MSPR 3: Principal Component Analysis

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Outline

This lecture follows closely lecture notes by

1. *Prof. Ulrich Kockelkorn (Emeritus Berlin Institute of Technolgy): Multivariate Statistics (unpublished)*
2. *and Prof. Siamac Fazli (Brain and Cognitive Engineering Department, Korea University): Introduction to Brain-Computer Interfacing, and the corresponding chapters of Strang: Introduction to Linear Algebra*

Outline

1 Vector Spaces
- Class Assignments

2 Expectation, Variance, and Covariance

3 Projections

4 Eigenvalues and Eigenvectors

5 Matrix Diagonalization

6 Eigenvalue Decomposition for Symmetric Matrices

Vector Spaces

DEFINITION:

The space $\mathbb{R}^n$ consists of all column vectors v with n components

- some vector spaces are: $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, ... , $\mathbb{R}^n$
- $\mathbb{R}^1$ is a line, $\mathbb{R}^2$ is a plane, etc.
- for example $\mathbb{R}^5$ contains all column vectors with five components
- we can add two vectors in $\mathbb{R}^n$ or multiply any vector $\mathbf{v}$ by any scalar c, the resulting vector will stay in the same vector space
- the *zero vector* is defined as: $\mathbf{0} + \mathbf{v} = \mathbf{v}$

Vector Spaces- Subspaces

DEFINITION:

A **subspace** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements:

If v and w are vectors in the subspace and c is any scalar, then

- $\mathbf{v} + \mathbf{w}$ **is in the subspace**
- $c\mathbf{v}$ **is in the subspace**

Vector Spaces- Subspaces

Examples:

- A plain through the origin in $\mathbb{R}^3$ is a subspace. If we add or scale any vectors in this plane they will stay in the plane.
- Also a line though the origin in $\mathbb{R}^3$ is a subspace. If we add two vectors on this line or scale one vector, we will stay on the line.
- Consider $\mathbb{R}^2$: keep only vectors (x, y) that are positive (a quarter plane). This is **not** a subspace: if $c = -1$, (x, y) becomes negative and leaves the quarter plane!

A subspace containing v and w must contain all linear combinations $c\mathbf{v} + d\mathbf{w}$

Class Assignments

1. Suppose $\mathbb{P}$ is a plane through $(0,0,0)$ and $\mathbb{L}$ is a line though $(0,0,0)$. The smallest vector space containing both $\mathbb{P}$ and $\mathbb{L}$ is either ____ or ____ .
2. Describe the subspace $\mathbb{R}^3$ (is it a line or plane in $\mathbb{R}^3$ spanned by:
   (a) the two vectors $(1,1,-1)$ and $(-1,-1,1)$
   (b) the three vectors $(0,1,1)$ and $(1,1,0)$ and $(0,0,0)$
   (c) all vectors in $\mathbb{R}^3$ with whole number components
   (d) all vectors with positive components

Class Assignments Solutions

- The smallest subspace containing a plane $\mathbb{P}$ and a line $\mathbb{L}$ is either $\mathbb{P}$ (when the line $\mathbb{L}$ is in the plane $\mathbb{P}$) or $\mathbb{R}^3$ (when $\mathbb{L}$ is not in the plane $\mathbb{P}$).
- a) Line in $\mathbb{R}^3$ b) Plane in $\mathbb{R}^3$ c) All of $\mathbb{R}^3$, d) All of $\mathbb{R}^3$

Data Matrix

$$\mathbf{X} := \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1J} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2J} \\ x_{31} & x_{32} & \cdots & x_{3j} & \cdots & x_{3J} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots & x_{iJ} \\ & & \ldots & & \ldots & \\ x_{I1} & x_{I2} & \cdots & x_{Ij} & \cdots & x_{IJ} \end{pmatrix}$$

- rows: I objects $i = 1, \cdots, I$
- columns: J features $j = 1, \cdots, J$
- $\mathbf{X}_{[i,j]} = x_{ij}$: feature j of object i
- $\mathbf{X}_{[i,:]} =$ i-th row represents object i by its features
- $\mathbf{X}_{[:,j]} =$ j-th column: feature i of all objects

