# Effect of Minimal Length on Landau Diamagnetism and de Haas-van Alphen Effect

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June 10, 2022



**Abstract**

We study Landau diamagnetism in the framework of generalised uncertainty principle(GUP). We calculate the correction to magnetisation and susceptibility by constructing the grand partition function of diamagnetic material in this framework. We explicitly show that Curie's law gets a temperature independent correction which vanishes when quantum gravity effects are neglected. We further consider the low temperature limit to find how GUP affects the de Haas-van Alphen effect.

## 1 Introduction

All the basic forces except gravity are well described in the quantum framework whereas the theory of gravity was understood based on Einstein's General theory of relativity which is formulated in classical physics. There were certain areas where we need to merge quantum mechanics and general relativity to develop the quantum theory of gravitation. The existence of a minimum possible length, the Planck length $l_{pl} \approx 10^{-35}m$ [1–7], was predicted by all approaches of quantum gravity theory, doubly special relativity(DSR), perturbative string theory, black hole physics, etc. The usual Heisenberg's uncertainty principle(HUP) needs essential modifications to incorporate the existence of minimum length scale. The new uncertainty principle, known as the generalised uncertainty principle(GUP) [1, 8–11], is as follows,

$$\Delta x \Delta p \quad \geq \quad \frac{1}{2}[1 + \alpha(\Delta p)^2 + \alpha\langle p\rangle^2], \tag{1}$$

where $\Delta x$ and $\Delta p$ are the uncertainties in position and momentum respectively. The positive parameter $\alpha = \frac{\alpha_0 l_{pl}^2}{2} = \frac{\alpha_0}{M_{pl}^2}$, does not depend on $\Delta x$ and $\Delta p$ but it may depend on the expectation values of $\mathbf{x}$ and $\mathbf{p}$. $\alpha_0$ is known as the GUP parameter which is a positive dimensionless parameter, and the Planck mass $M_{pl} \approx 10^{19} Gev$. According to Eq.(1), a nonzero minimal uncertainty in length is $\Delta x_{min} \approx l_{pl}\sqrt{\alpha_0}$. The parameter $\alpha_0$ is considered to be 1 in most of the GUP calculations. There are several ways to construct the modified Heisenberg algebra due to GUP and the most commonly used modified Heisenberg algebra due to GUP approach, via the Jacobi identity [11, 12] is given by,

$$[x_i, p_j] = i(\delta_{ij} + \alpha\delta_{ij}p^2 + 2\alpha p_i p_j), \quad [x_i, x_j] = 0, \quad [p_i, p_j] = 0. \tag{2}$$

To be consistent with Eq.(1) we can define the position and the momentum operators in the following way [7] [12] [13],

$$x_i \equiv x_{0i}, \quad p_i \equiv p_{0i}(1 + \alpha p_0^2), \tag{3}$$

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where, $p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j}$. The operators $x_{0i}$ and $p_{0i}(= -id/dx_{0i})$, at low energy, obey the commutation relation,

$$[x_{0i}, p_{0j}] = i\delta_{ij}. \tag{4}$$

In the presence of potential $V(\vec{r})$ the Hamiltonian of any quantum mechanical system should be modified due to GUP in the following form [13–36],

$$H = \frac{p_0^2}{2m} + V(\vec{r}) + \frac{\alpha}{m}p_0^4 + \mathcal{O}(\alpha^2). \tag{5}$$

The effects of GUP are investigated in almost every branches of physics. In quantum optics many results with the effect of quantum gravity were confirmed. In the region of minimum length, exact solutions to several relativistic and non relativistic problems have been obtained [14]. In this article we consider the effect of minimum length on diamagnetism.

In this paper, we take the velocity of light in vacuum, $c = 1$, reduced Planck constant, $\hbar = 1$, and Boltzmann constant, $k_B = 1$ throughout our calculation. In section 2, we have calculated the grand-Canonical partition function of a diamagnetic material and the corrections to the magnetisation and susceptibility due to minimal length at high temperature. The effect of GUP on the low temperature de Haas-van Alphen effect has been studied in section 3. We end with concluding remarks in section 4 where we summarise our main results and discuss some future directions.

