

ARTICLE TYPE

On Controller Design for Systems on Manifolds in Euclidean Space

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Summary

A new method is developed to design controllers in Euclidean space for systems defined on manifolds. The idea is to embed the state-space manifold M of a given control system into some Euclidean space \mathbb{R}^n , extend the system from M to the ambient space \mathbb{R}^n , and modify it outside M to add transversal stability to M in the final dynamics in \mathbb{R}^n . Controllers are designed for the final system in the ambient space \mathbb{R}^n . Then, their restriction to M produces controllers for the original system on M . This method has the merit that only one single global Cartesian coordinate system in the ambient space \mathbb{R}^n is used for controller synthesis, and any controller design method in \mathbb{R}^n , such as the linearization method, can be globally applied for the controller synthesis. The proposed method is successfully applied to the tracking problem for the following two benchmark systems: the fully actuated rigid body system and the quadcopter drone system.

KEYWORDS:

embedding, tracking, manifold, drone, quadcopter, rigid body

1 | INTRODUCTION

Many control systems are defined on manifolds that are not homeomorphic to Euclidean space, where we use the term ‘Euclidean space’ to mean some \mathbb{R}^n space, not imposing any metric on it. The geometric, or coordinate-free, approach has been developed to deal with those systems without being dependent on the choice of coordinates.^{1,4,23} However, a state-space manifold often appears as an embedded manifold in Euclidean space and the control system naturally extends from the manifold to the ambient Euclidean space: one example is the free rigid body system on $\text{SO}(3) \times \mathbb{R}^3$ which naturally extends to $\mathbb{R}^{3 \times 3} \times \mathbb{R}^3$. In such a case, it might be advantageous to use one single global Cartesian coordinate system in the ambient Euclidean space to design controllers for the original system on the manifold, eliminating the necessity to use rather complex tools from differential geometry or multiple local coordinate systems. For example, in the case of the free rigid body system, neither adding nor subtracting two rotation matrices is allowed in the geometric approach partly because the result does not lie on $\text{SO}(3)$, which may be mathematically orthodox, but would discourage control engineers from understanding or applying the geometric results. Since any two rotation matrices, as 3×3 matrices, can be conveniently added or subtracted in $\mathbb{R}^{3 \times 3}$, there is no reason to refrain from carrying out such basic and convenient operations as additions and subtractions. Moreover, since one can utilize one single global Cartesian coordinate system in the ambient Euclidean space $\mathbb{R}^{3 \times 3}$, he is free from such discontinuities as those that often occur due to the switching of local coordinate systems and chart-wise designed control laws. As such, in this paper we propose a new method that is an alternative to both the geometric approach, which adheres to differential geometric tools, and the classical approach, which employs local coordinates such as Euler angles for rigid bodies.

A brief summary of the proposed method is provided as follows. Given a control system Σ_M whose dynamics evolve on a manifold M , we embed M into some Euclidean space \mathbb{R}^n and extend the system Σ_M to a system $\Sigma_{\mathbb{R}^n}$ whose dynamics evolve

in \mathbb{R}^n or conservatively in a neighborhood of M in \mathbb{R}^n . We then legitimately modify the extended system $\Sigma_{\mathbb{R}^n}$ outside M to add transversal stability to M while the original dynamics on M are kept intact. It follows that M becomes an attractive invariant manifold of the resulting system denoted $\tilde{\Sigma}_{\mathbb{R}^n}$. We apply any controller design method available in Euclidean space to design controllers for $\tilde{\Sigma}_{\mathbb{R}^n}$ in \mathbb{R}^n for stabilization of a point on M or tracking of a reference trajectory on M , and then restrict the controllers to M which yield controllers for the original system Σ_M on M for the stabilization or tracking on M . To showcase this method, the linearization technique in \mathbb{R}^n is chosen in this paper to design tracking controllers although we could alternatively apply other techniques available in \mathbb{R}^n such as homogeneous approximation,¹⁰ model predictive control,³ iterative learning control,²⁴ differential flatness,¹² etc.

The theory of embedding of manifolds in Euclidean space has a long history in mathematics, including several famous theorems such as the Nash embedding theorems^{18,19} and the Whitney embedding theorem.² The embedding technique has been also applied in control theory. For example, it was used to produce a simple proof of the Pontryagin maximum principle on manifolds,⁵ and was combined with the transversal stabilization technique to yield feedback-based structure-preserving numerical integrators for simulation of dynamical systems.⁶ A series of relevant works have been made by Maggiore and his collaborators on local transverse feedback linearizability of control-invariant submanifolds and virtual holonomic constraints.^{17,20,21} The focus of Maggiore is placed on creation of a submanifold for a given system and its transversal stabilization via feedback for *path-following* controller synthesis, whereas our work in this paper is focused on embedding and extending a state space manifold of a given system into Euclidean space and its transversal stabilization for *tracking controller* synthesis. Moreover, our method has the merit to use one single global Euclidean coordinate system whereas the method by Maggiore does not. Another merit of our method is its openness to accommodate any existing control method developed in Euclidean space.

The paper is organized as follows. Section 2 is devoted to embedding into Euclidean space, transversal stabilization, tracking controller design via linearization, and their application to the rigid body system and the quadcopter drone system. Several tracking controllers are proposed for the two systems, and the exponential convergence of their tracking error dynamics is rigorously proven and numerical simulations are carried out to demonstrate the controllers' good tracking ability and robustness to unknown disturbances. The paper is concluded in Section 3. The contributions of the paper are summarized as follows: 1. the development of a new controller design methodology with the embedding and transversal stabilization technique which allows to convert difficult control problems on a manifold to tractable control problem in Euclidean space and to use one single global Euclidean coordinate system in controller synthesis; and 2. the design of exponentially tracking controllers with the developed method for the rigid body system and the quadcopter system which are designed via linearization in ambient Euclidean space but are still expressed geometrically, i.e. in a coordinate-free manner. It is noted that a presentation of preliminary results was given at the 56th IEEE Conference on Decision and Control.

2 | MAIN RESULTS

2.1 | Mathematical Preliminaries

The usual Euclidean inner product is exclusively used for vectors and matrices in this paper, i.e.

$$\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij} = \text{tr}(A^T B)$$

for any two matrices of equal size. The norm induced from this inner product, which is called the Frobenius or Euclidean norm, is exclusively used for vectors and matrices. Let Sym and Skew denote the symmetrization operator and the skew-symmetrization operator, respectively, on square matrices, which are defined by

$$\text{Sym}(A) = \frac{1}{2}(A + A^T), \quad \text{Skew}(A) = \frac{1}{2}(A - A^T)$$

for any square matrix A . Then,

$$A = \text{Sym}(A) + \text{Skew}(A), \quad \langle \text{Sym}(A), \text{Skew}(A) \rangle = 0.$$

Namely,

$$\mathbb{R}^{n \times n} = \text{Sym}(\mathbb{R}^{n \times n}) \oplus \text{Skew}(\mathbb{R}^{n \times n})$$

with respect to the Euclidean inner product. Let $[,]$ denote the usual matrix commutator that is defined by $[A, B] = AB - BA$ for any pair of square matrices A and B of equal size. It is easy to show that

$$\begin{aligned} [\text{Sym}(\mathbb{R}^{n \times n}), \text{Skew}(\mathbb{R}^{n \times n})] &\subset \text{Sym}(\mathbb{R}^{n \times n}), \\ [\text{Skew}(\mathbb{R}^{n \times n}), \text{Skew}(\mathbb{R}^{n \times n})] &\subset \text{Skew}(\mathbb{R}^{n \times n}), \\ [\text{Sym}(\mathbb{R}^{n \times n}), \text{Sym}(\mathbb{R}^{n \times n})] &\subset \text{Skew}(\mathbb{R}^{n \times n}). \end{aligned}$$

In other words, $[A, C] = [A, C]^T$ for all $A = A^T \in \mathbb{R}^{n \times n}$ and $C = -C^T \in \mathbb{R}^{n \times n}$; $[B, C] = -[B, C]^T$ for all $B = -B^T \in \mathbb{R}^{n \times n}$ and $C = -C^T \in \mathbb{R}^{n \times n}$; and $[B, C] = -[B, C]^T$ for all $B = B^T \in \mathbb{R}^{n \times n}$ and $C = C^T \in \mathbb{R}^{n \times n}$. Let $\text{SO}(3)$ denote the set of all 3×3 rotation matrices, which is defined as $\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R - I = 0, \det R > 0\}$. Let $\mathfrak{so}(3)$ denote the set of all 3×3 skew symmetric matrices, which is defined as $\mathfrak{so}(3) = \{A \in \mathbb{R}^{3 \times 3} \mid A^T + A = 0\}$. The hat map $\wedge : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined by

$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

for $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$. The inverse map of the hat map is called the vee map and denoted \vee such that $(\hat{\Omega})^\vee = \Omega$ for all $\Omega \in \mathbb{R}^3$ and $(A^\vee)^\wedge = A$ for all $A \in \mathfrak{so}(3)$.

Lemma 1. 1. $\langle RA, RB \rangle = \langle AR, BR \rangle = \langle A, B \rangle$ for all $R \in \text{SO}(3)$ and $A, B \in \mathbb{R}^{3 \times 3}$.

$$2. \max_{R_1, R_2 \in \text{SO}(3)} \|R_1 - R_2\| = 2\sqrt{2}.$$

$$3. \langle \hat{u}, \hat{v} \rangle = 2\langle u, v \rangle \text{ for all } u, v \in \mathbb{R}^3.$$

$$4. [\hat{u}, \hat{v}] = (u \times v)^\wedge \text{ and } \hat{u}\hat{v} = u \times v \text{ for all } u, v \in \mathbb{R}^3.$$

Given a function $f : A \rightarrow B$ and a subset C of B , the set $f^{-1}(C)$ is defined as $f^{-1}(C) = \{a \in A \mid f(a) \in C\}$. In particular, when C consists of a single point, say c , we just write $f^{-1}(c)$ to mean $f^{-1}(\{c\})$. Every function and manifold is assumed to be smooth in this paper unless stated otherwise. Stability, stabilization and tracking are all understood to be local unless globality is stated explicitly. The reader is referred to the book by Bloch¹ for more information on manifolds.

