QUANTUM INTEGRABILITY AND SYMMETRIC POLYNOMIALS

1. Exercise session 1

1.1. **Low-temperature expansion of the antiferroelectric six-vertex model.** Consider the six-vertex model with the usual Boltzmann weights a, b, c on a torus of size $K \times L$. We assume that K and L are even, and consider the regime where $c \gg a, b$.

- Describe the two ground states of the model, i.e., the configurations of the model with maximal Boltzmann weight.
- Use graphical notations to verify that to ninth order in a/c, b/c, the partition function is given by

  $$\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V+1)a^4b^4 + Va^2b^2(a^4+b^4) + \cdots ,$$

  where we have set $c = 1$ for convenience, and $V := KL$.
- Explain the symmetry of Z in a, b from the symmetries of the model.

1.2. **Determination of the six-vertex $\boldsymbol{R}$-matrix from $\boldsymbol{RLL}$ relations.** We encode the Boltzmann weights $a, b, c > 0$ of the six-vertex model into an L-matrix, and similarly for another set of weights a', b', c':

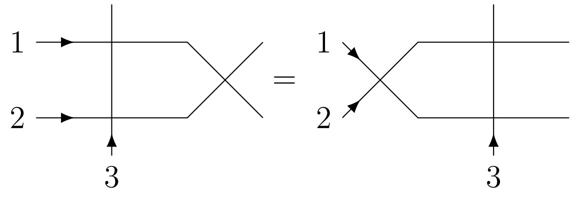
$$L = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}, \qquad L' = \begin{pmatrix} a' & & & \\ & b' & c' & \\ & c' & b' & \\ & & & a' \end{pmatrix}.$$

We want to find conditions on the weights a, b, c, a', b', c' such that there exists an invertible R-matrix of the same form

$$R = \begin{pmatrix} a'' & & & \\ & b'' & c'' & \\ & c'' & b'' & \\ & & & a'' \end{pmatrix}$$

such that the RLL relations hold, namely

$$R_{12}L_{13}L'_{23} = L'_{23}L_{13}R_{12},$$



- Show that due to line conservation out of these $2^6 = 64$ equations only $\sum_{k=0}^{3}\binom{3}{k}^2 = 20$ are nontrivial, which due to parity symmetry come in pairs of identical equations.
- Noting that the left and right hand sides are exchanged by $180°$ rotation of the corresponding pictures, conclude that out of the ten equations, four correspond to

180° rotation invariant boundary conditions and are therefore automatically satisfied, while six come in pairs of identical equations and read:

$$\begin{aligned} ab'c'' + cc'b'' &= ba'c'', \\ ac'b'' + cb'c'' &= bc'a'', \\ ac'c'' + cb'b'' &= ca'a''. \end{aligned}$$

Draw the corresponding configurations.

- Eliminating a'', b'', c'', show that a solution for R exists only if $\Delta(a,b,c) = \Delta(a',b',c')$ with
$$\Delta(a,b,c) = \frac{a^2+b^2-c^2}{2ab}.$$
- Show that if $\Delta(a,b,c) = \Delta(a',b',c')$ then one also has $\Delta(a'',b'',c'') = \Delta(a,b,c)$, i.e., the R-matrix is of the same form as the L-matrices.
- Assume $\Delta \neq \pm 1$. Recall that if one writes $\Delta = \frac{q+q^{-1}}{2}$, then the Boltzmann weights can be parameterized as

$$\begin{aligned} a(z) &= q\,z - q^{-1}z^{-1}, \\ b(z) &= z - z^{-1}, \\ c(z) &= q - q^{-1}, \end{aligned} \tag{1}$$

up to overall normalization. Here z is the *spectral parameter*.

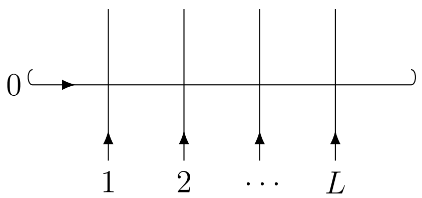
Show that if L is parameterized as above and L' similarly with z replaced by z', then R is of the same form with spectral parameter $z'' = z/z'$.