Class Assignment

- *From the Iris data set, calculate the sample means of the features sepal length, sepal width, petal length, petal width.*
- *From the Iris data set, calculate the sample variances and sample standard deviations for the features sepal length, sepal width, petal length, petal width.*

```
data_path='/Users/hendrik/teach/courses/mspr/data/';
fid = fopen([data_path 'iris.data']);
adata = textscan(fid,'%f%f%f%f%s','delimiter',',');
fclose(fid);
X=[adata{1} adata{2} adata{3} adata{4}];
[I J]=size(X);
featm=mean(X)% sample mean
% 5.8433    3.0540    3.7587    1.1987
var(X) % sample var
%  0.6857    0.1880    3.1132    0.5824
std(X)
% 0.8281    0.4336    1.7644    0.7632
```

Class Assignment

1. *Calculate the sample mean of each column (feature).*
2. *How to center the data, so that data points group around the origin (0), using matrix multiplication?*

1. Sample mean in column j $\mathbf{X}_{[:,j]}$:

$$\bar{\mathbf{X}}_{[:,j]} = \frac{1}{I} \sum_{i=1}^{I} x_{ij}$$

2. Centerize by substracting the column mean from each column component:

$$x_{ij} - \bar{\mathbf{X}}_{[:,j]}$$

- Centerize one column at once using vector subtraction:

$$\mathbf{X}_{[:,j]} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \bar{\mathbf{X}}_{[:,j]}$$

Matlab: `X(:,j)-ones(I,1)*xm(j)`

- Centerize matrix at once using column mean vector $\bar{\mathbf{x}}$:

$$\mathbf{X}_c = \mathbf{X} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \bar{\mathbf{x}}$$

Matlab: `Xc=X-ones(I,1)*xm`

Variance

Class Assignment

1. *Calculate the variances of each feature, using matrix multiplication (on paper and with Matlab)*

Variance

1. Sample variance for feature j using multiplications of vectors

$$\mathbf{s}_j = \frac{1}{I-1}\mathbf{X}^T_{c[:,j]}\mathbf{X}_{c[:,j]}$$

2. Matlab: s^2=`Xc(:,j)'*Xc(:,j)/(I-1)` or `s=var(Xc(:,j))`

Covariance Matrix I

- Covariance between feature j (j-th data matrix column) and k (k-th data matrix column):

$$\mathrm{cov}(\mathbf{X}_{[:,j]}, \mathbf{X}_{[:,k]}) = \frac{1}{I-1}\sum_{i=1}^{I}(x_{ij} - \bar{\mathbf{X}}_{[:,j]})(x_{ik} - \bar{\mathbf{X}}_{[:,k]})$$

- $J \times J$ covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathrm{var}(\mathbf{X}_{[:,1]}) & \mathrm{cov}(\mathbf{X}_{[:,1]}, \mathbf{X}_{[:,2]}) & \dots & \mathrm{cov}(\mathbf{X}_{[:,1]}, \mathbf{X}_{[:,J]}) \\ \mathrm{cov}(\mathbf{X}_{[:,2]}, \mathbf{X}_{[:,1]}) & \mathrm{var}(\mathbf{X}_{[:,2]}) & \dots & \mathrm{cov}(\mathbf{X}_{[:,2]}, \mathbf{X}_{[:,J]}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(\mathbf{X}_{[:,J]}, \mathbf{X}_{[:,1]}) & \mathrm{cov}(\mathbf{X}_{[:,J]}, \mathbf{X}_{[:,2]}) & \dots & \mathrm{var}(\mathbf{X}_{[:,J]}) \end{bmatrix}$$

The diagonal elements are the variances of each features $1, \dots, J$ Matrix is symmetric

Covariance Matrix

Class Assignment

1. *Calculate the covariance matrix of* $\mathbf{X}$*, using matrix multiplication (on paper and with Matlab)*

