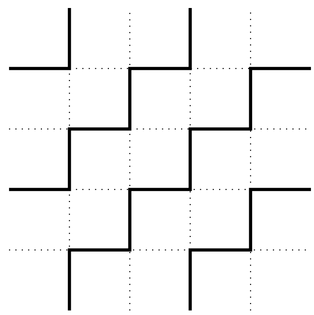
QUANTUM INTEGRABILITY AND SYMMETRIC POLYNOMIALS

Solutions

1.1. Low-temperature expansion of the antiferroelectric six-vertex model.

- *Describe the two ground states of the model, i.e., the configurations of the model with maximal Boltzmann weight.*

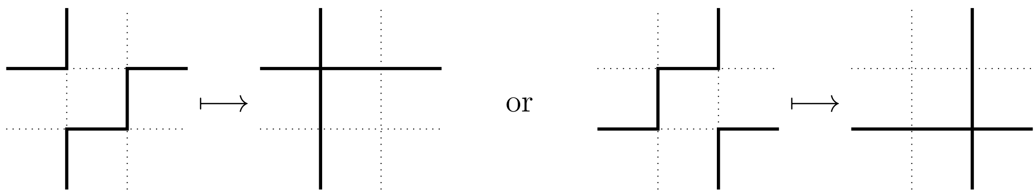
  The ground states involve only vertices with weight c. In the path picture the ground states look like staircases: one is

  and the other its parity reverse, which is also obtained by translation of the preceding by one in any direction. (Note that the above is only compatible with periodic boundary conditions when K and L are even.)
- *Use graphical notations to compute the partition function to ninth order in a/c, b/c.*

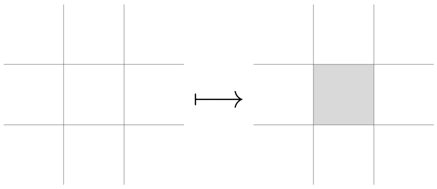
  By the previous part and parity symmetry we can focus on the 'sector' of configurations close to one of the two ground states, and multiply the result by 2 to get Z. Thus we compute $Z/2$. If we rescale the weights such that $c = 1$ the ground state in this sector has weight 1, so $\frac{1}{2}Z = 1 + h.d.$, where '*h.d.*' indicates subleading terms, of higher (total) degree in a, b.

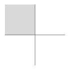
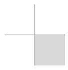
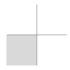
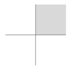
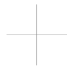
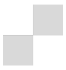
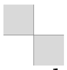
  The smallest change that the ice rule allows us to make is to pick any face of the lattice and reverse parity on the edges bounding that face:

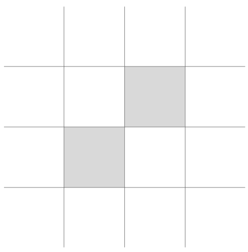
  In either case the resulting weight is a^2b^2. We can choose any of the $V = KL$ faces, so $\frac{1}{2}Z = 1 + V\,a^2b^2 + h.d.$

  Before we go on we notice that the graphical notation may be simplified for the purpose of this exercise. Indeed, it is somewhat cumbersome to have to distinguish the two cases as above, depending on which face we choose precisely. Both of the above, however, resulted in the same weight: this is a consequence of the fact, cf. the previous part, that for the ground state, translation by one in any direction is equivalent to reversing all spins. Let us therefore only indicate the face around which we reversed the spins, so that *both* of the preceding look like

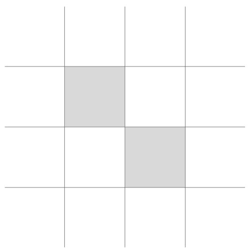


Moreover, it is easy to read off the weights:  $=$  $= a$,  $=$  $= b$,  $=$  $=$  $= c = 1$, while any configuration around a vertex such that two gray rectangles share an edges has weight zero. Importantly, one should convince oneself that the vertex weights can be read off from the grey-face configurations regardless of the where the thick and thin zigzag lines from the ground-state configuration run. This will simplify the combinatorics, and we're all set for a nice graphical perturbation theory to compute the partition function.

