Trees of Primitive Pythagorean Triples

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Abstract

All and only primitive Pythagorean triples are generated by three trees of Firstov, among which are the UAD tree of Berggren et al. and the Fibonacci boxes FB tree of Price.

Alternative proofs are offered here for the conditions on primitive Pythagorean triple preserving matrices and that there are only three trees with a fixed set of matrices and single root.

Some coordinate and area results are obtained for the UAD tree. Further trees with varying children are possible, such as filtering the Calkin-Wilf tree of rationals.

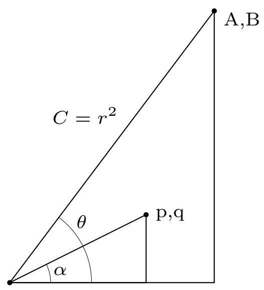
Contents

1 Pythagorean Triples

A Pythagorean triple is integers A,B,C satisfying $A^2 + B^2 = C^2$. These triples can be parameterized by two integers p,q as from Euclid[6],

$$\begin{aligned}
A &= p^2 - q^2 && A \text{ leg odd} \\
B &= 2pq && B \text{ leg even} \\
C &= p^2 + q^2 && \text{hypotenuse} \\
p = \sqrt{\tfrac{C+A}{2}} & \qquad q = \sqrt{\tfrac{C-A}{2}} && \text{converse}
\end{aligned} \tag{1}$$

If A,B and p,q are treated as points in the plane with polar coordinates C, θ and r, α respectively then they are related as



$$\begin{aligned}
C &= r^2 && \text{square distance} \\
\theta &= 2\alpha && \text{double angle}
\end{aligned} \tag{2}$$

This is from complex squaring giving 2α and r^2,

$$\begin{aligned}
(r\, e^{\alpha i})^2 &= r^2\, e^{2\alpha i} \\
(p + iq)^2 &= (p^2 - q^2) + 2pqi = A + Bi
\end{aligned}$$

Or the angle is since the ratio B/A written in terms of q/p is the tan double-angle formula,

$$\tan\theta = \frac{B}{A} = \frac{2pq}{p^2 - q^2} = \frac{2(q/p)}{1 - (q/p)^2} = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha \tag{3}$$

1.1 Primitive Pythagorean Triples

A primitive triple has $\gcd(A, B, C) = 1$. All triples are a multiple of some primitive triple. For A,B,C to be a primitive triple with positive A,B the parameters p,q must be

$$\begin{aligned}
p &> q && \text{so } p \geq 2 \\
q &\geq 1 \\
p + q &\equiv 1 \bmod 2 && \text{opposite parity} \\
\gcd(p, q) &= 1 && \text{coprime}
\end{aligned} \tag{4}$$

A 3×3 matrix can be applied by left multiplication to transform a triple to a new triple.

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$$

A 2×2 matrix can do the same on p,q.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Palmer, Ahuja and Tikoo[14] give the following formula for the 3×3 matrix corresponding to a 2×2. They show the correspondence is one-to-one.

$$\begin{pmatrix} \frac{(a^2-c^2)-(b^2-d^2)}{2} & ab-cd & \frac{(a^2-c^2)+(b^2-d^2)}{2} \\ ac-bd & ad+bc & ac+bd \\ \frac{(a^2+c^2)-(b^2+d^2)}{2} & ab+cd & \frac{(a^2+c^2)+(b^2+d^2)}{2} \end{pmatrix} \tag{5}$$

So finding triple preserving matrices can be done in either 3×3 or 2×2. Generally the 2×2 is more convenient and it reduces primitive triples to pairs p,q coprime and not both odd.

$$\begin{tikzcd}
\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} p^2-q^2 \\ 2pq \\ p^2+q^2 \end{pmatrix} \arrow[rr, "3\times 3"'] \arrow[d, leftrightarrow] & {} \arrow[d, leftrightarrow, dashed] & \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} p'^2-q'^2 \\ 2p'q' \\ p'^2+q'^2 \end{pmatrix} \arrow[d, leftrightarrow] \\
\begin{pmatrix} p \\ q \end{pmatrix} \arrow[rr, "2\times 2", "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] & {} & \begin{pmatrix} p' \\ q' \end{pmatrix}
\end{tikzcd}$$

2 UAD Tree

The UAD tree by Berggren (1934)[3] and independently Barning (1963)[2], Hall (1970)[8], Kanga (1990)[10] and Alperin (2005)[1], uses three matrices U,A,D to make a tree of all primitive Pythagorean triples.

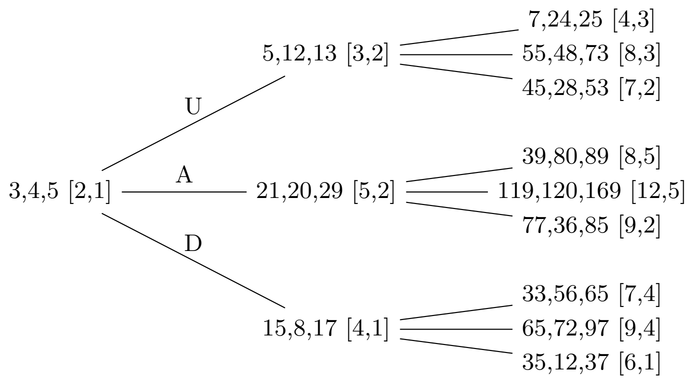


Figure 1: UAD tree, triples and [p,q] pairs

$$\begin{aligned} U &= \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix} & A &= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} & D &= \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} & &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} & &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{6}$$

The pairs shown $[2,1]$ etc are the p,q values corresponding to each triple. The matrices multiply onto a column vector of the triplet or pair, so for example the last descent from 15,8,17 and 4,1 has

$$U\begin{pmatrix}15\\8\\17\end{pmatrix}=\begin{pmatrix}33\\56\\65\end{pmatrix}\qquad U\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$$

The 2×2 UAD matrices send a given p,q to the disjoint regions of figure 2.

$$\begin{array}{lllr}
U & p'=2p-q\ <\ 2p=2q' & \text{so } p'<2q' & (7)\\
A & p'=2p+q\ >\ 2p=2q' & & \\
 & p'=2p+q\ <\ 3p=3q' & \text{so } 2q'<p'<3q' & \\
D & p'=p+2q\ >\ 3q=3q' & \text{so } p'>3q' & (8)
\end{array}$$

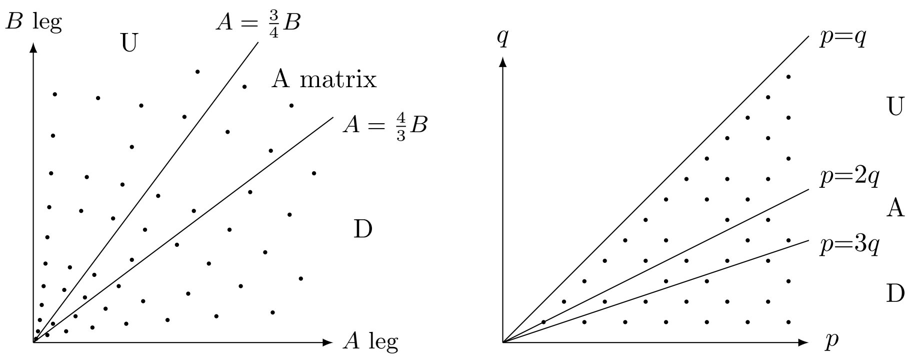


Figure 2: UAD tree regions of A,B legs and p,q points

Point 2,1 is on the $p=2q$ line and is the root. Thereafter $p=2q$ does not occur since it would have common factor q. The line $p=3q$ is never touched at all since 3,1 is not opposite parity and anything bigger would have common factor q.

A,B leg points are double the angle of a p,q as per (2) so a ratio for p,q becomes a ratio for the A,B legs too and hence the 3×3 UAD matrices fall in disjoint regions similarly.

If $p=kq$ then as from the double-angle (3),

$$B/A=\frac{2(q/p)}{1-(q/p)^2}=\frac{2/k}{1-(1/k)^2}\qquad\text{as } q/p=1/k$$

$$A=\frac{k^2-1}{2k}B$$

$$\begin{array}{lll}
k=2\text{ gives} & A=\frac{3}{4}B & \text{U matrix region}\\
k=3\text{ gives} & A=\frac{4}{3}B & \text{A matrix region}
\end{array}$$

Figure 3 shows how the 2×2 matrices transform a vertical line of p,q points. The dashed line $k,1$ through $k,k{-}1$ becomes the solid lines by the respective U,A,D matrices.

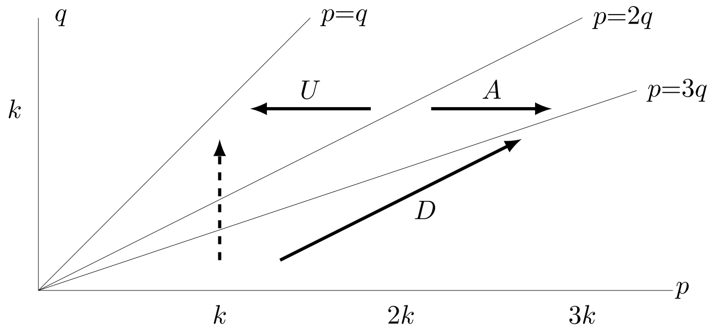


Figure 3: UAD tree, line transformations

D is a shear, as can be seen from its matrix $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$. The line for D is longer but has the same number of coprime not-both-odd points as the original dashed line.

U is a rotate $+90°$ then shear across to $p = 2q$. This shear is the same as D.

$$\begin{aligned} U &= \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} && \text{rotate then shear} \\ &= D\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \end{aligned} \tag{9}$$

A is the same as U but with a reflection to go to the right. This reflection $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ gives the negative determinant $\det(A) = -1$.

$$\begin{aligned} A &= U\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} && \text{mirror then same as } U \\ &= D\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} && \text{mirror, rotate, shear} \\ &= D\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \end{aligned} \tag{10}$$

2.1 UAD Tree Row Totals

Theorem 1. *In the UAD tree the total leg difference $d_n = \sum B - A$ across a row is $d_n = (-1)^n$.*

Proof. Depth 0 is the single point A=3, B=4 and its leg difference is $B - A = 1 = (-1)^0$.

Let $(A_1, B_1, C_1) \ldots (A_k, B_k, C_k)$ be the triples at depth $n-1$. The children of those points at depth n are

$$U\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, A\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, D\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, \ \ldots,\ U\begin{pmatrix}A_k\\B_k\\C_k\end{pmatrix}, A\begin{pmatrix}A_k\\B_k\\C_k\end{pmatrix}, D\begin{pmatrix}A_k\\B_k\\C_k\end{pmatrix}$$

Their total is

$$\begin{pmatrix}\alpha_n\\ \beta_n\\ \gamma_n\end{pmatrix} = (U{+}A{+}D)\left(\begin{pmatrix}A_1\\ B_1\\ C_1\end{pmatrix} + \cdots + \begin{pmatrix}A_k\\ B_k\\ C_k\end{pmatrix}\right) = \begin{pmatrix}1&2&6\\2&1&6\\2&2&9\end{pmatrix}\begin{pmatrix}\alpha_{n-1}\\ \beta_{n-1}\\ \gamma_{n-1}\end{pmatrix}$$

So the difference

$$\begin{aligned} d_n &= \beta_n - \alpha_n \\ &= (2\alpha_{n-1} + \beta_{n-1} + 6\gamma_{n-1}) - (\alpha_{n-1} + 2\beta_{n-1} + 6\gamma_{n-1}) \\ &= \alpha_{n-1} - \beta_{n-1} \\ &= -d_{n-1} \end{aligned}$$

□

Figure 4 shows the geometric interpretation of this in a plot of legs A and B. The leg difference $A - B$ is the distance from the leading diagonal $A{=}B$.

