The hitchhiker's guide to the (critical) planar Ising model. TA1.

Problem 1 (Kasteleyn's theorem). Recall that, for an antisymmetric $(2n)\times(2n)$ matrix A, the *Pfaffian* of A is defined as

$$\mathrm{Pf}\,A \;:=\; (2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}(-1)^{\mathrm{sign}(\pi)}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}.$$

(a) Prove the identity $(\mathrm{Pf}\,A)^2=|\det A\,|$.

Recall that a Kasteleyn orientation of edges of a planar graph is defined by the property that each face has odd number of edges oriented clockwise, and let $A=-A^\top$ be the signed (according to such an orientation) adjacency matrix of a finite planar graph.
(b) Prove the Kasteleyn theorem:

$$\mathcal{Z}_{\mathrm{dimers}}(G)\;:=\;(2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}\;=\;|\,\mathrm{Pf}\,A\,|.$$

Problem 2 (Kramers–Wannier duality for spins and disorders). Recall that $\mu_{v_1}\dots\mu_{v_n}$ can be viewed as a random variable $\prod_{(uw):(uw)\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset}x_e^{\sigma_u\sigma_w}$, where $\gamma^{[v_1,\dots,v_n]}$ is the union of disorder paths linking the vertices $v_1,\dots,v_n\in V(G)$ pairwise.
(a) Argue that $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]=Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)\cdot(Z(G))^{-1}$.
(b) Using the high-temperature expansion of the dual Ising model on the double-cover branching over $u_1,\dots,u_m$ prove that $\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}]=Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)\cdot(Z(G))^{-1}$.

Problem 3 (anti-commutativity of variables $\boldsymbol{\psi_c=\eta_c\mu_{v(c)}\sigma_{u(c)}}$). Recall that the spin-disorder correlations $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ are defined up to a sign which has the same branching structure as $\big[\prod_{p=1}^2\prod_{q=1}^2(v_p-u_q)\big]^{1/2}$. Argue that $\mathbb{E}[\psi_c\psi_d]$, $c\neq d$ (and, more generally, $\mathbb{E}[\psi_c\psi_d\mathcal{O}_\varpi^{[\mu,\sigma]}]$) is an *anti-symmetric* function of $c,d\in\Upsilon(G)$ (resp., $c,d\in\Upsilon_\varpi(G)$).

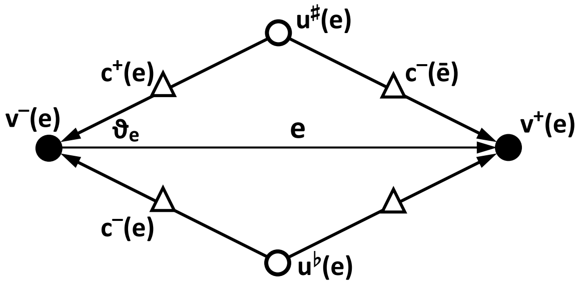
Problem 4 (propagation equation for fermions).
(a) Prove the propagation equation $X_\varpi(c_2)=X_\varpi(c_1)\cdot\cos\theta_e+X_\varpi(c_3)\cdot\sin\theta_e$.
(*Hint:* note that $\exp[-2\beta J_e\sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}]\cdot\sin\theta_e+\sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}\cdot\cos\theta_e=1$.)

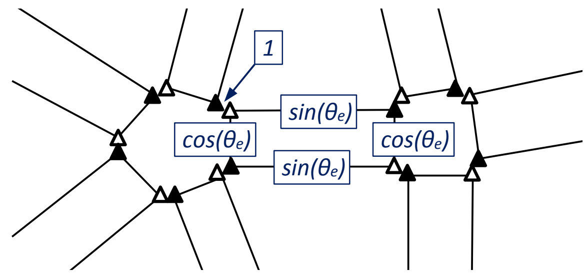
(b) Prove Smirnov's reformulation of the propagation equation for the critical model on isoradial graphs: provided that $z=(v^-(e)u^\flat(e)v^+(e)u^\sharp(e))$ is a *rhombus* with the half-angle θ_e and the Ising weights are chosen so that $x_e=\tan\frac{1}{2}\theta_e$, the propagation equation on this rhombus is equivalent to the existence of a value $\Psi_\varpi(z)\in\mathbb{C}$ such that

$$\Psi_\varpi(c)\;=\;\tfrac{1}{2}\big[\,\Psi_\varpi(z)+\eta_c^2\cdot\overline{\Psi_\varpi(z)}\,\big]=:\mathrm{Proj}[\Psi_\varpi(z)\,;\eta_c\mathbb{R}]\quad\text{for all}\;\;c=(u^\pm(e)v^\natural(e)).$$

[Turn the page for a bonus problem if you wish]

Problem 5*(bonus: interpretation of $\widehat{\mathrm{D}}$ as a discrete $\overline{\partial}$ operator).