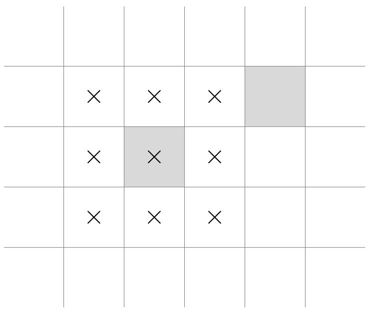
Let's see what happens when we colour *two* faces of the lattice in grey. Recalling that the two grey faces cannot share an edge there are two possibilities. Firstly the two grey squares can share one vertex:



or

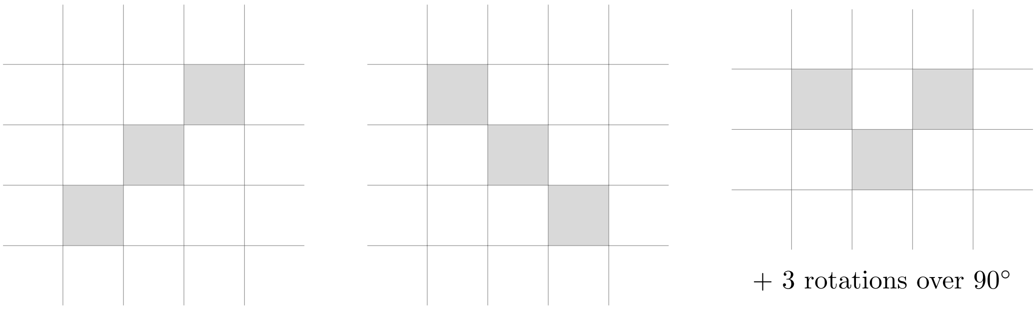


The left has weight a^4b^2 and the right has weight a^2b^4; clearly either shape can be put on V different locations on the lattice. The second possibility is when the two grey faces have no vertices in common:



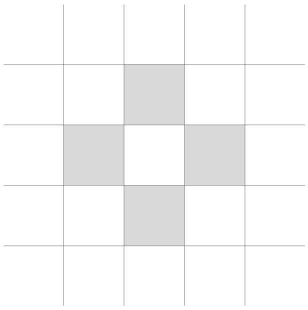
The weight is $(a^2b^2)^2$, and there are $\frac{1}{2}V(V-9)$ possibilities: choosing the first grey face excludes the faces marked by $\times$ for the second grey face, and the order in which we choose the faces does not matter so we have to include a factor of $\frac{1}{2}$ to avoid overcounting. So far we have $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V-9)a^4b^4 + \cdots$, though at this point it's not clear yet what the degree of the next terms in the expansion will be.

Next we consider *three* grey faces. When at least one of these faces has no vertex in common with either of the other two then the weight is a^2b^2 times what we got for two grey faces, yielding total degree ≥ 10 in a, b, which is further than we want to go. This leaves the following possibilities for us to consider:



The first has weight a^6b^2, the second a^2b^6 and the third a^4b^4 (by symmetry in a, b each rotation over 90° of the latter has the same weight). Clearly there are V ways to place each of these, so $4V$ of the latter if we include the rotated versions. Our tally thus is $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V-1)a^4b^4 + Va^2b^2(a^4+b^4) + \cdots$.

Next we consider *four* grey faces. The lowest possible total degree is attained when the grey faces share as many vertices as possible but have no edges in common:

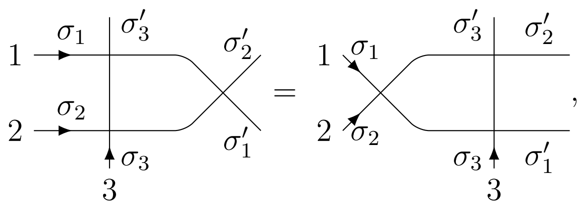


which has weight a^4b^4, and V ways to be placed on the lattice. Observe that any other way to distribute at least four grey faces will yield higher total degree. We therefore arrive at $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V+1)a^4b^4 + Va^2b^2(a^4+b^4) + h.d.$, where '*h.d.*' has total degree ≥ 10 in a, b. This is what we wanted to show.