2.2 | Embedding in Euclidean Space and Transversal Stabilization

2.2.1 | Theory

Let M be an m -dimensional regular manifold in \mathbb{R}^n , where $m < n$. Consider a control system Σ_M on M given by

$$\Sigma_M : \dot{x} = X(x, u), \quad x \in M, u \in \mathbb{R}^k. \quad (1)$$

Notice that

$$X(x, u) \in T_x M \quad \forall x \in M, u \in \mathbb{R}^k, \quad (2)$$

where $T_x M$ denotes the tangent space to M at x . Suppose that there is a control system $\Sigma_{\mathbb{R}^n}$ on \mathbb{R}^n given by

$$\Sigma_{\mathbb{R}^n} : \dot{x} = X_e(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k, \quad (3)$$

that satisfies

$$X_e(x, u) = X(x, u) \quad \forall x \in M, u \in \mathbb{R}^k. \quad (4)$$

In other words, $\Sigma_{\mathbb{R}^n}$ is an extension of Σ_M to \mathbb{R}^n and Σ_M becomes a restriction of $\Sigma_{\mathbb{R}^n}$ to M . By (2) and (4), M is an invariant manifold of $\Sigma_{\mathbb{R}^n}$.

Suppose that there is a function $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$M = \tilde{V}^{-1}(0) \quad (5)$$

and

$$\nabla \tilde{V}(x) \cdot X_e(x, u) = 0 \quad (6)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$. With this function, construct a system $\tilde{\Sigma}_{\mathbb{R}^n}$ in \mathbb{R}^n as

$$\tilde{\Sigma}_{\mathbb{R}^n} : \dot{x} = \tilde{X}_e(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k, \quad (7)$$

where the vector field \tilde{X}_e is defined by

$$\tilde{X}_e(x, u) = X_e(x, u) - \nabla \tilde{V}(x) \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^k. \quad (8)$$

Since every point in M is a minimum point of V , $\nabla V(x)$ vanishes on M identically. Hence, by (4) and (8)

$$\tilde{X}_e(x, u) = X(x, u) \quad \forall x \in M, u \in \mathbb{R}^k. \quad (9)$$

In other words, the system $\tilde{\Sigma}_{\mathbb{R}^n}$ coincides with the original system Σ_M on M . Hence, M is an invariant manifold of $\tilde{\Sigma}_{\mathbb{R}^n}$ as well. Along any flow of $\tilde{\Sigma}_{\mathbb{R}^n}$

$$\frac{d}{dt}\tilde{V} = \nabla \tilde{V} \cdot (X_e - \nabla \tilde{V}) = -\|\nabla \tilde{V}\|^2 \leq 0 \quad (10)$$

by (6).

Theorem 1. If there are positive numbers b and r such that

$$b\tilde{V}(x) \leq \|\nabla \tilde{V}(x)\|^2 \quad (11)$$

for all $x \in \tilde{V}^{-1}([0, r)) \subset \mathbb{R}^n$, then $\tilde{V}^{-1}([0, r))$ is positively invariant for $\tilde{\Sigma}_{\mathbb{R}^n}$ and every flow of $\tilde{\Sigma}_{\mathbb{R}^n}$ starting in $\tilde{V}^{-1}([0, r))$ converges to M as $t \rightarrow \infty$. In particular, $\tilde{V}(x(t)) \leq \tilde{V}(x(0))e^{-bt}$ for all $t \geq 0$ and $x(0) \in \tilde{V}^{-1}([0, r))$.

Proof. It follows from (10) and (11) that for any initial state $x(0) \in \tilde{V}^{-1}([0, r))$, $\tilde{V}(x(t)) \leq \tilde{V}(x(0))e^{-bt} < re^{-bt}$ for all $t \geq 0$, where $x(t)$ is the flow of $\tilde{\Sigma}_{\mathbb{R}^n}$ starting from $x(0)$. It implies that $\tilde{V}^{-1}([0, r))$ is a positively invariant set of $\tilde{\Sigma}_{\mathbb{R}^n}$ and that $\lim_{t \rightarrow \infty} \tilde{V}(x(t)) = 0$. From (5) and the continuity of \tilde{V} , it follows that $x(t)$ converges to M as $t \rightarrow \infty$. \square

The following corollary shows a typical situation in which to construct such a function \tilde{V} that satisfies (5), (6) and the hypothesis of Theorem 1.

Corollary 1. Suppose that there is a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$ such that $M = F^{-1}(0)$; that there is an open set $S \subset \mathbb{R}^n$ such that $M \subset S$ and every point in S is a regular point of F ; that $DF(x) \cdot X_e(x, u) = 0$ for all $(x, u) \in S \times \mathbb{R}^k$; and that there is a number $c > 0$ such that the smallest singular value of $\|DF(x)\|$ is larger than c for every $x \in S$. Suppose also that $\tilde{V}(x) = F(x)^T K F(x)$ is used to define the system $\tilde{\Sigma}_{\mathbb{R}^n}$ in (7) and (8), where K is an $(n-m) \times (n-m)$ positive definite symmetric matrix. Then, there is an open set W in \mathbb{R}^n with $M \subset W$ such that every trajectory of $\tilde{\Sigma}_{\mathbb{R}^n}$ starting in W remains in W for all future time and exponentially converges to M as $t \rightarrow \infty$.

Proof. Let $\tilde{V}(x) = F(x)^T K F(x)$, where K is an $(n-m) \times (n-m)$ positive definite symmetric matrix. Then, $\nabla \tilde{V}(x) = 2DF(x)^T K F(x)$ in column vector form. It is easy to show that this function \tilde{V} satisfies (5) and (6) for all $(x, u) \in S \times \mathbb{R}^k$. By hypothesis, $\|\nabla \tilde{V}(x)\| = \|2DF(x)^T K F(x)\| \geq 2c\|K F(x)\| \geq 2c\lambda_{\min}(K)\|F(x)\|$ for all $x \in S$. Hence, for any $x \in S$, $\|\tilde{V}(x)\| \leq \lambda_{\max}(K)\|F(x)\|^2 \leq (\lambda_{\max}(K)/4c^2\lambda_{\min}(K)^2)\|\nabla \tilde{V}(x)\|^2$. Let $b = 4c^2\lambda_{\min}(K)^2/\lambda_{\max}(K)$ and choose a number $r > 0$ such that $\tilde{V}^{-1}([0, r)) \subset S$ which is possible due to continuity of the function \tilde{V} . With these numbers b and r , the hypothesis of Theorem 1 holds true. Hence, by Theorem 1, $W := \tilde{V}^{-1}([0, r))$ is a positively invariant region of attraction for $\tilde{\Sigma}_{\mathbb{R}^n}$, and $\tilde{V}(x(t)) \leq \tilde{V}(x(0))e^{-bt}$ for all $x(0) \in W$ and $t \geq 0$. This inequality implies that

$$\|F(x(t))\| \leq A\|F(x(0))\|e^{-bt/2}$$

for all $x(0) \in W$ and all $t \geq 0$, where $A = \sqrt{\lambda_{\max}(K)/\lambda_{\min}(K)}$. Since every point of W is a regular point of F , $F(x)$ can be used as part of local coordinates such that $M = \{F(x) = 0\}$. Hence, the above inequality shows that the convergence of $x(t)$ to M is exponential. \square

Our goal is to design controllers for the system Σ_M whose dynamics evolve on the manifold M . Since the system $\tilde{\Sigma}_{\mathbb{R}^n}$ in \mathbb{R}^n coincides with Σ_M on M , and M is an invariant manifold of $\tilde{\Sigma}_{\mathbb{R}^n}$, we can first design controllers for $\tilde{\Sigma}_{\mathbb{R}^n}$ in one single global Cartesian coordinate system for \mathbb{R}^n and then restrict them to M to come up with controllers for the original system Σ_M . This method becomes much more tractable when M is an attractive invariant manifold of $\tilde{\Sigma}_{\mathbb{R}^n}$, which is guaranteed by the hypothesis in Theorem 1. Notice that the size of the region of attraction of M for the $\tilde{\Sigma}_{\mathbb{R}^n}$ dynamics is immaterial since the set $\mathbb{R}^n \setminus M$ is not a region of interest but only an auxiliary ambient region in which we take full advantage of the Euclidean structure of \mathbb{R}^n .

