Let $\Omega \subset \mathbb{C}$ be a bounded (not necessarily simply connected) domain, $a \in \Omega$ and $|\eta| = 1$. Recall that $f^{[\eta]}(a, \cdot) : \Omega \setminus \{a\} \to \mathbb{C}$ is defined as the unique[!] *holomorphic* function such that

$$f^{[\eta]}(a, z) = \frac{\overline{\eta}}{z - a} + O(1) \quad \text{as} \quad z \to a, \qquad f^{[\eta]}(a, \zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}, \quad \zeta \in \partial\Omega,$$

where $\tau(\zeta)$ denotes the tangent vector to Ω (oriented so that Ω remains to the left of $\tau(\zeta)$).

Problem 1. **(a)** Prove that there exists (unique) functions $f(a, z)$ and $f^\star(a, z)$ such that

$$f^{[\eta]}(a, z) = \tfrac{1}{2}[\,\overline{\eta} f(a, z) + \eta f^\star(a, z)\,] \quad \text{for all} \quad z, \eta.$$

(b) Denote $f^{[\eta,\mu]}(w, z) := \mathrm{Re}[\overline{\mu} f(w, z)]$, where $w \neq z$ and $|\eta| = |\mu| = 1$. Prove that

$$f^{[\mu,\eta]}(z, w) = -f^{[\eta,\mu]}(w, z).$$

Hint: Consider $\oint_{\partial\Omega} f^{[\eta]}(w, \zeta) f^{[\mu]}(z, \zeta) d\zeta$.

(c) Deduce that $f(z, w) = -f(w, z)$ and $f^\star(z, w) = -\overline{f^\star(w, z)}$. In particular, $f(z, w)$ is holomorphic in both variables whilst $f(w, z)$ is holomorphic in z and anti-holomorphic in w. Argue that the definition

$$\langle \varepsilon_w \rangle_\Omega^+ \;:=\; \tfrac{i}{2} f^\star(w, w)$$

makes sense and that $\langle \varepsilon_w \rangle_\Omega^+ \in \mathbb{R}$.

(d) Prove the conformal covariance rules: if $\varphi : \Omega \to \Omega'$ is a conformal map, then

$$f_\Omega(w, z) = f_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\varphi'(w))^{1/2} (\varphi'(z))^{1/2},$$
$$f^\star_\Omega(w, z) = f^\star_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\overline{\varphi'(w)})^{1/2} (\varphi'(z))^{1/2}.$$

In particular, one has $\langle \varepsilon_w \rangle_\Omega^+ = \langle \varepsilon_{\varphi(w)} \rangle_{\Omega'}^+ \cdot |\varphi'(w)|$.

Recall that the *holomorphic spinor* $g_{[v,u]}(z)$ (defined on the double cover of Ω ramified over $u, v \in \Omega$, $u \neq v$) is uniquely characterized by the following conditions:

$$g_{[v,u]}(z) \;=\; \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z-v}} \cdot [\,1 + O(z-v)\,] \quad \text{as} \quad z \to v; \tag{1}$$

$$g_{[v,u]}(z) \;=\; \frac{e^{i\frac{\pi}{4}}}{\sqrt{z-u}} \cdot [\,c + O(z-u)\,] \quad \text{as} \quad z \to u, \quad \text{with an unknown} \quad c \in \mathbb{R}, \tag{2}$$

and the boundary conditions $g_{[v,u]}(\zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}$ for $\zeta \in \partial\Omega$. Further, recall that $\mathcal{A}(v, u)$ is defined as the next coefficient in the expansion of $g_{[v,u]}(z)$ as $z \to v$:

$$g_{[v,u]}(z) \;=\; \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z - v}} \cdot [\,1 + 2\mathcal{A}(v, u)(z - v) + O((z - v)^2)\,],$$

and that

$$\langle \sigma_u \sigma_v \rangle_\Omega^+ \;:=\; \exp\Big[\int \mathrm{Re}\big[\mathcal{A}(v, u)dv + \mathcal{A}(u, v)du\big]\Big],$$

where the multiplicative normalization is chosen so that $\langle \sigma_u \sigma_v \rangle_\Omega^+ \sim |u - v|^{-1/4}$ as $u \to v$.

Problem 2. The goal is to prove the *fusion rule* $\sigma\sigma \rightsquigarrow 1 + \frac{1}{2}\varepsilon + \ldots$, more precisely:

$$\langle\sigma_v\sigma_u\rangle_\Omega^+ \;=\; |v-u|^{-1/4}\cdot[1+\tfrac{1}{2}\langle\varepsilon_v\rangle_\Omega^+\cdot|v-u| + o(|v-u|)] \quad \text{as} \quad v\to u \tag{3}$$

(not using explicit expressions available in simply connected Ω), where the correlation functions $\langle\sigma_u\sigma_v\rangle_\Omega^+$ and $\langle\varepsilon\rangle_\Omega^+$ are defined above.

Denote $\eta := e^{i\frac{\pi}{4}}\cdot(\overline{v}-\overline{u})^{1/2}/|v-u|^{1/2}$. First, take for granted that $\langle\varepsilon_v\rangle_\Omega^+ \to \langle\varepsilon_u\rangle_\Omega^+$ and

$$g_{[v,u]}(z) \;=\; |v-u|^{1/2}\cdot\left[f^{[\eta]}(v,z)\cdot\left(\frac{z-v}{z-u}\right)^{1/2} +\; o(1)\right] \quad \text{as} \quad v\to u, \tag{4}$$

uniformly on compact subsets $z\in\Omega\setminus\{u\}$.
Remark: The right-hand side of (4) is chosen so that the difference does *not* blow up at $z=v$ and approximately satisfies (2) and the boundary conditions, so it should be small.

(a) Deduce from (4) that

$$2\mathcal{A}_{[v,u]} + \frac{1}{2(v-u)} \;=\; \langle\varepsilon_v\rangle_\Omega^+\cdot\frac{|v-u|}{v-u} + o(1) \quad \text{as} \quad v\to u.$$

Hint: Consider $\oint g_{[v,u]}(z)\cdot(z-u)^{1/2}(z-v)^{-3/2}dz$.

(b) Deduce (3) from (4) and the asymptotics $\langle\sigma_u\sigma_v\rangle_\Omega^+ \sim |v-u|^{-1/4}$ as $v\to u$.

(c) Prove that $\langle\varepsilon_v\rangle_\Omega^+ \to \langle\varepsilon_u\rangle_\Omega^+$ as $v\to u$.
Hint: Argue that each subsequential limit of $f^{[\eta]}(v,\cdot)$ must coincide with $f^{[\eta]}(u,\cdot)$.

(d)* Prove (4).

More information on correlations of $\psi,\mu,\sigma,\varepsilon$ and fusion rules: [Section 4, `arXiv:1605.09035`]