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Using fuzzy sets theory and Black–Scholes formula to generate pricing boundaries of European options

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Abstract

The application by using extension principle in fuzzy sets theory to the Black–Scholes formula is proposed in this paper. Owing to the fluctuation of financial market from time to time, the risk-free interest rate, volatility and stock price may occur imprecisely in the real world. Therefore, it is natural to consider the fuzzy interest rate, fuzzy volatility and fuzzy stock price in the financial market. Under these assumptions, the European call and put option price will turn into the fuzzy numbers, and the extension principle will be invoked to generate the pricing boundaries of the European call and put options. This will make the financial analyst who can pick any reasonable European option price with an acceptable belief degree for his/her later financial analysis. In order to obtain the belief degree, an optimization problem has to be solved.

Keywords: Black–Scholes formula; European options; Fuzzy numbers; Financial market; Put-call parity

1. Introduction

In real world, the data sometimes cannot be recorded or collected precisely. For instance, let us consider interest rate. When the financial analyst tries to price an European option, the interest rate is sometimes assumed as a constant. However, the interest rate may have different values in the different commercial banks and financial institutions although the difference is so small. Therefore, the choice of a reasonable interest rate to price a European option may cause a dilemma. But one thing that can be sure is that the different interest rates may be around a fixed value within a short period of time. For instance, the interest rates may be around 5% in the different commercial banks and financial institutions. The phrase "around 5%" might have a problem to be modeled by using the probability theory. Therefore, the fuzzy sets theory plays an appropriate role to tackle this kind of fuzziness. In this case, the interest rate may be regarded as a fuzzy number $\tilde{5}\%$ when the financial analyst tries to price an European call option using the Black–Scholes formula.

The well-known closed form solution of the European call option was derived by Black and Scholes [4]. Because the fuzziness may occur in the financial market as we just described above, the fuzzy sets theory

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proposed by Zadeh [14] may become a useful tool for modeling this kind of problem. The book of collected papers edited by Ribeiro et al. [9] gave the interesting applications by using fuzzy sets theory to the discipline called financial engineering.

The expiry date and strike price in the options are always known precisely. Therefore, it will be enough to just consider the fuzzy interest rate, fuzzy volatility and fuzzy stock price. Although, Zmeškal [16] considered the expiry date and strike price in fuzzy sense to discuss the Black–Scholes formula by using the extension principle which was proposed by Zadeh [15]. We have to mention that, although the fuzzy expiry and fuzzy strike price are not considered in this paper, the technique developed in this paper is still valid if we insist of considering the expiry date and strike price in fuzzy sense. However, we have to emphasize again that, in real financial market, it may be a little weird to consider the expiry date and strike price in the fuzzy sense, because the expiry date and strike price should be written in the contract for buying the options. In this sense, the expiry date and strike price should be taken as real numbers. On the other hand, the European put option was not taken into account, and the detailed computational procedure was not provided by Zmeškal [16]. It makes the financial analysts having difficulty and hesitation to implement it, since a very complicated optimization problem for obtaining the global optimum has to be solved in Zmeškal's approach. In this paper, we propose another approach to generate the pricing boundaries of European call and put options. The detailed computational procedures are also provided by taking into account the sensitivity analysis of the Black–Scholes formula.

Yoshida [13] also discussed the valuation of European call and put options in a fuzzy environment. However, the price of European options depends on the fuzzy goal in Yoshida's approach; that is to say, a fuzzy goal has to be posted firstly in order to obtain the price of European options. In other words, under the same input data, the different fuzzy goals claimed by different financial analysts will lead to different prices of European options. This may also result the financial analysts having hesitation to use it, since most of us agree to have one price of European options under the same input data. In this paper, we have no this difficulty, since we shall obtain a (fuzzy) price of European options under the same (fuzzy) input data.

On the other hand, Simonelli [10] also provided a methodology in valuing financial instruments by using certainty equivalents that was proposed by Pratt [7]. However the Simonelli's approach was not based on the Black–Scholes formula. Pratt provided the definition of certain equivalent of a random variable by using the utility theory of von Neumann and Morgenstern. Simonelli then followed Pratt's approach to evaluate financial instruments under fuzzy conditions. Therefore the Black–Scholes formula was not invoked by Simonelli. The methodology proposed in this paper is based on the Black–Scholes formula. Wu [11] also considered the fuzzy pattern of Black–Scholes formula. When the arithmetics in the (conventional) Black–Scholes formula are replaced by the fuzzy arithmetics, Wu [11] obtain the so-called fuzzy pattern of Black–Scholes formula. Also, in Wu [12], the form of "Resolution Identity" in fuzzy sets theory was invoked to propose the fuzzy price of European options. However, the results obtained in this paper are better than the results obtained in Wu [11,12], which will be explained in the final section of conclusions.