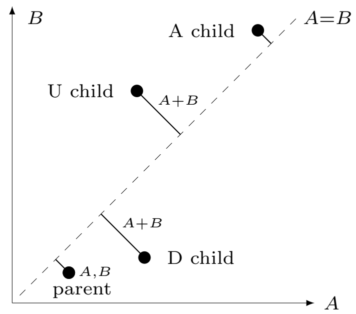


Figure 4: UAD children leg difference symmetry

For the U and D children the leg difference is $A + B$ of the original parent A,B point.

$$\begin{aligned} A'_U &= A - 2B + 2C && U \text{ matrix child legs} \\ B'_U &= 2A - B + 2C \\ B'_U - A'_U &= A + B && U \text{ child leg difference} \\[4pt] A'_D &= -A + 2B + 2C && D \text{ matrix child legs} \\ B'_D &= -2A + B + 2C \\ B'_D - A'_D &= -(A + B) && D \text{ child leg difference} \end{aligned}$$

The leg differences of U and D are opposite sign and so cancel out in the total,

$$(B'_U - A'_U) + (B'_D - A'_D) = 0$$

Matrix A mirrors a given A,B leg pair across the leading diagonal. It changes the sign of the leg difference but not the magnitude. On further descent this mirroring is applied to both the U and D children so their leg differences continue to cancel out in further tree levels.

$$\begin{aligned} A'_A &= A + 2B + 2C && A \text{ matrix child legs} \\ B'_A &= 2A + B + 2C \\ B'_A - A'_A &= -(B - A) \end{aligned}$$

The initial triple 3,4,5 with leg difference $4 - 3 = 1$ alternates 1 and -1 as the A matrix is repeatedly applied to it at each tree level. For all other children the leg differences cancel out.

Theorem 2. *In the UAD tree the total $P_n = \sum p$ and $Q_n = \sum q$ in row n are successive terms of the Pell sequence L_0=0, L_1=1, $L_i = 2L_{i-1} + L_{i-2}$.*

$$P_n = \sum_{depth=n} p \ = \ L_{2n+2} \qquad \textit{even Pell} \tag{11}$$

$$Q_n = \sum_{depth=n} q \ = \ L_{2n+1} \qquad \textit{odd Pell} \tag{12}$$

$$\begin{aligned} Q_0 &= 1 = L_1 \\ P_0 &= 2 = L_2 \\ Q_1 &= 2 + 2 + 1 = 5 = L_3 \\ P_1 &= 3 + 5 + 4 = 12 = L_4 \qquad \textit{etc} \end{aligned}$$

Proof. Depth 0 is the single point p=2,q=1 and its total is $p = 2 = L_2$ and $q = 1 = L_1$.

Let p_1,q_1 $\ldots p_k$,q_k be the points at depth n. The children of those points are

$$U\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, A\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, D\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \ \ldots, \ U\begin{pmatrix} p_k \\ q_k \end{pmatrix}, A\begin{pmatrix} p_k \\ q_k \end{pmatrix}, D\begin{pmatrix} p_k \\ q_k \end{pmatrix}$$

Their sum is

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = (U + A + D)\left(\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + \cdots + \begin{pmatrix} p_k \\ q_k \end{pmatrix}\right) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} P_n \\ Q_n \end{pmatrix}$$

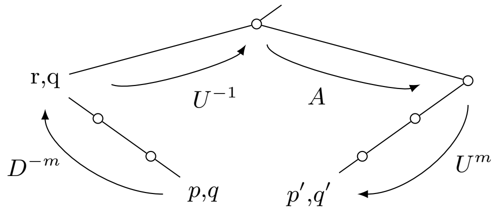
$Q_{n+1} = 2P_n + Q_n$ is the Pell recurrence $L_{2n+3} = 2L_{2n+2} + L_{2n+1}$.
$P_{n+1} = 5P_n + 2Q_n$ is the Pell recurrence L_{2n+4} in terms of L_{2n+2} and L_{2n+1},

$$\begin{aligned} L_{2n+4} &= 2L_{2n+3} + L_{2n+2} \\ &= 2(2L_{2n+2} + L_{2n+1}) + L_{2n+2} \\ &= 5L_{2n+2} + 2L_{2n+1} \end{aligned}$$

□

2.2 UAD Tree Iteration

The points of the UAD tree can be iterated row-wise by a method similar to that of Newman[13] for the Calkin-Wilf tree. A division finds how many trailing D legs are on a given p,q, and from that go up, across, and back down.



Theorem 3. *The next point after p,q going row-wise across the UAD tree is*

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{cases} \begin{pmatrix} m+3 \\ m+2 \end{pmatrix} & \textit{if } q = 1 \\ \begin{pmatrix} -(m+1) & 3m+4 \\ -m & 3m+1 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} & \textit{if } r < 2q \\ \begin{pmatrix} m+2 & -(m+3) \\ m+1 & -(m+2) \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} & \textit{if } r > 2q \end{cases}$$

where m and r are a quotient and remainder from dividing $p{-}q$ by $2q$

$$\begin{aligned} m &= \left\lfloor \frac{p-q}{2q} \right\rfloor \\ p &= (2q)m + r \qquad \textit{with } q < r < 3q \end{aligned}$$

Proof. As per (8) a final D leg has $p > 3q$ whereas a final U or A has $p < 3q$. Multiple final D steps are

$$D^m = \begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix} \qquad D^m \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p + (2m)q \\ q \end{pmatrix}$$

The division p by $2q$ finds the number of those final D steps on p,q. This is 0 when p,q ends on a U or A leg.

Point r,q is up the tree by those trailing D steps. r,q is either a U leg or an A leg. They are distinguished by $r < 2q$ for U or $r > 2q$ for A as per (7). For a U leg go across from U to A with $A.U^{-1}$ and then down U^m so

$$U^m.A.U^{-1} \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} -(m+1) & 3m+4 \\ -m & 3m+1 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix}$$

For an A leg step across from A to D with $D.A^{-1}$ and then down U^m so

$$U^m.D.A^{-1} \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} m+2 & -(m+3) \\ m+1 & -(m+2) \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix}$$

If p,q is the last point of a row then it is entirely D steps and has r=2, q=1 as from $\binom{p}{q} = D^m \binom{2}{1}$. This case is identified by q=1. This is the only time $r = 2q$ since otherwise q would be a common factor between p and q. The next point is the first of the next row m+1 so

$$U^{m+1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} m+2 & -(m+1) \\ m+1 & -m \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} m+3 \\ m+2 \end{pmatrix} \qquad \square$$

The calculation can also be made with a division $p/2q$ rather than $(p-q)/2q$ shown above.

$$
\begin{aligned}
m' &= \left\lfloor \frac{p}{2q} \right\rfloor \\
p &= (2q)m' + r' && \text{with } 0 \le r' \le 2q
\end{aligned}
$$

The effect is that when r,q is an A leg m' is 1 bigger than m and r' correspondingly $2q$ smaller than r. The distinguishing condition for U or A becomes $r' > q$ and the A case changes so

$$
\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{cases}
\begin{pmatrix} m'+3 \\ m'+2 \end{pmatrix} & \text{if } q = 1 \\
\begin{pmatrix} -(m'+1) & 3m'+4 \\ -m' & 3m'+1 \end{pmatrix} \begin{pmatrix} r' \\ q \end{pmatrix} & \text{if } r' > q \\
\begin{pmatrix} m'+1 & m' \\ m' & m'-1 \end{pmatrix} \begin{pmatrix} r' \\ q \end{pmatrix} & \text{if } r' < q
\end{cases}
$$

2.3 UAD Tree Low to High

A complete ternary tree has 3^n points at depth n. The position in the row can be written in ternary with n digits. The tree descent U,A,D at each node follows those digits from high to low, ie. most significant to least significant.

If the digits are instead taken low to high then figure 5 below shows how the tree branches remain in the separate regions from figure 2. This can be thought of recursively as the tree at a given point being a copy of the whole tree but with the matrices which reach that point applied to that whole tree.

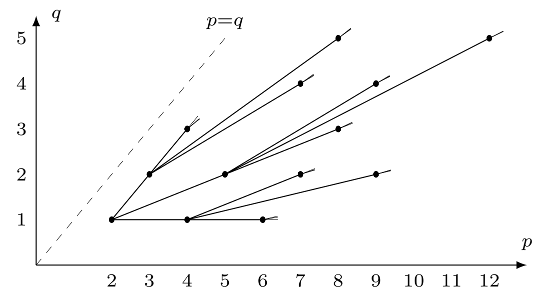


Figure 5: UAD tree branches, digits low to high

3 UArD Tree

A variation on the UAD tree can be made by applying a left-right mirror image reflection under each A matrix. Call this UArD. The three children at each node are the same but the order is reversed when under an odd number of A legs.

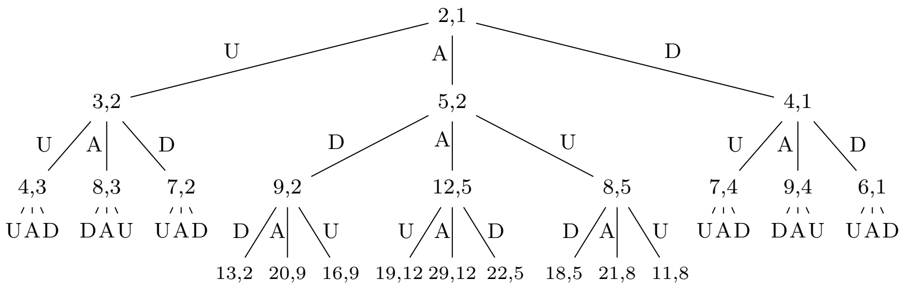


Figure 6: UArD tree, p,q pairs

The entire sub-tree under each A is mirrored. Under an even number of A matrices the mirrorings cancel out to be plain again. For example the middle 12,5 shown above is under A,A and so its children are U-A-D again.

The effect of these mirrorings is to apply matrices by ternary reflected Gray code digits high to low. For example the row shown at depth 2 goes

U-A-D, D-A-U, U-A-D

Theorem 4. *Across a row of the UArD tree the p,q points take steps alternately horizontal (q unchanged) and diagonal (same increment p and q).*

Figure 7 below illustrates the steps for the row at depth 2. The total distances H_2 and J_2 are for theorem 5 below.