2.2.2 | Application to the Rigid Body System

As a main example throughout the paper, we use the free rigid body system with full actuation whose equations of motion are given by

$$\dot{R} = R\hat{\Omega}, \quad (12a)$$

$$\dot{\Omega} = \mathbb{I}^{-1}(\mathbb{I}\Omega \times \Omega) + \mathbb{I}^{-1}\tau, \quad (12b)$$

where $(R, \Omega) \in \text{SO}(3) \times \mathbb{R}^3 \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^3$ is the state vector consisting of a rotation matrix R and a body angular velocity vector Ω ; $\tau \in \mathbb{R}^3$ is the control torque; and \mathbb{I} is the moment of inertial matrix of the rigid body. From here on, we regard the system (12) as a system defined on $\mathbb{R}^{3 \times 3} \times \mathbb{R}^3$, treating R as a 3×3 matrix. It is then easy to verify that $\text{SO}(3) \times \mathbb{R}^3$ is an invariant set of (12), i.e. every flow starting in M remains in M for all $t \in \mathbb{R}$. Assume that the full state of the system is available, which allows us to apply the following controller

$$\tau = \mathbb{I}(u - \mathbb{I}^{-1}(\mathbb{I}\Omega \times \Omega)) \quad (13)$$

to transform the above system to

$$\dot{R} = R\hat{\Omega}, \quad (14a)$$

$$\dot{\Omega} = u, \quad (14b)$$

where u is the new control vector. Note that $\text{SO}(3) \times \mathbb{R}^3$ is an invariant set of (14). Let $\text{GL}^+(3) = \{R \in \mathbb{R}^{3 \times 3} \mid \det R > 0\}$ and define a function $\tilde{V} : \text{GL}^+(3) \times \mathbb{R}^3 \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}$ by

$$\tilde{V}(R, \Omega) = \frac{k_e}{4} \|R^T R - I\|^2, \quad (15)$$

where $k_e > 0$ is a constant. It is easy to verify that $\tilde{V}^{-1}(0) = \text{SO}(3) \times \mathbb{R}^3$ and

$$\nabla_R \tilde{V} = -k_e R(R^T R - I), \quad \nabla_\Omega \tilde{V} = 0. \quad (16)$$

With this function \tilde{V} , the modified rigid body system corresponding to (7) and (8) is computed as

$$\dot{R} = R\hat{\Omega} - k_e R(R^T R - I), \quad (17a)$$

$$\dot{\Omega} = u, \quad (17b)$$

where $(R, \Omega) \in \text{GL}^+(3) \times \mathbb{R}^3 \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^3$.

We now show that Theorem 1 holds in the rigid body case.

Lemma 2. There are numbers $b > 0$ and $r > 0$ such that

$$b\tilde{V}(R, \Omega) \leq \|\nabla \tilde{V}(R, \Omega)\|^2$$

for all $(R, \Omega) \in \tilde{V}^{-1}([0, r])$.

Proof. Define an auxiliary function $f : \text{GL}^+(3) \rightarrow \mathbb{R}_{\geq 0}$ by

$$f(R) = \frac{k_e}{4} \|R^T R - I\|^2$$

for $R \in \text{GL}^+(3)$. Take any sufficiently small $\epsilon > 0$ such that every $A \in \mathbb{R}^{3 \times 3}$ satisfying $\|A - I\| \leq \epsilon$ is invertible. Let $r = k_e \epsilon^2 / 4$. Then, if $R \in f^{-1}([0, r])$, $\|R^T R - I\| \leq \epsilon$, so $R^T R$ is invertible, which implies that R is also invertible. Hence, $f^{-1}([0, r]) \subset \text{GL}^+(3)$. For each $i = 1, 2, 3$ and any $R \in \mathbb{R}^{3 \times 3}$,

$$\sum_{j=1}^3 R_{ji}^2 = |(R^T R)_{ii}| \leq \|R^T R\|$$

which implies

$$\|R\|^2 = \sum_{i=1}^3 \sum_{j=1}^3 R_{ji}^2 \leq 3\|R^T R\| \quad (18)$$

for any $R \in \mathbb{R}^{3 \times 3}$. Hence for any $R \in f^{-1}([0, r])$,

$$\|R^T R\| \leq \|R^T R - I\| + \|I\| \leq \epsilon + 3,$$

which implies by (18) that $\|R\| \leq \sqrt{3\epsilon + 9}$ for all $R \in f^{-1}([0, r])$. It follows that $f^{-1}([0, r])$ is compact in $\mathbb{R}^{3 \times 3}$, being closed and bounded. Since the matrix inversion operation is continuous, the image of $f^{-1}([0, r])$ under matrix inversion is also compact. Hence, there is a number $M > 0$ such that $\|R^{-1}\| \leq M$ for all $R \in f^{-1}([0, r])$. Hence, for any $(R, \Omega) \in \tilde{V}^{-1}([0, r])$

$$\|R^T R - I\| = \|R^{-1} R(R^T R - I)\| \leq \|R^{-1}\| \|R(R^T R - I)\| \leq M \|R(R^T R - I)\|$$

which implies $b\tilde{V}(R, \Omega) \leq \|\nabla \tilde{V}\|$ for all $(R, \Omega) \in \tilde{V}^{-1}([0, r])$ by (15) and (16), where $b = 4k_e/M^2$. This completes the proof. \square

Theorem 2. There is a number $r > 0$ such that every trajectory of (17) starting in $\tilde{V}^{-1}([0, r))$ remains in $\tilde{V}^{-1}([0, r))$ for all future time and converges exponentially to $\text{SO}(3) \times \mathbb{R}^3$ as $t \rightarrow \infty$.

Proof. Pick such numbers b and r as in the statement of Lemma 2. By Lemma 2 and Theorem 1, every trajectory of (17) starting in $\tilde{V}^{-1}([0, r))$ remains in $\tilde{V}^{-1}([0, r))$ for all future time and converges to $\text{SO}(3) \times \mathbb{R}^3$ as $t \rightarrow \infty$. Let $(R(t), \Omega(t))$ be an arbitrary trajectory starting in $\tilde{V}^{-1}([0, r))$ at $t = 0$. Then, by Theorem 1, it satisfies

$$\|R^T(t)R(t) - I\| \leq \|R^T(0)R(0) - I\|e^{-bt/2}$$

for all $t \geq 0$. It follows that the convergence of $(R(t), \Omega(t))$ to $\text{SO}(3) \times \mathbb{R}^3$ is exponential since the 3×3 zero matrix is a regular value of the map $g : \text{GL}^+(3) \rightarrow \text{Sym}(\mathbb{R}^{3 \times 3})$ defined by $g(R) = R^T R - I$ such that $\text{SO}(3) = \{R \in \text{GL}^+(3) \mid g(R) = 0\}$; refer to pp.22–23 of Guillemin and Pollack⁹ to see why the zero matrix is a regular value of g . \square

Remark 1. The technique of embedding into ambient Euclidean space and transversal stabilization was successfully tested in creating feedback integrators for structure-preserving numerical integration⁶ of the dynamics of uncontrolled dynamical systems. This technique is extended to control systems in this paper. In particular, Theorem 1, Corollary 1, Lemma 2 and Theorem 2 in this paper are new and powerful so as to guarantee exponential stability of M in the transversal direction.

2.3 | Tracking via Linearization in Ambient Euclidean Space

2.3.1 | Theory

Consider again the system $\tilde{\Sigma}_{\mathbb{R}^n}$ given in (7) and its restriction Σ_M to M given in (1). Choose a reference trajectory $x_0 : [0, \infty) \rightarrow M$ for Σ_M on M driven by a control signal $u_0 : [0, \infty) \rightarrow \mathbb{R}^k$ so that

$$\dot{x}_0(t) = \tilde{X}(x_0(t), u_0(t)) \quad \forall t \geq 0.$$

We can then linearize the ambient system $\tilde{\Sigma}_{\mathbb{R}^n}$ along the trajectory $(x_0(t), u_0(t))$ in \mathbb{R}^n as follows:

$$\tilde{\Sigma}'_{\mathbb{R}^n} : \quad \Delta \dot{x} = A(t)\Delta x + B(t)\Delta u, \quad (19)$$

where

$$A(t) = \frac{\partial \tilde{X}}{\partial x}(x_0(t), u_0(t)), \quad B(t) = \frac{\partial \tilde{X}}{\partial u}(x_0(t), u_0(t))$$

and

$$\Delta x = x - x_0(t) \in \mathbb{R}^n, \quad \Delta u = u - u_0(t) \in \mathbb{R}^k.$$

Refer to Section 4.6 of Khalil¹¹ about the linearization technique. Notice that the above linearization does not require any use of local charts on the state-space manifold M . In that sense the above linearization is conducted *globally* along the reference trajectory in one global coordinate system in \mathbb{R}^n . Also, in comparison with such a geometric linearization method as variational linearization in Lee et al.¹⁴ our Jacobian linearization is straightforward and simple to carry out. The following lemma is trivial but useful:

Lemma 3. If $u = u(t, x)$ is an exponentially tracking controller for the ambient system $\tilde{\Sigma}_{\mathbb{R}^n}$ for the reference trajectory $x_0(t)$, then it is also an exponentially tracking controller for the system Σ_M on M for the same reference trajectory.

The following theorem is an adaptation of Theorem 4.13 from the textbook by Khalil¹¹ in combination with Lemma 3 above.

Theorem 3. Suppose that a linear feedback controller $\Delta u = -K(t)\Delta x$ exponentially stabilizes the origin for the linearized system $\tilde{\Sigma}'_{\mathbb{R}^n}$ in \mathbb{R}^n . Let $B_r = \{z \in \mathbb{R}^n \mid \|z\| < r\}$ for some $r > 0$ and $f : [0, \infty) \times B_r \rightarrow \mathbb{R}$ be a function defined by

$$f(t, z) = \tilde{X}(x_0(t) + z, u_0(t) - K(t)z) - \tilde{X}(x_0(t), u_0(t)).$$

If the derivative $\frac{\partial f}{\partial z}(t, z)$ is bounded and Lipschitz on B_r uniformly in t , then the controller

$$u(t, x) = u_0(t) - K(t)(x - x_0(t))$$

enables the system Σ_M on M to track the reference trajectory $x_0(t)$ exponentially.

Notice that the key point in the above theorem is that the controller for the system Σ_M on M is designed in the ambient Euclidean space \mathbb{R}^n .

