QUANTUM INTEGRABILITY AND SYMMETRIC POLYNOMIALS

2. Exercise session 2

2.1. **The q-determinant.** Consider the *Yang–Baxter (bi)algebra* associated to the six-vertex model R-matrix, with the standard notation $\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ for the generating series of its generators.

- Prove the series of equalities

$$\begin{aligned}
\mathrm{qdet}(z) &:= A(qz)D(z) - B(qz)C(z) \\
&= D(qz)A(z) - C(qz)B(z) \\
&= A(z)D(qz) - C(z)B(qz) \\
&= D(z)A(qz) - B(z)C(qz).
\end{aligned}$$

- Prove that $\mathrm{qdet}(z)$ is central, i.e, commutes with all elements of the Yang–Baxter (bi)algebra, and prove that it is group like, i.e., $\Delta(\mathrm{qdet}(z)) = \mathrm{qdet}(z) \otimes \mathrm{qdet}(z)$.

2.2. **Commutation of twisted transfer matrices.** Consider as above the Yang–Baxter (bi)algebra associated to the six-vertex model R-matrix. Recall that its defining relations, the RTT relations, can be written in components as sixteen relations for its generators $A(z)$, $B(z)$, $C(z)$, $D(z)$.

- Write explicitly the components of the RTT relations involving $A(z)$ and $D(z)$ that you will need for the following part.
- Defining

$$T_\kappa(z) = A(z) + \kappa D(z), \qquad \kappa \in \mathbb{C},$$

conclude that $[T_\kappa(z), T_\kappa(z')] = 0$ for all z, z'.

In other words, for a fixed κ, $T_\kappa(z)$ is the generating series for a commutative subalgebra of the Yang–Baxter algebra.

2.3. **Bethe Ansatz equations as pole cancellations.** Consider the six-vertex model with periodic boundary conditions, and its transfer matrix $T(z) = A(z) + D(z)$ acting on $(\mathbb{C}^2)^{\otimes L}$.

- Consider an eigenvector $|\Psi\rangle$ of $T(z)$, with eigenvalue

$$T(z)\,|\Psi\rangle = t(z)\,|\Psi\rangle\,.$$

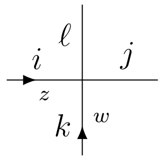
As a function of z, what can be said about $t(z)$?
- Now assume $|\Psi\rangle$ is a Bethe vector. Write the formula expressing the eigenvalue $t(z)$ as a function of the Bethe roots $z_1, \ldots, z_M$. What is its dependence on z? Comparing with the previous part, conclude that the residues of $t(z)$ at the would-be poles of this formula must vanish. Compute these residues and compare with Bethe Ansatz equations.

2.4. **Energy/momentum of XXZ eigenvectors.**

- Using the trace identities, cf. exercise 1.3, compute the momentum and XXZ energy of a Bethe vector in terms of the Bethe roots. (Recall that the shift operator U is unitary, so its eigenvalues are of the form e^{ip} where $p \in \mathbb{R}/2\pi\mathbb{Z}$ is the momentum.)
- Argue that these states can be viewed as consisting of quasiparticles called magnons, where each magnon can be associated with one Bethe root.
- How does the isotropy at $\Delta = 1$ show up in the spectrum?

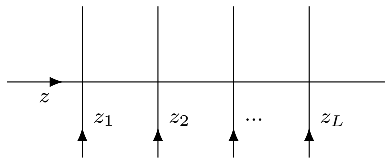
2.5. **Yang–Baxter algebra representations and inhomogeneous monodromy matrix.** We recall that a *representation* of an algebra $\mathcal{A}$ is the data of a vector space V and an algebra morphism $\rho : \mathcal{A} \to \mathrm{End}(V)$, i.e., a linear map preserving the multiplication. If $\mathcal{A}$ is a *bialgebra*, then one can take tensor products of representations using the coproduct Δ: given (V_1, ρ_1) and (V_2, ρ_2), define the representation $(V_1 \otimes V_2, \rho_{1\otimes 2})$ by $\rho_{1\otimes 2}(a) = (\rho_1 \otimes \rho_2)\Delta(a)$ for all $a \in \mathcal{A}$.

Consider the Yang–Baxter bialgebra $(\hat{\mathcal{T}}_i^j(z))_{i,j=1,\dots,n}$ associated to an invertible R-matrix $R(z) = (R_{ik}^{j\ell}(z))_{i,j,k,\ell=1,\dots,n}$:

$R_{ik}^{j\ell}(z/w) =$ 

satisfying the Yang–Baxter equation. One may limit oneself to the case of the six-vertex model R-matrix, with $n = 2$: $\hat{\mathcal{T}}_1^1(z) = \hat{A}(z)$, $\hat{\mathcal{T}}_1^2(z) = \hat{B}(z)$, $\hat{\mathcal{T}}_2^1(z) = \hat{C}(z)$, $\hat{\mathcal{T}}_2^2(z) = \hat{D}(z)$.

- Show that $\hat{\mathcal{T}}_i^j(z) \xmapsto{\rho_z} (R_{ik}^{j\ell}(z/w))_{k,\ell=1,\dots,n}$ defines a representation of $\mathcal{A}$ on the vector space $\mathbb{C}^n$. This representation is often denoted $\mathbb{C}^n(w)$.
- Define the *inhomogeneous* monodromy matrix

$\mathcal{T}(z; z_1, \dots, z_L) = R_{0L}(z/z_L) \dots R_{01}(z/z_1) =$ 

Note that the usual (homogeneous) monodromy matrix is the special case $\mathcal{T}(z) = \mathcal{T}(z; 1, \dots, 1)$.

Show that $\hat{\mathcal{T}}_i^j(z) \mapsto \mathcal{T}_i^j(z; z_1, \dots, z_L)$ is the tensor product representation $\mathbb{C}^n(z_1) \otimes \mathbb{C}^n(z_2) \otimes \cdots \otimes \mathbb{C}^n(z_L)$ of $\mathcal{A}$.