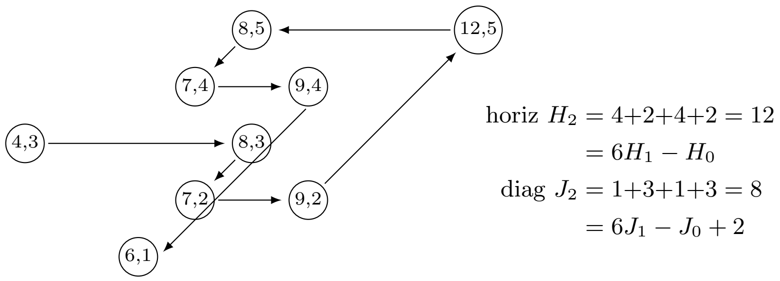


Figure 7: UArD steps of p,q in row at depth 2

Proof of Theorem 4. A step between U and A children is always horizontal since

$$A\begin{pmatrix}p\\q\end{pmatrix} - U\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2q\\0\end{pmatrix} \tag{13}$$

A step between A and D children is always diagonal since

$$A\begin{pmatrix}p\\q\end{pmatrix} - D\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}p-q\\p-q\end{pmatrix} \tag{14}$$

A UArD row steps between children in sequence either U-A-D or D-A-U and with each always followed by its reversal.

```
           p,q
    /       |       \
   U        A        D
  /|\      /|\      /|\
 U A D    D A U    U A D
```

When going across a D-D gap the common ancestor is U-A since D is at the right edge of the plain sub-tree beneath U and the left edge of the reflected sub-tree beneath A. This U-A ancestor is a horizontal step (13). The D descents multiply on the left and preserve the horizontal step and its distance since a horizontal p to $p+x$ has

$$D\begin{pmatrix}p+x\\q\end{pmatrix} - D\begin{pmatrix}p\\q\end{pmatrix} = D\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} \tag{15}$$

When going across a U-U gap the common ancestor is A-D since U is the right edge of the reflected sub-tree beneath A and the left edge of the plain sub-tree beneath D. This A-D ancestor is a diagonal step (14). The U descents multiply on the left and preserve the diagonal and its distance since

$$U\begin{pmatrix}p+x\\q+x\end{pmatrix} - U\begin{pmatrix}p\\q\end{pmatrix} = U\begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}2&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}x\\x\end{pmatrix} \tag{16}$$

So sequence U-A-D-D-A-U-U-A-D etc across a row takes steps alternately horizontal and diagonal. $\square$

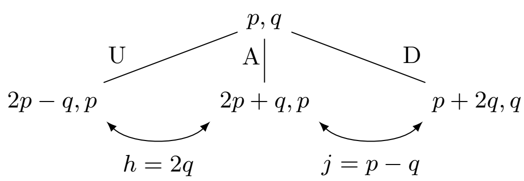
Theorem 5. *In row n of the UArD tree the total distance H_n of horizontal p,q steps is a cumulative $2Q_i$ and hence an even index Pell number,*

$$\begin{aligned} H_n &= 2\sum_{i=0}^{i=n-1} Q_i \;=\; P_{n-1} \;=\; L_{2n} && \textit{cumulative } Q\textit{, even Pell}\\ &= 6H_{n-2} - H_{n-1} \quad \textit{starting } H_0{=}0,\ H_1{=}2 && \textit{recurrence}\\ &= 2, 12, 70, 408, 2378, 13860, 80782, \ldots \end{aligned}$$

The total distance J_n of diagonal p,q steps is cumulative $P_i - Q_i$ and hence a cumulative difference of Pell numbers

$$\begin{aligned} J_n &= \sum_{i=0}^{i=n-1} P_i - Q_i \;=\; \sum_{i=1}^{i=n} L_{2i} - L_{2i-1} && \textit{cumulative diffs}\\ &= 6J_{n-2} - J_{n-1} + 2 \quad \textit{starting } J_0{=}0,\ J_1{=}1 && \textit{recurrence}\\ &= 1, 8, 49, 288, 1681, 9800, 57121, \ldots \end{aligned}$$

Proof. Let h and j be the horizontal and diagonal distances between the three children of a p,q point,



The total horizontal H_n at depth n is the U-A steps $h = 2q$ from the points at depth $n{-}1$, plus the D-D steps in depth $n{-}1$. The D-D steps are U-A propagated down from preceding depth levels per (15). This means $2q$ of all points in all preceding depth levels,

$$\begin{aligned}
H_n &= \sum_{\substack{\text{depths}\\ 0 \text{ to } n-1}} 2q && q \text{ of all earlier points} \\
&= 2\sum_{i=0}^{n-1} Q_i && Q_i \text{ as from theorem 2} \\
&= 2\sum_{i=0}^{n-1} L_{2i+1} && Q_i \text{ is odd Pells (12)} \\
&= L_{2n} && \text{usual odd Pell summation}
\end{aligned}$$

The total diagonal J_n at depth n is the A-D steps $j = p - q$ from the points at depth $n{-}1$, plus the U-U steps in depth $n{-}1$. The U-U steps are A-D propagated down from preceding depth levels per (16). This means $p - q$ of all points in all preceding depth levels,

$$\begin{aligned}
J_n &= \sum_{\substack{\text{depths}\\ 0 \text{ to } n-1}} p - q && p - q \text{ of all earlier points} \\
&= \sum_{i=0}^{n-1} P_i - Q_i && P_i, Q_i \text{ as from theorem 2} \\
&= \sum_{i=0}^{n-1} L_{2i+2} - L_{2i+1} && \text{Pells per (11)(12)} \qquad \square
\end{aligned}$$

The UArD row lines variously overlap. Figure 8 shows all row lines plotted together (truncated above and right).

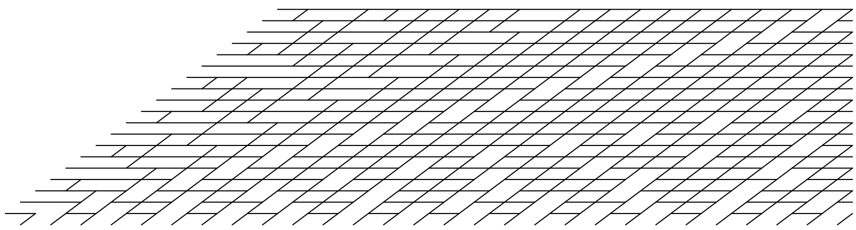
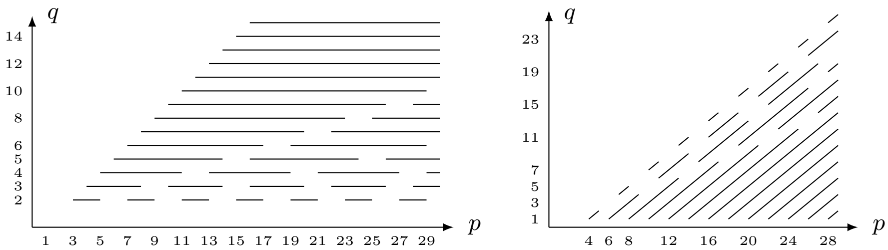


Figure 8: UArD tree row lines of p,q

The pattern of gaps can be seen by separating the horizontals and diagonals. The diagonals are always on odd differences $p{-}q$ since p,q must be opposite parity.



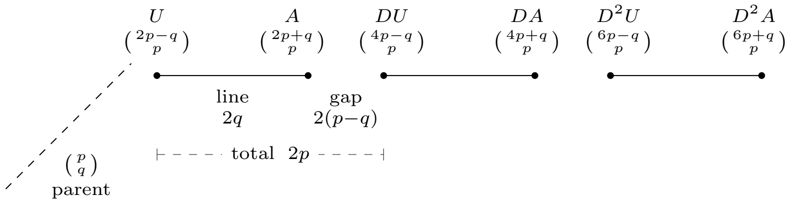
horizontals
lengths $2q{-}2 = 2, 4, 6, 8, \ldots$
with gaps 2

diagonals
lengths $p{-}q{-}2 = 1, 3, 5, 7, \ldots$
with gaps 2

Figure 9: UArD tree horizontal and diagonal steps of p,q

Theorem 6. *When all UArD tree row lines are plotted, the horizontals are length $2q' - 2$ with gap 2 in between and the diagonals are length $p' - q' - 2$ with gap 2 in between.*

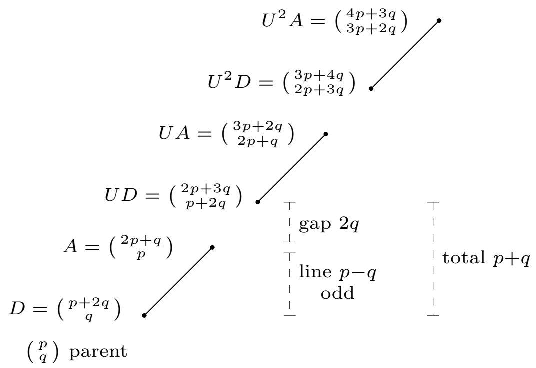
Proof. As from theorem 4 above, horizontal lines arise from the U child to A child of a p,q parent. The D matrix on them in subsequent tree levels preserves the length. The D matrix preserves the child q' but shears the line across so that it repeats in the following way,



When the parent point is $q = p{-}1$ the gap $2(p{-}q) = 2$ and the lines are length $2(p-1)$ which with $q' = p$ is $2q' - 2$.

Other parent points with the same p give line endings $2p \pm q$ so they are centered at $2p$, $4p$, etc. The longest line is $q = p - 1$ and it overlaps all others.

Also as from theorem 4, diagonal lines arise from the A child to D child of a given p,q parent. The U matrix on them in subsequent tree levels preserves the direction and length. U also preserves the $d' = p' - q'$ diagonal position but shifts it up in the following way.



When the parent is q=1 the gap $2q = 2$ and the lines are length $p-q = p-1$. Measured from the child $d' = p' - q'$ this is $p' - q' - 2 = p + 2q - q - 2 = p - 1$.

Lines from other parent p,q which fall on the same diagonal are centred on the line midpoint W. The longest is when $p - q$ is the maximum which is q=1 and this longest line overlaps all others.

$$\begin{aligned} W &= \begin{pmatrix} (3p+3q)/2 \\ (p+q)/2 \end{pmatrix} \\ D\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} p+2q \\ q \end{pmatrix} = W - \begin{pmatrix} (p-q)/2 \\ (p-q)/2 \end{pmatrix} \\ A\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 2p+q \\ p \end{pmatrix} = W + \begin{pmatrix} (p-q)/2 \\ (p-q)/2 \end{pmatrix} \end{aligned} \qquad \square$$

3.1 UArD Tree Low to High

The Gray code ordering of UArD can be taken low to high too. The Gray code is applied first, then its digits are interpreted low to high. The mirroring under each A compensates for the reflection in the A matrix (10).

n	ternary	Gray	matrices	
0	000	000	U.U.U $\binom{2}{1} = \binom{5}{4}$	
1	001	001	U.U.A	
2	002	002	U.U.D	
3	010	012	U.A.D	
4	011	011	U.A.A	reflect
5	012	010	U.A.U	
6	020	020	A.A.U	
7	021	021	A.A.A	
8	022	022	A.A.D	
...				
26	333	333	D.D.D $\binom{2}{1} = \binom{8}{1}$	

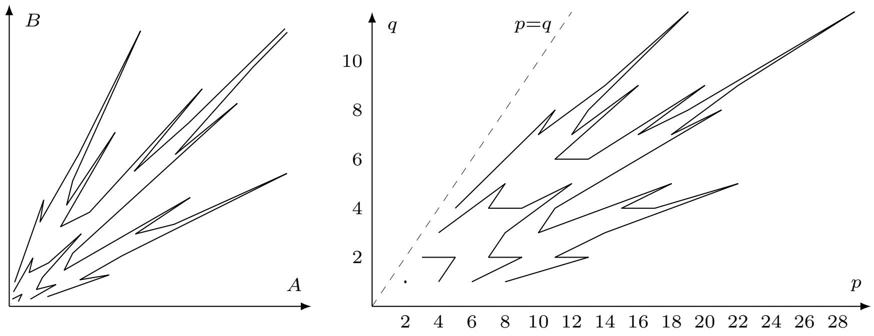
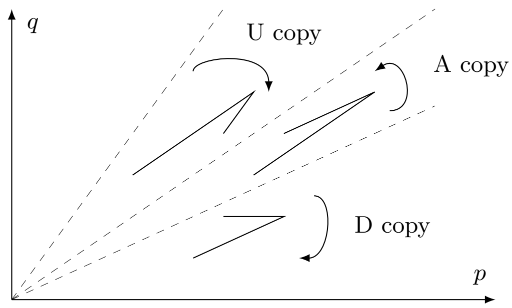


Figure 10: UArD tree rows, digits low to high

Theorem 7. *The A,B legs and the p,q points in a row of the low-to-high UArD tree go clockwise when plotted as points.*

Proof. In row 0 there is a single point p=2,q=1 so it goes clockwise.