Although we consider the fuzzy interest rate, fuzzy volatility and fuzzy stock price in this paper, some of those three input data can still be taken as the real numbers (the real numbers are also called the crisp numbers in the fuzzy literature) if the financial analysts can make sure that some of those three input data occur in a crisp sense. In this case, the methodology proposed in this paper is still applicable, since the real numbers (crisp numbers) are the degenerated case (special case) of fuzzy numbers. Now, under the considerations of fuzzy interest rate, fuzzy volatility and fuzzy stock price, the option price will turn into a fuzzy number. For instance, the price of a European call option can now be interpreted as "around \$3.5". The phrase "around \$3.5" is regarded as a fuzzy number $\widetilde{3.5}$. The fuzzy number $\widetilde{3.5}$ is, in fact, a function defined on $\mathbb{R}$ into $[0,1]$, and denoted by $\mu_{\widetilde{3.5}}(c)$. Given any value c_0, the function value $\mu_{\widetilde{3.5}}(c_0)$ will be interpreted as the belief degree of closeness to the value 3.5. The graph of this function $\mu_{\widetilde{3.5}}(c)$ will be bell-shaped. It means that the closer the value c_0 to 3.5 is, the higher the belief degree is. Therefore, the financial analyst can pick any value which is around 3.5 with an acceptable belief degree as the option price for his/her later financial analysis. In order to obtain the belief degree of any given option price, an optimization problem will be solved. The efficient computational procedures are proposed in this paper to solve this optimization problem.

This paper is organized as follows. In Section 2, the notions of fuzzy number and extension principle are introduced. In Section 3, the notion of fuzzy random variable is introduced for the purpose of considering fuzzy stock price. In Section 4, we apply the extension principle in fuzzy sets theory to the Black–Scholes formula. In Section 5, the computational procedures are provided in order to obtain the belief degrees of any given option prices. We also give an example to clarify the theoretical results and to show the potential applications by applying the fuzzy sets theory to the financial engineering.

2. Fuzzy numbers and extension principle

Let $\mathbb{R}$ be the set of all real numbers endowed with a usual topology. Then a fuzzy subset $\widetilde{A}$ of $\mathbb{R}$ is defined by its membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$. We denote by $\widetilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geqslant \alpha\}$ the α-level set of $\widetilde{A}$ for $\alpha \in (0,1]$. The 0-level set $\widetilde{A}_0$ of $\widetilde{A}$ is defined by the closure of the set $\{x : \mu_{\tilde{A}}(x) > 0\}$. Now $\widetilde{A}$ is called a *normal fuzzy set* if there exists an x such that $\mu_{\tilde{A}}(x) = 1$, and $\widetilde{A}$ is called a *convex fuzzy* set if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geqslant \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ for $\lambda \in [0,1]$; that is, $\mu_{\tilde{A}}$ is a quasi-concave function.

Let f be a real-valued function defined on $\mathbb{R}$. Then f is said to be *upper semicontinuous* if $\{x : f(x) \geqslant \alpha\}$ is a closed set for each α. Or, equivalently, f is upper semicontinuous at y if and only if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $f(x) < f(y) + \epsilon$.

Definition 2.1. Let $\tilde{a}$ be a fuzzy subset of $\mathbb{R}$. Then $\tilde{a}$ is called a *fuzzy number* if the following conditions are satisfied:

(i) $\tilde{a}$ is a normal and convex fuzzy set;
(ii) its membership function $\mu_{\tilde{a}}$ is upper semicontinuous;
(iii) the 0-level set $\tilde{a}_0$ is bounded.

The above condition (iii) also says that each α-level set $\tilde{a}_\alpha$ is bounded for $\alpha \in (0,1]$. From Zadeh [14], $\tilde{a}$ is a convex fuzzy set if and only if each of the α-level set $\tilde{a}_\alpha$ is a convex set. Therefore if $\tilde{a}$ is a fuzzy number, then the α-level set $\tilde{a}_\alpha$ is a compact (closed and bounded in $\mathbb{R}$) and convex set; that is, $\tilde{a}_\alpha$ is a closed interval. Then the α-level set of $\tilde{a}$ is denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. The following proposition is useful for further discussions.

Proposition 2.1 (Resolution identity [15]). *Let $\widetilde{A}$ be a fuzzy set with membership function $\mu_{\tilde{A}}$ and $\widetilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geqslant \alpha\}$. Then*

$$\mu_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\tilde{A}_\alpha}(x),$$

where 1_A is an indicator function of set A, i.e., $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$.

We denote by $\mathscr{F}$ the set of all fuzzy subsets of $\mathbb{R}$. Let $f(x_1, x_2, \ldots, x_n)$ be a real-valued function from $\mathbb{R}^n$ into $\mathbb{R}$ and $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n$ be n fuzzy subsets of $\mathbb{R}$. By the extension principle in Zadeh [15], we can induce a fuzzy-valued function $\tilde{f} : \mathscr{F}^n \to \mathscr{F}$ according to the real-valued function $f(x_1, x_2, \ldots, x_n)$. In other words, $\tilde{f}(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$ is a fuzzy subset of $\mathbb{R}$. Then the membership function of $\tilde{f}(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$ is defined by

$$\mu_{\tilde{f}(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)}(r) = \sup_{\{(x_1,\ldots,x_n): r = f(x_1,\ldots,x_n)\}} \min\left\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)\right\}. \tag{1}$$

The following propositions are very useful for discussing the Black–Scholes formula via the extension principle.

