assignment_02_assignment-02-b

September 17, 2020

Linear supervised regression

0. Import library

Import library

[34]:
```
# Import libraries

# math library
import numpy as np

# visualization library
%matplotlib inline
from IPython.display import set_matplotlib_formats
set_matplotlib_formats('png2x','pdf')
import matplotlib.pyplot as plt

# machine learning library
from sklearn.linear_model import LinearRegression

# 3d visualization
from mpl_toolkits.mplot3d import axes3d

# computational time
import time
```

1. Load dataset

Load a set of data pairs $\{x_i, y_i\}_{i=1}^{n}$ where x represents label and y represents target.

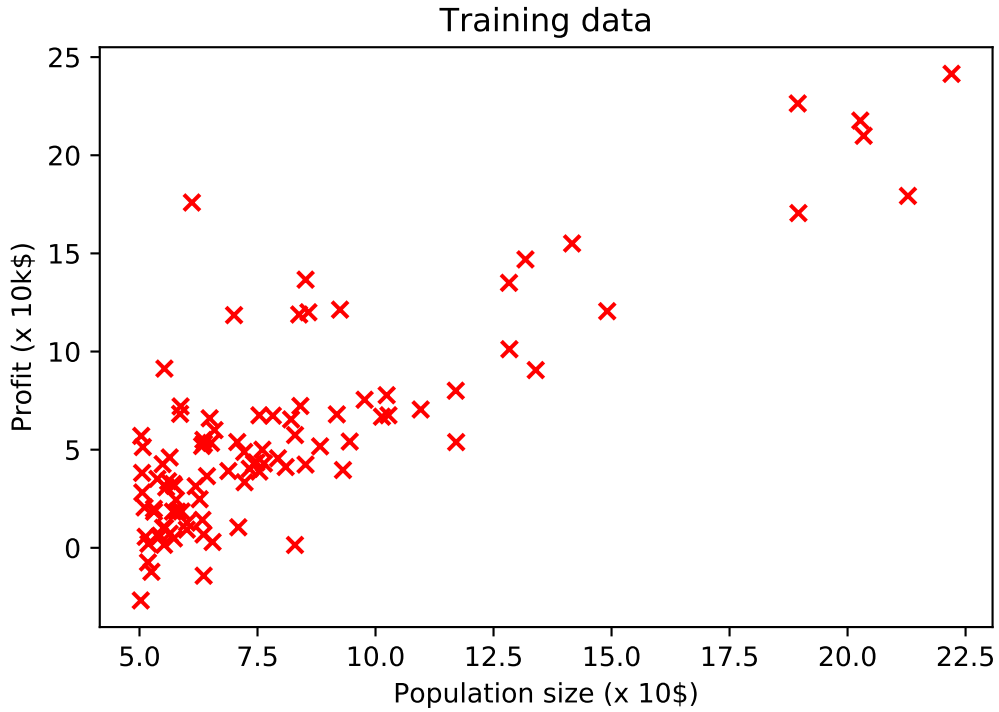
[35]:
```
# import data with numpy
data = np.loadtxt('profit_population.txt', delimiter=',')
```

2. Explore the dataset distribution

Plot the training data points.

[36]:
```
x_train = data[:,0]
y_train = data[:,1]
```

```
plt.title("Training data")
plt.xlabel("Population size (x 10$)")
plt.ylabel("Profit (x 10k$)")

plt.scatter(x_train, y_train , marker='x',color='r')
plt.show()
```



3. Define the linear prediction function

$$f_w(x) = w_0 + w_1 x$$

0.0.1 Vectorized implementation:

$$f_w(x) = Xw$$

with

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \quad \Rightarrow \quad f_w(x) = Xw = \begin{bmatrix} w_0 + w_1 x_1 \\ w_0 + w_1 x_2 \\ \vdots \\ w_0 + w_1 x_n \end{bmatrix}$$

Implement the vectorized version of the linear predictive function.