For a subsequent row the new matrices U,A,D multiply on the left and so copy the previous row p,q points into their respective matrix regions as from figure 2, which is also in the manner of the low-to-high branches figure 5.



The D copy is a shear and so maintains the clockwise order. The U copy is a rotate and shear (9) and so also maintains clockwise order.

The A copy includes a reflection so reverses to anti-clockwise. But for UArD the points under A are mirrored so clockwise order is restored.

The A,B legs as points are at double the angle of the corresponding p,q per (3). So A,B points go clockwise too. □

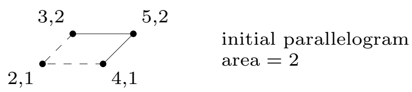
3.2 UArD Tree Row Area

The row lines of the UArD low to high (figure 10) do not intersect preceding rows and so give the shape of an expanding region of p,q or A,B coverage by the tree. This area is the same for all UAD tree forms, just the order of points within a row differs. The area of this expanding region can be calculated for p,q as follows.

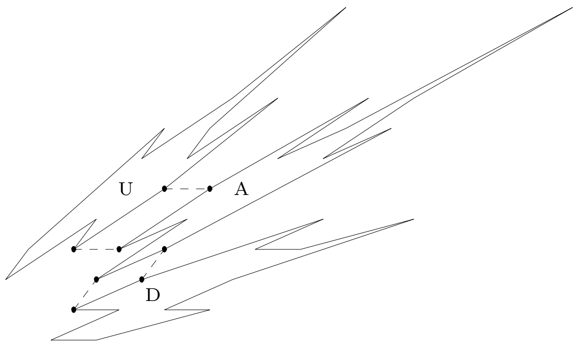
Theorem 8. *For the UAD tree in p,q coordinates, the area r_n between row n and $n+1$ and the total area R_n up to row n are*

$$\begin{aligned} r_n &= 5.3^n - 3 && \textit{area between rows } n \textit{ and } n+1 \\ &= 2, 12, 42, 132, 402, 1212, 3642, 10932, \ \ldots \\ R_n &= \sum_{i=0}^{n-1} r_i = \tfrac{5}{2}(3^n - 1) - 3n && \textit{total area to row } n \\ &= 0, 2, 14, 56, 188, 590, 1802, 5444, 16376, \ \ldots \end{aligned}$$

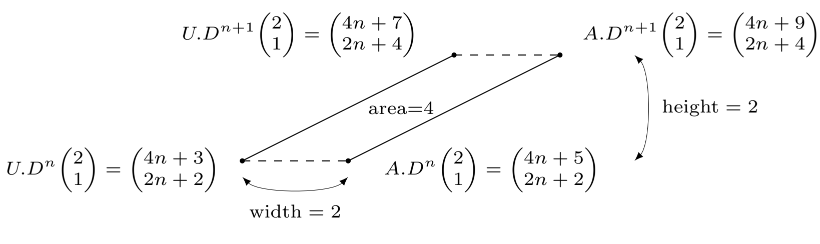
Proof. The area between row 0 and row 1 is the initial parallelogram 2,1–3,2–5,2–4,1 of area 2 which is $r_0 = 5.3^0 - 3 = 2$.



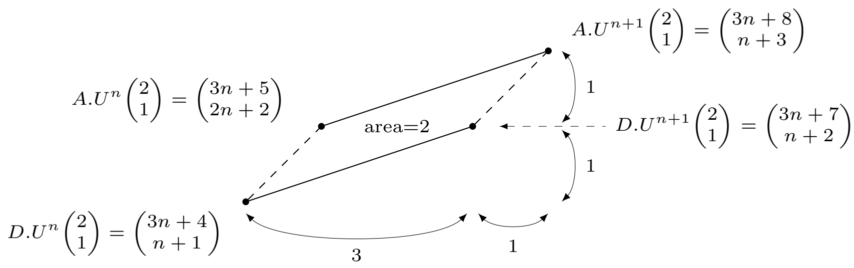
For a subsequent row the area between rows n and $n+1$ is three copies of the preceding $n-1$ to n area transformed by multiplication on the left by U,A,D. Those transformations don't change the area. Between the copies are two gaps show below by dashed lines.



The upper gap U to A is a parallelogram. The right edge of the U block is $U.D^n$. The left edge of the A block is $A.D^n$ since A is a reflection. The result is area 4.



The lower gap A to D is a parallelogram. The upper edge of the D block is $D.U^n$. The lower edge of the A block is $A.U^n$ since A is a reflection. The result is area $3.1 - 1.1 = 2$.



So the gaps are $4 + 2 = 6$ and the area between rows is thus

$$\begin{aligned} r_n &= 3r_{n-1} + 6 && \text{recurrence } n \geq 1 \\ &= 5.3^n - 3 \end{aligned}$$

Summing for the total area,

$$\begin{aligned} R_n &= \sum_{i=0}^{n-1} r_i \\ &= \tfrac{5}{2}(3^n - 1) - 3n \end{aligned} \qquad \square$$

Or total area as a recurrence,

$$\begin{aligned} R_n &= \sum_{i=0}^{n-1} r_i \\ &= r_0 + \sum_{i=1}^{n-1} (3r_{i-1} + 6) && r_i \text{ recurrence} \\ &= 2 + 6(n-1) + 3\sum_{i=0}^{n-2} r_i \\ &= 3R_{n-1} + 6n - 4 && \text{for } n \geq 1 \end{aligned}$$

The parallelograms making up the rows are a tiling of the eighth of the plane $p > q \geq 1$ using parallelograms of areas 2 and 4, but the repeated shears soon make them very elongated.

3.3 UArD as Filtered Stern-Brocot

The Stern-Brocot tree enumerates all rationals $p/q \geq 1$. It can be filtered to pairs $p > q \geq 1$ not-both-odd to give primitive Pythagorean triples. Katayama [11] shows this is the UArD tree low-to-high.

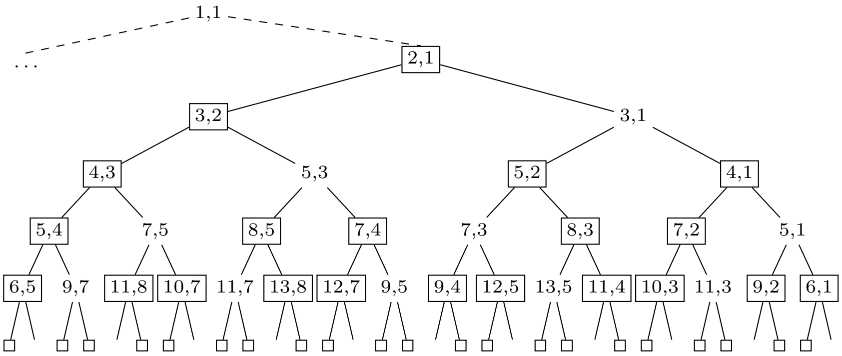


Figure 11: Stern-Brocot tree filtered to $p > q$ not-both-odd

Every third node is odd/odd when read by rows wrapping around at each row end. Those nodes can be removed to leave the nodes shown boxed in figure 11. The two children of a removed node are adopted by their grandparent to make a ternary tree.

The Stern-Brocot tree applies matrices by bits of the row position taken low-to-high. This means left and right sub-trees are defined recursively by a matrix L or R multiplied onto the points of the entire tree.

tree

L.tree $\qquad\qquad$ R.tree

$$\mathrm{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad\qquad \mathrm{R} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

1/(1 + 1/tree) $\qquad\qquad$ 1 + tree

The equivalence to UArD by Katayama can be outlined as follows. The sub-tree at 3,2 is $R.L$.tree and has same structure as the sub-tree at 2,1. Map from 2,1 to 3,2 using a matrix U. This is seen to be the U matrix of the UAD tree.

$$\begin{aligned} &U \;.\; R.\text{tree} = R.L.\text{tree} \\ &U \;.\; R = R.L \\ &U = R.L.R^{-1} = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right) \end{aligned}$$

Similarly mapping 2,1 to 4,1 is the D matrix of the UAD tree.

$$\begin{aligned} &D \;.\; R.\text{tree} = R.R.R.\text{tree} \\ &D = R.R = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \end{aligned}$$

The not-both-odd points of the sub-tree under 5,2 have a left-to-right mirroring. Swapping $p \leftrightarrow q$ performs such a mirroring in the Stern-Brocot tree. Apply that first with matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and then descend. Mapping 2,1 to 5,2 is then by a matrix A as follows and which is seen to be the A matrix of the UAD tree.

$$\begin{aligned}
&A\ .\ R.\text{tree} = R.R.L.S.\text{tree} \\
&A\,.\,R = R.R.L.S \\
&A = R.R.L.S.R^{-1} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)
\end{aligned}$$

If the filtered tree is read left to right then that reading includes the mirroring under each A. For example under 5,2 points 9,4–12,5–8,3 are sub-trees D–A–U. That mirroring is per UArD. The clockwise order of pairs in the Stern-Brocot rows corresponds to the clockwise order in the UArD rows of theorem 7.

4 FB Tree

Firstov[7] and independently Price[16] give another tree using a different set of three matrices, M1,M2,M3.

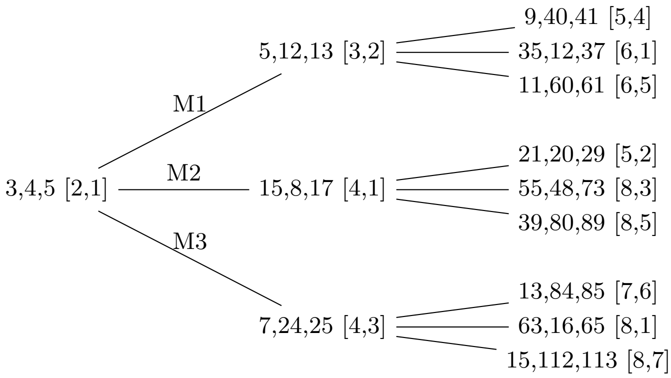


Figure 12: FB tree, triples and [p,q] pairs

$$\begin{aligned}
M1 &= \begin{pmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{pmatrix} & M2 &= \begin{pmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} & M3 &= \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} & &= \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} & &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}
\end{aligned} \tag{17}$$

Matrix M1 is all p odd. Matrices M2 and M3 are all p even and in disjoint regions $M3$ above and $M2$ below the $p = 2q$ line.

$$\begin{array}{llll}
M2 & p' = 2p \ > \ 2p - 2q = 2q' & \qquad p' > 2q' \\
M3 & p' = 2p \ < \ 2p + 2q = 2q' & \qquad p' < 2q'
\end{array} \tag{18}$$

When p is even leg $A \equiv 3 \bmod 4$. When p is odd leg $A \equiv 1 \bmod 4$.