Recall the operator

$$\mathrm{D}_{c,c'} = \begin{cases} -i & \text{if } c = c'; \\ \cos\theta_e \cdot \exp[\tfrac{i}{2}\mathrm{wind}(c,\overline{c}')] & \text{if } c = c^+(e) \text{ and } c' = c^-(e) \text{ for some } e; \\ \sin\theta_e \cdot \exp[\tfrac{i}{2}\mathrm{wind}(c,\overline{c}')] & \text{if } c = c^+(e) \text{ and } c' = c^-(\overline{e}) \text{ for some } e; \\ 0 & \text{otherwise,} \end{cases}$$

defined on $\Upsilon(G)$, and let $\widehat{\mathrm{D}} := i\mathrm{U}^*\mathrm{D}\mathrm{U}$ where $U := \mathrm{diag}\{\eta_c\}$; note that $\widehat{\mathrm{D}}$ is real-valued.

(a) Show that the matrix $\begin{pmatrix} 0 & \widehat{\mathrm{D}} \\ -\widehat{\mathrm{D}}^\top & 0 \end{pmatrix}$ is a *Kasteleyn matrix* (i.e., that the signs of its entries give a Kasteleyn orientation) of the bipartite graph G^{D}, provided that one interprets $\widehat{\mathrm{D}}$ as an operator sending functions defined on black vertices of G^{D} to those on white ones.

Assume now that we work with the critical Z-invariant model on isoradial graphs.

(b) Argue that the operator $\overline{\partial}_\bullet := \frac{1}{2}\mathrm{U}^*\widehat{\mathrm{D}}\mathrm{U}^* = \frac{i}{2}(\mathrm{U}^*)^2\mathrm{D}$ can be thought of as a discrete approximation to the Cauchy-Riemann operator $\overline{\partial} := \frac{1}{2}[\partial_x + i\partial_y]$.

Remark: Along the way, you might notice the mismatch by the factor $\sin\theta_e\cos\theta_e$ in the definitions. When arguing that discrete difference operators $\overline{\partial}_\bullet$ 'approximate' the continuous operator $\overline{\partial}$, one should think about scalar products $\langle f, \overline{\partial} g\rangle$ and their approximations by sums over the (edges of the rhombic) grid, this is why the *area of rhombii* become relevant.

(c) Further, let $\partial_\bullet := \frac{1}{2}\mathrm{U}\widehat{\mathrm{D}}\mathrm{U}$, $\overline{\partial}_\circ := -\frac{1}{2}\mathrm{U}\widehat{\mathrm{D}}^\top\mathrm{U} = -\partial_\bullet^*$, and $\partial_\circ := -\frac{1}{2}\mathrm{U}^*\widehat{\mathrm{D}}^\top\mathrm{U}^* = -\overline{\partial}_\bullet^*$. Argue that the operator

$$\frac{1}{4}\begin{pmatrix} \mathrm{U} & i\mathrm{U} \\ i\mathrm{U}^* & \mathrm{U}^* \end{pmatrix}\begin{pmatrix} 0 & \widehat{\mathrm{D}} \\ -\widehat{\mathrm{D}}^\top & 0 \end{pmatrix}\begin{pmatrix} \mathrm{U}^* & -i\mathrm{U} \\ -i\mathrm{U}^* & \mathrm{U} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix}$$

can be viewed as a discrete approximation to the massless Dirac operator on the domain Ω.

Remark: Note that all these operators are anti-self-adjoint, which suggest that the boundary conditions of the Dirac operator in continuum should also give rise to the anti-self-adjointness (see also the super-bonus question below).

Super-bonus: (d) (what happens at the boundary of Ω^δ?)** Staying in the discrete setup, provide a handwaving argument that this Dirac operator, acting on functions $(f\ g)^\top$, $f, g : \Omega \to \mathbb{C}$, should be equipped with the following boundary condition: $g = \tau \cdot f$ at $\partial\Omega$, where $\tau \in \mathbb{C}$, $|\tau| = 1$, denotes the (counterclockwise) tangent vector to Ω.