[44]:
```
m = len(x_train)

# construct data matrix
X = np.append(np.ones((m, 1)), np.array([x_train]).T, axis=1)
# parameters vector
w = np.array([[1], [1]])
# predictive function definition

def f_pred(X,w):
    f = X.dot(w)
    return f

# Test predicitive function
y_pred = f_pred(X,w)
```

4. Define the linear regression loss

$$L(w) = \frac{1}{n}\sum_{i=1}^{n} \left(f_w(x_i) - y_i\right)^2$$

0.0.2 Vectorized implementation:

$$L(w) = \frac{1}{n}(Xw - y)^T(Xw - y)$$

with

$$Xw = \begin{bmatrix} w_0 + w_1 x_1 \\ w_0 + w_1 x_2 \\ \vdots \\ w_0 + w_1 x_n \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Implement the vectorized version of the linear regression loss function.

[53]:
```
# loss function definition
def loss_mse(y_pred,y):
    loss = ( (y_pred - y).T ).dot( y_pred - y ) / m
    return loss.item()


# Test loss function
y = np.array([y_train]).T # label
y_pred = f_pred(X, w)# prediction

loss = loss_mse(y_pred,y)
```

5. Define the gradient of the linear regression loss

0.0.3 Vectorized implementation: Given the loss

$$L(w) = \frac{1}{n}(Xw - y)^T(Xw - y)$$

The gradient is given by

$$\frac{\partial}{\partial w}L(w) = \frac{2}{n}X^T(Xw - y)$$

Implement the vectorized version of the gradient of the linear regression loss function.

[54]:
```
# gradient function definition
def grad_loss(y_pred,y,X):
    global m
    grad = (2 / m) * (X.T).dot(y_pred-y)
    return grad


# Test grad function
y_pred = f_pred(X, w)
y = np.array([y_train]).T
grad = grad_loss(y_pred,y,X)
```

6. Implement the gradient descent algorithm

• Vectorized implementation:

$$w^{k+1} = w^k - \tau\frac{2}{n}X^T(Xw^k - y)$$

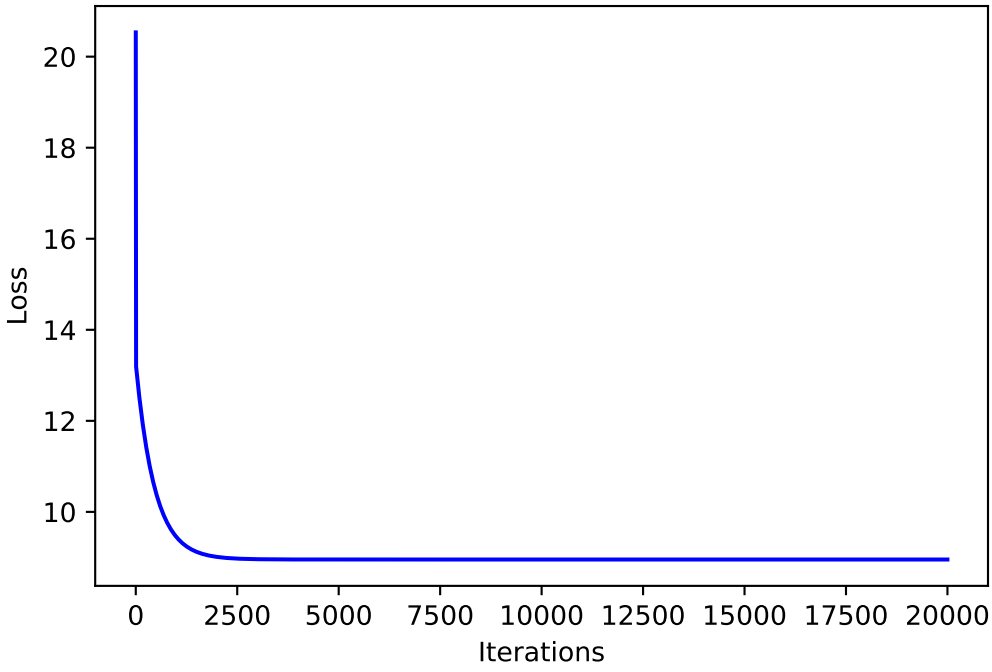
0.0.4 Implement the vectorized version of the gradient descent function.

