TU Wien

Department of Logic & Computation

Master Degree in Artificial Intelligence and Computer Science

Thesis

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Proof of my AMOSUM :)

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Contents

preliminars

Definition 1. *Increment function*

Given a set of literals L *then a group function* $inc : L \mapsto \mathbb{N}$ *is a function that maps a literal* l *to its increment.*

Definition 2. *Group function*

Given a set of literals L *and a set of groups* G *then a group function* $group : L \mapsto G$ *is a function that maps a literal* l *to its group.*

Definition 3. *max_lit function*

Given a group set G *and a set of literals* L *of literals then* $max\_lit : \mathcal{P}(L) \times G \mapsto L$ *is a function from a pair* $(S, g) \in \mathcal{P}(L) \times G \mapsto l \in S$ *such that* l *is the literal ,belonging to* g*, with the maximum increment, that is* $max\_lit(S, g) = \arg\max_{k \in S \cap g} inc(k)$*.*

Definition 4. *Active literals*

Given a set S *of literals the set* $A_S = \{l \in L | l = max\_lit(S, g) \wedge g = group(l)\}$ *is the set of active literals of* S*. If a literal* $l \notin S$ *then it is a non-active literal.*

Definition 5. *Below literal (blw) function*

Given a literal $l \in L$ *then* $blw : L \times \mathcal{P}(L)$ *is a function that given a literal* $l \in L$ *return all the literal below it, that is:* $blw(l) = \{k \in group(l) | inc(k) \leq inc(l)\}$*.*

Definition 6. *Sum function*

Given a set of literals L *then* $sum(L) = \sum_{l \in A_L} inc(l)$*.*

Definition 7. *Extension*

A extension of a set S *with a literal* l*, written as* $S' = S \xrightarrow{ext} l$*, is equal to* $S' = S \cup blw(l)$*.*

Definition 8. *Valid extension*

A extension of a set S *with a literal* l *is valid iff* $S \cap g = \emptyset$*, where* $g = group(l)$

Lemma 1. *Given a set of literals* L*, a literal* $\ell \in L$ *and a set* $S_{\bar{\ell}} \subseteq L$ *and* $S_\ell \subseteq S$ *and a sum* s*. Let* $S_{\bar{\ell}}$ *is the maximum cardinality subset giving* s *as sum with* ℓ *being an non-active literal and* S_ℓ *is the maximum cardinality subset with* ℓ *being an active literal giving* s *as sum.*
Let $S = \arg\max_{X \in \{S_\ell, S_{\bar{\ell}}\}} |X|$*.*
Then $S \subseteq L$ *is the maximum cardinality subset that gives as sum* s *with considering* L *literals.*

Proof. Let's assume by contradiction that S is not the maximum cardinality subset that gives as sum s. So there exists a set $S' \subseteq L$ such that $|S'| > |S|$. If $\ell \notin A_{S'}$ then $|S'| > |S_{\bar{\ell}}|$, that is $S_{\bar{\ell}}$ is not the maximum cardinality set with ℓ being a *non-active literal* giving as sum s, but this is a contradiction. So $\ell \in A_{S'}$ then $|S'| > |S_\ell|$, that is S_ℓ is not the maximum cardinality set with ℓ being a *active literal* giving as sum s, but this is a contradiction. So $\ell \in A_{S'} \cap \overline{A_{S'}}$ and this is a contradiction. □

Definition 9. *Maximum Subset Sum with Groups (MSG)*
$MSG_L = \{(S, s, n) \in \mathcal{P}(L) \times \mathbb{N} \times \mathbb{N} \,|\, sum(S) \leq s \wedge |S| \geq n\}$

Definition 10. *Decision problem*
The Maximum Subset Sum with Groups (MSG) it the following decision problem: Given a set $S \subseteq L$ of literals and $s, c \in \mathbb{N}$ decide if $(S, s, c) \in MSG_L$.

