MSPR 2: Linear Algebra

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Outline

This lecture follows closely lecture notes by Prof. Siamac Fazli (Brain and Cognitive Engineering Department, Korea University): Introduction to Brain-Computer Interfacing, and the corresponding chapters of Strang: Introduction to Linear Algebra

Outline

Vectors

Definition of a two-dimensional vector:

Column vector $\quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \begin{aligned} v_1 &= \text{first component} \\ v_2 &= \text{second component} \end{aligned}$

Vectors

Vector Addition:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

Class Assignment:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \mathbf{v} + \mathbf{w} = ?$$

$$\mathbf{w} + \mathbf{v} = ?$$

Vectors

Vector Addition:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad v + w = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

Class Assignment Solution:

$$v + w = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad w + v = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

Vectors

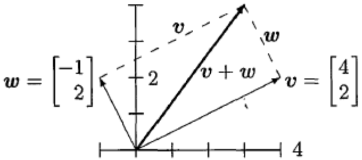
Class Assignment:

$$\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} + \mathbf{w} = ?$$

$$\mathbf{v} - \mathbf{w} = ?$$

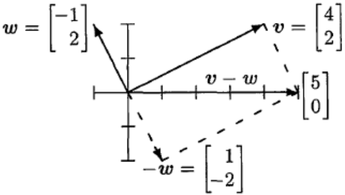
Vectors

Visualization of addition:



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Visualization of subtraction:



$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Vectors

Scalar multiplication:

$$2v = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad \text{and} \quad -v = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

Class Assignment:

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, -\pi\mathbf{v} = ?$$

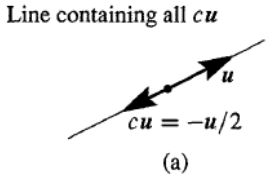
Vectors

Class Assignment Solution:

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, -\pi\mathbf{v} = \begin{bmatrix} -4\pi \\ 2\pi \end{bmatrix}$$

Vectors

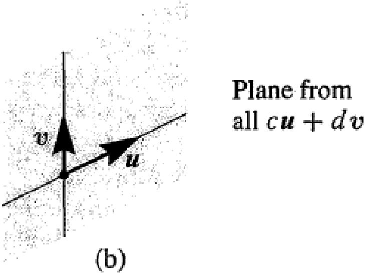
Visualization:



all scaled combinations of a single 2D vector form a line

Vectors

$c\mathbf{u} + d\mathbf{v}$ is known as a linear combination



(b)

for the 2D case: all linear combinations two vectors *can* span a plane

Class Assignment:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix},$$

Are there c, d so that

$$c\mathbf{u} + d\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}?$$

Vectors

Recap:

- a vector $\mathbf{v}$ in two-dimensional space has two components: v_1 and v_2
- a linear combination of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$
- Take *all* linear combinations of $\mathbf{u}$ (**line**), or $\mathbf{u}$ and $\mathbf{v}$ (**plane**) or $\mathbf{u}$,$\mathbf{v}$,$\mathbf{w}$ ($\mathbb{R}^3$).

Vectors

The scalar product (or dot product or inner product) of two vectors **v** and **w** is a number. For the two-dimensional case it is defined as:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

In 3D it is also a scalar:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Example:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

Class Assignment

Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$:

$$\mathbf{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Class Assignment

Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$:

$$\mathbf{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$-0.6 \cdot 3 + 0.8 \cdot 4 = -0.6 \cdot 3 + 0.8 \cdot 4 = 1.4$$

Vectors

The length of a vector $\mathbf{v}$ is defined as the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

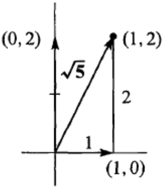
Class Assignments: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, Calculate the length of $\mathbf{u}, \mathbf{v}$

Vectors

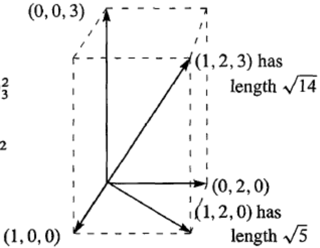
The length of a vector **v** is defined as the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Example in 2D and 3D:



$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{v} &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$



Class Assignment

$$\mathbf{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Compute the lengths $||\mathbf{u}||$. Find unit vectors in the directions of $\mathbf{v}$.