Proposition 2.2 [3]. *Let S be a compact set in $\mathbb{R}^n$. If f is upper semicontinuous on S then f attains maximum over S and if f is lower semicontinuous on S then f attains minimum over S.*

Proposition 2.3. *Let $f(x_1, \ldots, x_n)$ be a real-valued function defined on $\mathbb{R}^n$ and $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n$ be n fuzzy subsets of $\mathbb{R}$. Let $\tilde{f} : \mathscr{F}^n \to \mathscr{F}$ be a fuzzy-valued function induced by $f(x_1, \ldots, x_n)$ via the extension principle defined in (1). Suppose that each membership function $\mu_{\tilde{A}_i}$ is upper semicontinuous on $\mathbb{R}$ for $i = 1, \ldots, n$, and each $\{(x_1, \ldots, x_n) : r = f(x_1, \ldots, x_n)\}$ is a compact subset of $\mathbb{R}^n$ (it will be a closed and bounded set in $\mathbb{R}^n$) for r in the range of f. Then the α-level set of $\tilde{f}(\widetilde{A}_1, \ldots, \widetilde{A}_n)$ is*

$$(\tilde{f}(\widetilde{A}_1, \ldots, \widetilde{A}_n))_\alpha = \{f(x_1, \ldots, x_n) : x_1 \in (\widetilde{A}_1)_\alpha, \ldots, x_n \in (\widetilde{A}_n)_\alpha\}.$$

Proof. If

$$r \in \{f(x_1,\ldots,x_n) : x_1 \in (\widetilde{A}_1)_\alpha,\ldots,x_n \in (\widetilde{A}_n)_\alpha\}$$

then there exists an n-vector $(x_1,\ldots,x_n)$ such that $r = f(x_1,\ldots,x_n)$ and $x_i \in (\widetilde{A}_i)_\alpha$ for all $i = 1,\ldots,n$; that is, $\mu_{\tilde{A}_i}(x_i) \geqslant \alpha$ for all $i = 1,\ldots,n$. Thus $\min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\} \geqslant \alpha$. It says that

$$\mu_{\tilde{f}(\widetilde{A}_1,\ldots,\widetilde{A}_n)}(r) = \sup_{\{(x_1,\ldots,x_n):r=f(x_1,\ldots,x_n)\}} \min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\} \geqslant \alpha.$$

Therefore

$$\{f(x_1,\ldots,x_n) : x_1 \in (\widetilde{A}_1)_\alpha,\ldots,x_n \in (\widetilde{A}_n)_\alpha\} \subseteq (\tilde{f}(\widetilde{A}_1,\ldots,\widetilde{A}_n))_\alpha.$$

Conversely, since each $\mu_{\tilde{A}_i}$ is upper semicontinuous on $\mathbb{R}$ for $i = 1,\ldots,n$, we see that each $U_i = \{x : \mu_{\tilde{A}_i}(x) \geqslant \alpha\}$ is a closed subset of $\mathbb{R}$ for $i = 1,\ldots,n$. Therefore, each

$$C_i = \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{i-1} \times U_i \times \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n-i}$$

is a closed subset of $\mathbb{R}^n$ for $i = 1,\ldots,n$. Since

$$\left\{(x_1,\ldots,x_n) : \min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\} \geqslant \alpha\right\} = \{(x_1,\ldots,x_n) : \mu_{\tilde{A}_i}(x_i) \geqslant \alpha \text{ for all } i = 1,\ldots,n\} = \bigcap_{i=1}^n C_i$$

is a closed subset of $\mathbb{R}^n$, it shows that $\min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\}$ is upper semicontinuous on $\mathbb{R}^n$. Now suppose that $r \in (\tilde{f}(\widetilde{A}_1,\ldots,\widetilde{A}_n))_\alpha$. Then we have

$$\sup_{\{(x_1,\ldots,x_n):r=f(x_1,\ldots,x_n)\}} \min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\} \geqslant \alpha.$$

Since $\{(x_1,\ldots,x_n) : r = f(x_1,\ldots,x_n)\}$ is a compact set and $\min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\}$ is upper semicontinuous, by using Proposition 2.2, there exists an n-vector $(x_1,\ldots,x_n)$ such that $r = f(x_1,\ldots,x_n)$ and $\min_{1\leqslant i\leqslant n}\{\mu_{\tilde{A}_i}(x_i)\} \geqslant \alpha$. Therefore $\mu_{\tilde{A}_i}(x_i) \geqslant \alpha$ for all $i = 1,\ldots,n$; that is, $x_i \in (\widetilde{A}_i)_\alpha$ for all $i = 1,\ldots,n$. It shows that

$$r \in \left\{f(x_1,\ldots,x_n) : x_1 \in (\widetilde{A}_1)_\alpha,\ldots,x_n \in (\widetilde{A}_n)_\alpha\right\}.$$

This completes the proof. □

Proposition 2.4

(i) ([1, p. 82, Theorem 4.25]) *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). If f is continuous on a compace subset X of S, then the image $F(X)$ is a compact subset of T; in particular, $f(X)$ is a closed and bounded in T.*

(ii) ([1, p. 87, Theorem 4.37]) *Let $f : S \to T$ be a function from one metric space S to another metric space T. Let X be a connected subset of S. If f is continuous on X, then $f(X)$ is a connected subset of T.*

Proposition 2.5. *Let $f(x_1,\ldots,x_n)$ be a continuous real-valued function defined on $\mathbb{R}^n$ and $\tilde{a}_1,\ldots,\tilde{a}_n$ be n fuzzy numbers. Let $\tilde{f} : \mathscr{F}^n \to \mathscr{F}$ be a fuzzy-valued function induced by $f(x_1,\ldots,x_n)$ via the extension principle defined in (1). Suppose that each $\{(x_1,\ldots,x_n) : r = f(x_1,\ldots,x_n)\}$ is a compact subset of $\mathbb{R}^n$ for r in the range of f. Then $\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)$ is a fuzzy number and its α-level set is*

$$\begin{aligned}(\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n))_\alpha &= \{f(x_1,\ldots,x_n) : x_1 \in (\tilde{a}_1)_\alpha,\ldots,x_n \in (\tilde{a}_n)_\alpha\} \\ &= \{f(x_1,\ldots,x_n) : (\tilde{a}_1)^L_\alpha \leqslant x_1 \leqslant (\tilde{a}_1)^U_\alpha,\ldots,(\tilde{a}_n)^L_\alpha \leqslant x_n \leqslant (\tilde{a}_n)^U_\alpha\}.\end{aligned}$$

