The hitchhiker's guide to the (critical) planar Ising model. TA1. Solutions.

Problem 1 (Kasteleyn's theorem). Recall that, for an antisymmetric $(2n)\times(2n)$ matrix A, the *Pfaffian* of A is defined as

$$\mathrm{Pf}[A] \;:=\; (2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}(-1)^{\mathrm{sign}(\pi)}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}.$$

(a) Prove the identity $(\mathrm{Pf}[\,A\,])^2=|\det A|$.

Recall that a Kasteleyn orientation of edges of a planar graph is defined by the property that each face has odd number of edges oriented clockwise, and let $A=-A^\top$ be a signed (according to such an orientation) adjacency matrix of a finite planar graph.
(b) Prove the Kasteleyn theorem:

$$\mathcal{Z}_{\mathrm{dimers}}(G) \;:=\; (2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)} \;=\; |\,\mathrm{Pf}[A]\,|.$$

Solution. (a). Let us start with an "linear algebra" solution. Namely, given an antisymmetric matrix A one can consider the differential 2-form

$$\omega=\sum_{i,j=1}^{2n}A_{ij}dx_i\wedge dx_j.$$

Since A is antisymmetric, this form is correctly defined and one can see from the definition that

$$\omega^n=\mathrm{Pf}[A]\,dx_1\wedge\dots dx_n.$$

Using this representation one concludes that $\mathrm{Pf}[U^TAU]=\det U\mathrm{Pf}[A]$ for any matrix U. Now recall that any antisymmetric matrix is of the form

$$U^T\begin{pmatrix}0 & P\\ -P^T & 0\end{pmatrix}U$$

for some *orthogonal* matrix U. Using the definition of the Pfaffian one conclude that

$$\mathrm{Pf}[A]=\det P=\pm\sqrt{\det A}.$$

Another way to solve the problem is to play with the combinatorics of perfect matching. Let K_{2n} be the complete graph on n vertices enumerated by $1,2,\dots,2n$, a perfect matching D of K_{2n} is given by any collection of non-intersecting edges $e_1,\dots,e_n$ where an edge is just any pair (i,j) with $i\neq j$. Let us write $\pi\sim D$ if the perfect matching D is given by $(\pi(1),\pi(2)),\dots,(\pi(2n-1),\pi(2n))$. It is easy to see that the number of permutation that defines the given perfect matching is equal to $n!2^n$. If D_1,D_2 are two perfect matchings then $D_1\cup D_2$ defines a decomposition of K_{2n} into a disjoint union of even cycles and edges (an edge occurs if it belongs to both D_1 and D_2). Let us write that $\pi\sim D_1\cup D_2$ if the permutation π has the same cyclic decomposition as $D_1\cup D_2$ defined (edges are considered as cycles of length two, i.e. as transpositions). Note that one can find $\pi_1\sim D_1$ and $\pi_2\sim D_2$ such that $\pi=\pi_1\circ\pi_2^{-1}$. In particular, $(-1)^\pi=(-1)^{\pi_1}\cdot(-1)^{-\pi_2}$. Note that a permutation π

is given by $D_1 \cup D_2$ for some D_1, D_2 if and only if it has only even cycles. Let us call such π even. The discussion above produces the following equality

$$\mathrm{Pf}[A]^2 = \pm \sum_{\pi \text{ – even}} (-1)^\pi \prod_{j=1}^{2n} a_{j\pi(j)}$$

and it remains to show that the right-hand side is equal to $\pm \det A$. Let us write

$$\det A = \sum_{\pi \text{ – even}} (-1)^\pi \prod_{j=1}^{2n} a_{j\pi(j)} + \sum_{\pi \text{ – not even}} (-1)^\pi \prod_{j=1}^{2n} a_{j\pi(j)}.$$

Assume that π is not even. If π has a fixed point, say, $\pi(j) = j$, then $\prod_{j=1}^{2n} a_{j\pi(j)} = 0$ since $a_{j,j} = 0$ because A is antisymmetric. Now let $\pi = s_1 s_2 \dots s_k$ where s_j is a cyclic permutation and assume that s_1 is an odd cycle of length greater then 1. Then let $\pi' = s_1^{-1} s_2 \dots s_k$. Then the fact that A is antisymmetric implies that $(-1)^\pi \prod_{j=1}^{2n} a_{j\pi(j)} = -(-1)^{\pi'} \prod_{j=1}^{2n} a_{j\pi'(j)}$. Using these observations one can deduce that all the summands in the sum $\sum\limits_{\pi \text{ – not even}} (-1)^\pi \prod_{j=1}^{2n} a_{j\pi(j)}$ cancels out.