2.3.2 | Application to the Rigid Body System

We here apply Theorem 3 to the free rigid body system (17). Take a reference trajectory $(R_0(t), \Omega_0(t)) \in \text{SO}(3) \times \mathbb{R}^3$ and the corresponding control signal $u_0(t)$ such that

$$\dot{R}_0(t) = R_0(t)\hat{\Omega}_0(t), \quad \dot{\Omega}_0(t) = u_0(t), \quad \forall t \geq 0, \quad (20)$$

which can be also understood as equations that define $\Omega_0(t)$ and $u_0(t)$ in terms of $R_0(t)$ and its time derivatives. Assume that $(R_0(t), \Omega_0(t))$ and $u_0(t)$ are bounded over the time interval $[0, \infty)$.

Theorem 4. The linearization of (17) along the reference trajectory $(R_0(t), \Omega_0(t)) \in \text{SO}(3) \times \mathbb{R}^3$ and the reference control signal $u_0(t)$ is given by

$$\Delta\dot{R} = \Delta R\hat{\Omega}_0 + R_0\widehat{\Delta\Omega} - 2k_e R_0 \text{Sym}(R_0^T \Delta R), \quad (21a)$$

$$\Delta\dot{\Omega} = \Delta u, \quad (21b)$$

where

$$\Delta R = R - R_0(t) \in \mathbb{R}^{3 \times 3}, \quad \Delta\Omega = \Omega - \Omega_0(t) \in \mathbb{R}^3, \quad \Delta u = u - u_0(t) \in \mathbb{R}^3.$$

Proof. Equation (21a) can be easily derived by using the definition of derivative as follows. Let $c(s) = R_0 + s(R - R_0) = R_0 + s\Delta R$ and $d(s) = \Omega_0 + s(\Omega - \Omega_0) = \Omega_0 + s\Delta\Omega$, where $s \in \mathbb{R}$. Then

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} (c(s)\widehat{d(s)} - kc(s)(c(s)^T c(s) - I)) &= \Delta R\hat{\Omega}_0 + R_0\widehat{\Delta\Omega} - k_e R_0(\Delta R^T R_0 + R_0^T \Delta R) \\ &= \Delta R\hat{\Omega}_0 + R_0\widehat{\Delta\Omega} - 2k_e R_0 \text{Sym}(R_0^T \Delta R), \end{aligned}$$

which is equal to the expression on the right side of (21a). \square

We now introduce a new matrix variable Z replacing ΔR as follows:

$$Z = R_0^T(t)\Delta R. \quad (22)$$

Let

$$Z_s = \text{Sym}(Z), \quad Z_k = \text{Skew}(Z) \quad (23)$$

such that

$$Z = Z_s + Z_k. \quad (24)$$

Lemma 4. The system (21) is transformed to

$$\dot{Z}_s = [Z_s, \hat{\Omega}_0] - 2k_e Z_s, \quad (25a)$$

$$\dot{Z}_k^\vee = Z_k^\vee \times \Omega_0 + \Delta\Omega, \quad (25b)$$

$$\Delta\dot{\Omega} = \Delta u \quad (25c)$$

via the state transformation given in (22) – (24).

Proof. Differentiate (22) with respect to t and use (20) – (24) to obtain

$$\begin{aligned} \dot{Z} &= \dot{R}_0^T \Delta R + R_0^T \Delta \dot{R} \\ &= -\hat{\Omega}_0 R_0^T \Delta R + R_0^T \Delta R \hat{\Omega}_0 + \widehat{\Delta\Omega} - 2k_e \text{Sym}(R_0^T \Delta R) \\ &= [Z, \hat{\Omega}_0] + \widehat{\Delta\Omega} - 2k_e \text{Sym}(Z) \\ &= [Z_s, \hat{\Omega}_0] + [Z_k, \hat{\Omega}_0] + \widehat{\Delta\Omega} - 2k_e Z_s. \end{aligned}$$

Taking the symmetric and skew-symmetric parts, we get

$$\dot{Z}_s = [Z_s, \hat{\Omega}_0] - 2k_e Z_s, \quad \dot{Z}_k = [Z_k, \hat{\Omega}_0] + \widehat{\Delta\Omega},$$

where the second equation can be also written as (25b) by Lemma 1. This completes the proof. \square

Proposition 1. For any two matrices $K_P, K_D \in \mathbb{R}^{3 \times 3}$ such that the matrix

$$\begin{bmatrix} 0 & I \\ -K_P & -K_D \end{bmatrix} \quad (26)$$

is Hurwitz, the controller

$$\Delta u = -K_P \cdot Z_k^\vee - K_D(Z_k^\vee \times \Omega_0 + \Delta\Omega) - (Z_k^\vee \times \Omega_0 + \Delta\Omega) \times \Omega_0 - Z_k^\vee \times u_0 \quad (27)$$

exponentially stabilizes the origin for the system (25).

Proof. Let us first show the exponential stability of the subsystem (25a) that is decoupled from the rest of the system. Let $V(Z_s) = \|Z_s\|^2/2$. Along the trajectory of (25), $\frac{d}{dt}V = \langle Z_s, [Z_s, \hat{\Omega}_0] - 2k_e Z_s \rangle = -2k_e \|Z_s\|^2 = -4k_e V$, where it is easy to show $\langle Z_s, [Z_s, \hat{\Omega}_0] \rangle = 0$. Hence, $V(t) \leq e^{-4k_e t} V(0)$ for all $t \geq 0$, or

$$\|Z_s(t)\| \leq e^{-2k_e t} \|Z_s(0)\| \quad (28)$$

for all $t \geq 0$ and $Z_s(0) \in \text{Sym}(\mathbb{R}^{3 \times 3})$, which proves exponential stability of $Z_s = 0$ for (25a).

Differentiating (25b) and substituting (25c) transforms the subsystem (25b) and (25c) to the following second-order system:

$$\ddot{Z}_k^\vee = \dot{Z}_k^\vee \times \Omega_0 + Z_k^\vee \times u_0 + \Delta u$$

since $\dot{\Omega}(t) = u_0(t)$. This second-order system is exponentially stabilized by the controller

$$\Delta u = -K_P \cdot Z_k^\vee - K_D \dot{Z}_k^\vee - \dot{Z}_k^\vee \times \Omega_0 - Z_k^\vee \times u_0, \quad (29)$$

where the matrices $K_P, K_D \in \mathbb{R}^{3 \times 3}$ are any matrices such that the matrix in (26) becomes Hurwitz. So, there are positive constants C_1 and C_2 such that

$$\|Z_k^\vee(t)\| + \|\dot{Z}_k^\vee(t)\| \leq C_1 e^{-C_2 t} (\|Z_k^\vee(0)\| + \|\dot{Z}_k^\vee(0)\|)$$

for all $t \geq 0$ and $(Z_k^\vee(0), \dot{Z}_k^\vee(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$. Since $\Omega_0(t)$ is bounded by assumption, there is a constant $M > 0$ such that $\|\Omega_0(t)\| \leq M$ for all $t \geq 0$. By (25b) and the triangle inequality,

$$\|\dot{Z}_k^\vee(t)\| \leq M \|Z_k^\vee(t)\| + \|\Delta\Omega(t)\|$$

and

$$\|\Delta\Omega(t)\| \leq \|\dot{Z}_k^\vee(t)\| + M \|Z_k^\vee(t)\|$$

for all $t \geq 0$. It is then easy to show that

$$\|Z_k^\vee(t)\| + \|\Delta\Omega(t)\| \leq C_3 e^{-C_2 t} (\|Z_k^\vee(0)\| + \|\Delta\Omega(0)\|) \quad (30)$$

for all $t \geq 0$ and $(Z_k^\vee(0), \Delta\Omega(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $C_3 = C_1(1 + M)^2$. Notice that the controller given in (29) is the same as the one in (27). It follows from (28) and (30) that the controller (27) exponentially stabilizes the origin for the system (25). \square

Remark 2. The exponential stability of the subsystem (25a) is a consequence of adding the term $-k_e R(R^T R - I)$ in (17a), and it is consistent with Theorem 2.

The following proposition produces time-varying PID-like tracking controllers.

Proposition 2. For any three matrices $K_P, K_D, K_I \in \mathbb{R}^{3 \times 3}$ such that the polynomial

$$\det(\lambda^3 I + \lambda^2 K_D + \lambda K_P + K_I) = 0 \quad (31)$$

is Hurwitz, the controller

$$\Delta u = -K_P \cdot Z_k^\vee - K_D(Z_k^\vee \times \Omega_0 + \Delta\Omega) - K_I \int_0^t Z_k^\vee(\tau) d\tau - (Z_k^\vee \times \Omega_0 + \Delta\Omega) \times \Omega_0 - Z_k^\vee \times u_0 \quad (32)$$

exponentially stabilizes the origin for the system (25).

Proof. Apply the controller (32) to the system (25) and differentiate (25b) three times to transform the closed-loop system (25) to

$$\begin{aligned} \dot{Z}_s &= -2k_e Z_s, \\ \ddot{Z}_k^\vee + K_D \dot{Z}_k^\vee + K_P Z_k^\vee + K_I Z_k^\vee &= 0. \end{aligned}$$

It is easy to prove that this linear system is exponentially stable by the Hurwitz condition on the polynomial in (31). This proves the proposition. \square

The controllers proposed in (27) and (32) depend on the reference control signal $u_0(t)$. The following proposition proposes one that is independent of $u_0(t)$.

Proposition 3. For any positive number k_P and any positive definite symmetric matrix $K_D \in \mathbb{R}^{3 \times 3}$, the controller

$$\Delta u = -k_P Z_k^\vee - K_D \Delta \Omega \quad (33)$$

exponentially stabilizes the origin for the system (25).