Proof. Let $X_\alpha = \{(x_1,\ldots,x_n) : x_1 \in (\tilde{a}_1)_\alpha,\ldots,x_n \in (\tilde{a}_n)_\alpha\}$. Since $(\tilde{a}_i)_\alpha$ are closed intervals for all $i = 1,\ldots,n$, we see that X_α is a n-dimensional interval, i.e., X_α is a compact and connected subset of $\mathbb{R}^n$. By Propositions 2.3 and 2.4, we see that

$$\{\mu_{\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)}(r) \geqslant \alpha\} = (\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n))_\alpha = \{f(x_1,\ldots,x_n) : (x_1,\ldots,x_n) \in X_\alpha\}$$

is a closed interval, i.e., a convex set. This says that the membership function $\mu_{\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)}(r)$ is upper semicontinuous, the 0-level set of $\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)$ is bounded and $\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)$ is a convex fuzzy set. It is not hard to see that $\tilde{f}(\tilde{a}_1,\ldots,\tilde{a}_n)$ is a normal fuzzy set, since X_1 is nonempty. This completes the proof. □

3. Fuzzy random variables

The stock price S is a stochastic process, i.e., the stock price S_t at time t is a random variable. Therefore if the fuzzy stock price is considered at time t in the Black–Scholes formula, then it is natural to introduce the notion of fuzzy random variable. Roughly speaking, the usual random variable assumes real numbers and the fuzzy random variable assumes fuzzy numbers.

Let $(\Omega, \mathscr{A})$ be a measurable space and $(\mathbb{R}, \mathscr{B})$ be a Borel measurable space. Let $f : \Omega \to \mathscr{P}(\mathbb{R})$ (set of all subsets of $\mathbb{R}$) be a set-valued function. According to Aumann [2], f is called measurable if and only if each of the set $\{(x,y) : y \in f(x)\}$ is $\mathscr{A} \times \mathscr{B}$-measurable. Let $\mathscr{F}(\mathbb{R})$ be the set of all fuzzy numbers. If $\tilde{f} : \Omega \to \mathscr{F}(\mathbb{R})$ is a fuzzy-valued function, then $\tilde{f}_\alpha$ is a set-valued function for all $\alpha \in [0,1]$. Now $\tilde{f}$ is called (fuzzy-valued) measurable if and only if $\tilde{f}_\alpha$ is (set-valued) measurable for all $\alpha \in [0,1]$. Let $(\Omega, \mathscr{A}, P)$ be a probability space (a complete σ-finite measure space). Then $X : \Omega \to \mathbb{R}$ is a random variable if X is $(\mathscr{A}, \mathscr{B})$-measurable function. Thus, it is natural to propose the following definition.

Definition 3.1. Let $\widetilde{X} : \Omega \to \mathscr{F}(\mathbb{R})$ be a fuzzy-valued function. Then $\widetilde{X}$ is called a fuzzy random variable if $\widetilde{X}$ is measurable.

The following proposition holds true [8,5].

Proposition 3.1. *Let $\widetilde{X} : \Omega \to \mathscr{F}(\mathbb{R})$ be a fuzzy-valued function. Then $\widetilde{X}$ is a fuzzy random variable if and only if $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ are random variables for all α.*

4. Option pricing using extension principle

The well-known Black–Scholes formula for European call option written on a stock S with expiry date T and strike price K [4] is described as follows. Let the function f be given by the formula

$$f(s,t,K,r,\sigma) = s \cdot N(d_1) - K \cdot \mathrm{e}^{-rt} \cdot N(d_2),$$

where

$$d_1 = \frac{\ln(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma \cdot \sqrt{t}}$$

and

$$d_2 = d_1 - \sigma \cdot \sqrt{t}$$

and N stands for the cumulative distribution function of a standard normal random variable $N(0,1)$. Let C_t denote the price of a European call option at time $t \in [0,T]$. Then

$$C_t = f(S_t, T-t, K, r, \sigma) \tag{2}$$

for all $t \in [0,T]$. Furthermore, the price P_t of a European put option at time t with the same expiry date T and strike price K can be obtained by the following put-call parity relationship (ref. Musiela and Rutkowski [6])

$$C_t - P_t = S_t - K \cdot \mathrm{e}^{-r(T-t)} \tag{3}$$

for all $t \in [0,T]$.

Under the considerations of fuzzy interest rate $\tilde{r}$, fuzzy volatility $\tilde{\sigma}$ and fuzzy stock price $\widetilde{S}$, we can obtain the fuzzy price $\widetilde{C}_t$ of a European call option at time t according to (2) and extension principle. The membership function of $\widetilde{C}_t$ is then given by

$$\mu_{\widetilde{C}_t}(c) = \sup_{\{(s,r,\sigma):c=f(s,T-t,K,r,\sigma)\}} \min\{\mu_{\widetilde{S}_t}(s), \mu_{\tilde{r}}(r), \mu_{\tilde{\sigma}}(\sigma)\}. \tag{4}$$

From Proposition 2.5, the option price $\widetilde{C}_t$ at time t is a fuzzy number.