## 2 Diamagnetic properties at high temperatures

In this section, we study the diamagnetic property of material under the GUP effect. For this, we first calculate the grand-canonical partition function of a system of $N$ spinless electrons with mass $m$, confined in a volume $V$ under the action of a constant magnetic field $B$ along the $z$-direction. Using the grand-canonical partition function we systematically derive the susceptibility $\chi$ at a high temperature limit. When we turn on the magnetic field the electrons of the system start to move on a helical path with its axis parallel to the direction of the magnetic field. The projection of the helical trajectory of the electrons to the $(x, y)$ plane becomes circular [37] [38]. Now using the modified commutation relation due to quantum gravity, the single particle energy levels of electrons are given by [13] [32],

$$\epsilon_j = \frac{p_z^2}{2m} + 2\mu_B B(j + \frac{1}{2}) + \alpha\left[\frac{p_z^4}{m} + 16m\mu_B^2 B^2(j + \frac{1}{2})^2\right] + \mathcal{O}(\alpha^2). \tag{6}$$

These quantised energy levels with quantum number $j$ are degenerate. Now the density of the one particle states [39],

$$\begin{aligned} g_j &= \frac{1}{4\pi^2}\int_{p_j}^{p_{j+1}} \frac{dp_x dp_y dx dy}{(1 + \alpha p^2)^3} \\ &= \frac{V^{2/3}}{\pi} m\mu_B B[1 - 6\alpha m\mu_B B(2j + 1)] + \mathcal{O}(\alpha^2), \end{aligned} \tag{7}$$

where $p_j \equiv \sqrt{4m\mu_B B j}$ and the Bohr magneton, $\mu_B \equiv \frac{e}{2m}$. As opposed to the low energy case, we can see that $g_j$ is dependent on the quantum number $j$. At finite temperature $T$ the grand-canonical partition function of the electron gas becomes,

$$\ln \mathcal{Z}_G = \int_{-\infty}^{+\infty} \frac{V^{1/3} dp_z}{2\pi(1 + \alpha p_z^2)^2} \sum_{j=0}^{\infty} g_j \ln[1 + \mathsf{z}\ \exp(-\beta\epsilon_j)] \tag{8}$$

At a high temperature or equivalently at low $\beta(= \frac{1}{T})$, we can take the fugacity, $\mathsf{z} << 1$, and the system is effectively Boltzmannian, thus,

$$\begin{aligned} \ln \mathcal{Z}_G &= \int_{-\infty}^{+\infty} \frac{\mathsf{z}\ V^{1/3} dp_z}{2\pi(1 + \alpha p_z^2)^2} \sum_{j=0}^{\infty} g_j\ \exp(-\beta\epsilon_j) \\ &= \frac{V\mathsf{z}}{\lambda^3} x\ \text{cosech}\ x\left[1 - \frac{m\alpha}{\beta}\{5 + 6x\coth x + 4x^2(\coth^2 x + \text{cosech}^2 x)\}\right]. \end{aligned} \tag{9}$$

To make the above expression compact we have introced the thermal wavelength $\lambda \equiv \sqrt{\frac{2\pi}{mT}}$ and a new variable $x \equiv \frac{\mu_B B}{T}$. In case of $\alpha \to 0$, the above equation is reduced to the grand-canonical partition function of the diamagnetic material without GUP effect [37, 38]. Now the average number of electrons in the system is,

$$\bar{N} = \mathsf{z}\frac{\partial}{\partial \mathsf{z}} \ln \mathcal{Z}_G. \tag{10}$$

At high temperature limit, fugacity $\mathsf{z} \to 0$ and if the field intensity $B$ and the temperature $T$ are such that $\mu_B B << T$ then,

$$\bar{N} \simeq \frac{V\mathsf{z}}{\lambda^3}\big[1 - 19\frac{\alpha}{\beta}m\big]. \tag{11}$$

Magnetic moment $M$ of the gas is given by,

$$M = -\left\langle \frac{\partial H}{\partial B} \right\rangle = \frac{1}{\beta}\left(\frac{\partial}{\partial B} \ln \mathcal{Z}_G\right)_{T,V,z}. \tag{12}$$

For high temperature limit, the magnetic moment per unit volume, i.e the magnetisation is,

$$\mathcal{M} = \frac{M}{V} = -\tfrac{1}{3}\tfrac{\mathsf{z}}{\lambda^3}\mu_B x\left[1 + \tfrac{m\alpha}{\beta}\right]. \tag{13}$$

Now at high temperature, the magnetic susceptibility per unit volume as a function of temperature $T$ and specific volume $\mathcal{V} = \frac{V}{N}$ becomes,

$$\begin{aligned}\chi &= \left(\frac{\partial \mathcal{M}}{\partial B}\right) \\ &= -\frac{\mu_B^2}{3\mathcal{V}T}\left[1 + 20\alpha m T\right].\end{aligned} \tag{14}$$

The $1/T$ dependence in Figure 1 confirms that Curie's law still holds with the effect of minimum length. Also the Curie constant $(\frac{\mu_B^2}{3\mathcal{V}})$ reflects the quantization of the orbits. The negative sign in Eq.(14) implies the property of diamagnetism ($\chi < 0$) is not affected by the quantum gravity. In the case of $\alpha \to 0$, the above equation is reduced to the low energy susceptibility per unit volume of the electron gas.

