3. Exercise session 3

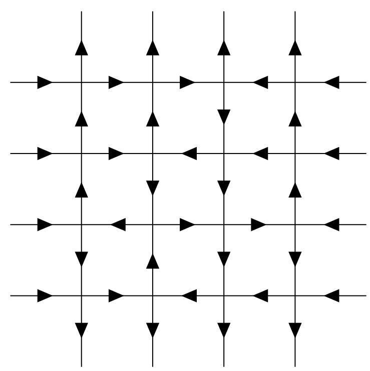
3.1. Alternating Sign Matrices, tableaux and Gelfand–Tseytlin patterns. Recall that there is a bijection between six-vertex configurations with Domain Wall Boundary Conditions (DWBC) and Alternating Sign Matrices (ASMs).

- As a warm up find the DWBC configuration, both in the arrow picture and the path picture, and the domino tilings of the Aztec diamond corresponding to the ASM

$$\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$$

The object of this exercise is to uncover more bijections. In the remainder one may use the following working example:

$$\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$$



and provide the image of the example under the various bijections below.

A *Gelfand–Tseytlin pattern* is a triangle of (integer) numbers $(\lambda_j^{(i)})_{1\leq j\leq i, 1\leq i\leq n}$ such that the inequalities hold $\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)}$; i.e.,

$$\begin{array}{ccccccccccccc}
\lambda_1^{(4)} & & & & \lambda_2^{(4)} & & & & \lambda_3^{(4)} & & & & \lambda_4^{(4)} \\
& \geq & & \leq & & \geq & & \leq & & \geq & & \leq & \\
& & \lambda_1^{(3)} & & & & \lambda_2^{(3)} & & & & \lambda_3^{(3)} & & \\
& & & \geq & & \leq & & \geq & & \leq & & & \\
& & & & \lambda_1^{(2)} & & & & \lambda_2^{(2)} & & & & \\
& & & & & \geq & & \leq & & & & & \\
& & & & & & \lambda_1^{(1)} & & & & & &
\end{array}$$

A *strict* Gelfand–Tseytlin pattern (a.k.a. monotone triangle) is defined identically, except we further impose the strict inequality $\lambda_j^{(i+1)} > \lambda_{j+1}^{(i+1)}$:

$$
\begin{array}{ccccccccccccc}
\lambda_1^{(4)} & & > & & \lambda_2^{(4)} & & > & & \lambda_3^{(4)} & & > & & \lambda_4^{(4)} \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \\
& & \lambda_1^{(3)} & & > & & \lambda_2^{(3)} & & > & & \lambda_3^{(3)} & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & \lambda_1^{(2)} & & > & & \lambda_2^{(2)} & & & & \\
& & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & & & \\
& & & & & & \lambda_1^{(1)} & & & & & &
\end{array}
$$

Finally, a Semi-Standard Young tableau (SSYT) is a filling $(T_{i,j})_{(i,j)\in Y}$ of a Young diagram Y with positive integers such $T_{i,j} < T_{i+1,j}$ and $T_{i,j} \leq T_{i,j+1}$, i.e.,

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 3 \\
\hline
2 & 3 \\
\cline{1-2}
4 \\
\cline{1-1}
\end{array}
$$

We identify Young diagram with partitions – the example above is $(4, 2, 1)$.

- Given a DWBC configuration of size n, define a triangular array of numbers as follows. The numbers are the columns of up-pointing arrows; more precisely, $\lambda_j^{(i)}$ is the column number (counted left to right from 1 to n) of the j^{th} up-arrow (counted from the *right*) of the i^{th} row (counted from the top).

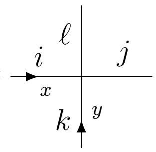
  Show that the resulting triangular array is a strict Gelfand–Tseytlin pattern, and that it provides a bijection between DWBC configurations of size n and strict Gelfand–Tseytlin patterns with top row $(n, \ldots, 2, 1)$.
- Given a (not necessarily strict) Gelfand–Tseytlin pattern $(\lambda_j^{(i)})$, one can produce a tableau as follows: each row $\lambda^{(i)}$ of the pattern is a partition, which can be drawn as a Young diagram; we obtain this way a sequence of Young diagrams which is weakly decreasing w.r.t. inclusion. In turn, this gives a tableau of the partition of the top row $\lambda^{(n)}$ as follows: a box of the Young diagram has label $i \in \{1, \ldots, n\}$ iff it belongs to (the Young diagram of) $\lambda^{(i)}$ but not to $\lambda^{(i-1)}$ (with the convention that $\lambda^{(0)}$ is the empty partition).