Covariance

1. Sample covariance Matrix $\mathbf{C}$ of all features (columns of centirized data matrix $\mathbf{X}_c$):

$$\mathbf{C} = \frac{1}{I-1}\mathbf{X}_c^T\mathbf{X}_c$$

2. Matlab: `C=Xc'*Xc/(I-1)` or `C=cov(X)`

Covariance Matrix

Class Assignment

Calculate the covariance matrix for the covariances between the features sepal length/width, petal length/width of the iris data set in Matlab with and without the Matlab function `cov`. *Discuss the results*

```
X=[adata{1} adata{2} adata{3} adata{4}];
[I J]=size(X);
Xm=mean(X); Xc=X-ones(I,1)*Xm; % centerize
covar=Xc'*Xc/(I-1); % covariance matrix
covar2=cov(X); chk=max(max(covar2-covar)); % check if both the same
```

COV	sepal l.	sepal w.	petal l.	petal w.
sepal l.	0.6857	-0.0393	1.2737	0.5169
sepal w.	-0.0393	0.1880	-0.3217	-0.1180
petal l.	1.2737	-0.3217	3.1132	1.2964
petal w.	0.5169	-0.1180	1.2964	0.5824

Higest variance for petal l., high covariance between sepal l. / petal l. and petal w. / petal l.

Class Assignment

Calculate the correlation matrix, using matrix multiplication and the Matlab function `diag` *that puts the elements of a vector on the diagonal of a matrix.*

- $$r_{jk} = \frac{cov(\mathbf{X}_{[:,j]}, \mathbf{X}_{[:,k]})}{s_{\mathbf{X}_{[:,j]}} s_{\mathbf{X}_{[:,k]}}}$$

- $$R = \mathrm{diag}(\frac{1}{s_1}, \ldots, \frac{1}{s_J}) \cdot \Sigma \cdot \mathrm{diag}(\frac{1}{s_1}, \ldots, \frac{1}{s_J})$$

- Matlab:

```
istd=1./std(X); R=diag(istd)*cov(X)*diag(istd);
```

or

```
R=corr(X)
```

Class Assignment

Calculate the correlation between the features sepal length, sepal width, petal length, petal width in the iris data set.

Correlation of Iris Data

```
R=corr(X)
```

R	sepal l.	sepal w.	petal l.	petal w.
sepal l.	1.0000	-0.1094	0.8718	0.8180
sepal w.	-0.1094	1.0000	-0.4205	-0.3565
petal l.	0.8718	-0.4205	1.0000	0.9628
petal w.	0.8180	-0.3565	0.9628	1.0000

- Correlation with itself always 1
- Very high correlation between petal width and petal length, also high correlation among sepal length, petal length, and petal width
- No correlation between sepal length and sepal width

Projections

$$\mathbf{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let's say we want to project **b** onto the z-axis:

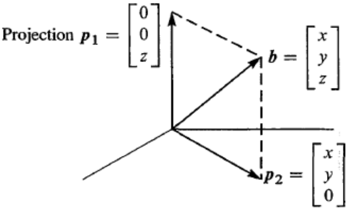
$$\mathbf{p_1} = P_1\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

Let's say we want to project **b** onto the x-y-plane:

$$\mathbf{p_1} = P_1\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Projections

graphical explanation:



- the *xy* plane and the *z*-axis are *orthogonal subspaces*
- the *xy* plane and the *z*-axis are *orthogonal complements*

Class assignment: What is the projection of $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ onto the x-z plane, onto y and onto the y-z plane?

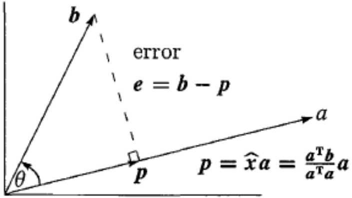
Projections

Class assignment solutions: What is the projection of $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ onto the x-z plane, onto the y axis and onto the y-z plane? $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Projection onto a Line

If we want to project vector **b** onto the vector **a**, the resulting vector **p** is defined as:

$$\boxed{\mathbf{p} = \mathbf{a}\hat{\mathbf{x}} = \mathbf{a}\frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}}$$



Let's call P *the projection matrix*: $\boxed{P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}}$ then the projection is given as: $\boxed{\mathbf{p} = P\mathbf{b}}$

Class Assignment

For $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ plot the two vectors and calculate $\hat{x}, \mathbf{P}, \mathbf{p}, \mathbf{e}$.