According to the put-call parity relationship in (3), the fuzzy price $\widetilde{P}_t$ of a European put option at time t can also be derived similarly. Let

$$g(s,t,K,r,\sigma) = f(s,t,K,r,\sigma) - s + K \cdot \mathrm{e}^{-rt}.$$

Then we can obtain the fuzzy price $\widetilde{P}_t$ of a European put option at time t. The membership function of $\widetilde{P}_t$ is then given by

$$\mu_{\widetilde{P}_t}(p) = \sup_{\{(s,r,\sigma):p=g(s,T-t,K,r,\sigma)\}} \min\{\mu_{\widetilde{S}_t}(s), \mu_{\tilde{r}}(r), \mu_{\tilde{\sigma}}(\sigma)\}.$$

From Proposition 2.5, the option price $\widetilde{P}_t$ at time t is also a fuzzy number. The computational procedures will be given in the next section.

5. Computational methods and example

In this section, we shall provide the computational procedures for the European call and put options. First of all, we discuss the computational procedure for European call option.

Given a European call option price c of the fuzzy price $\widetilde{C}_t$ at time t, we plan to know its membership value α, i.e., its belief degree. If the financial analysts are comfortable with this membership value α, then it will be reasonable to take the value c as the European option price at time t. In this case, the financial analysts can accept the value c as the European option price at time t with belief degree α.

The membership function of fuzzy price $\widetilde{C}_t$ of a European call option at time t is given in (4). From Proposition 2.1, the membership function of $\widetilde{C}_t$ can also be written as

$$\mu_{\widetilde{C}_t}(c) = \sup_{0\leqslant\alpha\leqslant 1} \alpha \cdot 1_{(\widetilde{C}_t)_\alpha}(c), \tag{5}$$

where $(\widetilde{C}_t)_\alpha$ is the α-level set of $\widetilde{C}_t$. From Proposition 2.5 and (4), $\widetilde{C}_t$ is a fuzzy number and its α-level set $(\widetilde{C}_t)_\alpha$ is then given by

$$(\widetilde{C}_t)_\alpha = t\left\{f(s,T-t,K,r,\sigma) : s \in (\widetilde{S}_t)_\alpha, r \in \tilde{r}_\alpha, \sigma \in \tilde{\sigma}_\alpha\right\}. \tag{6}$$

It also says that the α-level set $(\widetilde{C}_t)_\alpha$ is a closed interval

$$(\widetilde{C}_t)_\alpha = \left[(\widetilde{C}_t)^L_\alpha, (\widetilde{C}_t)^U_\alpha\right].$$

Now we see that the α-level sets of $\tilde{r}$, $\tilde{\sigma}$ and $\widetilde{S}$ are $\tilde{r}_\alpha = [\tilde{r}^L_\alpha, \tilde{r}^U_\alpha]$, $\tilde{\sigma}_\alpha = [\tilde{\sigma}^L_\alpha, \tilde{\sigma}^U_\alpha]$, and $\widetilde{S}_\alpha = [\widetilde{S}^L_\alpha, \widetilde{S}^U_\alpha]$, respectively. Therefore, from (6), the left-end point $(\widetilde{C}_t)^L_\alpha$ and right-end point $(\widetilde{C}_t)^U_\alpha$ can be displayed as

$$(\widetilde{C}_t)^L_\alpha = \min_{(\widetilde{S}_t)^L_\alpha \leqslant s \leqslant (\widetilde{S}_t)^U_\alpha, \tilde{r}^L_\alpha \leqslant r \leqslant \tilde{r}^U_\alpha, \tilde{\sigma}^L_\alpha \leqslant \sigma \leqslant \tilde{\sigma}^U_\alpha} f(s,T-t,K,r,\sigma) \tag{7}$$

and

$$(\widetilde{C}_t)^U_\alpha = \max_{(\widetilde{S}_t)^L_\alpha \leqslant s \leqslant (\widetilde{S}_t)^U_\alpha, \tilde{r}^L_\alpha \leqslant r \leqslant \tilde{r}^U_\alpha, \tilde{\sigma}^L_\alpha \leqslant \sigma \leqslant \tilde{\sigma}^U_\alpha} f(s,T-t,K,r,\sigma). \tag{8}$$

According to the sensitivity analysis of Black–Scholes formula, we have

$$\frac{\partial f}{\partial s} = N(d_1) > 0,$$
$$\frac{\partial f}{\partial r} = tK \cdot \mathrm{e}^{-rt} \cdot N(d_2) > 0,$$
$$\frac{\partial f}{\partial \sigma} = s \cdot \sqrt{t} \cdot n(d_1) > 0,$$

where $n(\cdot)$ stands for the probability density function of a standard normal random variable $N(0,1)$. It says that f is increasing with respect to s, r, σ. Therefore, from (7) and (8), we conclude that

$$(\widetilde{C}_t)^L_\alpha = f\left((\widetilde{S}_t)^L_\alpha, T-t, K, \tilde{r}^L_\alpha, \tilde{\sigma}^L_\alpha\right) \quad \text{and} \quad (\widetilde{C}_t)^U_\alpha = f\left((\widetilde{S}_t)^U_\alpha, T-t, K, \tilde{r}^U_\alpha, \tilde{\sigma}^U_\alpha\right). \tag{9}$$