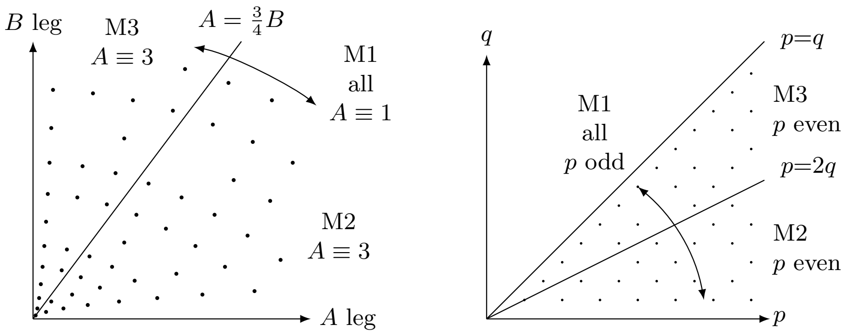


Figure 13: FB tree regions of p,q

Figure 14 shows how the matrices transform a vertical line of points $k, 1$ through $k, k-1$ out to bigger p,q.

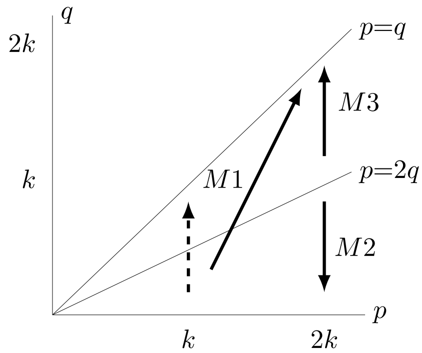


Figure 14: FB tree, line transformations

$M3$ is the simplest, just a shift up to point p,p.

$$M3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$M2$ is the same as $M3$ but negating q first so that it's mirrored to go downwards instead.

$$M2 = M3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{negate to mirror } q \text{ first}$$

5 UMT Tree

A third tree by Firstov[7] is formed by a further set of three matrices. Call it UMT. U is from the UAD tree. $M2$ is from the FB tree. The third matrix is $T = M1.D$.

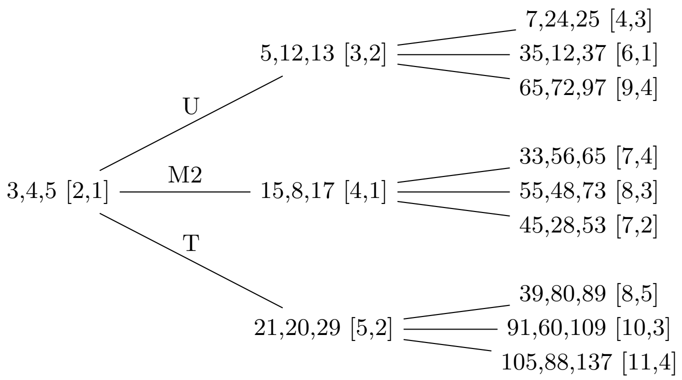


Figure 15: UMT tree, triples and [p,q] pairs

$$\begin{aligned}
U &= \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix} \qquad & M2 &= \begin{pmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \qquad & T &= \begin{pmatrix} -2 & 3 & 3 \\ -6 & 2 & 6 \\ -6 & 3 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} & &= \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} & &= \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \\
& & & & &= M1\,.\,D
\end{aligned} \tag{19}$$

The matrix sum is the same as in the UAD and so like theorem 2 the total p and q in depth n of the UMT are the Pell numbers.

$$U + M2 + T = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = U + A + D$$

Matrix U is all points $p < 2q$ (7). Matrix $M2$ is points $p > 2q$ with p even (18). Matrix T is points $p > 2q$ with p odd since

$$T \qquad p' = p + 3q \; > \; 2q = q' \qquad\qquad \text{so } p' > 2q' \tag{20}$$

Together $M2$ and T are all points $p > 2q$.

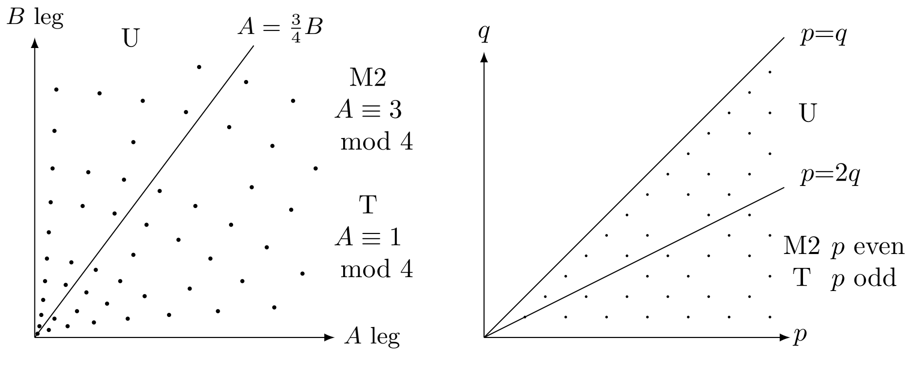


Figure 16: UMT tree regions of p,q

Theorem 9. *The UMT tree visits all and only primitive Pythagorean triples without duplication.*

Proof. It's convenient to work in p,q pairs. The tree visits only primitive Pythagorean triples because for a given p,q each of the three children p',q' satisfy the PQ conditions (4). For U,

$$\begin{aligned}
\begin{pmatrix} p' \\ q' \end{pmatrix} &= U \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2p-q \\ p \end{pmatrix} \\
p' &= 2p - q = p + (p-q) \ > \ p = q' && \text{since } p > q \\
q' &= p \ \geq \ 1 \\
\gcd(p', q') &= \gcd(2p-q, p) = \gcd(q, p) = 1 \\
p' + q' &= 3p - q \equiv p + q \equiv 1 \bmod 2
\end{aligned}$$

For $M2$,

$$\begin{aligned}
\begin{pmatrix} p' \\ q' \end{pmatrix} &= M2 \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2p \\ p-q \end{pmatrix} \\
p' &= 2p \ > \ p - q = q' \\
q' &= p - q \ \geq \ 1 && \text{since } p > q \\
\gcd(p', q') &= \gcd(2p, p-q) \\
&= \gcd(p, p-q) && \text{since } p-q \text{ odd} \\
&= \gcd(p, q) = 1 \\
p' + q' &= 3p - q \equiv p + q \equiv 1 \bmod 2
\end{aligned}$$

For T,

$$\begin{aligned}
\begin{pmatrix} p' \\ q' \end{pmatrix} &= T \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+3q \\ 2q \end{pmatrix} \\
p' &= p + 3q \ > \ 2q = q' \\
q' &= 2q \ \geq \ 1 && \text{since } q \geq 1 \\
\gcd(p', q') &= \gcd(p+3q, 2q) \\
&= \gcd(p+3q, q) && \text{since } p+3q \text{ odd} \\
&= \gcd(p, q) = 1 \\
p' + q' &= p + 5q \equiv p + q \equiv 1 \bmod 2
\end{aligned}$$

No pair is duplicated in the tree because the children of U are above the $p = 2q$ line (7) whereas $M2$ and T are below (18) (20). Then $M2$ gives p even whereas T gives p odd. Therefore two paths ending with a different matrix cannot reach the same point.

Conversely every pair p',q' occurs in the tree because it can be reversed according to its region and parity. The parent p,q satisfies the p,q conditions (4).

$$
\begin{array}{lllll}
\text{if } p' < 2q' \text{ then} & \begin{pmatrix} p \\ q \end{pmatrix} = U^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} & = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix} & = \begin{pmatrix} q' \\ 2q' - p' \end{pmatrix} \\
\text{if } p' < 2q',\ p' \text{ even, } q' \text{ odd} & \begin{pmatrix} p \\ q \end{pmatrix} = M2^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} & = \begin{pmatrix} 1/2 & 0 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix} & = \begin{pmatrix} p'/2 \\ p'/2 - q' \end{pmatrix} \\
\text{if } p' < 2q',\ p' \text{ odd, } q' \text{ even} & \begin{pmatrix} p \\ q \end{pmatrix} = T^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} & = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} & = \begin{pmatrix} p' - 3q'/2 \\ q'/2 \end{pmatrix}
\end{array}
$$

For reversing U have $p' < 2q'$ which is $p' \le 2q' - 1$.

$$
\begin{aligned}
q &= 2q' - p' = q' - (p' - q') \ < \ q' = p && \text{so } p > q \\
q &= 2q' - p' \ \ge \ 2q' - (2q' - 1) = 1 \\
\gcd(p,q) &= \gcd(q', 2q' - p') = \gcd(q', p') = 1 \\
p + q &= 3q' - p' \equiv p' + q' \equiv 1 \bmod 2 \\
p &= q' < p'
\end{aligned}
$$

For reversing $M2$ have $p' > 2q'$ which is $p' \ge 2q' + 2$ since p' is even.

$$
\begin{aligned}
q &= p'/2 - q' \ < \ p'/2 = p && \text{so } p > q \\
q &= p'/2 - q' \ \ge \ (2q' + 2)/2 - q' = 1 \\
\gcd(p,q) &= \gcd(p'/2, p'/2 - q') = \gcd(p'/2, q') \\
&= \gcd(p', q') && \text{since } p' \text{ even, } q' \text{ odd} \\
&= 1 \\
p + q &= p' - q' \equiv p' + q' \equiv 1 \bmod 2 \\
p &= p'/2 < p'
\end{aligned}
$$

For reversing T have $p' > 2q'$ and q' is even.

$$
\begin{aligned}
p &= p' - 3q'/2 \ > \ 2q' - 3q'/2 = q'/2 = q \\
q &= q'/2 \ \ge \ 1 && \text{since } q' \text{ even} \\
\gcd(p,q) &= \gcd(p' - 3q'/2, q'/2) = \gcd(p', q'/2) \\
&= \gcd(p', q') = 1 && \text{since q' even} \\
p + q &= p' + 3q'/2 - q'/2 = p' + q' \equiv 1 \bmod 2 \\
p &= p' - 3q'/2 < p' && \square
\end{aligned}
$$

Each ascent has $p < p'$ so repeatedly taking the parent this way is a sequence of strictly decreasing p which must eventually reach the root 2,1. The matrix to reverse goes according to the region and parity of p',q' which are where the respective matrices descend.

Figure 17 below shows how U,M2,T transform a vertical line of points $k,1$ through $k,k{-}1$ out to bigger p,q.

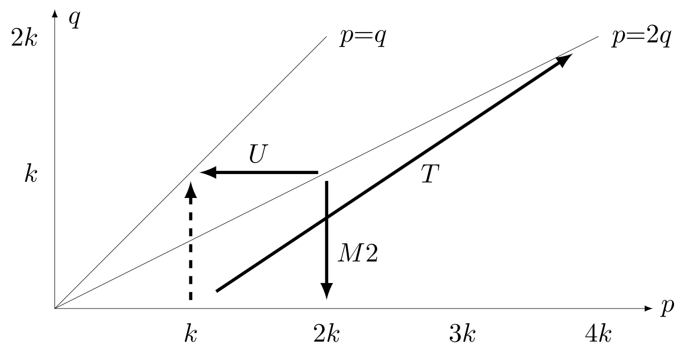


Figure 17: UMT tree, line transformations

U is described with the UAD tree figure 3. $M2$ is described with FB tree figure 14.

T shears p across by 3 to give $4k$ and then doubles q to put it on even points.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \text{shear then stretch } q$$

For T the line endpoint k,$k{-}1$ becomes $4k{-}3$,$2k{-}2$. This longer line still has the same number of coprime not-both-odd points as the original.