Class Assignment Solutions

$$\mathbf{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Compute the lengths $||\mathbf{u}||$. Find unit vectors in the directions of $\mathbf{v}$.

$\mathbf{u} \cdot \mathbf{u} = -0.6 \cdot 3 + 0.8 \cdot 4 = -1.8 + 3.2 = 1.4$

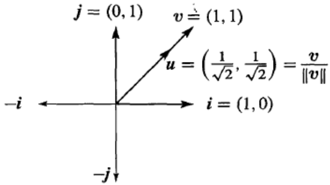
$||\mathbf{u}|| = \sqrt{(-0.6)^2 + 0.8^2} = \sqrt{0.36 + 0.64} = 1$ unit vector: $\frac{\mathbf{u}}{||\mathbf{u}||} = \mathbf{u}$

Vectors

The unit vector **u** of a given vector **v** is defined as:

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

Example:



the unit vector always has a length of 1, i.e. $\mathbf{u} \cdot \mathbf{u} = 1$

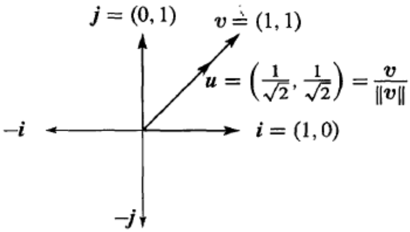
the unit vector **u** has the same direction as **v**.

Vectors

The unit vector $\mathbf{u}$ of a given vector $\mathbf{v}$ is defined as:

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

Example:



Vectors

If the dot product of two vectors is zero, i.e. $\mathbf{v} \cdot \mathbf{w} = 0$, they are perpendicular to each other.

The *Euclidean distance* between two vectors $\mathbf{u}, \mathbf{v}$ is defined as $||\mathbf{u} - \mathbf{v}||$

Class Assignment

A classifier ist defined as follows: if a point $\mathbf{x}$ is closer (Euclidean distance) to $\mathbf{a}1$, point $\mathbf{x}$ gets assigned to class 1, if it is closer to $\mathbf{a}2$ it gets assigned to class 2. Will $\mathbf{x}$ be assigned to class 1 or to class 2?

Class Assignment

A classifier ist defined as follows: if a point **x** is closer (Euclidean distance) to **a**1, point **x** gets assigned to class 1, if it is closer to **a**2 it gets assigned to class 2. Will **x** be assigned to class 1 or to class 2?

```
x=[1; 3; 2]; a1=[0; 1; 0]; a2=[2; 4; 2]
sqrt((x(1)-a1(1))^2+ (x(2)-a1(2))^2+(x(3)-a1(3))^2)
norm(x-a1)
% 3
norm(x-a2)
% 1.4142
```

Addition

Addition

- a matrix has *m* rows ans *n* columns - "*m* by *n* matrix"
- two matrices A and B can be added A+B, if they they have the same number of rows and columns
- addition of matrices is commutative: $A + B = B + A$
- and associative: $A + (B + C) = (A + B) + C$

Multiplication

Multiplication

- any matrix can be multiplied by a scalar: cA
- scalar multiplication is distributive: $c(A + B) = cA + cB$
- multiply AB: if A has n columns, B must have n rows

$$(\mathbf{m} \times n)(n \times \mathbf{p}) = (\mathbf{m} \times \mathbf{p})$$

- multiplication is associative: $A(BC) = (AB)C$
- but generally *not* commutative: $AB \neq BA$

Class Assignment

Create the mix of the two sounds 'vox.wav' and 'bass.wav' using matrix multiplication. The first mix should contain $0.1\cdot$ the vox sound and $0.9\cdot$ the bass sound. The second mix should contain $0.9\cdot$ the vox sound and $0.1\cdot$ the bass sound.

Example

Create the mix of the two sounds 'vox.wav' and 'bass.wav' using matrix multiplication. The first mix should contain $0.1\cdot$ the vox sound and $0.9\cdot$ the bass sound. The second mix should contain $0.9\cdot$ the vox sound and $0.1\cdot$ the bass sound.