## 3 Low temperature de Haas-van Alphen effect

At a very low temperature ($T \to 0$) the susceptibility of an ideal Fermi gas discontinuously changes with the varying magnetic field $B$, this phenomenon is known as the de Haas-van Alphen effect [37]. In this section, we study the effect of GUP on this quantum mechanical effect.

We consider $E_0$ to be the ground state energy of an ideal Fermi gas at absolute zero, and $E_0$ is a function of the field strength $B$. To simplify our calculation, we ignore the motion of the electrons along the direction of magnetic field $B$, i.e. we take $p_z = 0$. In other words, we consider a two dimensional Fermi gas whose single-particle energy levels are,

$$\epsilon_j = \mu_B B(2j+1)[1 + 4\alpha m \mu_B B(2j+1)], \tag{15}$$

which is $g_j$-fold degenerate, with,

$$g_j = g[1 - 6\alpha m \mu_B B(2j+1)], \tag{16}$$

where $g = \frac{V^{2/3}}{2\pi} eB$ is constant low energy density of states. The ground state energy $E_0$ is the sum of $\epsilon_j$ over the lowest $N$ single particle states. Since $g_j$ depends on B, the maximum number of particles that

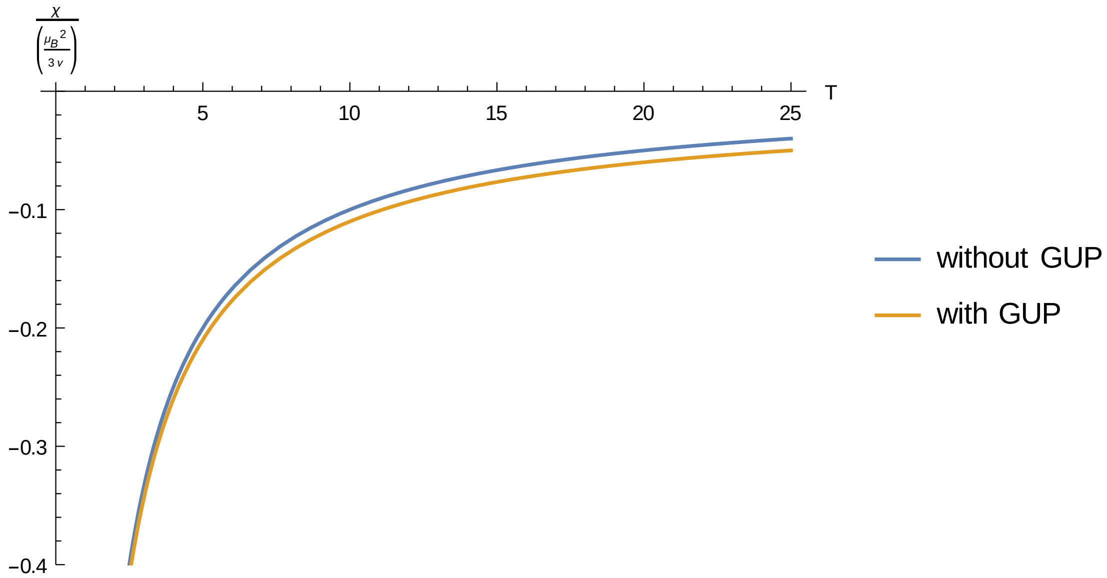


Figure 1: Reduced susceptibility, $\chi/(\frac{\mu_B^2}{3\mathcal{V}})$ vs temperature, $T$ plot of Landau diamagnetism with electron mass, $m = 0.5 \times 10^{-3} Gev$, and we set the GUP parameter, $\alpha = 1 Gev^{-2}$.