5.1 T Matrix Repeatedly

Repeatedly applying 2×2 matrix T to 2,1 gives a sequence

i	p_i	q_i	p_i binary	q_i binary
0	2	1	10	1
1	5	2	101	10
2	11	4	1011	100
3	23	8	10111	1000

$$\begin{aligned} q_i &= 2^i \\ p_i &= 2 + 3.1 + 3.2 + \cdots + 3.2^{i-1} \\ &= 3.2^i - 1 \end{aligned}$$

Repeatedly applying 3×3 matrix T to triple 3,4,5 gives

i	A_i	B_i	C_i	A_i binary	B_i binary	C_i binary
0	3	4	5	11	100	101
1	21	20	29	10101	10100	11101
2	105	88	137	1101001	1011000	10001001
3	465	368	593	111010001	101110000	1001010001

$$\begin{aligned} A_i &= p_i^2 - q_i^2 \\ &= 8.2^{2i} - 6.2^i + 1 \\ &= (4.2^i - 1)(2.2^i - 1) = \text{binomial } \tbinom{2^{i+2}-1}{2} \end{aligned}$$

$$
\begin{aligned}
B_i &= 2p_i q_i \\
&= (3.2^i - 1).2^{i+1} \\
C_i &= p_i^2 + q_i^2 \\
&= 10.2^{2i} - 6.2^i + 1
\end{aligned}
$$

The difference $C - A$ is successively bigger powers of 2.

$$C_i - A_i \;=\; 2q_i^2 \;=\; 2^{2i+1}$$

6 Triple Preserving Matrices

The set of all matrices which preserve primitive Pythagorean triples are characterised in 2×2 form by Firstov[7]. The conditions are stated in slightly different form here in theorem 10 below.

A primitive triple preserving matrix is to send a p,q point to a new p',q' point satisfying the PQ conditions (4), and without duplication meaning that two different p,q do not map to the same child p',q'.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Palmer, Ahuja and Tikoo[14] give a set of conditions on a,b,c,d which are sufficient, but not necessary. As they note their R-3 determinant condition $\Delta = ad - bc = \pm 1$ is sufficient, but not necessary. $\Delta = \pm 1$ ensures that $\gcd(p, q)$=1 implies $\gcd(p', q')$=1 over all integers but for Pythagorean triples only p,q of opposite parity need be considered.

For p,q opposite parity a determinant $\Delta = \pm 2^r$ (32) maintains $\gcd(p', q')$=1. The argument in this part of the proof largely follows an answer by Thomas Jager[9] for the all integers case (on $n\times n$ matrices). A little care is needed that the p,q constructed to induce a common factor obeys $p > q \geq 1$ not-both-odd.

Matrices $M1$, $M2$, $M3$ (17) and T (19) have $\Delta = \pm 2$. Higher powers of 2 occur from products of those matrices (and believe also from matrices not a product of others).

$$\det(M^r) = (\det M)^r = \pm 2^r$$

Theorem 10 (variation of Firstov)**.** *Conditions* (21) *through* (32) *are necessary and sufficient for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to preserve p,q pairs without duplication, and hence for its corresponding* 3×3 *matrix to preserve primitive Pythagorean triples without duplication.*

$$
\begin{aligned}
& a, b, c, d \textit{ integers} && && (21) \\
& a \geq 1 && && (22) \\
& c \geq 0 && && (23) \\
& a > c && && (24) \\
& a + b \geq 1 && \textit{so } b \geq -a + 1 && (25) \\
& c + d \geq 0 && \textit{so } d \geq -c && (26) \\
& a + b \geq c + d && && (27)
\end{aligned}
$$

$$\gcd(a,c) = 1 \tag{28}$$
$$\gcd(b,d) = 1 \tag{29}$$
$$a + c \equiv 1 \bmod 2 \qquad \textit{opposite parity} \tag{30}$$
$$b + d \equiv 1 \bmod 2 \qquad \textit{opposite parity} \tag{31}$$
$$ad - bc = \pm 2^r,\ r \geq 0 \qquad \textit{determinant} \tag{32}$$

The identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ preserves p,q pairs without duplication and satisfies the conditions.

The GCD is taken as $\gcd(n, 0) = |n|$ in the usual way. So $\gcd(a, c) = 1$ and $a \geq 1$ together mean

$$\text{if } c = 0 \text{ then can only have } a = 1 \tag{33}$$

Similarly $\gcd(b, d) = 1$ means

$$\text{if } d = 0 \text{ then can only have } b = \pm 1$$

Proof of Theorem 10. Take first the necessity, that when a matrix sends all good p,q to good p',q' and never duplicates p',q', then its a,b,c,d are as described.

Consider p=2, q=1 and p=3, q=2,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a + b \\ 2c + d \end{pmatrix} = \begin{pmatrix} p'_1 \\ q'_1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3a + 2b \\ 3c + 2d \end{pmatrix} = \begin{pmatrix} p'_2 \\ q'_2 \end{pmatrix}$$

These are solved for a,b,c,d in terms of p'_1,p'_2,q'_1,q'_2

$$a = 2p'_1 - p'_2 = \text{integer} \qquad b = -3p'_1 + 2p'_2 = \text{integer}$$
$$c = 2q'_1 - q'_2 = \text{integer} \qquad d = -3q'_1 + 2q'_2 = \text{integer}$$

p'_1,p'_2,q'_1,q'_2 are all integers so a,b,c,d are all integers (21).

Consider $p = 2k$, $q = 1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2k \\ 1 \end{pmatrix} = \begin{pmatrix} 2ka + b \\ 2kc + d \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Must have $a \geq 0$ otherwise big enough k gives $p' < 2$. Similarly $c \geq 0$ (23) otherwise $q' < 1$. Must have $a \geq c$ otherwise big enough k gives $p' \leq q'$. But cannot have both $a = 0$ and $c = 0$ otherwise constant $p' = b$, $q' = d$ is duplicated, so $a \geq 1$ (22). b and d must be opposite parity (31) so that p' and q' are opposite parity. If b,d have a common factor $g = \gcd(b, d) > 1$ then $k = g$ gives that common factor in p',q', so must have $\gcd(b, d) = 1$ (29).

Consider $p = 2k+1$, $q = 2k$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2k + 1 \\ 2k \end{pmatrix} = \begin{pmatrix} 2k(a + b) + a \\ 2k(c + d) + c \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Must have $a + b \geq 0$ otherwise big enough k gives $p' < 2$. Similarly $c + d \geq 0$ (26) otherwise $q' < 1$. Must have $a + b \geq c + d$ (27) otherwise big enough k gives $p' \leq q'$. But cannot have both $a + b = 0$ and $c + d = 0$ otherwise constant $p' = c$, $q' = d$ is duplicated, so $a + b \geq 1$ (25). a and c must be opposite parity

(30) so that p' and q' are opposite parity. Since a,c are opposite parity cannot have $a = c$ so $a \geq c$ above becomes $a > c$ (24). If a,c have a common factor $g = \gcd(a, c) > 1$ then $k = g$ gives that common factor in p',q', so must have $\gcd(a, c) = 1$ (28).

The determinant $\Delta = ad - bc \neq 0$ because if it was 0 then $ad = bc$ and with $a \geq 1$, $\gcd(a, c) = 1$ and $\gcd(b, d) = 1$ could only have $d = c$ and $b = a$. In that case

$$\begin{pmatrix} a & a \\ c & c \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3a \\ 3c \end{pmatrix}$$

has common factor 3 in p',q' if $c \neq 0$ or has $q' = 0$ if $c = 0$. Therefore $\Delta \neq 0$.

Let δ be the odd part of the determinant so

$$\Delta = ad - bc = \delta 2^r \qquad \delta \text{ odd}$$

Since $\gcd(a, c) = 1$ there exist integers x,y with

$$-xc + ya = 1 \qquad \text{since } a,c \text{ coprime} \tag{34}$$

Consider p,q pair

$$\begin{aligned} p &= xd - yb + \delta k && \text{some integer } k \\ q &= 1 \end{aligned}$$

Choose k the same parity as $xd - yb$ so as to make p even (since δ is odd). Choose k big enough positive or negative to give $p \geq 2$ (possible since $\delta \neq 0$). The resulting p,q gives

$$\begin{aligned} p' &= a(xd - yb + \delta k) + b \\ &= xad - yab + a\delta k + b \\ &= xad - (1 + xc)b + a\delta k + b && \text{since } ya = 1 + xc \text{ (34)} \\ &= x(ad - bc) + a\delta k \\ &= x\delta 2^r + a\delta k && \text{multiple of } \delta \end{aligned}$$

$$\begin{aligned} q' &= c(xd - yb + \delta k) + d \\ &= xcd - ybc + c\delta k + d \\ &= (ya - 1)d - ybc + c\delta k + d && \text{since } xc = ya - 1 \text{ (34)} \\ &= y(ad - bc) + c\delta k \\ &= y\delta 2^r + c\delta k && \text{multiple of } \delta \end{aligned}$$

δ is a common factor in p',q' and so must have δ=± 1 and therefore $\Delta = ad - bc = \pm 2^r$ (32).

Remark. This p,q pair arises from M and its adjoint (inverse times determinant),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\Delta \\ y\Delta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta k \\ 0 \end{pmatrix} \right) = \begin{pmatrix} x\Delta + a\delta k \\ y\Delta + c\delta k \end{pmatrix}$$

This is common factor δ in p',q' on the right, provided the vector part on the left

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta k \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

is an acceptable p,q pair or can be made so. One way to make it so is q=1 by x,y from $\gcd(a,c) = 1$ and then $p \geq 2$ and even using k.

$\gcd(a,c) = 1$ gives a whole class of solutions to $-cx + ay = 1 = q$,

$$\begin{aligned} x &= x_0 + fa & &\text{integer } f \\ y &= y_0 + fc \end{aligned}$$

Taking a different f adds $f(ad - bc)$ to p. This is a multiple of the determinant and so maintains factor δ in the resulting p'. Could choose f to make $p \geq 2$, but if Δ is even then f cannot force p to even. It's necessary to have k and the odd part δ for that.

Turn now to the sufficiency, ie. that if the above conditions hold then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends all good p,q to good p',q' (4) without duplication.

$$\begin{aligned} p' &= ap + bq \\ &= ap + (1-a)q & & a + b \geq 1 \text{ (25) so } b \geq 1 - a \\ &= a(p-q) + q \\ &\geq 2 & & \text{as } a \geq 1 \text{ (22)} \end{aligned}$$

If $c = 0$ then $c + d \geq 0$ (26) means $d \geq 0$. Determinant $ad - bc = \pm 2^r \neq 0$ means not c=0, d=0, so $d \geq 1$ giving

$$\begin{aligned} q' &= cp + dq \\ &= dq \ \geq \ 1 \end{aligned}$$

If $c > 0$ then

$$\begin{aligned} q' &= cp + dq \\ &= cp + (-c)q & & c + d \geq 0 \text{ (26) so } d \geq -c \\ &= c(p-q) \ \geq \ 1 \end{aligned}$$

For the relative magnitude of p' and q',

$$\begin{aligned} p' &= ap + bq \\ &= a(p-q) + (a+b)q \\ &> c(p-q) + (c+d)q && a > c \ (24) \text{ and } a+b \geq c+d \ (27) \\ &= cp + dq = q' \end{aligned}$$

For the parity of p' and q',

$$\begin{aligned} p' + q' &= (a+c)p + (b+d)q \\ &\equiv p + q \bmod 2 && \text{by } a+c \equiv 1 \ (30),\ b+d \equiv 1 (31) \\ &\equiv 1 && (35) \end{aligned}$$

Any p',q' is reached from just one p,q since $\Delta \neq 0$ means M is invertible so p,q is uniquely determined by p',q'.

