

**Problem 1 (Kasteleyn's theorem).** Recall that, for an antisymmetric  $(2n) \times (2n)$  matrix  $A$ , the *Pfaffian* of  $A$  is defined as

$$\text{Pf } A := (2^n n!)^{-1} \sum_{\pi \in S_{2n}} (-1)^{\text{sign}(\pi)} a_{\pi(1)\pi(2)} \cdots a_{\pi(2n-1)\pi(2n)}.$$

(a) Prove the identity  $(\text{Pf } A)^2 = |\det A|$ .

Recall that a Kasteleyn orientation of edges of a planar graph is defined by the property that each face has odd number of edges oriented clockwise, and let  $A = -A^\top$  be the signed (according to such an orientation) adjacency matrix of a finite planar graph.

(b) Prove the Kasteleyn theorem:

$$\mathcal{Z}_{\text{dimers}}(G) := (2^n n!)^{-1} \sum_{\pi \in S_{2n}} a_{\pi(1)\pi(2)} \cdots a_{\pi(2n-1)\pi(2n)} = |\text{Pf } A|.$$

**Problem 2 (Kramers–Wannier duality for spins and disorders).** Recall that  $\mu_{v_1} \cdots \mu_{v_n}$  can be viewed as a random variable  $\prod_{(uw):(uw) \cap \gamma[v_1, \dots, v_n] \neq \emptyset} x_e^{-2\sigma_u \sigma_w}$ , where  $\gamma[v_1, \dots, v_n]$  is the union of disorder paths linking the vertices  $v_1, \dots, v_n \in V(G)$  pairwise.

(a) Argue that  $\mathbb{E}[\mu_{v_1} \cdots \mu_{v_n} \sigma_{u_1} \cdots \sigma_{u_m}] = Z_{[u_1, \dots, u_m]}^{[v_1, \dots, v_n]}(G) \cdot (Z(G))^{-1}$ .

(b) Using the high-temperature expansion of the dual Ising model on the double-cover branching over  $u_1, \dots, u_m$  prove that  $\mathbb{E}^*[\sigma_{v_1}^* \cdots \sigma_{v_n}^* \mu_{u_1}^* \cdots \mu_{u_m}^*] = Z_{[u_1, \dots, u_m]}^{[v_1, \dots, v_n]}(G) \cdot (Z(G))^{-1}$ .

**Problem 3 (anti-commutativity of variables  $\psi_c = \eta_c \mu_{v(c)} \sigma_{u(c)}$ ).** Recall that the spin-disorder correlations  $\mathbb{E}[\mu_{v_1} \mu_{v_2} \sigma_{u_1} \sigma_{u_2}]$  are defined up to a sign which has the same branching structure as  $[\prod_{p=1}^2 \prod_{q=1}^2 (v_p - u_q)]^{1/2}$ . Argue that  $\mathbb{E}[\psi_c \psi_d]$ ,  $c \neq d$  (and, more generally,  $\mathbb{E}[\psi_c \psi_d \mathcal{O}_{\varpi}^{[\mu, \sigma]}]$ ) is an *anti-symmetric* function of  $c, d \in \Upsilon(G)$  (resp.,  $c, d \in \Upsilon_{\varpi}(G)$ ).

**Problem 4 (propagation equation for fermions).**

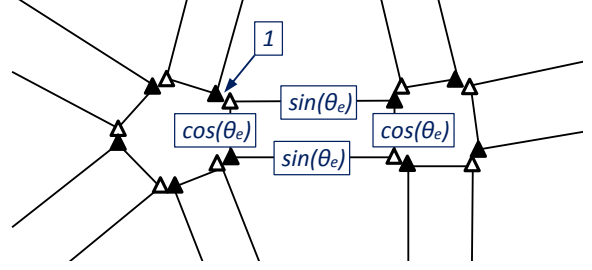
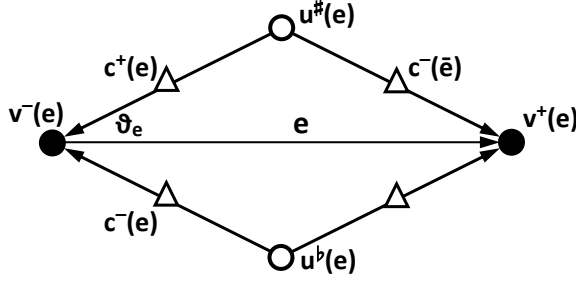
(a) Prove the propagation equation  $X_{\varpi}(c_2) = X_{\varpi}(c_1) \cdot \cos \theta_e + X_{\varpi}(c_3) \cdot \sin \theta_e$ .