Now from (5), given any European call option price c, its belief degree can be obtained by solving the following optimization problem

$$\begin{array}{lll} (\text{MP1}) & \max & \alpha \\ & \text{subject to} & (\widetilde{C}_t)^L_\alpha \leqslant c \leqslant (\widetilde{C}_t)^U_\alpha, \\ & & 0 \leqslant \alpha \leqslant 1. \end{array}$$

Now time t is fixed, let $\eta(\alpha) = (\widetilde{C}_t)^L_\alpha$ and $\zeta(\alpha) = (\widetilde{C}_t)^U_\alpha$. The optimization problem (MP1) can be rewritten as

$$\begin{array}{lll} (\text{MP2}) & \max & \alpha \\ & \text{subject to} & \eta(\alpha) \leqslant c, \\ & & \zeta(\alpha) \geqslant c, \\ & & 0 \leqslant \alpha \leqslant 1. \end{array}$$

Since $\eta(\alpha) = (\widetilde{C}_t)^L_\alpha \leqslant (\widetilde{C}_t)^U_\alpha = \zeta(\alpha)$, we can discard one of the constraints $\eta(\alpha) \leqslant c$ or $\zeta(\alpha) \geqslant c$ which will be described below.

(i) If $\eta(1) \leqslant c \leqslant \zeta(1)$ then $\mu_{\widetilde{C}_t}(c) = 1$.
(ii) If $c < \eta(1)$ then the constraint $\zeta(\alpha) \geqslant c$ is redundant, since $\zeta(\alpha) \geqslant \zeta(1) \geqslant \eta(1) \geqslant c$ for all $\alpha \in [0,1]$ using the facts that $\zeta(\alpha)$ is decreasing and $\eta(\alpha) \leqslant \zeta(\alpha)$ for all $\alpha \in [0,1]$. Thus the following easier optimization problem will be solved

$$\begin{array}{lll} (\text{MP3}) & \max & \alpha \\ & \text{subject to} & \eta(\alpha) \leqslant c, \\ & & 0 \leqslant \alpha \leqslant 1. \end{array}$$

(iii) If $c > \zeta(1)$ then the constraint $\eta(\alpha) \leqslant c$ is redundant, since $\eta(\alpha) \leqslant \eta(1) \leqslant \zeta(1) \leqslant c$ for all $\alpha \in [0,1]$ using the facts that $\eta(\alpha)$ is increasing and $\eta(\alpha) \leqslant \zeta(\alpha)$ for all $\alpha \in [0,1]$. Thus the following easier optimization problem will be solved

$$\begin{array}{lll} (\text{MP4}) & \max & \alpha \\ & \text{subject to} & \zeta(\alpha) \geqslant c, \\ & & 0 \leqslant \alpha \leqslant 1. \end{array}$$

Since $\eta(\alpha)$ is increasing, problem (MP3) can be solved using the following algorithm (bisection search).

5.1. Bisection search algorithm

Step 1. Let ϵ be the tolerance and α_0 be the initial value. Set $\alpha \leftarrow \alpha_0$, *low* $\leftarrow 0$ and *up* $\leftarrow 1$.
Step 2. Find $\eta(\alpha)$. If $\eta(\alpha) \leqslant c$ then go to Step 3, otherwise go to Step 4.

Step 3. If $c - \eta(\alpha) < \epsilon$ then EXIT and the maximum is α, otherwise set $low \leftarrow \alpha$, $\alpha \leftarrow \frac{low+up}{2}$ and go to Step 2.
Step 4. Set $up \leftarrow \alpha$, $\alpha \leftarrow \frac{low+up}{2}$ and go to Step 2.

For problem (MP4), it is enough to consider the equivalent constraint

$$-\zeta(\alpha) \leqslant -c$$

since $\zeta(\alpha)$ is decreasing, i.e., $-\zeta(\alpha)$ is increasing. Thus the above algorithm is still applicable.

In the sequel, we discuss the computational procedure for European put option. Let $\widetilde{P}_t$ be the fuzzy price of a European put option at time t. Using the same arguments, we see that the α-level set of $\widetilde{P}_t$ is also a closed interval and is denoted by $(\widetilde{P}_t)_\alpha = [(\widetilde{P}_t)_\alpha^L, (\widetilde{P}_t)_\alpha^U]$, where

$$(\widetilde{P}_t)_\alpha^L = \min_{(\widetilde{S}_t)_\alpha^L \leqslant s \leqslant (\widetilde{S}_t)_\alpha^U, \tilde{r}_\alpha^L \leqslant r \leqslant \tilde{r}_\alpha^U, \tilde{\sigma}_\alpha^L \leqslant \sigma \leqslant \tilde{\sigma}_\alpha^U} g(s, T-t, K, r, \sigma) \tag{10}$$

and

$$(\widetilde{P}_t)_\alpha^U = \max_{(\widetilde{S}_t)_\alpha^L \leqslant s \leqslant (\widetilde{S}_t)_\alpha^U, \tilde{r}_\alpha^L \leqslant r \leqslant \tilde{r}_\alpha^U, \tilde{\sigma}_\alpha^L \leqslant \sigma \leqslant \tilde{\sigma}_\alpha^U} g(s, T-t, K, r, \sigma). \tag{11}$$