1.3. **XXZ Hamiltonians.** Parameterize the six-vertex R-matrix for $\Delta = \frac{q+q^{-1}}{2} \neq \pm 1$ as in the previous exercise:

$$R(z) = \begin{pmatrix} a(z) & & & \\ & b(z) & c(z) & \\ & c(z) & b(z) & \\ & & & a(z) \end{pmatrix}$$

with weights given by (1). The (homogeneous) transfer matrix is defined by

$$T(z) = \mathrm{tr}_0(R_{0L}(z)\ldots R_{01}(z)) =$$



- Show that up to normalization $T(1)$ is the *translation operator* U, i.e., $U\, e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_L} = e_{k_L} \otimes e_{k_1} \otimes \cdots \otimes e_{k_2}$ where e_1, e_2 are standard basis vectors.
- Show that up to an additive constant and normalization $(\log T)'(1) = \frac{dT}{dz}T^{-1}|_{z=1}$ is the *XXZ Hamiltonian*

$$H_{\mathrm{XXZ}} = \sum_{i=1}^{L} \left(2\,(\sigma_i^+\sigma_{i+1}^- + \sigma_i^+\sigma_{i+1}^-) + \Delta\,\sigma_i^z\sigma_{i+1}^z\right),$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where periodic boundary conditions are implicit ($L+1 \equiv 1$), and $\sigma_i^\bullet$ means $\sigma^\bullet$ acting on the i^{th} factor of the tensor product $(\mathbb{C}^2)^{\otimes L}$.

Hint. One may show first that $\frac{dT}{dz}T^{-1}|_{z=1} = (q-q^{-1})^{-1}\sum_{i=1}^{L}\frac{d\check{R}_{i\,i+1}}{dz}|_{z=1}$ where $\check{R}_{i\,i+1}(z) = P_{i\,i+1}R_{i\,i+1}(z)$ and $P_{i\,i+1}$ exchanges the spins at sites i and $i+1$, i.e., $P_{i\,i+1}\, e_{k_1}\otimes\cdots\otimes e_{k_i}\otimes e_{k_{i+1}}\otimes\cdots\otimes e_{k_L} = e_{k_1}\otimes\cdots\otimes e_{k_{i+1}}\otimes e_{k_i}\otimes\cdots\otimes e_{k_L}$.

- (*Optional*) Compute the second XXZ Hamiltonian $(\log T)''(1)$.

1.4. **The 1D Ising model.** With the same notations as in the previous exercise, consider the Hamiltonian

$$H_{\text{Ising}} = J\sum_{i=1}^{L}\sigma_i^z\sigma_{i+1}^z + h\sum_{i=1}^{L}\sigma_i^z.$$

- How is H_{Ising} related to H_{XXZ} of the previous exercise?
- Show that $Z = \operatorname{tr}(e^{-\beta H_{\text{Ising}}})$ is the partition function of the *classical one-dimensional Ising model*:

$$Z = \sum_{\sigma\colon \mathbb{Z}/L\mathbb{Z}\to\{\pm 1\}} e^{-\beta J\sum_{i=1}^{L}\sigma_i\sigma_{i+1}-\beta h\sum_{i=1}^{L}\sigma_i}.$$

- Denoting $K=-\beta J$, $B=-\beta h$, show that

$$Z = \operatorname{tr}(T^L), \qquad T = \begin{pmatrix} e^{K+B} & e^{-K} \\ e^{-K} & e^{K-B}\end{pmatrix},$$

where T plays the role of a *one-dimensional* transfer matrix.
- Conclude that

$$Z = \Lambda_+^L + \Lambda_-^L, \qquad \Lambda_\pm = e^K\cosh B \pm \sqrt{e^{2K}\sinh^2 B + e^{-2K}}.$$

- Compute $\lim_{L\to\infty}\frac{\log Z}{L}$. Is there any phase transition in temperature, i.e., in the parameter β (the inverse temperature) as it varies from 0 to $+\infty$?