Math 285 Lecture Note: Week 2

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Lecture 4. First-order Linear Equations: Integrating Factors (Sec 2.1)

In this section, we focus on how to find an explicit solution to the first order linear ODE of the form

$$F(t, y, y') = 0$$

where F is linear. In other words, we consider the following form

$$P(t)\frac{dy}{dt} + Q(t)y = R(t)$$

where $P(t), Q(t), R(t)$ are given functions. For example,

$$t\frac{dy}{dt} - y = t^2e^{-t}.$$

In this case $P(t) = t$, $Q(t) = -1$, and $R(t) = t^2e^{-t}$. The idea of solving this type of ODEs is to use the product rule:

$$\frac{d}{dt}(P(t)y) = P(t)\frac{dy}{dt} + P'(t)y.$$

If we have $P'(t) = Q(t)$, then

$$\begin{aligned} P(t)\frac{dy}{dt} + Q(t)y = \frac{d}{dt}(P(t)y) &= R(t) \\ P(t)y &= \int R(t)\,dt \\ y &= \frac{1}{P(t)}\int R(t)\,dt. \end{aligned}$$

Example 1. Consider an ODE

$$(t^3+1)\frac{dy}{dt} + 3t^2y = \sin t.$$

Since $P(t) = (t^3 + 1)$ and $Q(t) = 3t^2 = P'(t)$, it follows from the previous argument that

$$y = \frac{1}{t^3+1}\int \sin t\,dt = \frac{-\cos t + C}{t^3+1}$$

is a solution to the ODE.

In general, $Q(t)$ may not be the derivative of $P(t)$. Before dealing with general cases, we consider the case where $P(t)$ and $Q(t)$ are constants.

Example 2. Consider an ODE

$$\frac{dy}{dt} + 2y = t.$$

The idea is to multiply a new function $\mu(t)$

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = t\mu(t).$$

If we have $\mu'(t) = 2\mu(t)$, then we can apply the previous technique. To find such a function μ, we solve the ODE

$$\begin{aligned} \frac{1}{\mu}\frac{d\mu}{dt} &= 2 \\ \ln|\mu(t)| &= 2t + C \\ \mu(t) &= Ce^{2t}. \end{aligned}$$

Let $\mu(t) = e^{2t}$, then the original ODE can be written as

$$\begin{aligned} e^{2t}\frac{dy}{dt} + 2e^{2t}y = \frac{d}{dt}(e^{2t}y) &= te^{2t} \\ e^{2t}y &= \int te^{2t}\,dt \\ &= \frac{1}{2}(te^{2t} - \int e^{2t}\,dt) \\ &= \frac{1}{4}(2te^{2t} - e^{2t} + C) \end{aligned}$$

and so

$$y = \frac{1}{4}(2t - 1 + Ce^{-2t}).$$

Example 3. Consider an ODE

$$y' - 3y = \cos t, \qquad y(0) = 0.$$

Solving the auxiliary ODE

$$\frac{d\mu}{dt} = -3\mu,$$

we let $\mu(t) = e^{-3t}$. Then the original ODE gives

$$\begin{aligned} \mu(t)y' - 3\mu(t)y = \frac{d}{dt}(\mu(t)y) &= \mu(t)\cos t \\ e^{-3t}y &= \int e^{-3t}\cos t\,dt \\ &= \frac{1}{10}e^{-3t}(\sin t - 3\cos t) + C \\ y(t) &= \frac{1}{10}(\sin t - 3\cos t) + Ce^{3t}. \end{aligned}$$

Since

$$y(0) = -\frac{3}{10} + C = 0,$$

we get

$$y(t) = \frac{1}{10}(\sin t - 3\cos t + 3e^{3t}).$$

We are ready to discuss how to solve a first order linear ODE

$$P(t)\frac{dy}{dt} + Q(t)y = R(t).$$

By dividing $P(t)$ of both sides, we consider a first order linear ODE of the standard form

$$\frac{dy}{dt} + p(t)y = r(t)$$

where $p(t), r(t)$ are given. We introduce a new function $\mu(t)$ and multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)r(t).$$