0.0.5 Plot the loss values $L(w^k)$ with respect to iteration k the number of iterations.

[70]:
```
# gradient descent function definition
def grad_desc(X, y, w_init, tau, max_iter):

    L_iters = []# record the loss values
    w_iters = []# record the parameter values
    w = w_init # initialization

    for i in range(max_iter): # loop over the iterations

        y_pred = f_pred(X, w) # linear predicition function
        grad_f = grad_loss(y_pred, y, X) # gradient of the loss
        w = w - (tau * grad_f) # update rule of gradient descent
        L_iters.append(loss_mse(y_pred,y)) # save the current loss value
        w_iters.append(w)
    return w, L_iters, w_iters
```

```
# run gradient descent algorithm
start = time.time()
w_init = np.array([[1], [1]])
tau = 0.003
max_iter = 20000

w, L_iters, w_iters = grad_desc(X,y,w_init,tau,max_iter)

print('Time=',time.time() - start) # plot the computational cost
print( L_iters[-1] ) # plot the last value of the loss
print( w_iters[-1] ) # plot the last value of the parameter w


# plot
plt.figure(2)
plt.plot(L_iters, color='b') # plot the loss curve
plt.xlabel('Iterations')
plt.ylabel('Loss')
plt.show()
```

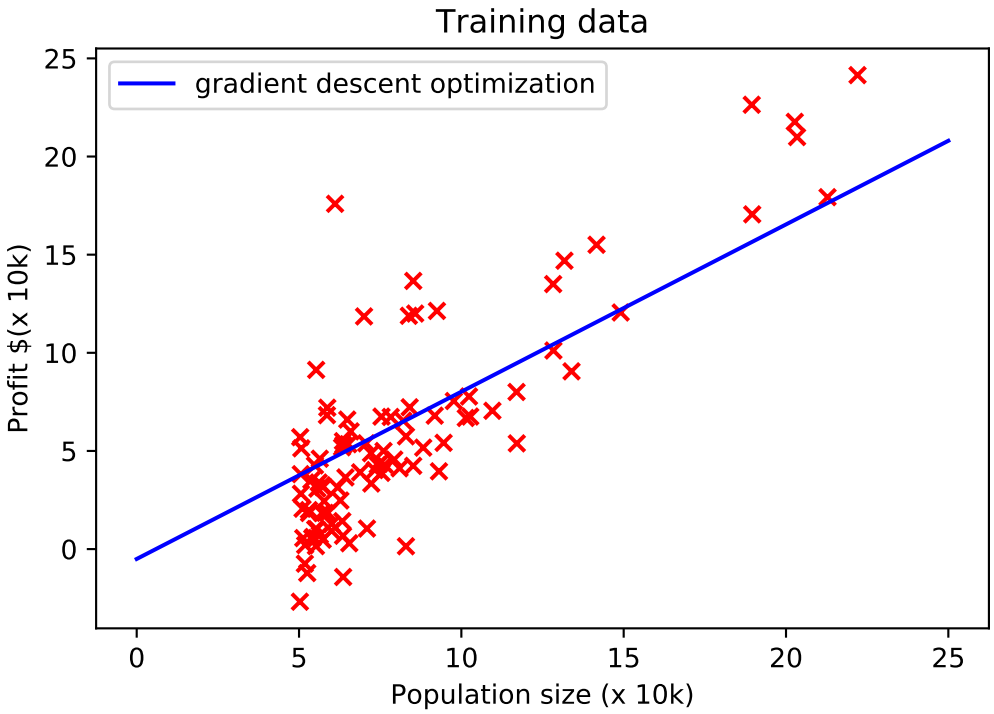
```
Time= 0.1923840045928955
8.953942751950358
[[-3.89578088]
 [ 1.19303364]]
```



