $\square$

More generally, let  $\mathcal{K} \subset \mathcal{O}_p(\mathbb{C}^m)$  be an ideal of finite multiplicity in the ring of germs at zero of holomorphic functions on  $\mathbb{C}^m$ .

**Lemma 6.** Assume that  $\mathcal{K}' \supset \mathcal{K}^n$ . Then

$$(4) \quad \text{mult } \mathcal{K}' \leq n^m \text{ mult } \mathcal{K}$$

*Proof.* For monomial ideals, the claim is a trivial combinatorial statement. To prove claim in the general case, let  $\text{LT}(K)$  denote the ideal of leading terms of  $K$  with respect to an arbitrary monomial ordering. It is well known that

$$(5) \quad \text{mult } K = \text{mult } \text{LT}(K),$$

and that

$$(6) \quad \text{LT}(K)^n \subset \text{LT}(K^n),$$

see e.g. [2]. Thus the general claim follows from the case of monomial ideals.  $\square$

**3.2. Poisson extension.** Given a pair  $(I, \mathcal{J})$  and two functions  $F, G \in I$  we define a new pair  $(J, \mathcal{J})$  by letting  $J = \langle I, \{F, G\} \rangle$  and  $\mathcal{J} = \langle \mathcal{J}, \{F, G\} \rangle$ . Assume that complexity of  $F, G$  is known, e.g. they are linear combinations of generators of  $I$ . We claim that  $(J, \mathcal{J})$  is a controllable extension of  $(I, \mathcal{J})$ . The complexity of the ideal  $J$  can clearly be estimated from that of  $I$ .

It is well-known that for a map  $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  of finite multiplicity the Jacobian  $J(f)$  generates a one-dimensional ideal in the local algebra  $Q_f$ , see [1]. As a corollary, we conclude that the multiplicity of  $\mathcal{J}$  can be estimated from that of  $\mathcal{J}$ .

**Lemma 7.** *Let  $\mathcal{K} \subset \mathcal{O}_p(\mathcal{L})$  be an ideal of finite multiplicity, and suppose  $f, g \in \mathcal{K}$ . Then*

$$(7) \quad \text{mult } \mathcal{K} \leq \text{mult } \langle \mathcal{K}, \{f, g\} \rangle + 1$$

**3.3. Jacobian extension.** Let  $(I, \mathcal{J})$  be a pair and  $F \in I$  be of known complexity. Assume that  $I$  is a radical ideal and that all intersections of  $I$  and  $\mathcal{F}$  are non-isolated. Write  $F|_{\mathcal{L}} = fh$  where  $h$  consists of the factors of  $F|_{\mathcal{L}}$  that vanish on  $V(I)|_{\mathcal{L}}$  and  $f$  consists of the other factors. Finally assume that  $f \in \mathcal{J}$ .

Let  $h'$  denote the reduced form of  $h$ . Denote by  $k$  the minimal multiplicity of a factor of  $h$ . Let  $K$  be the maximal multiplicity of each factor of  $h$ , so that  $h$  divides  $(h')^K$ .  $K$  can be effectively bounded from above. Indeed, let us take a generic point  $q$  of  $h = 0$ , and let  $\ell$  be a generic linear function on  $X$  vanishing at  $q$ . Then  $K = \text{mult}_q \langle h, \ell \rangle$ , and multiplicity of this isolated intersection can be bounded from above either using the considerations above, or (with better bound) using the main result of [4].

Note that the same upper bound holds for the total number of branches of  $h$ , counted with multiplicities: one should take  $p$  instead of  $q$ .

Given the conditions above, we define a new pair  $(J, \mathcal{J})$  by letting

$$(8) \quad J = \langle I, \{V_1^{\alpha_1} V_2^{\alpha_2} F \mid \alpha_1 + \alpha_2 = k\} \rangle$$

$$(9) \quad \mathcal{J} = \langle \mathcal{J}, h' \rangle.$$

We claim that  $(J, \mathcal{J})$  is a controllable extension of  $(I, \mathcal{J})$ .

First, to prove that this is a pair it suffices to show that all derivatives of order  $k$  of  $F|_{\mathcal{L}}$  belong to  $\mathcal{J}$ . This follows from the Leibnitz rule,

$$(10) \quad \begin{aligned} V_1^{\alpha_1} V_2^{\alpha_2} F|_{\mathcal{L}} &= V_1^{\alpha_1} V_2^{\alpha_2} (fh) = \\ &= (\text{derivatives of order } \leq k-1 \text{ of } h) \cdot (\dots) + (\dots) f \end{aligned}$$

since  $f \in \mathcal{J} \subset \mathcal{J}$  and derivatives of order  $\leq k-1$  of  $h$  are divisible by  $h' \in \mathcal{J}$ .