- *Explain the symmetry of Z in a, b from the symmetries of the model.*

  As is clear in the arrow picture or the grey-face picture a rotation over 90° exchanges $a \leftrightarrow b$ and preserves c. (In particular, a rotation over 180° preserves all weights, cf. the second part of the next exercise.) In the above we saw that contributions are either invariant under such rotations, or come in pairs related by such a rotation.

1.2. Determination of the six-vertex R-matrix from RLL relations.



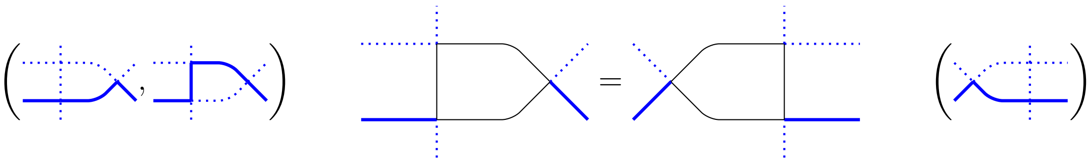
, $\qquad \sigma_1, \sigma_2, \sigma_3, \sigma'_1, \sigma'_2, \sigma'_3 \in \{\pm 1\}$

- *Show that due to line conservation out of these $2^6 = 64$ equations only $\sum_{k=0}^{3}\binom{3}{k}^2 = 20$ are nontrivial, which due to parity symmetry come in pairs of identical equations.*

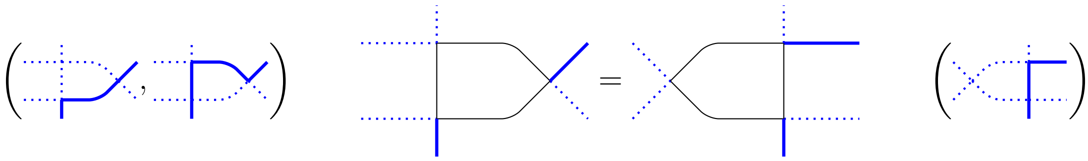
  This is a consequence of the ice rule: at each vertex we must have $\sum_{\text{incoming}} \sigma = \sum_{\text{outgoing}} \sigma$ for the Boltzmann weight to be nonzero. By summing over the three vertices, we find that both l.h.s. and r.h.s. are zero unless $\sum_{i=1}^{3} \sigma_i = \sum_{i=1}^{3} \sigma'_i$. If $k = \#\{i : \sigma_i = +1\}$ this gives the desired counting.

- *Noting that the left and right hand sides are exchanged by 180° rotation of the corresponding pictures, conclude that out of the ten equations, four correspond to 180° rotation invariant boundary conditions and are therefore automatically satisfied, while six come in pairs of identical equations and read:*

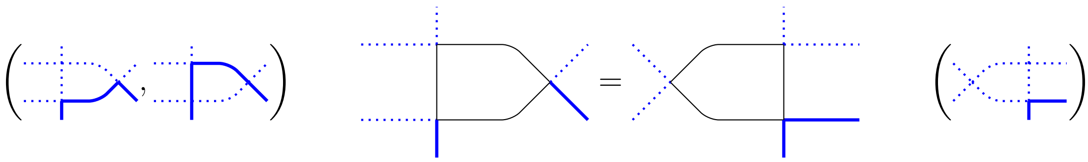
$$ab'c'' + cc'b'' = ba'c'',$$



$$ac'b'' + cb'c'' = bc'a'',$$



$$ac'c'' + cb'b'' = ca'a''.$$



We have indicated below each equation the corresponding choice of boundary conditions, as well as the actual configurations.