(b). Note that if A is the adjacency matrix then $a_{\pi(1)\pi(2)} \dots a_{\pi(2n-1)\pi(2n)}$ is non-zero only if the pair of vertices $(\pi(2n-1), \pi(2n))$ form an edge in the graph, so we find that

$$\begin{aligned}
\mathrm{Pf}[A] &= (2^n n!)^{-1} \sum_{\pi \in S_{2n}} (-1)^{\mathrm{sign}(\pi)} a_{\pi(1)\pi(2)} \dots a_{\pi(2n-1)\pi(2n)} \\
&= \sum_{\substack{D \text{ is perfect} \\ \text{matching}}} (2^n n!)^{-1} \sum_{\pi \sim D} (-1)^{\mathrm{sign}(\pi)} a_{\pi(1)\pi(2)} \dots a_{\pi(2n-1)\pi(2n)}.
\end{aligned}$$

As we already mentioned, there are precisely $2^n n!$ permutations that correspond to a given perfect matching, so what we need to show is that if A is the Kasteleyn matrix, then the sing of $(-1)^{\mathrm{sign}(\pi)} a_{\pi(1)\pi(2)} \dots a_{\pi(2n-1)\pi(2n)}$ is the same for any choice of π.

First, if π and π' corresponds to the same perfect matching then $\pi' \circ \pi^{-1}$ is a product of transposition, so $(-1)^\pi (-1)^{\pi'}$ is -1 iff the number of transpositions is odd. Using that $a_{\pi(2j-1)\pi(2j)} = -a_{\pi(2j)\pi(2j-1)}$ one get that the same is true for the sign of $\frac{a_{\pi(1)\pi(2)} \cdots a_{\pi(2n-1)\pi(2n)}}{a_{\pi'(1)\pi'(2)} \cdots a_{\pi'(2n-1)\pi'(2n)}}$ and the sign of the product remains the same for π and π'.

Now, let $\pi \sim D$ and $\pi' \sim D'$. By the discussion in the previous case we can choose π and π' so that $\pi' \circ \pi^{-1} \sim D' \cup D$. Let us write $\pi' \circ \pi^{-1} = s_1 \circ \dots \circ s_k$ where s_j is a cycle permutation. Since s_j is even for each j we get that $(-1)^{\pi' \circ \pi^{-1}} = (-1)^k$. Each s_j can be considered as an oriented cycle on our graph G. Let $\overline{e}(s_k)$ denote the number of edges of G lying on this cycle and whose orientation is opposite the orientation of the cycle. Then it follows that

$$a_{\sigma(1)\sigma(2)} \dots a_{\sigma(2n-1)\sigma(2n)} = \prod_{j=1}^{k} (-1)^{\overline{e}(s_j)}$$

where we write $\sigma = \pi' \circ \pi$ to simplify the formula. Let us show that $\overline{e}(s_j)$ is odd for any j. Let G_j be the subgraph of G that consists of all vertices lying *inside* s_j or belong to s_j (here we use the planar structure of G!!). Since all the cycles s_j are non-intersecting we find that all the cycles s_i that lies inside s_j (and s_j itself) cover all vertices of G_j, thus the number of vertices of G_j is even. Applying the Euler formula to G_j we find that

$$\#E(G_j) + 1 = \#F(G_j) \mod 2.$$

Recall that for each face $f \in F(G_j)$ there is odd number of edges oriented clockwise with respect to this edge. Denote the number of such edges by $\overline{e}(f)$; note that each edge contribute to $\overline{e}(f)$ for exactly one of its neighboring faces. Then we see that the following identities holds modulo 2:

$$\begin{aligned} \#E(G_j) + 1 &\equiv \#F(G_j) \equiv \sum_{f\in F} \overline{e}(f) \equiv \\ &\equiv \#\{\text{edges lying inside } s_j\} + \text{length}(s_j) + \overline{e}(s_j) = E(G_j) + \overline{e}(s_j) \end{aligned}$$

where we use that $\text{length}(s_j)$ is even and thus $\overline{e}(s_j)$ is of the same parity as the number of edges whose orientation does not agree with the orientation of s_j. It follows that $\overline{e}(s_j) = 1 \mod 2$ and we are done.

Problem 2 (Kramers–Wannier duality for spins and disorders). Recall that $\mu_{v_1} \dots \mu_{v_n}$ can be viewed as a random variable $\prod_{(ww')\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset} x_e^{\sigma_w\sigma_{w'}}$, where $\gamma^{[v_1,\dots,v_n]}$ is a collection of disorder paths linking the vertices $v_1, \dots, v_n \in V(G)$ pairwise.

(a) Argue that $\mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}] = Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)/Z(G)$.

(b) Using the high-temperature expansion of the dual Ising model on the double-cover branching over $u_1, \dots, u_m$ prove that $\mathbb{E}^*[\sigma^*_{v_1} \dots \sigma^*_{v_n} \mu^*_{u_1} \dots \mu^*_{u_m}] = \mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}]$.