It is clear that the complexity of  $J$  can be estimated in terms of the complexity of  $I$ . It remains to show that the multiplicity of  $\mathcal{J}$  can be estimated in terms of the multiplicity of  $\mathcal{J}$ . For this it will suffice to show that for some explicit number  $N$ , we have  $(h')^N \in \mathcal{J}$ . Since  $(h')^K$  is divisible by  $h$ , it will suffice to find  $N$  such that  $h^N \in \mathcal{J}$ . We in fact have a stronger statement. Lemma 10 from the Appendix shows that there exists a global analytic function  $H$  such that  $V(I) \subset V(H)$  and  $H|_{\mathcal{L}}$  divides  $= h^n$  for  $n = 2^K$ . Since  $I$  is radical,  $H \in I_{\text{an}}$  and

certainly  $h^n = H|_{\mathcal{L}} \in I|_{\mathcal{L}} \subset \mathcal{J}$  ( $I_{\text{an}}$  is the ideal generated by  $I$  in the ring of germs at  $p$  of functions holomorphic near  $p$ ).

To end this section we remark that crucially,  $V(J) \subsetneq V(I)$ . Indeed, the derivatives of order  $k$  of  $F$  added to  $I$  insure that the lowest order factors of  $h$  are not contained in  $V(J)$ .

#### 4. INTERSECTION MULTIPLICITIES

In this section we study the intersection multiplicities of the restrictions of polynomial functions to the leaf  $\mathcal{L}$ . In the first subsection we consider the case of isolated intersection multiplicities, and in the following subsection we extend the result to the case of non-isolated intersections.

**4.1. Isolated intersection multiplicities.** Let  $I \subset \mathcal{O}(X)$  be a polynomial ideal. We will be particularly interested in the case that  $I$  is generated by two polynomial functions  $F, G$ .

We will say that the intersection of  $I$  (or  $V(I)$ ) and  $\mathcal{F}$  is isolated at a point  $p \in V(I)$  if there exists an open neighborhood  $U$  of  $p$  such that  $U \cap V(I) \cap \mathcal{L} = \{p\}$ , where  $\mathcal{L}$  is the leaf of  $\mathcal{F}$ . In this case, we are interested in bounding  $\text{mult}_p I|_{\mathcal{L}}$ .

**Proposition 8.** *Let  $I$  be a radical ideal and suppose that it has isolated intersections with  $\mathcal{F}$ . Then there exist  $F, G \in I$  of bounded complexity such that  $V(\langle I, \{F, G\} \rangle) \subsetneq V(I)$ .*

*Proof.* We may assume without loss of generality that  $V(I)$  is irreducible. Since the condition of having an isolated intersection is open, the generic point of  $V(I)$  is an isolated intersection.

We claim that at a generic point  $p$  of  $V(I)$ ,  $T_p V(I) \pitchfork \mathcal{F}$ . Indeed, otherwise the intersection of these spaces defines a line field whose integral trajectories lie in the intersection  $V(I) \cap \mathcal{L}$  where  $\mathcal{L}$  is the leaf containing  $p$ , in contradiction to the assumption that  $p$  is an isolated point of intersection.

The tangent space  $T_p V(I)$  is defined by differentials of functions in  $I$ . By transversality, there exist  $F, G \in I$  such that  $dF|_{\mathcal{L}}, dG|_{\mathcal{L}}$  are linearly independent at the point  $p$ . One can assume that  $F, G$  are linear combinations of generators of  $I$ . Thus  $\{F, G\}$  is non-vanishing at  $p$  and the claim is proved.  $\square$

**Corollary 9.** *For any pair  $(I, \mathcal{J})$  there exists a controllable inclusion  $(I, \mathcal{J}) \subset (J, \mathcal{J})$  such that  $J$  is radical and  $V(J)$  is the variety of non-isolated intersections between  $I$  and  $\mathcal{F}$ .*

*Proof.* We obtain this inclusion using Proposition 8 by forming an alternating sequence of controllable inclusions of radical and Poisson type. Each pair of inclusions in this sequence reduces the dimension of the set of isolated intersections, so after at most  $n$  steps the sequence stabilizes on the variety of non-isolated intersections.  $\square$

Suppose now that  $I = \langle F, G \rangle$  and  $p$  is an isolated intersection point of  $I$  and  $\mathcal{F}$ . Consider the pair  $(I, I|_{\mathcal{L}})$ . Applying Corollary 9 and Proposition 4 we immediately obtain an upper bound for  $\text{mult}_p I|_{\mathcal{L}}$ .