Solution: The sound arrays of length n are the rows of X :

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \end{pmatrix} \tag{1}$$

M is the mixing matrix:

$$M = \begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$$

The result is stored in the remix matrix with the two remixes as the rows:

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \end{pmatrix}$$

Then the remix is calculated as follows:

$$R = M \cdot X$$

```
clear
[chans(:,1) sr]=audioread('../../snd/bigstone/vox.wav');
[chans(:,2) sr]=audioread('../../snd/bigstone/bass.wav');
sound(chans(:,1),sr);pause
sound(chans(:,2),sr);
A=[0.1 0.9;
     0.9 0.1;];
remix=chans*A;
sound(remix(:,1),sr);pause
sound(remix(:,2),sr);
```

Inverse Matrices

Definition of an *inverse matrix*:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

some properties of inverse matrices:

- the inverse A^{-1} of A is **unique**
- if A is invertible there is only one solution of the equation $\mathbf{Ax} = b$: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- if $\mathbf{x}$ is non-zero and gives: $\mathbf{Ax} = 0$ then $\mathbf{A}$ is not invertible
- if a matrix is invertible, its *determinant* is non-zero
- if A and B are invertible, then AB is invertible
- Matlab: `inv(A), rank(A)` if rank(A) smaller than n for n x n matrix $\mathbf{A}$ then $\mathbf{A}$ is not invertible

Class Assignment

$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & -1 \\ 1 & -3 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & -1 \\ -1 & -2 & -3 \end{bmatrix}$ Use Matlab to check if $\mathbf{A}, B$ have full rank, if yes invert them and multiply the matrix with its inverse and sove the equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Bx} = \mathbf{b}$.

Class Assignment Solution

$\mathbf{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{A} = \begin{bmatrix}2 & 4 & 5\\1 & 3 & -1\\1 & -3 & 5\end{bmatrix}, \mathbf{B} = \begin{bmatrix}2 & 4 & 6\\1 & 0 & -1\\-1 & -2 & -3\end{bmatrix}$ Use Matlab to check if $\mathbf{A}$, B have full rank, if yes invert them and multiply the matrix with its inverse and sove the equation $\mathbf{Ax} = \mathbf{b}$.

```
b=[1; 2; 3]
A=[2 4 5;
   1 3 -1;
   1 -3 5]
>rank(A)  % 3
>inv(A)
   -0.4000    1.1667    0.6333
    0.2000   -0.1667   -0.2333
    0.2000   -0.3333   -0.0667
x=inv(A)*b; A*inv(A)
```

Class Assignment Solution

$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & -1 \\ 1 & -3 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & -1 \\ -1 & -2 & -3 \end{bmatrix}$ Use Matlab to check if $\mathbf{A}, B$ have full rank, if yes invert them and multiply the matrix with its inverse and sove the equation $\mathbf{Bx} = \mathbf{b}$.

```
B=[2 4 6;
   1 0 -1;
   -1 -2 -3]
>rank(B)
  2
>inv(B)
   Inf     Inf     Inf
   Inf     Inf     Inf
   Inf     Inf     Inf
```

Transposes

Transpose: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$

Exchange row and columns: $(A^T)_{ij} = A_{ji}$

Sum: $(A + B)^T = A^T + B^T$
Product: $(AB)^T = B^T A^T$
Inverse: $(A^{-1})^T = (A^T)^{-1}$

Symmetric Matrices

For any symmetric matrix:

$$A^T = A$$

or:

$$a_{ji} = a_{ij}$$

the *ji*-entry of A is equal to the *ij* entry of A^T

Example of a symmetric matrix:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix}$$

the inverse of a symmetric matrix is also symmetric

Symmetric Products

Any matrix R that is multiplied by R^T will become a symmetric matrix!