Projection onto a Subspace

Let's say we have a number of (independent) vectors:

$$\mathbf{a_1}, \ldots, \mathbf{a_n}$$

we want to find the combination $\mathbf{p} = \hat{x}_1 \mathbf{a_1} + \ldots + \hat{x}_n \mathbf{a_n}$ closest to a given vector $\mathbf{b}$.

In other words we are trying to find $\mathbf{p} = A\hat{\mathbf{x}}$ that is closest to $\mathbf{b}$ (the vectors $\mathbf{a_1}, \ldots, \mathbf{a_n}$ are the columns of A)

Projections

In the 1D case we had: $\mathbf{e} = \mathbf{b} - \hat{\mathbf{x}}\mathbf{a}$

Now we get: $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$

and same as before, the error vector makes right-angles with all vectors $\mathbf{a_1}, \ldots, \mathbf{a_n}$ (if we multiply it gives zero!!):

$$\begin{matrix} \mathbf{a_1^T}(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \\ \vdots \\ \mathbf{a_n^T}(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \end{matrix} \quad \text{or} \quad \begin{bmatrix} - & \mathbf{a_1^T} & - \\ & \vdots & \\ - & \mathbf{a_1^T} & - \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{b} - A\hat{\mathbf{x}} \end{bmatrix}}_{\mathbf{e}} = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

or:

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$$

Projections

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$$

we can rewrite this as:

$$A^T\mathbf{b} - A^T A\hat{\mathbf{x}} = \mathbf{0}$$

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

$$\boxed{\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}}$$

and so the projection of $\mathbf{b}$ onto the subspace is $\mathbf{p}$:

$$\boxed{\mathbf{p} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b}}$$

and with $\mathbf{p} = P\mathbf{b}$ finally our *Projection matrix P:*

$$\boxed{P = A(A^T A)^{-1} A^T}$$

Projection Example

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find $\hat{\mathbf{x}}$, $\mathbf{p}$ and the projection matrix P.

Projection Example

$$(A^T A)\hat{\mathbf{x}} = A^T\mathbf{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Projections

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Project **b** onto the column space of A:

$$\mathbf{p} = A\hat{\mathbf{x}}$$

column view:

$$\mathbf{p} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

The *error* is given as: $\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Projections

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{e} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

e should be perpendicular to both columns of A!

$$\mathbf{e}^\mathbf{T} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\mathbf{e}^\mathbf{T} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$$

Projections

Find the projection matrix P:

$$P = A(A^T A)^{-1} A^T$$

$$(A^T A)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \quad \text{then} \quad P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Strang: Chapter 6.1: Introduction to Eigenvalues pp. 283-297

Eigenvalues and Eigenvectors

Generally the vector $\mathbf{x}$ *changes direction* when multiplied by $\mathbf{A}$...

However there are some special vectors $\mathbf{x}$ that are in the same direction as $\mathbf{Ax}$:

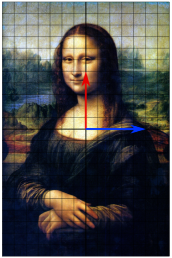
$$\mathbf{Ax} = \lambda \mathbf{x}$$

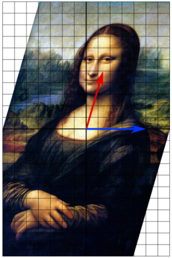
- such a vector $\mathbf{x}$ is an *eigenvector*
- λ is the *eigenvalue* corresponding to $\mathbf{x}$
- λ can is a number, it scales $\mathbf{x}$
- if $\mathbf{A} = \mathbf{I}$, $\mathbf{Ax} = \mathbf{x}$, i.e. all vectors are eigenvectors of $\mathbf{I}$

Example I

- Shearing: $\mathbf{A} = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1
- $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is another eigenvector with eigenvalue 1

http://en.wikipedia.org/wiki/File:Mona_Lisa_eigenvector_grid.png





Example II

Video: `http://en.wikipedia.org/wiki/File:Eigenvectors.gif`

- Transformation matrix
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
- Example of eigenvectors: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with corresponding eigenvalues $1, 3$.