**4.2. Non-isolated intersection multiplicities.** We now consider the case of non-isolated intersection multiplicities. Let  $I = \langle F, G \rangle$  and suppose that the intersection is not isolated on  $\mathcal{L}$  at a point  $p$ .

Near  $p$  we may write

$$(11) \quad F|_{\mathcal{L}} = h_f f$$

$$(12) \quad G|_{\mathcal{L}} = h_g g$$

where  $h_f, h_g$  are the factors which are common to  $F|_{\mathcal{L}}$  and  $G|_{\mathcal{L}}$  (note that they may appear with different multiplicities). We stress that  $h_{f,g}, f, g$  are defined only on  $\mathcal{L}$ , and the decomposition of  $F$  and  $G$  as products of these factors need not extend outside of  $\mathcal{L}$ . We let

$$(13) \quad \mathcal{J} = \langle f, g \rangle.$$

**Theorem 1.** *The multiplicity  $\text{mult}_p \mathcal{J}$  can be bounded from above in terms of degrees of  $V_1, V_2, F, G$  and the dimension  $n$ .*

We begin in the same manner as in the isolated-intersection case.

Let  $(I_0, \mathcal{J}_0) = (I, \mathcal{J})$ . We apply Corollary 9 to obtain an controllable inclusion  $(I_0, \mathcal{J}_0) \subset (I_1, \mathcal{J}_1)$  such that  $I_1$  is radical and  $V(I_1)$  is the variety of non-isolated intersections between  $I$  and  $\mathcal{F}$ .

We are now in position to apply the Jacobian extension controllable inclusion. Indeed,  $V(I_1)$  is the locus of non-isolated intersections, and with  $h_1 = h_f$  we have:

- $V(I_1)|_{\mathcal{L}} = \{h_1 = 0\}$ .
- $F|_{\mathcal{L}} = h_1 f$ .
- $f \in \mathcal{J}_0 \subset \mathcal{J}_1$ .

We thus obtain an inclusion  $(I_1, \mathcal{J}_1) \subset (I_2, \mathcal{J}_2)$ , and as remarked at the end of subsection 3.3,  $V(I_2) \not\subseteq V(I_1)$ .

Applying Corollary 9 again, we obtain an controllable inclusion  $(I_2, \mathcal{J}_2) \subset (I_3, \mathcal{J}_3)$  such that  $I_3$  is radical and  $V(I_3)$  is the variety of non-isolated intersections between  $I_2$  and  $\mathcal{F}$ .

We are now again in position to apply the Jacobian extension controllable inclusion. Indeed,  $V(I_3)$  has only non-isolated intersections with  $\mathcal{F}$ , and letting  $h_3$  denote the factors of  $h_f$  which remain in  $V(I_3)|_{\mathcal{L}}$  we have:

- $V(I_3)|_{\mathcal{L}} = \{h_3 = 0\}$ .
- $F|_{\mathcal{L}} = h_3 f \cdot (h_f/h_3)$ .
- $f \cdot (h_f/h_3) \in \mathcal{J}_0 \subset \mathcal{J}_1$ .

We thus obtain an inclusion  $(I_3, \mathcal{J}_3) \subset (I_4, \mathcal{J}_4)$ , and as remarked at the end of subsection 3.3,  $V(I_4) \not\subseteq V(I_3)$ .

Repeating these two alternating types of controllable inclusions and applying Proposition 3 we obtain an inclusion  $(I_0, \mathcal{J}_0) \subset (I_{2k+1}, \mathcal{J}_{2k+1})$  where at least  $k$  factors of  $h_f$  have been removed from  $V(I_{2k+1})$ . In particular, for  $K$  equal to the number of factors of  $h_f$ , we know that  $p \notin V(I_{2K+1})$ . Thus, applying Proposition 4 we obtain the required upper bound for  $\text{mult}_p \mathcal{J}_0 = \text{mult}_p \mathcal{J}$ .

## 5. APPENDIX

In this appendix we prove the lemma used in subsection 3.3 which may be of some independent interest.