Since $\gcd(p,q) = 1$ there exist integers x,y satisfying

$$xp + yq = 1 \qquad \text{as } p,q \text{ coprime}$$

and the inverses for p,q in terms of p',q' give

$$\begin{aligned} x(dp' - bq')/\Delta + y(-cp' + aq')/\Delta &= 1 \\ (xd - yc)p' + (-xb + ya)q' &= \pm 2^r && \text{by} \Delta = \pm 2^r \ (32) \end{aligned}$$

which is integer multiples of p',q' adding up to $\pm 2^r$. So $\gcd(p',q')$ must be a divisor of 2^r. But p',q' are opposite parity (35) so one of them is odd which means $\gcd(p',q')$ is odd and the only odd divisor of 2^r is $\gcd(p',q') = 1$. $\square$

Since a,c are opposite parity (30) and b,d are opposite parity (31), there are four combinations of odd/even among the matrix terms. All four occur in the trees.

The parity of the terms control whether the child p',q' either keeps, swaps, or has fixed parity.

Combination	$\begin{pmatrix} E & O \\ O & E \end{pmatrix}$	$\begin{pmatrix} O & E \\ E & O \end{pmatrix}$	$\begin{pmatrix} E & E \\ O & O \end{pmatrix}$	$\begin{pmatrix} O & O \\ E & E \end{pmatrix}$
Matrices	U,A	D	$M2$,$M3$	$M1$,T
PQ	swap parity $p' \equiv q$ $q' \equiv p$	keep parity $p' \equiv p$ $q' \equiv q$	always p' even q' odd	always p' odd q' even
Determinant	odd $\Delta = \pm 1$	odd $\Delta = \pm 1$	even $\Delta = \pm 2^r$ $r \geq 1$	even $\Delta = \pm 2^r$ $r \geq 1$

7 No Other Trees

Firstov[7] shows that the only trees which can be made from a fixed set of matrices are the UAD, FB and UMT. The proof offered here takes points according to increasing p whereas Firstov goes by sum p+q. The conditions of the theorem allow any number of matrices but it happens that the trees all have 3 matrices each.

Lemma 1. *A non-identity triple preserving matrix advances p, ie. $p' = ap+bq$ gives $p' > p$.*

Proof. If $a = 1$ then $a > c \geq 0$ (24)(23) means $c = 0$, and $a + b \geq 1$ (25) means $b \geq 0$. If $b = 0$ then $\gcd(b, d) = 1$ (29) and $c + d \geq 0$ (26) together mean $d = 1$ which is the identity matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. So if $a = 1$ then a non-identity matrix has $b \geq 1$ and

$$\begin{aligned} p' &= ap + bq \\ &\geq p + q \\ &> p \end{aligned}$$

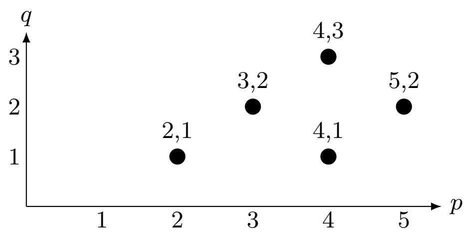
Otherwise if $a \geq 2$ then

$$\begin{aligned} p' &= ap + bq \\ &\geq ap + (1 - a)(p - 1) && \text{by } b \geq 1 - a \text{ (25) and } p - 1 \geq q > 0 \\ &= p + a - 1 \\ &> p && a \geq 2 \end{aligned}$$

□

Theorem 11 (Firstov)**.** *The UAD, FB and UMT trees are the only trees which generate all and only primitive Pythagorean triples in least terms without duplicates using a fixed set of matrices from a single root.*

By lemma 1 the matrices always advance p. So a given p',q' at 3,2 onwards must have as its parent some p,q with smaller p. The following cases consider the ways points up to 5,2 can be reached, culminating in the diagram of figure 18 on page 34.



Case. *2,1 to 3,2 — U and M1*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{36}$$

$$\begin{aligned} a + (a + b) &= 3 \\ c + (c + d) &= 2 \end{aligned} \tag{37}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	2	1	0	2	2	matrix $M1$
2	1	-1	0			fails $\gcd(a, c) = 1$ (28)
2	1	-1	1	1	0	matrix U

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is to send $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$ as per (36). This is rewritten in terms of a, $a+b$ and c, $c+d$ (37) since those terms are ≥ 1 and ≥ 0. The table headings are reminders of the conditions on those quantities (22),(25), (23),(26).

The solutions are listed in the table by increasing a and then increasing c as long as $a > c$ (24). The b column is derived from a and $a+b$. The d column is derived from c and $c+d$.

The note beside each combination is either the matrix name or how the values fail the matrix conditions. For example the middle row of the table above fails $\gcd(a, c) = 1$ (28). This GCD failure is of the "if c=0 then only a=1" kind (33). This a and c also fail opposite parity (30).

Case. *2,1 to 4,1 — $M2$ and D*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{matrix} a + (a+b) = 4 \\ c + (c+d) = 1 \end{matrix}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	3	2	0	1	1	matrix D
2	2	0	0			fails $\gcd(a, c) = 1$ (28)
2	2	0	1	0	-1	matrix $M2$
3	1	-2	0			fails $\gcd(a, c) = 1$ (28)
3	1	-2	1	0		fails $a + c \equiv 1 \bmod 2$ (30)

Case. *2,1 to 4,3 — $M3$ and $X1$*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \begin{matrix} a + (a+b) = 4 \\ c + (c+d) = 3 \end{matrix}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	3	2	0	3	3	fails $ad - bc = \pm 2^r$ (32)
2	2	0	0			fails $\gcd(a, c) = 1$ (28)
2	2	0	1	2	1	matrix $M3$
3	1	-2	0	3		fails $\gcd(a, c) = 1$ (28)
3	1	-2	1	2		fails $a + b \geq c + d$ (27)
3	1	-2	2	1	-1	matrix $X1 = U.U$

$$X1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = U.U \tag{38}$$

Remark. In general any matrix with c=0 such as $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right)$ in the first line of the table must not have $g = \gcd(a+b, d) > 1$ odd (here $g = 3$), otherwise $p = g+1$,

$q=1$ leads to a common factor g in p',q'.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} g+1 \\ 1 \end{pmatrix} = \begin{pmatrix} ag+(a+b) \\ d \end{pmatrix}$$
$$= \begin{pmatrix} ag+s'g \\ d'g \end{pmatrix} \qquad a+b=s'g \text{ and } d=d'g$$

Case. *2,1 to 5,2 — matrices A, T, and X2*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a+(a+b)&=5 \\ c+(c+d)&=2 \end{aligned}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	4	3	0	2	2	matrix T
2	3	1	0			fails $\gcd(a,c)=1$ (28)
2	3	1	1	1	0	matrix A
3	2	-1	0			fails $\gcd(a,c)=1$ (28)
3	2	-1	1			fails $a+c \equiv 1 \bmod 2$ (30)
3	2	-1	2	0	-2	matrix $X2 = M1.M2$
4	1	-3	0			fails $\gcd(a,c)=1$ (28)
4	1	-3	1	1	0	fails $\gcd(b,d)=1$ (29)
4	1	-3	2			fails $a+c \equiv 1 \bmod 2$ (30)

$$X2 = \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix} = M1.M2 \tag{39}$$

Case. *3,2 to 4,1 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{aligned} a+2(a+b)&=4 \\ c+2(c+d)&=1 \end{aligned}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
2	1	-1	1	0	-1	fails $b+d \equiv 1 \bmod 2$ (31)

Case. *3,2 to 4,3 — matrix U only*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \begin{aligned} a+2(a+b)&=4 \\ c+2(c+d)&=3 \end{aligned}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
2	1	-1	1	1	0	matrix U

Case. *3,2 to 5,2 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a+2(a+b)&=5 \\ c+2(c+d)&=2 \end{aligned}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	2	-1	0	1	1	fails $b+d \equiv 1 \bmod 2$ (31)
3	1	-2	0			fails $\gcd(a,c) = 1$ (28)
3	1	-2	2	0	-2	fails $b+d \equiv 1 \bmod 2$ (31)

Case. *4,1 to 5,2 — M1 only*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} 3a + (a+b) &= 5 \\ 3c + (c+d) &= 2 \end{aligned}$$

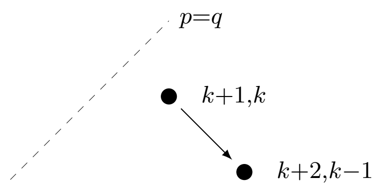
$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
1	2	1	0	2	2	matrix $M1$

Case. *4,3 to 5,2 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a + 3(a+b) &= 5 \\ c + 3(c+d) &= 2 \end{aligned}$$

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d	
2	1	-1	2	0		fails $a > c$ (24)

Remark. In general no matrix can take a unit step diagonally down from just under the leading diagonal like 4,3 to 5,2 here and 3,2 to 4,1 above.



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k+1 \\ k \end{pmatrix} = \begin{pmatrix} k+2 \\ k-1 \end{pmatrix} \qquad \begin{aligned} a + (a+b)k &= k+2 \\ c + (c+d)k &= k-1 \end{aligned} \qquad \text{with } k \geq 3$$

There is a single solution a,b,c,d.

$a \geq 1$	$a+b \geq 1$	b	$c \geq 0$	$c+d \geq 0$	d
2	1	-1	k-1	0	-(k-1)

If $k \geq 3$ then this solution fails $a > c$ (24), as per 4,3 to 5,2 here. If $k = 2$ (or for that matter any k even) then this solution fails $b + d \equiv 1 \bmod 2$ (31) as per 3,2 to 4,1 above.

Figure 18 summarises the above cases. These are the only possible ways to reach the points shown.

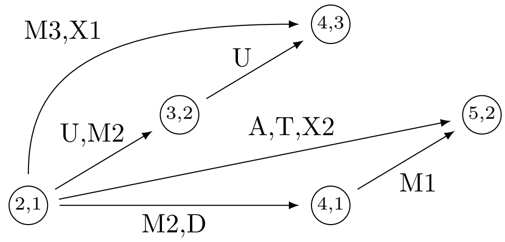


Figure 18: Possible p,q point descents

$X1 = U.U$ (38) goes 2,1 to 4,3 by U twice. $X2 = M1.M2$ (39) goes 2,1 to 5,2 by $M2$ then $M1$. It is in that order because the matrix multiplication is on the left so $X2\binom{p}{q} = M1.M2\binom{p}{q}$ means apply $M2$ first then $M1$.

Proof of Theorem 11. 3,2 is reached from 2,1 by either U or $M1$. Suppose firstly it is $M1$. $M1$ repeated goes

$$2,1 \to 3,2 \to 5,4 \to \cdots$$

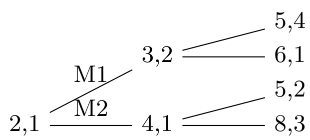
So 4,1 and 4,3 are not visited. Consider 4,1 first. The candidates for its parent are

2,1 to 4,1	$M2$ or D
3,2 to 4,1	none

It cannot be D since $M1$ and D overlap at 7,2.

$$M1\begin{pmatrix}6\\1\end{pmatrix} = D\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}7\\2\end{pmatrix} \qquad\qquad M1 \text{ and } D \text{ overlap}$$

M1 and M2 together visit



Point 4,3 is still not visited. The candidates for its parent are

2,1 to 4,3	$M3$ or $X1$
3,2 to 4,3	U

Its parent cannot be U since U overlaps with $M1$ at 3,2 (both U and $M1$ send 2,1 to 3,2).

$$M1\begin{pmatrix}2\\1\end{pmatrix} = X1\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \qquad\qquad U \text{ and } M1 \text{ overlap} \tag{40}$$

Neither can it be $X1$ because $X1$ overlaps $M1$ at 5,4,

$$M1\begin{pmatrix}3\\2\end{pmatrix} = X1\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix} \qquad\qquad M1 \text{ and } X1 \text{ overlap}$$

So when M1 is included the only possible combination is $M1$,$M2$,$M3$ which is the FB tree by Price.