- *Eliminating a'', b'', c'', show that a solution for R exists only if $\Delta(a,b,c) = \Delta(a',b',c')$ with*

$$\Delta(a,b,c) = \frac{a^2+b^2-c^2}{2ab}.$$

a'', b'', c'' satisfy the following set of linear equations

$$\begin{pmatrix} 0 & cc' & ab'-ba' \\ -bc' & ac' & cb' \\ -ca' & cb' & ac' \end{pmatrix} \begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} = 0$$

For R to be invertible, one must have in particular $(a'', b'', c'') \neq (0,0,0)$, which means the linear system above is degenerate. Computing the determinant of the matrix yields

$$\begin{aligned} \begin{vmatrix} 0 & cc' & ab'-ba' \\ -bc' & ac' & cb' \\ -ca' & cb' & ac' \end{vmatrix} &= cc'(a'b'(a^2+b^2-c^2) - ab(a'^2+b'^2-c'^2)) \\ &= 2aa'bb'cc'(\Delta(a,b,c) - \Delta(a',b',c')) \end{aligned}$$

Recalling that we have assumed $a,b,c,a',b',c' > 0$, we conclude that $\Delta(a,b,c) = \Delta(a',b',c')$.

- *Show that if $\Delta(a,b,c) = \Delta(a',b',c')$ then one also has $\Delta(a'',b'',c'') = \Delta(a,b,c)$, i.e., the R-matrix is of the same form as the L-matrices.*

Note that the first two equations are swapped by exchanging primes and double primes, whereas the third equation is left invariant. We can therefore apply the same reasoning as above to conclude that $\Delta(a,b,c) = \Delta(a'',b'',c'')$.

- *Show that if L is parameterized as*

$$\begin{aligned} a(z) &= q\,z - q^{-1}z^{-1}, \\ b(z) &= z - z^{-1}, \\ c(z) &= q - q^{-1}, \end{aligned}$$

and L' similarly with z replaced by z', then R is of the same form with spectral parameter $z'' = z/z'$.

Any row of the comatrix of the degenerate linear system above solves it. For example taking the last two rows leads to

$$\begin{aligned} \begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} \propto \begin{pmatrix} a^2c'^2 - b'^2c^2 \\ abc'^2 - a'b'c^2 \\ -bb'cc' + aa'cc' \end{pmatrix} &= (q-q^{-1})^2 \begin{pmatrix} a^2 - b'^2 \\ ab - a'b' \\ -bb' + aa' \end{pmatrix} \\ &= (q-q^{-1})^2(qzz' - (qzz')^{-1}) \begin{pmatrix} q\,z/z' - q^{-1}z'/z \\ z/z' - z'/z \\ q - q^{-1} \end{pmatrix} \end{aligned}$$

1.3. **XXZ Hamiltonians.**

- *Show that up to normalization $T(1)$ is the translation operator U, i.e., $U\,e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_L} = e_{k_L} \otimes e_{k_1} \otimes \cdots \otimes e_{k_{L-1}}$ where e_+, e_- are standard basis vectors.*

First note that $R(1) = (q-q^{-1})\,P$ with P the permutation on $\mathbb{C}^2 \otimes \mathbb{C}^2$, i.e. $R_{0i} = (q-q^{-1})\,P_{0i}$ in the notation $P_{0i}\,e_{k_0} \otimes e_{k_1} \cdots \otimes e_{k_i} \otimes \cdots \otimes e_{k_L} = e_{k_i} \otimes e_{k_1} \cdots \otimes e_{k_0} \otimes \cdots \otimes e_{k_L}$. Therefore $T(1) = (q-q^{-1})^L\,\mathrm{tr}_0[P_{0L} \cdots P_{02}P_{01}]$. To compute the partial trace we note that, if P_π denotes the permutation on $(\mathbb{C}^2)^{\otimes(L+1)}$ corresponding to $\pi \in S_{L+1}$ so that $P_{(0i)} = P_{0i}$, then $P_{0L} \cdots P_{02}P_{01} = P_{(012\cdots L)} = P_{01}\,P_{(12\cdots L)}$. Since moreover $\mathrm{tr}_0 P_{0i} = \mathrm{id}$ we conclude that $T(1) = (q-q^{-1})^L\,\mathrm{tr}_0[P_{01}]\,P_{(12\cdots L)}$, with $P_{(12\cdots L)} = U$ the right shift. (All of these identities can be verified by acting on a basis vector $e_{k_1} \otimes \cdots \otimes e_{k_L}$.)