**Lemma 10.** *Let  $I \subset \mathcal{O}(X)$  be an ideal and  $F \in I$ , and suppose that every intersection of  $I$  and  $\mathcal{F}$  is non-isolated. Write  $F|_{\mathcal{L}} = fh$  where  $h$  consists of the factors of  $F|_{\mathcal{L}}$  that vanish on  $V(I)|_{\mathcal{L}}$  and  $f$  consists of the other factors. Finally denote by  $\mu$  the multiplicity of  $h$  at  $p$ .*

*Then there exists a function  $H \in \mathcal{O}_{an}(X)$  such that  $H|_{\mathcal{L}}$  divides  $h^{2^{\mu}}$  and  $V(I) \subset V(H)$ .*

*Proof.* Let  $(x, y)$  denote a system of coordinates on  $\mathcal{L}$ , and  $(z)$  denote the coordinates parameterizing the leafs, with  $p$  being the origin and  $\mathcal{L} = \{z = 0\}$ . Possibly making a linear change of coordinates in  $(x, y)$ , we may assume that the projection  $\pi : (x, y, z) \rightarrow (x, z)$  restricted to  $\{F = 0\}$  defines a *ramified* covering map in a neighborhood of the origin.

Fix an annulus  $A_x$  around the origin in  $(x)$  and some arbitrary point  $x_0 \in A_x$ , and a sufficiently small disc  $D_y$  in  $(y)$ . Then the fibre of  $\pi$  restricted to  $\{F = 0\} \cap \mathcal{L}$  is a discrete set  $B$  parametrizing the branches of  $F$  on  $\mathcal{L}$ . In case  $F$  has repeated factors, we view  $B$  as a multiset with the appropriate multiplicities. Finally we write  $B$  as the disjoint union of branches corresponding to  $h$  and  $f$ ,  $B = B_h \amalg B_f$ .

On  $\mathcal{L}$ , one may express the branches of  $\{F = 0\}$  as ramified functions  $y_b(x)$ ,  $b \in B$  defined on  $A_x$ . Furthermore, for  $z$  in a sufficiently small polydisc  $D_z$ , these functions extends as holomorphic functions  $y_b(x, z)$ , possibly ramified over a ramification locus  $\Sigma_z \subset D_z$ . We note that since the different branches of  $F$  remain far apart in  $A_x$  (on  $\mathcal{L}$ , and hence also for sufficiently small  $z$ ), the only ramification in  $z$  occurs when  $B$  is a multi-set. In this case, several branches corresponding to the same branch on  $\mathcal{L}$  may be permuted by the  $z$ -monodromy.

We will call a set  $S \subset B$  monodromic if it is invariant as a multi-set under the monodromy of  $A_x$  on  $\mathcal{L}$ , in the sense that after applying the monodromy, each branch appears with the same multiplicity as it originally did. We stress that this is only a condition on  $\mathcal{L}$ , and does not imply that the corresponding set  $\{y_b : b \in S\}$  is monodromic for non-zero values of  $z$ . For  $z \neq 0$ , the  $A_x$  monodromy may replace one branch by another, as long as the two branches correspond to the same branch on  $\mathcal{L}$ .

Assume now that  $S$  is monodromic and  $S \subset B_h$ , and consider the function

$$(14) \quad F_S(x, y, z) = \prod_{y_b : b \in S} (y - y_b(x, z)).$$

On  $\mathcal{L}$ , the functions  $y_b$  actually extend from  $A_x$  to the punctured disc of the same radius, as the only ramification point occurs at the origin. Furthermore, since  $S$  is assumed to be monodromic on  $\mathcal{L}$ , it follows that  $F_S|_{\mathcal{L}}$  is univalued and holomorphic outside the origin. Since it is bounded, it extends holomorphically at the origin well. Since  $S \subset B_h$ , we see that  $F_S|_{\mathcal{L}}$  divides  $h$ .

On nearby leafs, the set  $S$  need no longer be monodromic even in  $A_x$ , and the function  $F_S$  is not necessarily univalued. However, for each fixed value of  $z$  it is a well-defined function in  $A_x$ . We develop  $F_S$  as a Puiseux series in  $A_x$  and let  $F'_S$  denote the holomorphic part obtained by removing all negative and fractional terms. The result is a holomorphic function in  $(x, y, z)$ , possibly ramified over  $\Sigma_z$  but unramified in  $(x, y)$ .

