MSPR 4: Appendix: Maximum Likelihood Parameter Estimation for Gaussians

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This lecture is based on presentations made in my course 'Advanced Topics in in Music Technology' at Music Technology Group, Universitat Pompeu Fabra between 2007-2011, Barcelona, especially the ones by Stefan Kersten and Srikanth Cherla, closely following

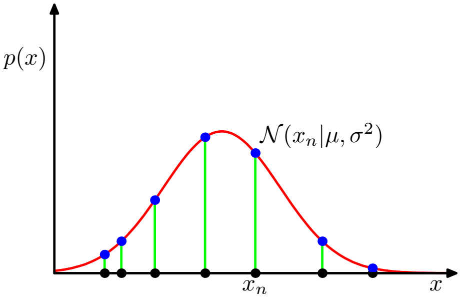
- *Andrew Moore's machine learning tutorial lectures: Gaussians, Gaussian Mixture Models,* *http://www.autonlab.org/tutorials/*
- *Christopher Bishop: Pattern Recognition and Machine Learning: Chapter 1 (Introduction) 1.2.3 (The Gaussian Distribution) p. 24 - 27, 2.3 (The Gaussian Distribution) p. 78, 84 bottom - 85 top.*

Outline

1 Gaussian Parameter Estimation
- ML

- Let's consider a vector $\mathbf{x}$ of N samples from a random distribution
- Elements of the vector are drawn independently and are identically distributed (i.i.d)
- Then the probability of $\mathbf{x}$ being produced by a Gaussian with parameters μ and σ^2 is

$$p(\mathbf{x}|\mu, \sigma^2) = \prod_{n=1}^{N} \mathcal{N}(x_n|\mu, \sigma^2)$$



Maximum Likelihood Estimation

- Given data $\mathbf{x}$, we want to find the most probable μ_{MAP} and σ^2_{MAP} that have generated $\mathbf{x}$: (*maximum a posteriori (MAP) estimation*)

$$(\hat{\mu}_{MAP}, \hat{\sigma}^2_{MAP}) = \arg\max_{\mu,\sigma^2} p(\mu, \sigma^2 \mid \mathbf{x})$$

- Bayes' Theorem: $p(\mu, \sigma^2 \mid \mathbf{x}) = \frac{p(\mathbf{x}|\mu,\sigma^2)}{p(\mathbf{x})} \cdot p(\mu, \sigma^2)$
- Marginalization $p(\mathbf{x}) = \sum_{\theta_1,\theta_2} p(\mathbf{x}|\theta_1, \theta_2) \cdot p(\theta_1, \theta_2)$ across all (θ_1, θ_2) does not dep. on $(\mu, \sigma^2) \Rightarrow p(\mathbf{x})$ const.
- Assume no prior knowledge about $(\mu, \sigma^2) \Rightarrow p(\mu, \sigma^2)$ const:

$$\arg\max_{\mu,\sigma^2} p(\mathbf{x} \mid \mu, \sigma^2) = \arg\max_{\mu,\sigma^2} p(\mu, \sigma^2 \mid \mathbf{x})$$

- *Maximum Likelihood Estimation*:

$$\boxed{(\hat{\mu}_{ML}, \hat{\sigma}^2_{ML}) = \max_{\mu,\sigma^2}\{p(\mathbf{x} \mid \mu, \sigma^2)\}}$$

- The joint probability of (two) independent variables factorizes into the product of each marginal probability:

$$p(X, Y) = p(X)p(Y)$$

- Assume data points are *independent and identically distributed (i.id.)*
- $\Rightarrow$ Likelihood function of the Gaussian distribution:

$$\begin{aligned} p(\mathbf{x} \mid \mu, \sigma^2) &= \prod_{n=1}^{N} N(x_n \mid \mu, \sigma^2) \\ &= N(x_1 \mid \mu, \sigma^2) \cdot N(x_2 \mid \mu, \sigma^2) \ldots \cdot N(x_N \mid \mu, \sigma^2) \end{aligned}$$

- ML adjusts mean μ and variance σ^2 to Gaussian distribution
- It is preferred to maximise the *log-likelihood* $\ln p(\mathbf{x}|\mu, \sigma^2)$, because
  - The natural logarithm is a monotonically increasing function. So maximizing log of a function is equivalent to maximizing function itself.
  - $\prod$ turns into $\sum$ and makes math simpler (Remember: $\ln(x \cdot y) = \ln x + \ln y$)
  - It gets rid of the exponentials.
  - A sum of logarithms is less likely to underflow a machine representation than a product of small probabilities
- Find parameter values maximizing the likelihood function, in two stages:
  1. Maximize with respect to mean
  2. Maximize with respect to variance