Theorem 1. *The algorithm is sound, that is $ALG_L \subseteq MSG_L$.*

Proof. Let L be a set of literals and $s, c \in \mathbb{N}$ and $n = |L|$.
Let G is the sets of groups of the literals inside L.
Let $L_g = \{l \in L \mid group(l) = g\}$ and L_g is ordered w.r.t. the *inc* function, that is $L_g = \{l_{g,1}, l_{g,2}, \ldots, l_{g,n_g}\}$ such that $inc(l_{g,1}) \leq inc(l_{g,2}) \leq \cdots \leq inc(l_{g,n_g})$.
Let $L_o = \bigcup_{g \in G} L_g$.
Let L_{o_j} being the first j literals of L_o. The algorithm constructs a matrix M of size $s \times n + 1$ where $Dom(M_{i,j}) = \mathcal{P}(L) \cup \{\square\}$ where $'\square'$ means that there are no subsets $S \subseteq L_{o_j}$ such that $sum(S) \leq s$ and $|S| \geq c$ and $|\square| = -1$.
The setup of this matrix is done in this way:

- $M_{0,0} = \{\}$
- $M_{0,j} = \{l | l \in L_{o_j} \wedge inc(l) = 0\} \quad \forall j \in \{1, \ldots, n\}$
- $M_{i,0} = \square \quad \forall i \in \{1, \ldots, s\}$

After the setup is completed: $\forall i \in \{1, \ldots, s\}, j \in \{1, \ldots, n\}$

- if $inc(l_j) \leq i$ then $M_{i,j} = \arg\max\{|M_{i,j-1}|, |M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j|\}$ where $k = \max\{d \,| d \in \{0, \ldots, j-1\} \wedge M_{(i-inc(l_j)),d} \neq \square \wedge \nexists u \in group(l) \cap M_{(i-inc(l_j)),d}\}$. It is easy to see that $|M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j|$ is a valid extension.
- if $inc(l_j) > i$ then $M_{i,j} = M_{i,j-1}$.

Now a claim has to be reach

- *claim:* $\forall i \in \{0,\dots,s\}, j \in \{0,\dots,n\} \quad M_{i,j}$ *is the cardinality-maximal subset of* L_{o_j} *with* $sum(M_{i,j}) \leq i$. From now on, to refer to the fact that $M_{i,j}$ is the cardinality-maximal subset of L_{o_j} with $sum(M_{i,j}) = i$ we will say that $M_{i,j}$ is *correct*.

- *proof*:

  This proof is done by induction.

  - Base case: $M_{0,j}$ is correct for $0 \leq j \leq n$ and $M_{i,0}$ is correct for $1 \leq i \leq s$
  - Induction hypothesis: given a sum $0 \leq i \leq s$ and a $0 \leq j \leq n-1$ all the cells $M_{i',j'}$ are correct, with $0 \leq i' \leq i$ and $0 \leq j' \leq j$

