$$\cancel{p^k(1-p)^{n-k} = \left(p^k - p^{k+1}\right)^{n-k} = \left(p^{k-(k+1)}\right)^{n-k} \left(p^{k-k+1}\right)^{n-k} = p^{n-k}}$$

1. a. $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

stosujemy wzór $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

i otrzymujemy:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (1-p+p)^n = 1^n = 1$$

b. Stosujemy poprzedni wzór oraz $k\binom{n}{k} = n\binom{n-1}{k-1}$

$$\sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} np\binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} =$$

$$= np \sum_{k=0}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} = np(p+(1-p))^{n-1} = np$$

2. $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$

b. $\sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^k}{\lambda (k-1)!} = e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} \lambda$
$= \lambda$

3. $\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\, dt = \int_0^{\infty} t^{n-1} (-e^{-t})'\, dt = [t^{n-1} e^{-t}]_0^{\infty} - \int_0^{\infty} -(n-1) t^{n-2} e^{-t}\, dt$

$= [t^{n-1} e^{-t}]_0^{\infty} + \int_0^{\infty} (n-1) t^{n-2} e^{-t}\, dt$

Skoro $e^{-\infty} = 0$ i $0^{n-1} = 0$ to pierwsze wyrażenie to 0

$$\Gamma(n) = \int_0^{\infty} (n-1) t^{n-2} e^{-t}\, dt = (n-1) \int_0^{\infty} t^{n-2} e^{-t}\, dt = (n-1)\Gamma(n-1)$$

Sprawdźmy jeszcze $\Gamma(1)$:

$\Gamma(1) = \int_0^{\infty} 1 e^{-t}\, dt = 1$

A więc mamy:

$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) = (n-1)!$

4. $f(x) = \lambda e^{-\lambda x}$ $\quad x > 0$

a) $\int_0^\infty f(x)\,dx$

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-\lambda x}\,dx = \lambda\left(-\frac{1}{\lambda}e^{-\lambda x}\right) = \left[-e^{-\lambda x}\right]_0^\infty$$

$$= -e^{-\lambda\cdot\infty} + e^{-\lambda\cdot 0} = 0 + 1 = 1$$

b) $\lambda \int_0^\infty x \cdot e^{-\lambda x}\,dx = \lambda\left(x\left(-\frac{1}{\lambda}e^{-\lambda x}\right) - \int_0^\infty -\frac{1}{\lambda}e^{-\lambda x}\right)$

$f(x) = x$

$f'(x) = 1$ $\quad \int f(x)\cdot g'(x) = f(x)\cdot g(x) - \int f'(x)\cdot g(x)$

$g'(x) = e^{-\lambda x}$

$g(x) = \frac{e^{-\lambda x}}{\lambda}$

$$= \left[-xe^{\lambda x}\right]_0^\infty + \lambda \int_0^\infty \frac{1}{\lambda} e^{-\lambda x}\,dx$$

$= 0$

$$= \frac{1}{\lambda}\lambda \int \frac{1}{\lambda} e^{-\lambda x}\,dx \overset{\text{poprzedni}}{=} \frac{1}{\lambda}\cdot 1 = \frac{1}{\lambda}$$

5. Wykaż $D_n = n$

$$D_n = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}$$

Do pierwszego wiersza dodajemy pozostałe, co daje nam:

$$\begin{bmatrix} n & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 1 & 0 & & & 1 \end{bmatrix}$$

Jest to macierz dolnotrójkątna. Jej wyznacznikiem jest n.

gdzie ... ojca w tablicy

zaczynamy więc od znalezienia ojca w tablicy

7 a. $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} (x_k^2 - 2x_k\bar{x} + \bar{x}^2) =$

$= \sum_{k=1}^{n} x_k^2 - (2x_1\bar{x} - 2x_2\bar{x} - \dots - 2x_n\bar{x}) + n\bar{x}^2 =$

$= \sum_{k=1}^{n} x_k^2 - 2\bar{x} \frac{x_1 + x_2 + \dots + x_n}{n} \cdot n + n \cdot \bar{x}^2 =$

$= \sum_{k=1}^{n} x_k^2 - 2\bar{x}^2 \cdot n + n \cdot \bar{x}^2 = \sum_{k=1}^{n} x_k^2 - n\bar{x}^2$ c.n.u.

b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} (x_k y_k - \bar{x} y_k - \bar{y} x_k + \bar{x}\bar{y})$

$= \sum_{k=1}^{n} x_k y_k - \bar{x}\sum_{k=1}^{n} y_k - \bar{y}\sum_{k=1}^{n} x_k + n\bar{x}\bar{y} =$

$= \sum_{k=1}^{n} x_k y_k - \bar{x}\bar{y} \cdot n - \bar{y}\bar{x} \cdot n + n\bar{y}\bar{x} = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$ c.n.u