On nearby leafs, the functions  $y_b$  may exhibit several ramification points (the ramification at the origin on  $\mathcal{L}$  may bifurcate into several ramification points). But suppose that for a certain nearby leaf  $\mathcal{L}_{z_0}$  the set  $S$  happens to be monodromic (as

a multiset) with respect to the full monodromy in the  $(x)$  variable. Then, arguing just as we did for the leaf  $\mathcal{L}$ , we deduce that  $F_S$  is in fact holomorphic on  $\mathcal{L}_{z_0}$ . Hence for such leafs we have  $F_S|_{\mathcal{L}_{z_0}} \equiv F'_S|_{\mathcal{L}_{z_0}}$ .

Now define the function  $H$  as follows,

$$(15) \quad H = \prod_{S \subset B_h \text{ monodromic}} F'_S.$$

We claim that  $H$  is in fact an unramified holomorphic function. Indeed, we have already seen that each of the factors  $F'_S$  is holomorphic in  $(x, y)$ . Furthermore, the monodromy in the  $z$  variables permutes the set  $\{S : S \subset B_h \text{ monodromic}\}$ , and hence only permutes the factors in the product defining  $H$ . Thus  $H$  is univalued, and since it is also bounded near its singular locus, it extends holomorphically there as well.

We claim that  $H$  vanishes on the  $V(I)$  (at least in a sufficiently small neighborhood of the origin). Indeed, consider some fixed value of  $z_0$  of  $z$ . By assumption, all intersections of  $V(I)$  with  $\mathcal{L}_{z_0}$  are non-isolated, so  $V(I)|_{\mathcal{L}_{z_0}}$  necessarily consists of some set  $S_{z_0}$  of branches  $y_b$ . This set, being the set of branches of an analytic set on  $\mathcal{L}_{z_0}$ , is necessarily monodromic. This implies that it is monodromic on  $\mathcal{L}$  as a multiset as well.

We claim that for sufficiently small  $z_0$ ,  $S_{z_0} \subset B_h$ . Otherwise  $V(I)|_{\mathcal{L}_{z_0}}$  contains branches from  $B_f$  for arbitrarily small valued of  $z_0$ , and since  $V(I)$  is closed we see that  $V(I)|_{\mathcal{L}}$  contains branches of  $f = 0$ , in contradiciton to the assumptions of the lemma.

To conclude,  $S$  is monodromic and  $S \subset B_h$ , and hence as we have seen  $F_S|_{\mathcal{L}_{z_0}} \equiv F'_S|_{\mathcal{L}_{z_0}}$ . Since  $F_S$  vanishes on  $V(I)|_{\mathcal{L}_{z_0}}$  by definition, and  $F'_S$  is a factor of  $H$ , we deduce that  $H$  vanishes on  $V(I)|_{\mathcal{L}_{z_0}}$ . Since this is true for any sufficiently small  $z_0$ , the claim is proved.

Finally, on  $\mathcal{L}$  we have shown that for monodromic  $S \subset B_h$ ,  $F'_S \equiv F_S$  and  $F_S|_{\mathcal{L}}$  divides  $h$ . Since  $H$  is the product over at most  $2^\mu$  sets of the factors  $F'_S$ , we deduce that  $H|_{\mathcal{L}}$  divides  $h^{2^\mu}$ , concluding the proof of the lemma.  $\square$

## REFERENCES

- [1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps*, Vol. I. The classification of critical points, caustics and wave fronts. Translated from the Russian by Ian Porteous and Mark Reynolds. Monographs in Mathematics, **82**. Birkhäuser Boston, Inc., Boston, MA, 1985. xi+382 pp.
- [2] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- [3] A. Gabrielov, *Multiplicities of Pfaffian intersections, and the ojasiewicz inequality*. Selecta Math. (N.S.) **1** (1995), no. 1, p. 113–127.
- [4] A. Gabrielov, *Multiplicity of a zero of an analytic function on a trajectory of a vector field*. The Arnoldfest (Toronto, ON, 1997), 191–200, Fields Inst. Commun., 24, Amer. Math. Soc., Providence, RI, 1999.
- [5] A. Gabrielov, A. G. Khovanskii, *Multiplicity of a Noetherian intersection*. Geometry of differential equations, 119–130, Amer. Math. Soc. Transl. Ser. 2, 186, Amer. Math. Soc., Providence, RI, 1998.
- [6] A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs, **vol. 88**, AMS, Providence, RI, 1991, 139 pp.