We want to find $\mu(t)$ such that

$$\frac{d}{dt}\mu(t) = \mu(t)p(t).$$

Indeed, we have

$$\frac{1}{\mu(t)}\frac{d}{dt}\mu(t) = \frac{d}{dt}(\ln|\mu(t)|) = p(t)$$

and

$$\ln|\mu(t)| = \int p(t)\,dt.$$

Let $\mu(t) = \exp(\int p(t)\,dt)$, then $\mu'(t) = \mu(t)p(t)$. Thus,

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)\frac{dy}{dt} + \mu'(t)y = (\mu(t)y)' = \mu(t)r(t).$$

Therefore, we get

$$y = \frac{1}{\mu(t)}\int \mu(t)r(t)\,dt.$$

Example 4. Consider

$$t\frac{dy}{dt} - y = t^2e^{-t}.$$

Dividing by t of both sides, we get

$$\frac{dy}{dt} - \frac{1}{t}y = te^{-t}$$

and so $p(t) = -\frac{1}{t}$ and $r(t) = te^{-t}$. The previous argument yields

$$\begin{aligned}
\ln|\mu(t)| &= \int p(t)\,dt = -\int \frac{1}{t}\,dt = -\ln|t| + C,\\
\mu(t) &= \frac{C}{t}
\end{aligned}$$

where C is an arbitrary constant. Thus, solutions of the equation are

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\int \mu(t)r(t)\,dt\\
&= \frac{1}{C}t\int \frac{C}{t}te^{-t}\,dt\\
&= t\int e^{-t}\,dt\\
&= t(-e^{-t} + C).
\end{aligned}$$

Example 5. Consider an ODE

$$(\cos t)\frac{dy}{dt} + (\sin t)y = \cos^3 t, \qquad y(0) = 2$$

for $t \in (-\pi/2, \pi/2)$. As before, we consider

$$\mu(t)\frac{dy}{dt} + \mu(t)\tan t y = \mu(t)\cos^2 t.$$

Then, the auxiliary ODE is

$$\begin{aligned}
\frac{d\mu}{dt} &= \mu(t)\tan t \\
\ln|\mu(t)| &= \int \tan t\, dt \\
&= \ln|\sec t| + C \\
\mu(t) &= C\sec t.
\end{aligned}$$

Simply, we put $mu(t) = \sec t$ then

$$\begin{aligned}
\mu(t)\frac{dy}{dt} + \mu(t)\tan t y = \frac{d}{dt}(\mu(t)y) &= \sec t\cos^2 t = \cos t \\
y = \cos t \int \cos t\, dt &= \cos t(\sin t + C).
\end{aligned}$$

Since $y(0) = C = 2$, we obtain

$$y = (\sin t + 2)\cos t.$$

Lecture 5. First-order Nonlinear Equations: Separable Equations (Sec 2.2)

In this section, we discuss how to solve nonlinear first order ODEs. Previously, we have seen an ODE of the form

$$\frac{dy}{dt} = F(y).$$

The idea was to bring $F(y)$ to the other side and apply the Chain rule, which leads to

$$\frac{d}{dt}(G(y)) = \frac{1}{F(y)}\frac{dy}{dt} = 1$$

and so $G(y) = t + C$. This method indeed works for a more general ODE. Consider a first order ODE of the form

$$\frac{dy}{dt} = F(t, y).$$

where $F(t, y)$ is a product of functions $F_1(t)$ and $F_2(y)$. Then,

$$\frac{1}{F_2(y)}\frac{dy}{dt} = F_1(t).$$

If we find a function G such that $G'(y) = \frac{1}{F_2(y)}$, then

$$\begin{aligned}
\frac{d}{dt}(G(y)) &= F_1(t), \\
G(y) &= \int F_1(t)\, dt.
\end{aligned}$$

Example 6. Consider an ODE

$$y' = \frac{x^2 y}{1+x^3}.$$

Then,

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2}{1+x^3}.$$

To apply the chain rule, we find a function $G(y)$ such that

$$G'(y) = \frac{1}{y}.$$

By integrating of the both sides, we get

$$G(y) = \ln|y| + C.$$

Let $C = 0$, then

$$\ln|y| = \int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\ln|1+x^3| + C = \ln(e^C|1+x^3|^{\frac{1}{3}}).$$