- Logarithm of the likelihood function:

$$\ln(p(\mathbf{x} \mid \mu, \sigma^2)) = \ln(\prod_{n=1}^{N} N(x_n \mid \mu, \sigma^2))$$

$$= \ln(\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \cdot e^{-\frac{(x_n-\mu)^2}{2\sigma^2}})$$

$$\ln\,(\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi \cdot \sigma^2}}) + \ln\,(\prod_{n=1}^{N} e^{-\frac{(x_n-\mu)^2}{2\sigma^2}})$$

  - First term:

$$\ln\,((\frac{1}{\sqrt{2\pi \cdot \sigma^2}})^N) = \ln(1^N) - ln(2\pi \cdot \sigma^2)^{\frac{N}{2}} = -\frac{N}{2} ln(2\pi) - \frac{N}{2} ln(\sigma^2)$$

  - Second term:

$$\sum_{n=1}^{N} -\frac{(x_n-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n-\mu)^2$$

- Log likelihood function:

$$\ln(p(\mathbf{x} \mid \mu, \sigma^2)) = -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - \mu)^2 - \frac{N}{2} ln(2\pi) - \frac{N}{2} ln(\sigma^2) \qquad (1)$$

Estimation of the Mean

- Partial derivative with respect to μ:

$$\frac{\partial \ln(p(\mathbf{x} \mid \mu, \sigma^2))}{\partial \mu} = \frac{\partial}{\partial \mu}\{(-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - \mu)^2 - \frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2)\})$$

$$= -\frac{1}{2\sigma^2} \cdot \frac{\partial}{\partial \mu}(\sum_{n=1}^{N}(x_n - \mu)^2) = \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n - \mu)$$

- Find maximum by setting derivative to 0:

$$0 = \frac{\partial \ln(p(\mathbf{x} \mid \mu, \sigma^2))}{\partial \mu} = \sum_{n=1}^{N}(x_n - \mu) = \sum_{n=1}^{N} x_n - N \cdot \mu$$

- Solution for μ:

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

Estimation of the Variance

- Maximize for variance (for which $\theta = (\mu, \sigma^2)$ is most likely):
- Derive for σ^2 (mean is known):

$$\frac{\partial \ln(p(\mathbf{x} \mid \mu, \sigma^2))}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\{(-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2)\})$$
$$= \frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2\sigma^2}$$

- First term (derivation rule: a $\frac{d}{dy}\frac{1}{y} = -a\frac{1}{y^2}$):

$$-\frac{1}{2}\sum_{n=1}^{N}(x_n-\mu)^2\frac{\partial}{\partial\sigma^2}\frac{1}{\sigma^2} = \frac{1}{2}\sum_{n=1}^{N}(x_n-\mu)^2\frac{1}{\sigma^4}$$

- Second term (derivation rule: $\frac{d}{dy}\ln y = \frac{1}{x}$):

$$-\frac{N}{2}\frac{\partial}{\partial\sigma^2}\ln(\sigma^2) - = -\frac{N}{2}\cdot\frac{1}{\sigma^2} = -\frac{N}{2\sigma^2} \tag{2}$$

Estimation of the Variance

$$\frac{\partial \ln(p(\mathbf{x} \mid \mu, \sigma^2))}{\partial \sigma^2} = 0$$

$$\Leftrightarrow \frac{1}{2\sigma^4} \sum_{n=1}^{N} (x_n - \mu)^2 - \frac{N}{2\sigma^2} = 0 \qquad \backslash \cdot 2\sigma^2$$

$$\Leftrightarrow \frac{1}{\sigma^2} \sum_{n=1}^{N} (x_n - \mu)^2 - N = 0 \qquad \backslash + N$$

$$\Leftrightarrow \frac{1}{\sigma^2} \sum_{n=1}^{N} (x_n - \mu)^2 = N \qquad \Leftrightarrow \sigma^2_{ML} = \tfrac{1}{N} \textstyle\sum_{n=1}^{N} (x_n - \mu)^2$$

- Solving $\ln p(\mathbf{x}|\mu, \sigma^2)$ yields maximum likelihood solutions for mean and variance

$$\mu_{ML} = \frac{1}{N} \sum_{n=1}^{N} x_n$$

$$\sigma^2_{ML} = \frac{1}{N} \sum_{n=1}^{N} (x_n - \mu_{ML})^2$$

- Same as our simple estimators for mean and variance above!