(Hint: note that  $\exp[-2\beta J_e \sigma_{u^b(e)} \sigma_{u^\sharp(e)}] \cdot \sin \theta_e + \sigma_{u^b(e)} \sigma_{u^\sharp(e)} \cdot \cos \theta_e = 1$ .)

(b) Prove Smirnov's reformulation of the propagation equation for the critical model on isoradial graphs: provided that  $z = (v^-(e)u^b(e)v^+(e)u^\sharp(e))$  is a *rhombus* with the half-angle  $\theta_e$  and the Ising weights are chosen so that  $x_e = \tan \frac{1}{2}\theta_e$ , the propagation equation on this rhombus is equivalent to the existence of a value  $\Psi_{\varpi}(z) \in \mathbb{C}$  such that

$$\Psi_{\varpi}(c) = \frac{1}{2} [\Psi_{\varpi}(z) + \eta_c^2 \cdot \overline{\Psi_{\varpi}(z)}] =: \text{Proj}[\Psi_{\varpi}(z); \eta_c \mathbb{R}] \quad \text{for all } c = (u^\pm(e)v^\sharp(e)).$$

**Problem 5\*(bonus: interpretation of  $\widehat{D}$  as a discrete  $\bar{\partial}$  operator).**



Recall the operator

$$D_{c,c'} = \begin{cases} -i & \text{if } c = c'; \\ \cos \theta_e \cdot \exp[\frac{i}{2} \text{wind}(c, \bar{c}')] & \text{if } c = c^+(e) \text{ and } c' = c^-(e) \text{ for some } e; \\ \sin \theta_e \cdot \exp[\frac{i}{2} \text{wind}(c, \bar{c}')] & \text{if } c = c^+(e) \text{ and } c' = c^-(\bar{e}) \text{ for some } e; \\ 0 & \text{otherwise,} \end{cases}$$

defined on  $\Upsilon(G)$ , and let  $\widehat{D} := iU^*DU$  where  $U := \text{diag}\{\eta_c\}$ ; note that  $\widehat{D}$  is real-valued.

(a) Show that the matrix  $\begin{pmatrix} 0 & \widehat{D} \\ -\widehat{D}^\top & 0 \end{pmatrix}$  is a *Kasteleyn matrix* (i.e., that the signs of its entries give a Kasteleyn orientation) of the bipartite graph  $G^D$ , provided that one interprets  $\widehat{D}$  as an operator sending functions defined on black vertices of  $G^D$  to those on white ones.

*Assume now that we work with the critical Z-invariant model on isoradial graphs.*

(b) Argue that the operator  $\bar{\partial}_\bullet := \frac{1}{2}U^*\widehat{D}U^* = \frac{i}{2}(U^*)^2D$  can be thought of as a discrete approximation to the Cauchy-Riemann operator  $\bar{\partial} := \frac{1}{2}[\partial_x + i\partial_y]$ .

*Remark:* Along the way, you might notice the mismatch by the factor  $\sin \theta_e \cos \theta_e$  in the definitions. When arguing that discrete difference operators  $\bar{\partial}_\bullet$  ‘approximate’ the continuous operator  $\bar{\partial}$ , one should think about scalar products  $\langle f, \bar{\partial}g \rangle$  and their approximations by sums over the (edges of the rhombic) grid, this is why the *area of rhombii* become relevant.

(c) Further, let  $\partial_\bullet := \frac{1}{2}U\widehat{D}U$ ,  $\bar{\partial}_\circ := -\frac{1}{2}U\widehat{D}^\top U = -\partial_\bullet^*$ , and  $\partial_\circ := -\frac{1}{2}U^*\widehat{D}^\top U^* = -\bar{\partial}_\bullet^*$ . Argue that the operator

$$\frac{1}{4} \begin{pmatrix} U & iU \\ iU^* & U^* \end{pmatrix} \begin{pmatrix} 0 & \widehat{D} \\ -\widehat{D}^\top & 0 \end{pmatrix} \begin{pmatrix} U^* & -iU \\ -iU^* & U \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}$$

can be viewed as a discrete approximation to the massless Dirac operator on the domain  $\Omega$ .

*Remark:* Note that all these operators are anti-self-adjoint, which suggest that the boundary conditions of the Dirac operator in continuum should also give rise to the anti-self-adjointness (see also the super-bonus question below).

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**Super-bonus: (d)\*\* (what happens at the boundary of  $\Omega^\delta$ ?)** Staying in the discrete setup, provide a handwaving argument that this Dirac operator, acting on functions  $(f \ g)^\top$ ,  $f, g : \Omega \rightarrow \mathbb{C}$ , should be equipped with the following boundary condition:  $g = \tau \cdot f$  at  $\partial\Omega$ , where  $\tau \in \mathbb{C}$ ,  $|\tau| = 1$ , denotes the (counterclockwise) tangent vector to  $\Omega$ .