  This proof will explain how, given a $0 \leq i \leq s$ and $0 \leq j \leq n-1$, to construct the cell $M_{i,j+1}$ proving that it is also correct. Let $0 \leq i \leq s$ and $1 \leq j \leq n$. $M_{i,j-1}$ by I.H. it is correct, so it is the cardinality-maximum subset such that sum is s considering the first L_{o_j} literals and $l_j \in L_o$ is not present into $M_{i,j-1}$ so it is not active, that is $l_j \notin A_{M_{i,j-1}}$. $M_{(i-inc(l_j)),k}$ since $k = \max\{d \,|d \in \{0,\dots,j-1\} \wedge M_{(i-inc(l_j)),d} \neq \Box \wedge \nexists u \in group(l) \cap M_{(i-inc(l_j)),d}\}$ and since by I.H. $M_{(i-inc(l_j)),k}$ is correct it means that it the cardinality-maximal subset such that $sum(M_{(i-inc(l_j)),k}) = i$ considering the first L_{o_k} literals and $l_j \in L_o$ is not present into $M_{i,k}$. The valid extension $S = M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j$ is the cardinality-maximal subset such that sum is less the s considering the first L_{o_j} literals and $l_j \in L_o$ is active, that is $l_j \in A_{M_{(i-inc(l_j)),k}}$. If $inc(l_j) > i$ then $M_{i,j} = M_{i,j-1}$, since $inc(l_j)$ is greater than i it means that cannot be an active literal, otherwise the sum would be greater then s. Hence every subset considering j literals do not have l_j, in this case since $M_{i,j-1}$ is correct then is the cardinality-maximum subset of those subsets, so given that $M_{i,j} = M_{i,j-1}$ then $M_{i,j}$ is correct. On the other hand if $inc(l_j) \leq i$ then $M_{i,j} = \arg\max\{|M_{i,j-1}|, |M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j|\}$ where $k = \max\{d \,|d \in \{0,\dots,j-1\} \wedge M_{(i-inc(l_j)),d} \neq \Box \wedge \nexists u \in group(l) \cap M_{(i-inc(l_j)),d}\}$. $M_{(i-inc(l_j)),k}$ is correct so $sum(M_{(i-inc(l_j)),k}) = i - inc(l_j)$, extending this set with l_j will generate $S = M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j$. Since by construction $\nexists u \in group(l) \cap M_{(i-inc(l_j)),k}$ it means that the extension is a valid one, so

$l_j \in A_S$. Thus $sum(S) = sum(M_{(i-inc(l_j)),k}) + inc(l_j) = i - inc(l_j) + inc(l_j) = i$. S is the cardinality-maximum subset of L_{o_j} where l_j is active. To see it, let's assume the contrary, that is: there is a subset S' of L_{o_j} such that $|S'| > |S|$, $sum(S') = i - inc(l_j)$ and $l_j \in A_{S'}$. In this case since $l_j \in A_S \cap A_{S'}$ then $|S' \setminus blw(l_j)| > |S \setminus blw(l_j)|$. Given that $S = M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j = M_{(i-inc(l_j)),k} \cup blw(l_j)$ then $S \setminus blw(l_j) = (M_{(i-inc(l_j)),k} \cup blw(l_j)) \setminus blw(l_j) = M_{(i-inc(l_j)),k}$. Let $S'' = S' \setminus blw(l_j)$ Since S' cardinality-maximum subset of L_{o_j} where $sum(S') = i - inc(l_j)$ and $l_j \in A_{S'}$ then $sum(S' \setminus blw(l_j)) = sum(S') - inc(i) = i - inc(i)$. So, $S' \setminus blw(l_j)$ is the cardinality-maximum subset of L_{o_k} where $k = \max\{d \,|\, d \in 0, \ldots, |L_o| and L_{o_d} \cap group(l_j) = \emptyset\}$. But is this case $M_{(i-inc(l_j)),k}$ would not be correct, but this is a contradiction. So, $M_{i,j-1}$ is the maximum cardinality subset giving s as sum with ℓ being an *non-active* literal and $M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j$ is the maximum cardinality subset with ℓ being an *active literal* giving s as sum. By lemma 1 it implies that $M_{i_j} = max(M_{i,j-1}, M_{(i-inc(l_j)),k} \xrightarrow{\text{ext}} l_j)$ is the cardinality-maximal subset of L_{o_j} with $sum(M_{i,j}) = i$ so it is correct. $\square$

So every cell is correct, then taking the cardinality-maximum subset in the last column will give the cardinality-maximum subset with a sum less of equal to s, let S be such subset. Let a tuple $(S, s, v) \in ALG_L$ then $|S| \geq v$ and $sum(S) \leq s$. Since S is the cardinality-maximum subset with $sum(S) \leq s$ and $|S| \geq v$ then $(S, s, v) \in MSG$. Thus $ALG_L \subseteq MSG$. $\square$