Proof. Since the exponential stability of the subsystem (25a) has been shown in the proof of Proposition 1, it remains to prove the exponential stability of the subsystem (25b) and (25c) with the control law given above. Since $\Omega_0(t)$ is bounded by assumption, there is a number M such that $\|\Omega_0(t)\| \leq M$ for all $t \geq 0$. Choose a number ϵ such that

$$0 < \epsilon < \min \left\{ \sqrt{k_P}, \frac{4k_P \lambda_{\min}(K_D)}{4k_P + (M + \lambda_{\max}(K_D))^2} \right\}. \quad (34)$$

Define two functions V_1 and V_2 by

$$\begin{aligned} V_1 &= \frac{k_P}{2} \|Z_k^\vee\|^2 + \frac{1}{2} \|\Delta \Omega\|^2 + \epsilon \|Z_k^\vee\| \|\Omega\|, \\ V_2 &= \epsilon k_p \|Z_k^\vee\|^2 + (\lambda_{\min}(K_D) - \epsilon) \|\Delta \Omega\|^2 - \epsilon(M + \lambda_{\max}(K_D)) \|Z_k^\vee\| \|\Omega\|. \end{aligned} \quad (35)$$

These two functions are all positive definite quadratic functions of $(\|Z_k^\vee\|, \|\Omega\|)$ by (34), so there exists a constant $C > 0$ such that

$$CV_1 \leq V_2. \quad (36)$$

Define a function V by

$$V = \frac{k_P}{2} \|Z_k^\vee\|^2 + \frac{1}{2} \|\Delta \Omega\|^2 + \epsilon \langle Z_k^\vee, \Delta \Omega \rangle, \quad (37)$$

which is a positive definite quadratic function of $(Z_k^\vee, \Delta \Omega)$ and satisfies

$$V \leq V_1. \quad (38)$$

Along any trajectory of the subsystem (25b) and (25c) with the control (33),

$$\begin{aligned} \frac{d}{dt} V &= k_P \langle Z_k^\vee, Z_k^\vee \times \Omega_0 + \Delta \Omega \rangle + \langle \Delta \Omega, u \rangle + \epsilon (\langle Z_k^\vee \times \Omega_0 + \Delta \Omega, \Delta \Omega \rangle + \langle Z_k^\vee, u \rangle) \\ &\leq -\epsilon k_P \|Z_k^\vee\|^2 - (\lambda_{\min}(K_D) - \epsilon) \|\Delta \Omega\|^2 + \epsilon(M + \lambda_{\max}(K_D)) \|Z_k^\vee\| \|\Delta \Omega\| \\ &= -V_2 \leq -CV_1 \leq -CV \end{aligned}$$

by (36) and (38). Hence, $V(t) \leq e^{-Ct} V(0)$ for all $t \geq 0$, which implies that the closed-loop subsystem (25b) and (25c) is exponentially stable with the control (33). This completes the proof. \square

The following proposition is a variant of Proposition 3.

Proposition 4. For any two positive numbers k_P and ϵ and any positive definite symmetric matrix $K_D \in \mathbb{R}^{3 \times 3}$ such that

$$0 < \epsilon < \min \left\{ \sqrt{k_P}, \frac{4k_P \lambda_{\min}(K_D)}{4k_P + (\lambda_{\max}(K_D))^2} \right\}, \quad (39)$$

the controller

$$\Delta u = -k_P Z_k^\vee - K_D \Delta \Omega - \epsilon (Z_k^\vee \times \Omega_0) \quad (40)$$

exponentially stabilizes the origin for the system (25).

Proof. The exponential stability of (25a) has already been shown in the proof of Theorem 1, so we now focus on the stability of (25b) and (25c) with the feedback (40). Consider the same function V_1 as that defined in (35). Let

$$V_2 = \epsilon k_p \|Z_k^\vee\|^2 + (\lambda_{\min}(K_D) - \epsilon) \|\Delta \Omega\|^2 - \epsilon \lambda_{\max}(K_D) \|Z_k^\vee\| \|\Delta \Omega\|.$$

By (39), the two functions V_1 and V_2 are both positive definite quadratic functions of $(\|Z_k^\vee\|, \|\Omega\|)$, so there exists a constant $C > 0$ such that (36) holds. Consider the function V defined in (37), which is a positive definite quadratic function of $(Z_k^\vee, \Delta \Omega)$ and satisfies (38). It is then straightforward to show that along any trajectory of the subsystem (25b) and (25c) with the control (40), $\frac{d}{dt} V \leq -V_2 \leq -CV_1 \leq -CV$ by (36) and (38). Hence, $V(t) \leq e^{-Ct} V(0)$ for all $t \geq 0$, which implies that the closed-loop subsystem (25b) and (25c) is exponentially stable with the control (40). This completes the proof.

□

The following proposition essentially derives the control law in equation (13) of Lee et al.¹⁵ which was derived using geometric control theory therein, but is easily derived here with the linearized dynamics (25).

Proposition 5. For any $k_R > 0$ and $k_\Omega > 0$, the controller

$$\Delta u = -k_R Z_k^\vee - k_\Omega \Delta \Omega + \Delta \Omega \times \Omega_0 \quad (41)$$

exponentially stabilizes the origin for the system (25).

Proof. Choose any number ϵ that satisfies $0 < \epsilon < \min\{\sqrt{k_R}, 4k_R k_\Omega / (4k_R + k_\Omega^2)\}$. Then, the function $V(Z_k^\vee, \Delta \Omega) = k_R \|Z_k^\vee\|^2/2 + \epsilon \langle Z_k^\vee, \Delta \Omega \rangle + \|\Delta \Omega\|^2/2$ is a positive definite quadratic function of $(Z_k^\vee, \Delta \Omega)$. Along any flow of (25b) and (25c), the derivative of V can be easily computed as $dV/dt = -\epsilon k_R \|Z_k^\vee\|^2 - \epsilon k_\Omega \langle Z_k^\vee, \Delta \Omega \rangle - (k_\Omega - \epsilon) \|\Delta \Omega\|^2$, which can be easily shown to be a negative definite quadratic function of $(Z_k^\vee, \Delta \Omega)$, which proves the closed-loop exponential stability of the origin for the system (25). □

The following theorem puts together the five preceding propositions to provide various exponentially tracking controllers for the rigid body system (14).

Theorem 5. The following controller

$$u = u_0 + \Delta u, \quad (42)$$

where Δu is any of (27), (32), (33), (40) and (41) with

$$Z_k = \text{Skew}(R_0^T \Delta R)^\vee = \text{Skew}(R_0^T R)^\vee, \quad (43)$$

enables the rigid body system (14) to track the reference trajectory $(R_0(t), \Omega_0(t))$ exponentially.

Proof. By (22), $\|\Delta R(t)\| = \|R_0(t)Z(t)\| = \|Z(t)\|$, so exponential stability of (25) implies that of (21). Hence, this theorem follows from Theorem 3 and Propositions 1 – 5. □

Remark 3. As can be seen in (43), Z_k can be computed without computing $\Delta R = R - R_0(t)$. As a result, all the control laws for the rigid body system (14) on $\text{SO}(3) \times \mathbb{R}^3$ provided in Theorem 5 can be computed using matrix multiplications on $\text{SO}(3)$ although they have been derived with ΔR in $\mathbb{R}^{3 \times 3}$. In other words, all the control laws in Theorem 5 are intrinsic on $\text{SO}(3) \times \mathbb{R}^3$ though they are derived in the ambient Euclidean space $\mathbb{R}^{3 \times 3} \times \mathbb{R}^3$.

Remark 4. One can observe that the subsystem (25b) coincides with the $\dot{\eta}$ equation in (16) in the paper by Lee et al,¹⁴ where equation (16) therein is derived through so-called variational linearization. Since we have extended the rigid body system to ambient Euclidean space, our linearization is the usual Jacobian linearization taken in Euclidean space, which is not only simpler than the variational one, but also allows us to rigorously and easily apply the Lyapunov linearization method in one single global Cartesian coordinate system with the transversal dynamics (25a) taken into account. Also, thanks to the added term $-\nabla \tilde{V}$, the Z_s -subsystem (25a), which is decoupled from the subsystem (25b) and (25c), is exponentially stable by itself. Without it, i.e. if $k_e = 0$, the Z_s -dynamics would be only neutrally stable, not enabling us to directly apply the Lyapunov linearization method.

We carry out a simulation to show a good tracking performance of the controller (42) with (40) for the rigid body system (14) or (17) with $k_e = 1$. The control parameters are chosen as

$$k_P = 4, \quad K_D = 2I, \quad \epsilon = 1.$$

The reference trajectory $(R_0(t), \Omega_0(t)) \in \text{SO}(3) \times \mathbb{R}^3$ with the reference control signal $u_0(t) \in \mathbb{R}^3$ are chosen as

$$R_0(t) = \begin{bmatrix} \cos^2 t & (1 + \sin t) \cos t \sin t & (\sin t - \cos^2 t) \sin t \\ -\sin t \cos t & \cos^2 t - \sin^3 t & (1 + \sin t) \cos t \sin t \\ \sin t & -\cos t \sin t & \cos^2 t \end{bmatrix}, \quad (44)$$

$$\Omega_0(t) = [-1 - \sin t, (-1 + \sin t) \cos t, -\sin t - \cos^2 t]^T, \quad (45)$$

$$u_0(t) = \dot{\Omega}_0(t) = [-\cos t, \sin t + \cos^2 t - \sin^2 t, -\cos t + 2 \cos t \sin t]^T, \quad (46)$$