- *Show that up to an additive constant and normalization $(\log T)'(1) = \frac{dT}{dz}T^{-1}|_{z=1}$ is the XXZ Hamiltonian.*

Since the transfer matrix forms a one-parameter family of commuting operators we have $(\log T)'(1) = T^{-1}\frac{dT}{dz}|_{z=1} = \frac{dT}{dz}T^{-1}|_{z=1}$. We compute

$$\begin{aligned}(q-q^{-1})^{1-L}\,\frac{dT}{dz}\Big|_{z=1} &= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,R'_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}]\\ &= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,P_{0i}\,\check{R}'_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}]\\ &= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,\check{R}'_{i\,0}(1)\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}]\\ &= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots \check{R}'_{i\,i+1}(1)P_{0\,i+1}\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}]\\ &= \sum_{i=1}^{L} \check{R}'_{i\,i+1}(1)\,\mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}].\end{aligned}$$

Recognize the shift operator from above to get $(\log T)'(1) = (q-q^{-1})^{-1}\sum_{i=1}^{L}\check{R}'_{i\,i+1}(1)$. As

$$\check{R}'_{i\,i+1}(1) = \begin{pmatrix} 2\Delta & 0 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 2\Delta\end{pmatrix} = \big(2\,(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + \Delta\,(\sigma_i^z\sigma_{i+1}^z + 1)\big)$$

we conclude that $(\log T)'(1) = (q-q^{-1})^{-1}\,(H_{\mathrm{XXZ}} + L\,\Delta)$.

- (*Optional*) *Compute the second XXZ Hamiltonian* $(\log T)''(1)$.

  Note that $(\log T)''(1) = (T'\,T^{-1})'(1) = T''(1)\,T(1)^{-1} - (T'(1)\,T(1)^{-1})^2$. Proceeding like before we get

$$\begin{aligned}T''(1) = (q-q^{-1})^{L-1}&\sum_{i=1}^{L}\mathrm{tr}_0[P_{0L}\cdots P_{0i}\,\check{R}''_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}]\\ &+ 2\,(q-q^{-1})^{L-2}\sum_{i<j}^{L}\mathrm{tr}_0[P_{0L}\cdots P_{0i}\,\check{R}'_{0i}(1)\,P_{0\,i-1}\cdots\\ &\qquad\qquad \cdots P_{0j}\,\check{R}'_{0j}(1)\,P_{0\,j-1}\cdots P_{02}P_{01}],\end{aligned}$$

where we used $\sum_{i<j}\cdots + \sum_{j>i}\cdots = 2\sum_{i<j}\cdots$. Therefore

$$\begin{aligned}T''(1)T(1)^{-1} = (q-q^{-1})^{-1}&\sum_{i=1}^{L}\check{R}''_{i\,i+1}(1) + 2\,(q-q^{-1})^{-2}\sum_{i<j-1}^{L}\check{R}'_{i\,i+1}(1)\,\check{R}'_{j\,j+1}(1)\\ &+ 2\,(q-q^{-1})^{-2}\sum_{i=1}^{L}\check{R}'_{i\,i+1}(1)\,\check{R}'_{i+1\,i+2}(1).\end{aligned}$$

On the other hand, by the previous part

$$(T'(1)T(1)^{-1})^2 = (q-q^{-1})^{-2}\Big(2\sum_{i<j-1}^{L} \check{R}'_{i\,i+1}(1)\,\check{R}'_{j\,j+1}(1) + \sum_{i=1}^{L} \check{R}'_{i\,i+1}(1)\check{R}'_{i+1\,i+2}(1)$$
$$+ \sum_{i=1}^{L} \check{R}'_{i\,i+1}(1)^2 + \sum_{i=1}^{L} \check{R}'_{i\,i+1}(1)\check{R}'_{i-1\,i}(1)\Big).$$