7. Plot the linear prediction function

$$f_w(x) = w_0 + w_1 x$$

[64]:
```
# linear regression model
x_pred = np.linspace(0,25,100) # define the domain of the prediction function
y_pred = w[0] + w[1] * x_pred # compute the prediction values within the given␣
 ↪domain x_pred

# plot
plt.figure(3)
plt.scatter(x_train, y_train, color = 'r', marker='x')
plt.plot(x_pred, y_pred, color = 'blue', label = 'gradient descent␣
 ↪optimization')
plt.legend(loc='best')
plt.title('Training data')
plt.xlabel('Population size (x 10k)')
plt.ylabel('Profit $(x 10k)')
plt.show()
```

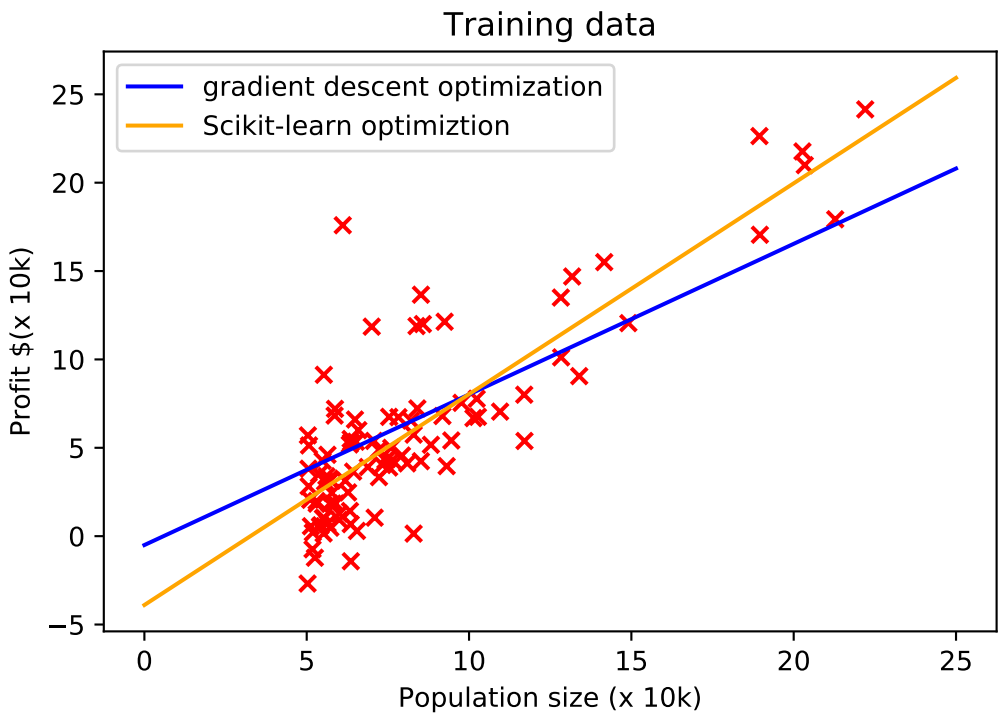


8. Comparison with Scikit-learn linear regression algorithm

0.0.6 Compare with the Scikit-learn solution

[65]:
```
# run linear regression with scikit-learn
start = time.time()
lin_reg_sklearn = LinearRegression()
lin_reg_sklearn.fit(np.array([x_train]).T, y) # learn the model parameters
print('Time=',time.time() - start)


# compute loss value
w_sklearn = np.zeros([2,1])
w_sklearn[0,0] = lin_reg_sklearn.intercept_
w_sklearn[1,0] = lin_reg_sklearn.coef_

print(w_sklearn)

loss_sklearn = loss_mse(f_pred(X, w_sklearn), y) # compute the loss from the␣
 ↪sklearn solution

print('loss sklearn=',loss_sklearn)
print('loss gradient descent=',L_iters[-1])


# plot
y_pred_sklearn = w_sklearn[0, 0] + w_sklearn[1, 0] * x_pred  # prediction␣
 ↪obtained by the sklearn library

plt.figure(3)

plt.scatter(x_train, y_train, color = 'r', marker='x')
plt.plot(x_pred, y_pred, color = 'blue', label = 'gradient descent␣
 ↪optimization')
plt.plot(x_pred, y_pred_sklearn, color = 'orange', label = 'Scikit-learn␣
 ↪optimiztion')
plt.legend(loc='best')
plt.title('Training data')
plt.xlabel('Population size (x 10k)')
plt.ylabel('Profit $(x 10k)')
plt.show()
```

```
Time= 0.0008718967437744141
[[-3.89578088]
 [ 1.19303364]]
loss sklearn= 8.953942751950358
loss gradient descent= 11.05034451544502
```