can have the energy $\epsilon_j$ depends on $B$. If the field $B$ is such that $g_0 \geq N$ then all particles can occupy the lowest energy level and $E_0$ becomes,

$$\begin{aligned} E_0(B) &= N\epsilon_0, \\ \frac{E_0(B)}{N} &= \mu_B B_0 y + 4\alpha m \mu_B^2 B_0^2 y^2; \quad \text{for } g_0 \geq N, \end{aligned} \tag{17}$$

where, we have defined two new variables $B_0 := \frac{NB}{g}$ and $y := \frac{B}{B_0}$. For a particular value of field $B$, suppose $\sum_{i=0}^{j} g_i \ < N < \ \sum_{i=0}^{j+1} g_i$ , then the $(j+1)$ number of lowest levels are completely filled with $g_i$ number of particles in every $i$th level. Also, the $(j+1)$st level is only partially filled, and the higher levels are empty. In this case, the ground state energy becomes,

$$E_0(B) = \sum_{i=0}^{j} g_i \epsilon_i + \left[ N - \sum_{i=0}^{j} g_i \right] \epsilon_{j+1},$$

$$\frac{E_0(B)}{N} = \mu_B B_0 y[(2j+3) - y(j+1)(j+2)] + \frac{2}{3}\alpha m B_0^2 \mu_B^2 y^2 \left[ 6(2j+3)^2 - 5y(j+1)(j+2)(2j+3) \right]; \quad \text{for } \sum_{i=0}^{j} g_i \ < N < \ \sum_{i=0}^{j+1} g_i. \tag{18}$$

As calculated in the previous section, the magnetisation of the system is given by,

$$\mathcal{M} = -\frac{1}{V}\frac{\partial E_0}{\partial B}. \tag{19}$$

Now there are two situations when the total number of electrons is greater than the degeneracy of the lowest level with $j = 0$, or greater than the degeneracy of any $j$-th level of the system as discussed above. So, the magnetisation,

$$\mathcal{M} \ = -\tfrac{\mu_B}{\mathcal{V}} \left[ 1 + 8\alpha m B_0 \mu_B y \right]; \quad \text{for } g_0 \geq N \tag{20}$$

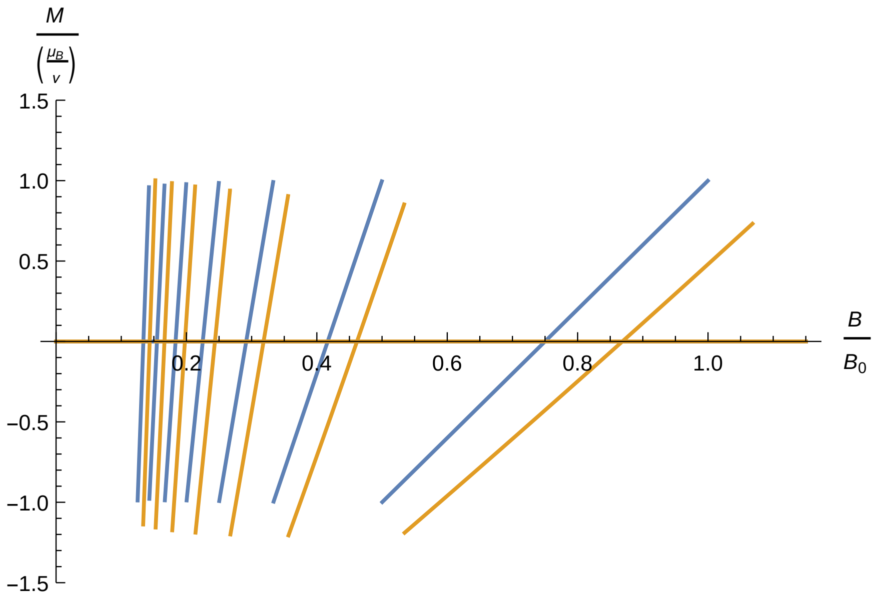


Figure 2: Reduced magnetisation, $\mathcal{M}/(\frac{\mu_B}{\mathcal{V}})$ vs magnetic field, $y = B/B_0$ plot in de Haas-Van Alphen effect, with the GUP dependent parameter, $\alpha m B_0 \mu_B = 10^{-2}$. The blue curve indicates without GUP correction, and the yellow one indicates the result with GUP correction.

$$\mathcal{M} = \frac{\mu_B}{\mathcal{V}}\left[\left(2(j+1)(j+2)y-(2j+3)\right)-2\alpha m B_0\mu_B y\left(4(2j+3)^2-5y(j+1)(j+2)(2j+3)\right)\right]; \quad \text{for} \sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i. \tag{21}$$

Figure 2, shows that the GUP corrected magnetisation still changes discontinuously with the magnetic field. Also interestingly the curves approach the low energy results in higher $j$ values. Now the susceptibility per unit volume is,

$$\chi = \frac{\partial \mathcal{M}}{\partial B} = -\frac{1}{V}\frac{\partial^2 E_0}{\partial B^2}. \tag{22}$$