Now since

$$\frac{\partial g}{\partial s} = -N(-d_1) < 0,$$
$$\frac{\partial g}{\partial r} = tK \cdot \mathrm{e}^{-rt} \cdot (N(d_2) - 1) < 0,$$
$$\frac{\partial g}{\partial \sigma} = s \cdot \sqrt{t} \cdot n(d_1) > 0,$$

(10) and (11) cannot be reduced as the form in (9). Therefore we have to solve the optimization problems (10) and (11) directly for the purpose of obtaining $(\widetilde{P}_t)_\alpha^L$ and $(\widetilde{P}_t)_\alpha^U$. However, the bisection search algorithm is still applicable for obtaining the belief degree, since $(\widetilde{P}_t)_\alpha^L$ is increasing with respect to α, $(\widetilde{P}_t)_\alpha^U$ is decreasing with respect to α and $(\widetilde{P}_t)_\alpha^L \leqslant (\widetilde{P}_t)_\alpha^U$ for all $\alpha \in [0,1]$. Let $\eta(\alpha) = (\widetilde{P}_t)_\alpha^L$ and $\zeta(\alpha) = (\widetilde{P}_t)_\alpha^U$. Then, given a European put option price p of $\widetilde{P}_t$, we can obtain the membership value $\mu_{\widetilde{P}_t}(p)$ of p by applying the bisection search algorithm. As a matter of fact, in Step 2, we shall encounter a difficulty for solving $\eta(\alpha) = (\widetilde{P}_t)_\alpha^L$ or $\zeta(\alpha) = (\widetilde{P}_t)_\alpha^U$ which are presented in (10) and (11). Therefore, we are going to provide a more convenient computational procedure. Before doing that, we adopt the following notations:

$$l(\alpha) = \min\left\{ g\left((\widetilde{S}_t)_\alpha^L, T-t, K, \tilde{r}_\alpha^L, \tilde{\sigma}_\alpha^L\right), g\left((\widetilde{S}_t)_\alpha^U, T-t, K, \tilde{r}_\alpha^U, \tilde{\sigma}_\alpha^U\right)\right\}$$

and

$$u(\alpha) = \max\left\{ g\left((\widetilde{S}_t)_\alpha^L, T-t, K, \tilde{r}_\alpha^L, \tilde{\sigma}_\alpha^L\right), g\left((\widetilde{S}_t)_\alpha^U, T-t, K, \tilde{r}_\alpha^U, \tilde{\sigma}_\alpha^U\right)\right\}.$$

Let us consider the closed interval $A_\alpha = [l(\alpha), u(\alpha)]$. Then we have the following result.

Proposition 5.1. *We have that* $A_\alpha \subseteq (\widetilde{P}_t)_\alpha = [(\widetilde{P}_t)_\alpha^L, (\widetilde{P}_t)_\alpha^U]$.

Proof. Since $(\widetilde{P}_t)_\alpha^L$ is presented in (10), we see that $(\widetilde{P}_t)_\alpha^L \leqslant g((\widetilde{S}_t)_\alpha^L, T-t, K, \tilde{r}_\alpha^L, \tilde{\sigma}_\alpha^L)$ and $(\widetilde{P}_t)_\alpha^L \leqslant g((\widetilde{S}_t)_\alpha^U, T-t, K, \tilde{r}_\alpha^U, \tilde{\sigma}_\alpha^U)$. Therefore, we have $(\widetilde{P}_t)_\alpha^L \leqslant l(\alpha)$. Similarly, we can show that $(\widetilde{P}_t)_\alpha^U \geqslant u(\alpha)$. This completes the proof. □

If there exists an α' such that the family of closed intervals

$$\{A_\alpha = [l(\alpha), u(\alpha)] : \alpha' \leqslant \alpha \leqslant 1\} \text{ is decreasing with respect to } \alpha, \tag{12}$$

i.e., $A_\alpha \subseteq A_\beta$ for $\alpha' \leqslant \beta < \alpha \leqslant 1$, then we are going to invoke the bisection search algorithm by setting $\eta(\alpha) = l(\alpha)$ and $\zeta(\alpha) = u(\alpha)$, and using the initial value $\alpha_0 = \alpha'$. If the final value α^* is obtained from the bisection search algorithm, then it means that

$$\alpha^* = \sup_{\alpha_0 \leqslant \alpha \leqslant 1} \alpha \cdot 1_{A_\alpha}(p).$$

Since the closed interval $(\widetilde{P}_t)_\alpha$ is decreasing with respect to α and the membership value of p is given by

$$\mu_{\tilde{P}_t}(p) = \sup_{0 \leqslant \alpha \leqslant 1} \alpha \cdot 1_{(\widetilde{P}_t)_\alpha}(p).$$

From Proposition 5.1, we have $\mu_{\tilde{P}_t}(p) \geqslant \alpha^*$. Now, we conclude that, instead of solving the difficult problems $\eta(\alpha) = (\widetilde{P}_t)^L_\alpha$ or $\zeta(\alpha) = (\widetilde{P}_t)^U_\alpha$ in Step 2 to obtain $\mu_{\tilde{P}_t}(p)$, we are going to solve the easier problems $\eta(\alpha) = l(\alpha)$ or $\zeta(\alpha) = u(\alpha)$ to obtain α^*. However, we can sure that $\mu_{\tilde{P}_t}(p) \geqslant \alpha^*$. In this case, if the financial analysts are comfortable with this value α^*, then they can take this value p as the European put option price of $\widetilde{P}_t$ for their later decision-making, since the membership value (or belief degree) of p is greater than α^*.