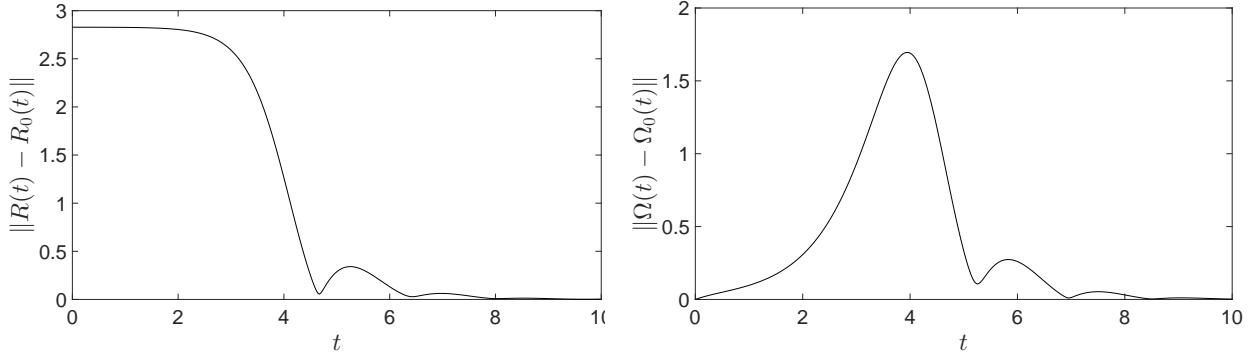


FIGURE 1 The simulation result for tracking of the reference $(R_0(t), \Omega_0(t))$ by the linear controller (42) with (40) for the rigid body system.

which satisfy (20). Notice that if the reference trajectory $R_0(t)$ is parameterized by the $Z - Y - X$ Euler angles, then the parameterization will become singular at $t = \pi/2 + k\pi$, $k \in \mathbb{Z}$. Hence, the use of Euler angles for controller design is not desirable. The initial condition is chosen as

$$R(0) = \exp(0.99\pi\hat{e}_2), \quad \Omega(0) = (-1, -1, -1),$$

where $R(0)$ is a rotation around $e_2 = (0, 1, 0)$ through 0.99π radians. The initial orientation tracking error is almost $2\sqrt{2}$ that is the maximum possible orientation error. The tracking errors are plotted in Figure 1, which shows a good tracking performance of the controller for the *nonlinear* system (14).

We now carry out a simulation to compare the controller (42) and (40) with the controller proposed by Lee¹³ which is modified for the system (14) as follows:

$$u_{\text{Lee}} = -k_R e_R - k_\Omega e_\Omega - \hat{\Omega} R^T R_0 \Omega_0 + R^T R_0 \dot{\Omega}_0,$$

where

$$e_R = \frac{1}{\sqrt{1 + \text{trace}(R_0^T R)}} \text{Skew}(R_0^T R)^\vee, \quad e_\Omega = \Omega - R^T R_0 \Omega_0.$$

For the controller (42) with (40), we use the parameter values: $k_P = 4$, $K_D = 2I$ and $\epsilon = 1$. To make a fair comparison, we choose for the controller u_{Lee} the following parameter values: $k_R = 4$ and $k_\Omega = 2$. The two controllers are applied to the system (14) with the initial condition $R(0) = \exp(0.9\pi\hat{e}_2)$ and $\Omega(0) = (-1, -1, -1)$ for the reference trajectory given in (44) – (46). The simulation results are plotted in Figure 2. We can see that there is a difference between the two controllers in the transient response. The controller by Lee initially performs better than our controller in attitude tracking but it has a large overshoot in angular velocity tracking and has a huge initial value of control, which is due to the nonlinear term $1/\sqrt{1 + \text{trace}(R_0^T R)}$ present in Lee's controller, u_{Lee} . After about $t = 5$, both controllers behave similarly, and the responses of the system are similar to each other. From these observations, we can draw the conclusion that our linear controller (42) with (40) is on par with the nonlinear controller u_{Lee} by Lee. However, our controller has been easily obtained with a linear technique whereas the controller by Lee was obtained with a nonlinear technique that is not as easy to use as the linear technique.

2.4 | Tracking Controller Design for the Quadcopter System

The equations of motion of the quadcopter system are given by

$$\dot{R} = R\hat{\Omega}, \tag{47a}$$

$$\dot{\Omega} = \mathbb{I}\Omega \times \Omega + \tau, \tag{47b}$$

$$\ddot{x} = -ge_3 + fRe_3, \tag{47c}$$

where x is the \mathbb{R}^3 -vector for the position of the quadcopter, R is the 3×3 rotation matrix for orientation, and $\Omega \in \mathbb{R}^3$ is the \mathbb{R}^3 -vector for body angular velocity. Here, $f \geq 0$ is the upward control thrust *per mass* and $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ is the control torque on the quadcopter expressed in the body frame. The parameter g denotes the gravitational acceleration; \mathbb{I} is the 3×3

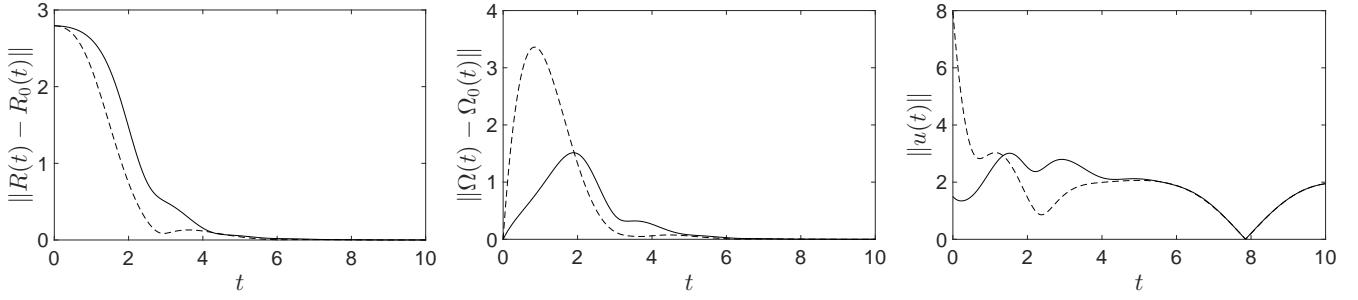


FIGURE 2 The simulation results for tracking the reference $(R_0(t), \Omega_0(t))$ by the linear controller (42) with (40) (solid) and the nonlinear controller by Lee (dashed) for the rigid body system.

moment of inertia matrix; and $e_3 = (0, 0, 1)$. Although f is a thrust per mass unit-wise, it shall be simply called a thrust in this paper. Refer to the book by Lee et al.¹⁶ for the derivation of (47).

Assume that the full state is available and apply the feedback

$$\tau = -\|\Omega \times \Omega + \|u \quad (48)$$

to transform the subsystem (47b) to

$$\dot{\Omega} = u,$$

where $u \in \mathbb{R}^3$ is the new control sub-vector replacing $\tau \in \mathbb{R}^3$. Extend dynamically the subsystem (47c) by introducing a double integrator through the thrust variable as follows:

$$\ddot{f} = q, \quad (49)$$

where $q \in \mathbb{R}$ is now a new control variable, and f and \dot{f} are now regarded as state variables. As done for the rigid body system, we embed SO(3) to $\mathbb{R}^{3 \times 3}$ and subtract $\nabla \tilde{V}$, with \tilde{V} given in (15), from the equations of motion of the quadcopter to get the following equations of motion in the ambient Euclidean space:

$$\dot{R} = R\hat{\Omega} - k_e R(R^T R - I), \quad (50a)$$

$$\dot{\Omega} = u, \quad (50b)$$

$$\ddot{x} = -ge_3 + fRe_3, \quad (50c)$$

$$\ddot{f} = q. \quad (50d)$$

Choose a reference trajectory

$$(R_0(t), \Omega_0(t), x_0(t), \dot{x}_0(t), f_0(t), \dot{f}_0(t))$$

with $R_0(t) \in \text{SO}(3)$ for all $t \geq 0$, and a reference control signal

$$(u_0(t), q_0(t))$$

such that they satisfy the equations of motion (50). It is understood that $\dot{x}_0(t)$ and $\dot{f}_0(t)$ are the time derivatives of $x_0(t)$ and $f_0(t)$, respectively. It is further assumed that $\Omega_0(t), \dot{\Omega}_0(t), f_0(t), \dot{f}_0(t)$ and $\ddot{f}_0(t)$ are bounded for $t \geq 0$, and there is a constant $\delta > 0$ such that

$$f_0(t) \geq \delta \quad \forall t \geq 0.$$

Define the tracking error variables:

$$\begin{aligned} \Delta R &= R - R_0(t), \quad \Delta \Omega = \Omega - \Omega_0(t), \quad \Delta x = x - x_0(t), \\ \Delta f &= f - f_0(t), \quad \Delta u = u - u_0(t), \quad \Delta q = q - q_0(t). \end{aligned}$$

Then, linearize the system (50) along the reference trajectory and use the state transformation given in (22) – (24) replacing ΔR , to obtain the following linearized system:

$$\dot{Z}_s = [Z_s, \hat{\Omega}_0] - 2k_e Z_s, \quad (51a)$$

$$\dot{Z}_k^\vee = Z_k^\vee \times \Omega_0 + \Delta\Omega, \quad (51b)$$

$$\Delta\dot{\Omega} = \Delta u, \quad (51c)$$

$$\Delta\ddot{x} = \Delta f R_0 e + f_0 R_0 (Z_s + Z_k) e_3, \quad (51d)$$

$$\Delta\ddot{f} = \Delta q. \quad (51e)$$

Retaining all the other state variables, we replace the state variable $\Delta\Omega \in \mathbb{R}^3$, via (51b), with $\dot{Z}_k^\vee \in \mathbb{R}^3$ or $\dot{Z}_k \in \mathfrak{so}(3)$. Apply the feedback

$$\Delta u = -(Z_k^\vee \times \Omega_0 + \Delta\Omega) \times \Omega_0 - Z_k^\vee \times \dot{\Omega}_0 + \tilde{u}, \quad (52)$$

so as to replace (51b) and (51c) with the following second-order equation:

$$\ddot{Z}_k^\vee = \tilde{u},$$

where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \mathbb{R}^3$ is the new control sub-vector replacing Δu . Then, the system (51) is transformed to the following:

$$\dot{Z}_s = [Z_s, \hat{\Omega}_0] - 2k_e Z_s, \quad (53a)$$

$$\ddot{Z}_k^\vee = \tilde{u}, \quad (53b)$$

$$\Delta\ddot{x} = \Delta f R_0 e + A_0 (Z_s + Z_k) e_3, \quad (53c)$$

$$\Delta\ddot{f} = \Delta q, \quad (53d)$$

where the matrix-valued signal

$$A_0(t) = f_0(t) R_0(t) \in \mathbb{R}^{3 \times 3}$$

is introduced for convenience. Let

$$z_k = (z_{k1}, z_{k2}, z_{k3}) := Z_k^\vee \in \mathbb{R}^3 \quad (54)$$

so that

$$Z_k = \begin{bmatrix} 0 & -z_{k3} & z_{k2} \\ z_{k3} & 0 & -z_{k1} \\ -z_{k2} & z_{k1} & 0 \end{bmatrix}. \quad (55)$$