Eigenvalue Decomposition for Symmetric Matrices

Eigenvalue Decomposition for Symmetric Matrices

Strang: Chapter 6.2: Diagonalizing a Matrix pp. 298-311, Chapter 6.4: Symmetric Matrices

Eigenvalue Decomposition for Symmetric Matrices

Definition

An *orthonormal* matrix V is a matrix in which

- all column vectors (row vectors) are orthogonal to each other, i.e. $\mathbf{v} \cdot \mathbf{w} = 0$ for $\mathbf{v} \neq \mathbf{w}$
- all column vectors (row vectors have length 1, i.e. $\|\mathbf{v}\| = 1$
- $\Rightarrow \mathbf{V}^T\mathbf{V} = \mathbf{I} = \mathbf{V}^{-1}\mathbf{V} \Rightarrow \mathbf{V}^{-1} = \mathbf{V}^T$

Theorem

(*Eigenvalue decomposition for symmetric matrices:*) Every symmetric matrix has the matrix decomposition $\mathbf{A} = \mathbf{V\Lambda V}^T$ with real eigenvalues in $\mathbf{\Lambda}$ and orthogonal eigenvectors in $\mathbf{V}$:

$$\mathbf{A} = \mathbf{V\Lambda V}^{-1} = \mathbf{V\Lambda V}^T \qquad \text{with} \qquad \mathbf{V}^{-1} = \mathbf{V}^T$$

Eigenvalue Decomposition for Symmetric Matrices

The *eigenvector matrix* $\mathbf{V}$ has the eigenvectors $\mathbf{v}_1 \ldots \mathbf{v}_n$ as columns.

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

The *eigenvalue matrix* $\mathbf{\Lambda}$ has the eigenvalues as diagonal elements in corresponding order:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Eigenvalue Decomposition for Symmetric Matrices

Proof

Let's multiply any square matrix **A** with the eigenvector matrix **V**:

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \dots & \lambda_n \mathbf{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \mathbf{V\Lambda}$$

$$\mathbf{AV} = \mathbf{V\Lambda} \qquad \Rightarrow \boxed{\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda}} \qquad \text{or} \qquad \boxed{\mathbf{A} = \mathbf{V\Lambda V}^{-1}}$$

Eigenvalue Decomposition for Symmetric Matrices

Remark on Diagonalization

- The order of eigenvectors in $\mathbf{V}$ corresponds to the order of eigenvalues in $\mathbf{\Lambda}$ (see proof on diagonalization)

Exercise Solution

- Find the eigenvectors and eigenvalue(s) λ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Matlab

```
A=[0 1; 1 0];
[V D]=eig(A);
```

Output:

```
V =     -0.7071     0.7071
         0.7071     0.7071
D =      -1       0
          0       1
```

Measuring how Much Variance a Projection Preserves

The *Inertia* for set of points $\mathbf{X}_{[1,:]}, \mathbf{X}_{[2,:]}, \ldots, \mathbf{X}_{[I,:]}$ is a measure for the variability in the data:

$$\mathcal{IN} = \frac{1}{I} \sum_{i=1}^{I} \left\| \mathbf{X}_{[i,:]} \right\|^2 .$$

We can measure the quality of a projection $\mathbf{P}$ by measuring much inertia (variance) in the data is preserved: Calculate the quotient of inertia of the projected points and the original points:

$$\tau := \frac{\sum_{i=1}^{I} \left\| \mathbf{P}\mathbf{X}_{[i,:]} \right\|^2}{\sum_{i=1}^{I} \left\| \mathbf{X}_{[i,:]} \right\|^2}$$

PCA provides the best projection, in the sense of optimal intertia preservation.