Sometimes, it is really hard to show that the family of closed intervals in (12) is decreasing with respect to α for some α_0. However, (12) can be checked numerically by evaluating many different α values. Although this technique is not so rigorous, it is really helpful in applying the bisection search algorithm. Fortunately, in many cases, (12) is satisfied for $\alpha_0 \geqslant 0.9$. In this case, it will be reasonable and comfortable to accept it, since we are really interested in the higher α-level cases.

For the computational convenience in the following example, we consider the fuzzy input data (parameters) as triangular fuzzy numbers, i.e., the membership functions look like triangles. Let us remark that, in fact, any bell-shaped membership functions are also valid for our computational procedure. The membership function of a triangular fuzzy number $\tilde{a}$ is defined by

$$\mu_{\tilde{a}}(r) = \begin{cases} (r - a_1)/(a_2 - a_1) & \text{if } a_1 \leqslant r \leqslant a_2 \\ (a_3 - r)/(a_3 - a_2) & \text{if } a_2 < r \leqslant a_3 \\ 0 & \text{otherwise,} \end{cases}$$

which is denoted by $\tilde{a} = (a_1, a_2, a_3)$. The triangular fuzzy number $\tilde{a}$ can be expressed as "around a_2" or "being approximately equal to a_2", where a_2 is called the core value of $\tilde{a}$, and a_1 and a_3 are called the left and right spread values of $\tilde{a}$, respectively. The α-level set (a closed interval) of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1-\alpha)a_1 + \alpha a_2, (1-\alpha)a_3 + \alpha a_2];$$

that is,

$$\tilde{a}^L_\alpha = (1-\alpha)a_1 + \alpha a_2 \quad \text{and} \quad \tilde{a}^U_\alpha = (1-\alpha)a_3 + \alpha a_2.$$

Example 5.1. Consider a European call option on a stock with strike price \$30 and with 3 months to expiry. Suppose that the current stock price is around \$33, the stock price volatility is around 10% and the risk-free interest rate is around 5% per annum with continuous compounding. Assume that $t = 0$ and $T = 0.25$. The fuzzy interest rate $\tilde{r}$, fuzzy volatility $\tilde{\sigma}$ and fuzzy stock price $\widetilde{S}_0$ are assumed as triangular fuzzy numbers $\tilde{r} = (0.048, 0.05, 0.052)$, $\widetilde{S}_0 = (32, 33, 34)$ and $\tilde{\sigma} = (0.08, 0.1, 0.12)$, respectively. The current fuzzy price $\widetilde{C}_0$ of a European call option can be obtained. Then the following table gives the belief degrees $\mu_{\widetilde{C}_0}(c)$ for different European call option prices c by solving the optimization problems (MP3) and (MP4) using the computational procedure proposed above.

c	3.18	3.23	3.28	3.33	3.38	3.39	3.44	3.49	3.54	3.59
$\mu_{\widetilde{C}_0}(c)$	0.8010	0.8505	0.8998	0.9492	0.9987	0.9913	0.9420	0.8926	0.8432	0.7938

For example, if the European call option price \$3.33 is taken, then its belief degree is 0.9492. Therefore if the financial analyst is comfortable with this belief degree 0.9492 then, he/she can take this option price \$3.33 for his/her later decision-making. If the European call option price is taken as $c = 3.3813$, then its belief degree will be 1.00. As a matter of fact, if we consider the crisp case by taking $r = 0.05$, $S_0 = 33$ and $\sigma = 0.1$, then the European call option price will be \$3.3813. This situation matches the observation that $c = 3.3813$ will have belief degree 1.00.

The following table gives the α-level closed intervals $(\widetilde{C}_0)_\alpha$ of the fuzzy price $\widetilde{C}_0$ of a European call option.

α	$(\widetilde{C}_0)_\alpha$
0.99	$[3.3712, 3.3914]$
0.98	$[3.3611, 3.4016]$
0.97	$[3.3509, 3.4117]$
0.96	$[3.3408, 3.4218]$
0.95	$[3.3307, 3.4319]$
0.94	$[3.3206, 3.4420]$
0.93	$[3.3105, 3.4522]$
0.92	$[3.3003, 3.4623]$
0.91	$[3.2902, 3.4724]$
0.90	$[3.2801, 3.4825]$

For $\alpha = 0.95$, it means that the call option price will lie in the closed interval $[3.3307, 3.4319]$ with belief degree 0.95.

6. Conclusions

Let us remark that the numerical example adopted in this paper and in Wu [11,12] use the same input fuzzy data. We see that the results obtained in this paper is the same as the results obtained in Wu [12]. It shows that, although the approaches proposed in this paper and Wu [12] are different, the results are identical. On the other hand, the results shown in this paper (or in Wu [12]) is better than that of Wu [11], since the α-level closed interval $(\widetilde{C}_0)_\alpha$ obtained in Wu [11] is wider than the α-level closed interval obtained in this paper. In other words, the uncertaity of fuzzy price obtained in Wu [11] is higher than the uncertainty of fuzzy price obtained in this paper. We prefer to obtain the fuzzy price of Europenan options with lower uncertainty. In the future research, we shall investigate another methodology to obtain the narrower α-level closed intervals for fuzzy price of European options.

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