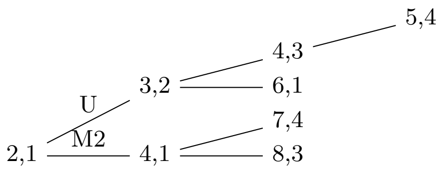
Return to instead take U for 2,1 to 3,2. Repeated U goes

$$2,1 \to 3,2 \to 4,3 \to 5,4 \to \cdots$$

Point 4,1 is the smallest p not visited. The candidates for its parent are

2,1 to 4,1	$M2$ or D
3,2 to 4,1	none

Take first U and M2. Together they visit



The smallest p not visited yet is 5,2. The candidates for its parent are

2,1 to 5,2	A, T or $X2$
3,2 to 5,2	none
4,1 to 5,2	$M1$
4,3 to 5,2	none

(41)

$M1$ is excluded because it overlaps with U at 3,2 per (40).
A is excluded because it overlaps with $M2$ at 8,3.

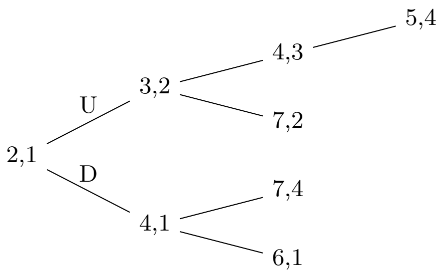
$$A\begin{pmatrix}3\\2\end{pmatrix} = M2\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}8\\3\end{pmatrix} \qquad \text{A and $M2$ overlap}$$

$X2$ is excluded because it overlaps with U at 11,6,

$$U\begin{pmatrix}6\\1\end{pmatrix} = X2\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}11\\6\end{pmatrix} \qquad \text{$X2$ and U overlap} \tag{42}$$

So when U and $M2$ are in the tree the only combination is U,$M2$,T which is the UMT tree.

Return to U and D. They together visit



Again the smallest p not visited is 5,2 and its parent candidates are per (41).

$M1$ and $X2$ are excluded because they overlap U by (40) and (42) respectively. T is excluded because it overlaps D at 11,6,

$$D\begin{pmatrix}3\\2\end{pmatrix} = T\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}11\\6\end{pmatrix} \qquad\qquad D \text{ and } T \text{ overlap}$$

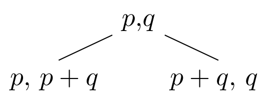
So when U and D are included the only combination is UAD by Berggren et al. □

Remark. It might be noted for the proof that only some of the conditions on triple preserving a,b,c,d from theorem 10 are needed.

It would be enough here to have the ranges of a,b,c,d giving the 29 matrices which are the rows of the case tables, then for each of the 18 failing matrices exhibit a particular p,q which makes a bad p',q'. Those p,q would follow the general conditions but be just particular integer values.

8 Calkin-Wilf Tree Filtered

The tree by Calkin and Wilf[5] arranges the Stern diatomic sequence into tree rows which descend as



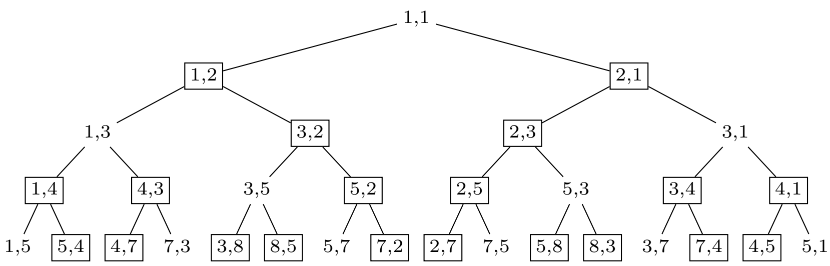
The diatomic sequence goes in a repeating pattern of odd and even

O O E   O O E   O O E   O O E   . . .

So when taking adjacent pairs every third is odd/odd and to be filtered out.

O/O, O/E, E/O,   O/O, O/E, E/O,   O/O, O/E, E/O   . . .

Removing the odd/odd points leaves two tree roots 1,2 and 2,1.



The orphaned children can be adopted by their grandparent to make a 3-point descent. The rule for the children then varies according to whether the parent p is odd or even. As it happens the D matrix (6) is one of the legs.

if p odd, q even

$$\begin{array}{ccc} & p,q & \\ p,2p+q & 2p+q,p+q & p+q,q \\ \begin{pmatrix}1&0\\2&1\end{pmatrix} & \begin{pmatrix}2&1\\1&1\end{pmatrix} & \begin{pmatrix}1&1\\0&1\end{pmatrix} \\ = D \text{ matrix} & & \\ \text{transposed} & & \end{array}$$

if p even, q odd

$$\begin{array}{ccc} & p,q & \\ p,\ p+q & p+q,\ p+2q & p+2q,\ q \\ \begin{pmatrix}1&0\\1&1\end{pmatrix} & \begin{pmatrix}1&1\\1&2\end{pmatrix} & \begin{pmatrix}1&2\\0&1\end{pmatrix} \\ & & = D \text{ matrix} \end{array}$$

Points $p > q$ and $p < q$ are intermingled in the tree. Each left leg has $p < q$ and each right leg has $p > q$.

The $p < q$ points give leg $A = p^2 - q^2 < 0$. For example 1,2 becomes -3,4,5. So two copies of the primitive triples are obtained, one with A positive and the other A negative. If desired the negatives could instead be taken to mean swap A and B giving triples such as 4,3,5 which is A even, B odd. That would give all positive primitive triples with A,B both ways around.

9 Parameter Variations

9.1 Parameter Difference

Triples can also be parameterized by d,q where d is a difference $d = p - q$.

$$\begin{aligned} &d \geq 1, \ \text{odd integer} \\ &q \geq 1, \ \text{any integer} \\ &\gcd(d, q) = 1 \end{aligned}$$

$$\begin{array}{lll} A = d^2 + 2dq & & A \text{ leg odd} \\ B = \qquad 2dq + 2q^2 & & B \text{ leg even} \\ C = d^2 + 2dq + 2q^2 & & \text{hypotenuse} \\ d = \sqrt{C - B} \qquad q = \sqrt{\frac{C-A}{2}} & & \end{array}$$

The effect is to shear p,q coordinates left to use the whole first quadrant. This for example changes the UArD tree steps (theorem 4, page 10) from diagonals to verticals.

$$\text{shear} \quad H = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} \qquad\qquad \begin{pmatrix}d\\ q\end{pmatrix} = H \begin{pmatrix}p\\ q\end{pmatrix}$$

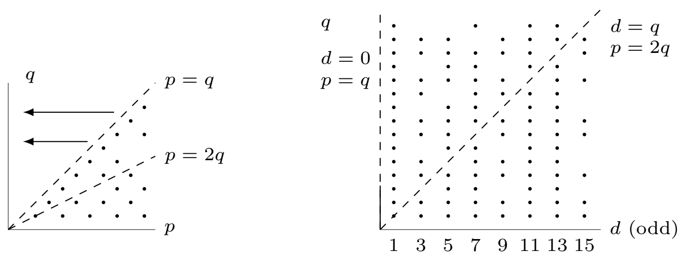


Figure 19: Shear for $d = p - q$

Tong[17] takes d,q as a symmetric parameterization of all triples. It is symmetric in that d and q can be swapped to give a conjugate triple.

$$\text{conjugate swap} \quad J_{dq} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} q \\ d \end{pmatrix} = J_{dq} \begin{pmatrix} d \\ q \end{pmatrix}$$

Primitive triples with q odd have a primitive conjugate. Primitive triples with q even do not have a primitive conjugate since it would be d even.

Braza, Tong and Zhan[4] consider 3×3 matrix transformations on Pythagorean triples and show that the only invertible transformation is an L which corresponds to the d,q conjugate operation.

$$L = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix}$$

L can be had from the $2{\times}2 \to 3{\times}3$ formula (5) by expressing the conjugate operation on p,q

$$\begin{aligned} J_{pq} &= H^{-1}.J_{dq}.H = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \\ \text{2to3}(J_{pq}) &= L \end{aligned}$$

J_{pq} is p,$p{-}q$. Geometrically this is a reversal of the q values within a column. The swap of d,q is an reversal in an anti-diagonal and those diagonals correspond to columns in p,q.

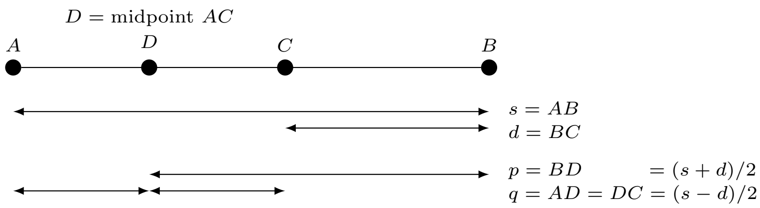
9.2 Parameters Sum and Difference

Triples can also be parameterized by sum $s = p + q$ and difference $d = p - q$.

$$\begin{aligned} & s > d \geq 1, \text{ integers, both odd} \\ & \gcd(s, d) = 1 \\ & A = sd && A \text{ leg odd} \\ & B = (s^2 - d^2)/2 && B \text{ leg even} \qquad (43) \\ & C = (s^2 + d^2)/2 && \text{hypotenuse} \\ & s = \sqrt{C + B} \qquad d = \sqrt{C - B} \end{aligned}$$

s,d is the original parameterization in Euclid[6] and also noted for instance by Mitchell[12]. In Euclid s is a length AB and a point C on that line defines

d. From there p is the midpoint (mean) of A and C at D and q is the distance to that midpoint.



Algebraically the difference of two squares in the A and B legs (1) (43) each suggest taking sum and difference which is the other parameterization.

$$\begin{array}{rcl} A &=& p^2 - q^2 &=& (p-q)(p+q) \\ B &=& \frac{1}{2}(s^2 - d^2) &=& \frac{1}{2}(s-d)(s+d) \end{array}$$

The geometric interpretation of s,d is to transform a column in p,q to a downward diagonal as per figure 20 below.

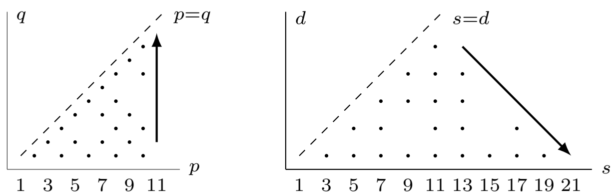


Figure 20: Transformation for $s = p + q$, $d = p - q$

Points in the eighth of the plane $x > y \geq 1$ with $\gcd(x, y) = 1$ are either opposite parity or both odd. Both even does not occur as that would be common factor 2. p,q are the opposite parity points. s,d are the both-odd points. The sum and difference is a one-to-one mapping between the two classes.

Write the mapping as a matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{so} \quad \begin{pmatrix} s \\ d \end{pmatrix} = F \begin{pmatrix} p \\ q \end{pmatrix}$$

The U,A,D matrices transformed to act on s,d are D,A,U.

$$U_{sd} = F.U.F^{-1} = D$$
$$A_{sd} = F.A.F^{-1} = A$$
$$D_{sd} = F.D.F^{-1} = U$$

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