  Show that the resulting tableau is semi-standard, and that this forms a bijection between Gelfand–Tseytlin pattern with fixed first row $\lambda^{(n)}$ and the SSYTs of the partition $\lambda^{(n)}$ with labels in $\{1, \ldots, n\}$.
- Given a DWBC configuration, apply successively the bijections of the two previous questions to produce a SSYT $(T_{i,j})$. What is its shape? Define a new triangular array by $(\lambda_j^{(i)})$ by $\lambda_j^{(i)} = T_{i+1-j,n+1-i}$. This corresponds to rotating and deforming the SSYT:

$$
\begin{array}{|c|c|c|c|}
\hline
T_{1,1} & T_{1,2} & T_{1,3} & T_{1,4} \\
\hline
T_{2,1} & T_{2,2} & T_{2,3} \\
\cline{1-3}
T_{3,1} & T_{3,2} \\
\cline{1-2}
T_{4,1} \\
\cline{1-1}
\end{array}
\quad \mapsto \quad
\begin{array}{ccccccc}
T_{4,1} & & T_{3,1} & & T_{2,1} & & T_{1,1} \\
& T_{3,2} & & T_{2,2} & & T_{1,2} & \\
& & T_{2,3} & & T_{1,3} & & \\
& & & T_{1,4} & & &
\end{array}
$$

Show that $(\lambda^{(i)}_j)$ is again a strict Gelfand–Tseytlin pattern, and that it is associated via the first bijection to the $\pi/2$ clockwise rotation of the original DWBC configuration with all arrows reversed (or equivalently, to the $\pi/2$ clockwise rotation of the original ASM).

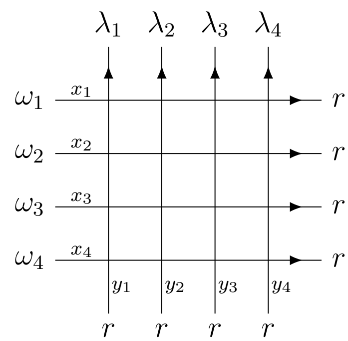
3.2. **NilHecke solution of Yang–Baxter equation and Schubert polynomials.** Let $r \in \mathbb{Z}_{>0}$, and consider the following *rational* R-matrix:

$R^{j\ell}_{ik}(x-y) =$ 

$$= \begin{cases} 1 & i=\ell, \quad k=j \\ x-y & i=j<k=\ell \\ 0 & \text{else} \end{cases}, \qquad 1 \le i,j,k,\ell \le r$$

Viewed as an operator, it acts on $\mathbb{C}^r \otimes \mathbb{C}^r$.

Given a n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \{1, \dots, r\}^n$, we define ω to be its "sort", i.e., the only weakly increasing permutation of λ. E.g., if $\lambda = (2,1,3,2)$, then $\omega = (1,2,2,3)$. We define the *Schubert polynomial* S_λ associated to λ to be the following partition function: (on this example, $n=4$)

$S_\lambda =$ 

- Show that R satisfies the (rational) Yang–Baxter equation.
- When $r=2$, what is the R-matrix? Still at $r=2$, what is S_λ when all ys are 0?
- For general r, show that S_λ is a *homogeneous polynomial* in the xs and ys. What is its degree?
- Denote $I_a = \{i : \omega_i = a\}$ for $a = 1, \dots, r$. Show that S_λ does not depend on the x_i, $i \in I_r$.
- Show that for each $a \in \{1, \dots, r\}$, S_λ is invariant by permutation of the x_i, $i \in I_a$.
- Given a n-tuple λ, define its *standardization* μ to be the unique n-tuple such that each integer in $\{1, \dots, n\}$ occurs once (i.e., μ is a permutation) and for all $i<j$ $\lambda_i \le \lambda_j$ iff $\mu_i < \mu_j$. E.g., if $\lambda = (2,1,3,2)$ then $\mu = (2,1,4,3)$.

  Show that $S_\lambda = S_\mu$ (where S_μ is defined as S_λ, but by choosing the value of r to be n)
- Define the *inversion code* $\underline{\lambda}$ of a n-tuple λ to be the sequence
$$\underline{\lambda} = (\#\{j > i : \ \lambda_j < \lambda_i\})_{i=1,\dots,n}$$
  e.g., if $\lambda = (5,2,1,3,2)$ then $\underline{\lambda} = (4,1,0,1,0)$. Show that if $\underline{\lambda}$ is weakly decreasing, then
$$S_\lambda = \prod_{j=1}^{n} \prod_{i=1}^{\underline{\lambda}_j} (x_i - y_j)$$
  If $r=2$, what are the λ satisfying this condition?