Lemma 5. The coordinate system

$$(Z_s, Z_k^\vee, \dot{Z}_k^\vee, \Delta x, \Delta \dot{x}, \Delta f, \Delta \dot{f}) \quad (56)$$

can be globally replaced with

$$(Z_s, \Delta x, \Delta \dot{x}, \Delta \ddot{x}, z_{k3}, \dot{z}_{k3}). \quad (57)$$

The coordinates $\Delta \ddot{x}$ and $\Delta \ddot{\dot{x}}$ in (57) are expressed in terms of the coordinates (56) as

$$\Delta \ddot{x} = (\Delta f R_0 + A_0 Z_k + A_0 Z_s) e_3, \quad (58)$$

$$\Delta \ddot{\dot{x}} = (\Delta \dot{f} R_0 + A_0 \dot{Z}_k + \Delta f \dot{R}_0 + \dot{A}_0 (Z_s + Z_k) + A_0 ([Z_s, \hat{\Omega}_0] - 2k_e Z_s)) e_3. \quad (59)$$

The coordinates $\Delta f, \Delta \dot{f}, z_{ki}, \dot{z}_{ki}, i = 1, 2$, in (56) are expressed in terms of the coordinates (57) as

$$[z_{k2}, z_{k1}, \Delta f]^T = B_0^{-1} R_0^T (\Delta \ddot{x} - A_0 Z_s e_3), \quad (60)$$

$$[\dot{z}_{k2}, \dot{z}_{k1}, \Delta \dot{f}]^T = B_0^{-1} R_0^T (\Delta \ddot{\dot{x}} - (\Delta f \dot{R}_0 + \dot{A}_0 (Z_s + Z_k) + A_0 ([Z_s, \hat{\Omega}_0] - 2k_e Z_s)) e_3), \quad (61)$$

where

$$B_0(t) = \text{diag}[f_0(t), -f_0(t), 1] \in \mathbb{R}^{3 \times 3}. \quad (62)$$

Proof. Differentiate (53c) with respect to t and use (53a) to replace \dot{Z}_s with $[Z_s, \hat{\Omega}_0] - 2k_e Z_s$, so as to obtain the expression for $\Delta \ddot{\dot{x}}$ in (59). From the definition of the vector z_k in (54) or (55), $Z_k e_3 = z_{k2} e_1 - z_{k1} e_2$, where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Hence, it is straightforward to get (60) and (61) from (58) and (59), respectively. \square

We express the system (53) in the new coordinates (57) and transform it via feedback to simple integrators as in the following theorem.

Theorem 6. The system (53) is transformed to

$$\dot{Z}_s = [Z_s, \hat{\Omega}_0] - 2k_e Z_s, \quad (63a)$$

$$\Delta x^{(4)} = v, \quad (63b)$$

$$\ddot{z}_{k3} = w, \quad (63c)$$

where $(v, w) \in \mathbb{R}^3 \times \mathbb{R}$ is the new control vector, by the feedback

$$\tilde{u}_3 = w, \quad (64a)$$

$$[\tilde{u}_2, \tilde{u}_1, \Delta q]^T = B_0^{-1} R_0^T (v - C e_3) \quad (64b)$$

where

$$C = 2\Delta f \dot{R}_0 + 2\dot{A}_0(\dot{Z}_s + \dot{Z}_k) + \Delta f \ddot{R}_0 + \ddot{A}_0(Z_s + Z_k) + A_0([Z_s, \hat{\Omega}_0] + [Z_s, \hat{\Omega}_0] - 2k_e \dot{Z}_s). \quad (65)$$

In the above expression of C , \dot{Z}_s is understood as $[Z_s, \hat{\Omega}_0] - 2k_e Z_s$.

Proof. Differentiate (59) with respect to t and simplify the result using the equations of motion in (53) to obtain

$$\begin{aligned} \Delta x^{(4)} &= \Delta \ddot{f} R_0 e_3 + A_0 \ddot{Z}_k e_3 + C e_3 \\ &= R_0 B_0 (\tilde{u}_2 e_1 + \tilde{u}_1 e_2 + \Delta q e_3) + C e_3, \end{aligned}$$

with B_0 and C defined in (62) and (65), respectively. It is transformed to (63b) by the feedback (64b). Equation (63c) is obtained by taking the inner product of (53b) with e_3 and using (64a). \square

Proposition 6. Take any four matrices $K_0, K_1, K_2, K_3 \in \mathbb{R}^{3 \times 3}$ such that the polynomial

$$\det(\lambda^4 I + \lambda^3 K_3 + \lambda^2 K_2 + \lambda K_1 + K_0)$$

is a Hurwitz polynomial in λ , and take any two positive numbers a_1 and a_0 . Then, the feedback controller

$$v = -K_3 \Delta \ddot{x} - K_2 \Delta \ddot{x} - K_1 \Delta \dot{x} - K_0 \Delta x, \quad (66)$$

$$w = -a_1 \dot{z}_{k3} - a_0 z_{k3} \quad (67)$$

makes the origin exponentially stable for the system (63).

Proof. The exponential stability of the Z_s dynamics (63a) has been already shown in the proof of Proposition 1. It is trivial to show the exponential stability of the origin for the subsystem (63b) and (63c) with the proposed controller. \square

Notice that the controller in (66) and (67) can be expressed in terms of the original variables via Lemma 5 and equations (22), (23) and (51b).

Proposition 7. Take any five matrices $K_0, K_1, K_2, K_3, K_I \in \mathbb{R}^{3 \times 3}$ such that the polynomial

$$\det(\lambda^5 I + \lambda^4 K_3 + \lambda^3 K_2 + \lambda^2 K_1 + \lambda K_0 + K_I)$$

is a Hurwitz polynomial in λ , and take any three numbers a_1, a_0, a_I such that the polynomial

$$\lambda^3 + a_1 \lambda^2 + a_0 \lambda + a_I$$

is Hurwitz. Then, the feedback controller

$$v = -K_3 \Delta \ddot{x} - K_2 \Delta \ddot{x} - K_1 \Delta \dot{x} - K_0 \Delta x - K_I \int_0^t \Delta x(\tau) d\tau,$$

$$w = -a_1 \dot{z}_{k3} - a_0 z_{k3} - a_I \int_0^t z_{k3}(\tau) d\tau$$

makes the origin exponentially stable for the system (63).

Proof. Trivial. \square

After a controller (v, w) is designed as in Propositions 6 and 7, the controller $(\tilde{u}, \Delta q)$ in (64) is computed. Then, Δu is computed via (52), which produces the control torque τ in (48) with $u = u_0(t) + \Delta u$ and the control thrust f via (49) with $q = q_0(t) + \Delta q$.

Theorem 7. The controller (τ, f) designed as above enables the quadcopter system (47) to exponentially track the reference trajectory $(R_0(t), \Omega_0(t), x_0(t), \dot{x}_0(t))$.

Proof. It is easy to prove that the origin is exponentially stable for the linear system (51) with the controller $(\Delta u, \Delta q)$ designed as described above. By Theorem 3, the controller (u, q) designed as described above enables the extended quadcopter system (50) to exponentially track the reference trajectory $(R_0(t), \Omega_0(t), x_0(t), \dot{x}_0(t), f_0(t), \dot{f}_0(t))$ from which the present theorem follows. \square

Remark 5. The controllers proposed in the paper by Goodarzi et al.⁸ have two separate modes: attitude controlled flight mode and position controlled flight mode. In contrast, our controllers have the merit to simultaneously control both the attitude and the position of quadcopter.

Remark 6. Our controllers have no singularity since we use only one single global Cartesian coordinate system, whereas the controller proposed by Mellinger and Kumar²² would become singular when the roll angle becomes $\pm\pi/2$, which purely comes from the use of an Euler angle coordinate system. This shows the merit of our method that utilizes one single global Cartesian coordinate system in the ambient Euclidean space. It will be interesting to re-do the work by Mellinger and Kumar²² in this framework.

Remark 7. Although the dynamic extension (49) is simple, it has the drawback that the non-negative sign of $f(t)$ may not be preserved along the trajectory even with a positive initial value $f(0) > 0$. To remedy this, the following dynamic extension

$$\dot{f} = fh, \quad h = q \quad (68)$$

was proposed in the paper by Chang and Eun⁷ to replace (49), where h is an added state variable replacing \dot{f} . It is easy to verify that this extension preserves the positive sign of $f(t)$ when $f(0) > 0$. The linearization of (68) along the reference trajectory is computed as

$$\Delta \dot{f} = \Delta f h_0 + f_0 \Delta h, \quad \Delta \dot{h} = \Delta q,$$

and it shall replace (51e) in the linearization of the quadcopter dynamics, where $h_0(t) = \dot{f}_0(t)/f_0(t)$ and $\Delta h = h - h_0(t)$. It is left to the reader to verify that with the extension (68) the consequent linearized quadcopter system can also be transformed to (63) via an appropriate feedback control law.