Exercises

1. Calculate $AB + AC$
$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$
2. Find the matrix P that multiplies $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to give $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$.
3. Show that $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the inverse of $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Exercises

Calculate $AB + AC$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

Solutions:

$$AB + AC = \begin{bmatrix} 38 \\ 69 \end{bmatrix}$$

Exercises

Find the matrix P that multiplies $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to give $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$. **Solutions:**

$$\begin{bmatrix} 010 \\ 001 \\ 100 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

Show that $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the inverse of $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ **Solution:**

$$A \cdot B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot \frac{1}{2} + (-1) \cdot (-\frac{1}{2}) & 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} \\ 1 \cdot \frac{1}{2} + 1 \cdot (-\frac{1}{2}) & 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

MATLAB:

```
A=[1 -1; 1  1]; B=[1/2 1/2; -1/2 1/2]
C= A*B
```

RESULT:

```
C =   1      0
         0      1
```

Invert B :

```
C =   1      0
         0      1

inv(A)
=    0.5000    0.5000
   -0.5000    0.5000
inv(B)
=    1     -1
      1      1
```

Vector Spaces

DEFINITION:

The space $\mathbb{R}^n$ consists of all column vectors v with n components

- some vector spaces are: $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, ... , $\mathbb{R}^n$
- $\mathbb{R}^1$ is a line, $\mathbb{R}^2$ is a plane, etc.
- for example $\mathbb{R}^5$ contains all column vectors with five components
- we can add two vectors in $\mathbb{R}^n$ or multiply any vector $\mathbf{v}$ by any scalar c, the resulting vector will stay in the same vector space
- the *zero vector* is defined as: $\mathbf{0} + \mathbf{v} = \mathbf{v}$

Vector Spaces- Subspaces

DEFINITION:

A **subspace** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements:

If v and w are vectors in the subspace and c is any scalar, then

- $\mathbf{v} + \mathbf{w}$ **is in the subspace**
- $c\mathbf{v}$ **is in the subspace**

Vector Spaces- Subspaces

Examples:

- A plain through the origin in $\mathbb{R}^3$ is a subspace. If we add or scale any vectors in this plane they will stay in the plane.
- Also a line though the origin in $\mathbb{R}^3$ is a subspace. If we add two vectors on this line or scale one vector, we will stay on the line.
- Consider $\mathbb{R}^2$: keep only vectors (x, y) that are positive (a quarter plane). This is **not** a subspace: if $c = -1$, (x, y) becomes negative and leaves the quarter plane!

A subspace containing v and w must contain all linear combinations $c\mathbf{v} + d\mathbf{w}$

Class Assignments

1. Suppose $\mathbb{P}$ is a plane through $(0,0,0)$ and $\mathbb{L}$ is a line though $(0,0,0)$. The smallest vector space containing both $\mathbb{P}$ and $\mathbb{L}$ is either ____ or ____ .
2. Describe the subspace $\mathbb{R}^3$ (is it a line or plane in $\mathbb{R}^3$ spanned by:
   (a) the two vectors $(1,1,-1)$ and $(-1,-1,1)$
   (b) the three vectors $(0,1,1)$ and $(1,1,0)$ and $(0,0,0)$
   (c) all vectors in $\mathbb{R}^3$ with whole number components
   (d) all vectors with positive components

Class Assignments Solutions

- The smallest subspace containing a plane $\mathbb{P}$ and a line $\mathbb{L}$ is either $\mathbb{P}$ (when the line $\mathbb{L}$ is in the plane $\mathbb{P}$) or $\mathbb{R}^3$ (when $\mathbb{L}$ is not in the plane $\mathbb{P}$).
- a) Line in $\mathbb{R}^3$ b) Plane in $\mathbb{R}^3$ c) All of $\mathbb{R}^3$, d) All of $\mathbb{R}^3$

Projections

$$\mathbf{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let's say we want to project **b** onto the z-axis:

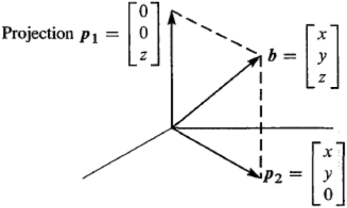
$$\mathbf{p_1} = P_1\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

Let's say we want to project **b** onto the x-y-plane:

$$\mathbf{p_1} = P_1\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Projections

graphical explanation:



- the xy plane and the z-axis are *orthogonal subspaces*
- the xy plane and the z-axis are *orthogonal complements*

Class assignment: What is the projection of $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ onto the x-z plane, onto y and onto the y-z plane?

Projections

Class assignment solutions: What is the projection of $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ onto the x-z plane, onto the y axis and onto the y-z plane? $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$