Depending upon the electron numbers we have,

$$\chi = -\frac{8\alpha}{\mathcal{V}} m\mu_B^2; \quad \text{for } g_0 \geq N \tag{23}$$

$$\chi = \frac{2\mu_B}{\mathcal{V}B_0}(j+1)(j+2) - \frac{2\alpha}{\mathcal{V}} m\mu_B^2\left[4(2j+3)^2-10y(j+1)(j+2)(2j+3)\right]; \quad \text{for} \sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i. \tag{24}$$

As opposed to the magnetisation case, in Figure 3 the susceptibility approaches to the low energy values in higher $j$ values. From the above results, it is straightforward to see that the magnetisation $(\mathcal{M})$ and the magnetic susceptibility per unit volume $(\chi)$ are reduced to the result without minimal length effect as $\alpha \to 0$.

# 4 Conclusions

From HUP, we can infer that essential quantum phenomena are observed at sufficiently high energy (high momenta). On the other hand, gravity also becomes very interesting at an extremely small length scale, which is at very high energy. Thus to address fundamental areas of physics we must resort to quantum

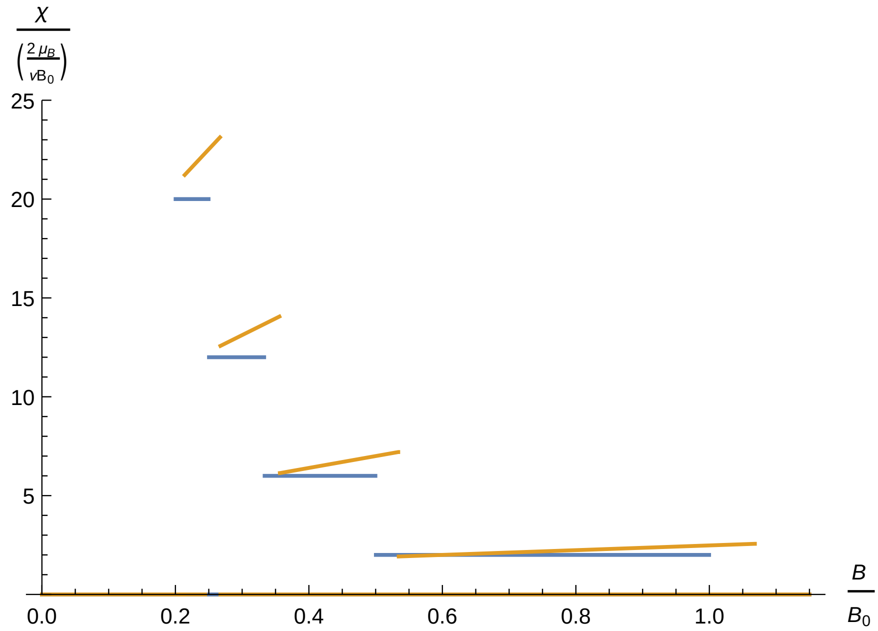


Figure 3: Reduced susceptibility, $\chi/(\frac{2\mu_B}{VB_0})$ vs magnetic field, $y = B/B_0$ plot in de Haas-Van Alphen effect, with the GUP dependent parameter, $\alpha m B_0 \mu_B = 10^{-2}$. The blue curve indicates without GUP correction, and the yellow one indicates the result with GUP correction.

gravity theories. In this article, we have studied how the theory of Landau diamagnetism gets modified when we consider the effect of quantum gravity theories. Our analysis shows that the overall behavior of the magnetic susceptibility for diamagnetic material is the same. It gets only a temperature independent constant shift in the first order (first order in $\alpha$) correction due to GUP. In absence of quantum gravity effects, it will reduce to $\frac{1}{T}$ dependence (usual Curie's law). At a very low temperature, the susceptibility of an ideal Fermi gas changes discontinuously with the magnetic field, commonly known as the de Haas-van Alphen effect. We observed a modified de Haas-van Alphen effect in presence of GUP as explained in Figure 3. A similar analysis using non commutative algebra(NC) was done in [40], which may help to further investigate the relationship between NC algebra and GUP for this simple system of an ideal Fermi gas.

## Acknowledgements

MA thanks Aditya Dwivedi and Kajal Singh for useful discussions. BPM acknowledges the research Grant for faculty under IoE Scheme (Number 6031) of Banaras Hindu University, Varanasi.

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