The double sums (with terms with $|i-j|>2$) cancel, so obtain

$$(\log T)''(1) = (q-q^{-1})^{-1}\sum_{i=1}^{L}\big(\check{R}''_{i\,i+1}(1) - \check{R}'_{i\,i+1}(1)^2\big)$$
$$+ (q-q^{-1})^{-2}\sum_{i=1}^{L}\big[\check{R}'_{i\,i+1}(1), \check{R}'_{i+1\,i+2}(1)\big]$$

where we recognised a commutator that acts nontrivially at *three* neighbouring sites. (This pattern continues, and the higher Hamiltonians computed in this way will be less and less local. A better set of conserved charges that are *quasilocal* can be obtained from transfer matrices with *higher-spin* auxiliary spaces, by fusion, but this is beyond the scope of this course.)

1.4. **The 1D Ising model.**

- *How is H_{Ising} related to H_{XXZ} of the previous exercise?*

  We have: $H_{\text{Ising}} = J\lim_{\Delta\to\infty}\Delta^{-1}H_{\text{XXZ}} + h\sum_{i=1}^{L}\sigma_i^z$. Because $[H_{\text{XXZ}}, \sum_{i=1}^{L}\sigma_i^z] = 0$, diagonalization of H_{Ising} is equivalent to that of $\lim_{\Delta\to\infty}\Delta^{-1}H_{\text{XXZ}}$. Furthermore, if one takes $h=0$ and J and Δ are of the same sign, the two models (XXZ at $\Delta\to\pm\infty$ and Ising) are therefore equivalent.
- *Show that $Z = tr(e^{-\beta H_{Ising}})$ is the partition function of the classical one-dimensional Ising model.*

  H_{Ising} is diagonal in the standard basis $\bigotimes_{i=1}^{L} e_{\sigma_i}$ of $(\mathbb{C}^2)^{\otimes L}$, where $\sigma\in\{\pm\}^L$, with corresponding eigenvalue

  $$E_{\text{Ising}}(\sigma) = \sum_{i=1}^{L}\sigma_i\sigma_{i+1} - \beta h\sum_{i=1}^{L}\sigma_i$$

  Therefore $\mathrm{tr}(e^{-\beta H_{\text{Ising}}})$ is the sum of $e^{-\beta\ \text{eigenvalues}}$, which is the desired expression.
- *Denoting $K=-\beta J$, $B=-\beta h$, show that*

  $$Z = tr(T^L), \qquad T = \begin{pmatrix} e^{K+B} & e^{-K} \\ e^{-K} & e^{K-B}\end{pmatrix},$$

We use the transfer matrix approach (this is a spin model, but in 1D, vertex models and spin models are actually equivalent by exchanging edges and vertices). We first split the energy in terms of edges only as

$$E_{\text{Ising}}(\sigma) = J\sum_{i=1}^{L}(\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i+\sigma_{i+1})).$$

and then consider the matrix encoding a single edge:

$$T = \overset{\sigma}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma'}{\bullet} = (e^{-\beta J\sigma\sigma' - \beta\frac{h}{2}(\sigma+\sigma')})_{\sigma,\sigma'\in\{\pm\}}$$

Concatenating successive edges corresponds to taking powers of T, and the periodic boundary conditions correspond to taking the trace, so that we find $Z = \mathrm{tr}(T^L)$ as expected.

- *Conclude that*

$$Z = \Lambda_+^L + \Lambda_-^L, \qquad \Lambda_\pm = e^K \cosh B \pm \sqrt{e^{2K} \sinh^2 B + e^{-2K}}.$$

$\Lambda_\pm$ are the eigenvalues of T.

- *Compute* $\lim_{L\to\infty} \frac{\log Z}{L}$. *Is there any phase transition in temperature, i.e., in the parameter* β *(the inverse temperature) as it varies from* 0 *to* $+\infty$*?*

Note $\Lambda_+ > \Lambda_- > 0$. So $\lim_{L\to\infty} \frac{\log Z}{L} = \log \Lambda_+$. This function is analytic in β, so no phase transition occurs.