9. Plot the loss surface, the contours of the loss and the gradient descent steps

[66]:
```
# plot gradient descent
def plot_gradient_descent(X,y,w_init,tau,max_iter):

    def f_pred(X,w):

        f = X.dot(w)

        return f

    def loss_mse(y_pred,y):

        loss = ((y_pred - y).T).dot(y_pred - y) / m

        return loss

    # gradient descent function definition
    def grad_desc(X, y, w_init, tau, max_iter):

        L_iters = [] # record the loss values
        w_iters = [] # record the parameter values
        w = w_init # initialization
```

```
        for i in range(max_iter): # loop over the iterations

            y_pred = f_pred(X, w)# linear predicition function
            grad_f = grad_loss(y_pred, y, X)# gradient of the loss
            w =  w - (tau * grad_f) # update rule of gradient descent
            L_iters.append(loss_mse(y_pred,y) ) # save the current loss value
            w_iters.append(w) # save the current w value

        return w, L_iters, w_iters

    # run gradient descent
    w, L_iters, w_iters = grad_desc(X, y, w_init, tau, max_iter)

    # Create grid coordinates for plotting a range of L(w0,w1)-values
    B0 = np.linspace(-10, 10, 50)
    B1 = np.linspace(-1, 4, 50)

    xx, yy = np.meshgrid(B0, B1, indexing='xy')
    Z = np.zeros((B0.size,B1.size))

    # Calculate loss values based on L(w0,w1)-values
    for (i,j),v in np.ndenumerate(Z):
        Z[i,j] = loss_mse(f_pred(X, np.array([[xx[i][j]], [yy[i][j]]])), y)

    # 3D visualization
    fig = plt.figure(figsize=(15,6))
    ax1 = fig.add_subplot(121)
    ax2 = fig.add_subplot(122, projection='3d')

    # Left plot
    CS = ax1.contour(xx, yy, Z, np.logspace(-2, 3, 20), cmap=plt.cm.jet)
    w_iters = np.array(w_iters)
    ax1.scatter(w[0, 0], w[1, 0], color = 'r')
    ax1.plot(w_iters[:, 0, 0], w_iters[:, 1, 0])

    # Right plot
    ax2.plot_surface(xx, yy, Z, rstride=1, cstride=1, alpha=0.6, cmap=plt.cm.
↪jet)
    ax2.set_zlabel('Loss $L(w_0,w_1)$')
    ax2.set_zlim(Z.min(),Z.max())

    # plot gradient descent
    Z2 = np.zeros([max_iter])

    for i in range(max_iter):
        w0 = w_iters[i, 0, 0]
        w1 = w_iters[i, 1, 0]
```

```
        Z2[i] = loss_mse( f_pred( X, np.array([[w0], [w1]])), y)

    ax2.plot(w_iters[:, 0, 0], w_iters[:, 1, 0])
    ax2.scatter( w[0,0], w[1,0], color = 'r')

    # settings common to both plots
    for ax in fig.axes:
        ax.set_xlabel(r'$w_0$', fontsize=17)
        ax.set_ylabel(r'$w_1$', fontsize=17)
```

[71]:
```
# run plot_gradient_descent function
w_init = np.array([[1], [1]])
tau = 0.00001
max_iter = 100000

plot_gradient_descent(X,y,w_init,tau,max_iter)
```