We now run a simulation to demonstrate a good tracking performance of the proposed controller $u = u_0(t) + \Delta u$ and $q = q_0(t) + \Delta q$ with $\Delta u, \tilde{u}, \Delta q, v$ and w given in (52), (64), (66) and (67), for the extended quadcopter system (50) with $k_e = 1$. Choose a reference trajectory for (50) as follows: $R_0(t)$, $\Omega_0(t)$ and $u_0(t)$ are given in (44)–(46), and $x_0(t)$ and $f_0(t)$ are given as

$$x_0(t) = g \begin{bmatrix} \frac{1}{2}t^2 + \frac{4}{9}\sin t - \frac{1}{2}\sin^2 t + \frac{2}{9}\sin t \cos^2 t \\ \frac{4}{9} - \frac{4}{9}\cos t - \frac{1}{2}\cos t \sin t - \frac{2}{9}\cos t \sin^2 t \\ \frac{1}{2}\sin^2 t \end{bmatrix},$$

$$f_0(t) = 2g.$$

Choose the following initial condition for (50):

$$R(0) = \exp(0.25\pi\hat{e}_2), \quad \Omega(0) = (0, 0, 0),$$

$$x(0) = (-0.5g, -0.5g, 0), \quad \dot{x}(0) = (0, 0, 0),$$

$$f(0) = 2g, \quad \dot{f}(0) = 0,$$

where $R(0)$ is a rotation through $\pi/4$ radians about the axis $e_2 = (0, 1, 0)$. By scaling x by g , we may assume that $g = 1$. Choose the following values of control parameters:

$$K_3 = 8I, \quad K_2 = 32I, \quad K_1 = K_0 = 64I, \quad a_1 = 8, \quad a_0 = 20$$

for (66) and (67), so that the poles of the tracking error dynamics (63b) for Δx are all located at $-2 \pm j2$ and the poles of (63c) for z_{k3} are located at $-4 \pm j2$. Apply the resulting controller (u, q) to (50). The tracking errors and the control thrust f are plotted in

Figure 3. The tracking errors all converge to zero as $t \rightarrow \infty$, and the control thrust f converges to the reference thrust $f_0(t) = 2$ as $t \rightarrow \infty$. To test robustness of the controller to disturbance, we now add disturbance terms to (50b) and (50c) as follows:

$$\dot{\Omega} = u + R^T d,$$

$$\ddot{x} = -g e_3 + f R e_3 + d,$$

where $d(t) = \sin(2\pi(t-3))(1, 1, 1)$ if $3 \leq t \leq 4$, and $d(t) = 0$ otherwise. We run a simulation with the same controller without any compensation for the disturbance. The simulation result is plotted in Figure 4, where the two dotted vertical lines denote the start time and end time of the disturbance. We can see that the tracking degrades from $t = 3$ till approximately $t = 4.2$ due to the effect of disturbance and then gets back to the exponentially convergent mode. This result shows robustness of our tracking controller to disturbance.

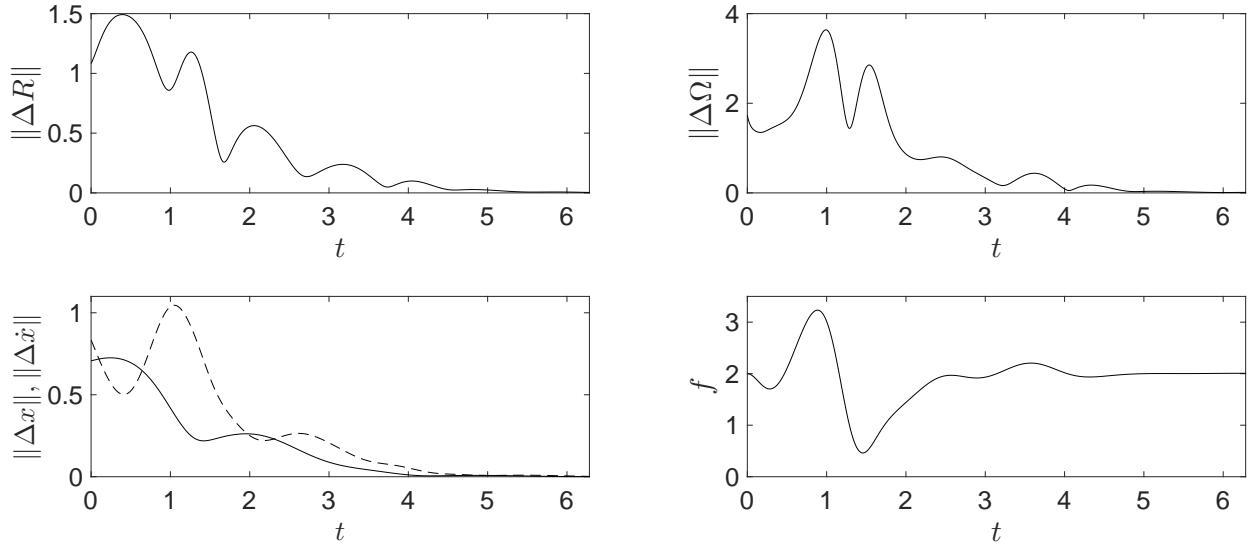


FIGURE 3 The trajectory of the tracking errors and the thrust variable of the quadcopter system (50) with the linear controller described in Theorem 7. In the left bottom plot, the solid line is the trajectory of $\|\Delta x(t)\|$ and the dashed line that of $\|\Delta \dot{x}(t)\|$.

3 | CONCLUSION

We have presented a method to design controllers in Euclidean space for systems defined on manifolds. The idea is to embed the state-space manifold M of a given control system to some Euclidean space \mathbb{R}^n , extend the system from M to the ambient space \mathbb{R}^n , and modify it outside M to add transversal stability to M in the final dynamics in \mathbb{R}^n . We then design controllers for the final system in the ambient Euclidean space \mathbb{R}^n and restrict the controllers to M after the synthesis. Since the controller synthesis is carried out in Euclidean space in this framework, it has the merit that only one single global Cartesian coordinate system in the ambient Euclidean space is used and all possible controller design methods on \mathbb{R}^n , including the linearization method, can be rigorously applied for controller synthesis. This method is successfully applied to the tracking problem for the following two benchmark systems: the fully actuated rigid body system and the quadcopter drone system. As future work, we plan to consider control constraints such as saturation in the proposed method for which the technique developed by Su et al.²⁵ is expected to be effective. We also plan to study robustness of the proposed method with respect to measurement errors.

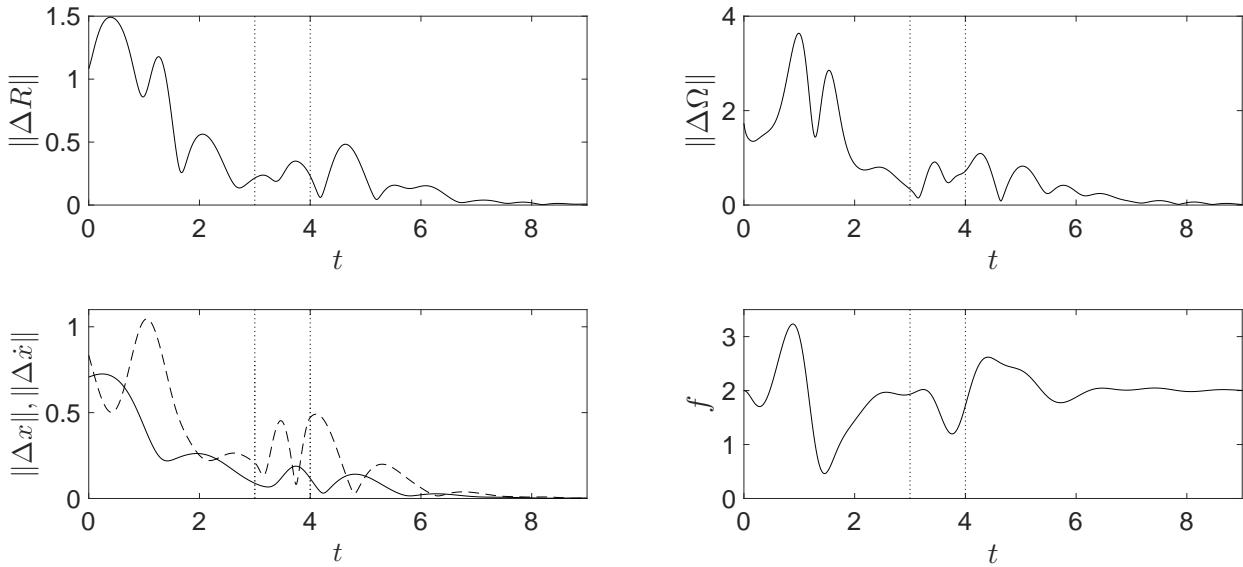


FIGURE 4 The trajectory of the tracking errors and the thrust variable of the quadcopter system (50) with the linear controller described in Theorem 7 in the presence of an unknown disturbance during the time interval, $3 \leq t \leq 4$. The two dotted vertical lines denote the time interval $[3, 4]$. In the left bottom plot, the solid line is the trajectory of $\|\Delta x(t)\|$ and the dashed line that of $\|\Delta \dot{x}(t)\|$.

ACKNOWLEDGMENT

This research has been in part supported by KAIST under grant G04170001 and by the ICT R&D program of MSIP/IITP [2016-0-00563, Research on Adaptive Machine Learning Technology Development for Intelligent Autonomous Digital Companion].

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