Thus, the solution is

$$y = C|1+x^3|^{\frac{1}{3}}$$

This method can be understood in terms of differential forms. We can rewrite the previous form of ODEs as

$$\begin{aligned}\frac{dy}{dx} &= F_1(x)F_2(y)\\ \frac{1}{F_2(y)}\,dy &= F_1(x)\,dx\\ -F_1(x)\,dx + \frac{1}{F_2(y)}\,dy &= 0.\end{aligned}$$

So, we simply consider an ODE of the form

$$M(x)\,dx = N(y)\,dy.$$

In this case, we take integration of both sides with respect to x and y respectively, which yields

$$\int M(x)\,dx = \int N(y)\,dy.$$

Such an equation is said to be separable.

Example 7. Consider an ODE

$$xdx + ye^{-x}dy = 0, \qquad y(0) = 1.$$

Then,

$$\begin{aligned}xe^x dx &= -ydy\\ \int xe^x dx &= -\int ydy\\ (x-1)e^x &= -\frac{1}{2}y^2 + C\\ y^2 &= 2(1-x)e^x + C\\ y &= \pm\sqrt{2(1-x)e^x + C}.\end{aligned}$$

Since $y(0) = 1$, the sign is plus and we get

$$y(0) = 1 = \sqrt{2 + C},$$

which yields $C = -1$. Therefore, the solution is

$$y = \sqrt{2(1 - x)e^x - 1}.$$

Lecture 6. First-order Nonlinear Equations: Further Discussion (Sec 2.2, 2.4)

Last time, we have seen that if we have a separable equation $y' = F(x)G(y)$ or $M(x)dx + N(y)dy = 0$, then we can find a solution.

Example 8. Consider

$$y' = \frac{2x}{y + x^2 y} = \frac{2x}{1 + x^2} y.$$

Then,

$$y dy = \frac{2x}{1 + x^2} dx$$

and so $y^2 = 2\ln(1 + x^2) + C$.

We have seen how to find a solution of an ODE that is separable. In this section, we discuss other cases where we can find a solution even though the ODE is not separable nor linear.

Example 9 (Homogeneous equations)**.** We call an ODE $y' = F(x, y)$ is homogenous if $F(tx, ty) = F(x, y)$ for all $t \neq 0$. In this case, we can replace $F(x, y)$ with $F(1, y/x)$. Consider an ODE

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}.$$

This is not separable but we can make it separable by introducing a new variable. Let $v = y/x$, then the RHS can be written as

$$\frac{x^2 + xy + y^2}{x^2} = 1 + v + v^2.$$

On the other hands, we have $xv = y$ and so

$$v + x\frac{dv}{dx} = \frac{dy}{dx}.$$

Thus, we get

$$\begin{aligned}
x\frac{dv}{dx} &= 1 + v^2 \\
\frac{1}{1 + v^2} dv &= \frac{1}{x} dx \\
\arctan(v) &= \ln|x| + C \\
v(x) &= \tan(\ln|x| + C) \\
y(x) &= x\tan(\ln|x| + C).
\end{aligned}$$

Example 10 (Bernoulli equations)**.** Consider an ODE

$$y' + p(t)y = q(t)y^n.$$

If $n = 0, 1$, then it is linear so that we can solve it. Suppose $n \neq 0, 1$. First, $y(t) = 0$ is a trivial solution. Suppose $y(t) \neq 0$. Dividing y^n of the both sides,

$$y^{-n}y' + p(t)y^{1-n} = q(t).$$

Let $v = y^{1-n}$, then $v' = (1-n)y^{-n}y'$ and so the ODE can be written as

$$\frac{1}{1-n}v' + p(t)v = q(t),$$

which is solvable. For example, let $y' + y = xy^2$, then for $v = y^{-1}$ we have

$$v' - v = -x.$$

Thus,

$$v = -e^t \int xe^{-t}\,dt = x + 1 + Ce^t.$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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