Solution. (a) Recall that in order to define $Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)$ we need to fix a choice of the collection $\gamma_{[u_1,\dots,u_m]}$ of paths linking the faces $u_1, \dots, u_m$. Then we set $(x_{[u_1,\dots,u_m]})_e = x_e$ if $e \cap \gamma_{[u_1,\dots,u_m]} = \varnothing$ and $(x_{[u_1,\dots,u_m]})_e = -x_e$ in the opposite case and the quantity $Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)$ is given by

$$Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G) := \sum_{C\in\mathcal{E}(G;v_1,\dots,v_n)} x_{[u_1,\dots,u_m]}(C).$$

We claim that

$$\mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}] = (-1)^{\gamma^{[v_1,\dots,v_n]}\cdot\gamma_{[u_1,\dots,u_m]}} Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)/Z(G) \qquad (1)$$

where $\gamma^{[v_1,\dots,v_n]} \cdot \gamma_{[u_1,\dots,u_m]}$ stands for the total number of intersections between paths from $\gamma^{[v_1,\dots,v_n]}$ and paths from $\gamma_{[u_1,\dots,u_m]}$. We will show that

$$\mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}] = Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)/Z(G)$$

in the case when $\gamma^{[v_1,\dots,v_n]} \cdot \gamma_{[u_1,\dots,u_m]} = 0$ and leave the general case as an exercise.

first step. Given an even subgraph $C \subset \mathcal{E}(G)$ let us denote by $C\Delta\gamma^{[v_1,\dots,v_n]}$ the symmetric difference (i.e. $e \in C\Delta\gamma^{[v_1,\dots,v_n]}$ iff e belongs only to one of C and $\gamma^{[v_1,\dots,v_n]}$). Observe that $C \mapsto C\Delta\gamma^{[v_1,\dots,v_n]}$ is a bijection between $\mathcal{E}(G)$ and $\mathcal{E}(G; v_1, \dots, v_n)$.

second step. Let $\sigma = \{\sigma_u\}_{u\in F(G)}$ be a spin configuration and let $C \in \mathcal{E}(G)$ be the corresponding domain wall. Observe that

$$\left(\prod_{j=1}^m \sigma_{u_j}\right) \exp(\beta \sum_{e=(w,w')} J_e\sigma_w\sigma_{w'}) = \exp(\beta \sum_{e=(w,w')} J_e\sigma_w\sigma_{w'}) x_{[u_1,\dots,u_m]}(C).$$

third step. Let $\sigma = \{\sigma_u\}_{u\in F(G)}$ be a spin configuration and let $C \in \mathcal{E}(G)$ be the corresponding domain wall. Observe that

$$\left(\prod_{(ww')\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset} x_e^{\sigma_w\sigma_{w'}}\right)\left(\prod_{j=1}^{m}\sigma_{u_j}\right)\exp(\beta\sum_{e=(w,w‘)} J_e\sigma_w\sigma_{w'}) =$$
$$= \exp(\beta\sum_{e=(w,w‘)} J_e\sigma_w\sigma_{w'})x_{[u_1,\dots,u_m]}(C\Delta\gamma^{[v_1,\dots,v_n]})$$

if $\gamma^{[v_1,\dots,v_n]}\cdot\gamma_{[u_1,\dots,u_m]} = 0$.

final step. Using the previous steps write

$$\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}] = \mathcal{Z}^{-1}\sum_{\sigma}\left(\prod_{(ww')\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset} x_e^{\sigma_w\sigma_{w'}}\right)\left(\prod_{j=1}^{m}\sigma_{u_j}\right)\exp(\beta\sum_{e=(w,w‘)} J_e\sigma_w\sigma_{w'})$$
$$= Z_{[u_1,\dots,u_m]}^{[v_1,\dots,v_n]}(G)/Z(G)$$

(b) Recall that we have fixed paths $\gamma^{[v_1,\dots,v_n]}$ and $\gamma_{[u_1,\dots,u_m]}$ such that $\gamma^{[v_1,\dots,v_n]}\cdot\gamma_{[u_1,\dots,u_m]} = 0$. Our goal is to expand $\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}]$ via the high-temperature expansion. It is convenient to introduce an additional notation $s(e)$ such that $s(e) = 1$ if $e\cap\gamma_{[u_1,\dots,u_m]} = \varnothing$ and $s(e) = -1$ in the opposite case. Using this notation we can write

$$\prod_{(vv')\cap\gamma_{[u_1,\dots,u_n]}\neq\emptyset}(x_e^*)^{\sigma_v\sigma_{v'}}\cdot\exp(\beta^*\sum_{e=(v,v')} J_e^*\sigma_v\sigma_{v'}) = \exp(\beta^*\sum_{e=(v,v')} s(e)J_e^*\sigma_v\sigma_{v'}).$$

Observe also that

$$\exp(\beta^* J_e^* s(e)\sigma_v\sigma_{v'}) = \cosh(\beta^* J_e^*) + \sinh(\beta^* J_e^*)s(e)\sigma_v\sigma_{v'}.$$

Given a subgraph C we set $s(C) = \prod_{e\in C} s(e)$. Observe that

$$x_{[u_1,\dots,u_m]}(C) = x(C)s(C).$$

Finally, recall that $x_e = \tanh(\beta^* J_e^*)$. Now let us expand $\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}]$ using the definition of $\mu^*_{u_1}\dots\mu^*_{u_m}$ via random variables:

$$\begin{aligned}
\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}] &= \mathcal{Z}^{-1}\sum_{\sigma^*}\prod_{j=1}^{n}\sigma^*_{v_j}\exp(\beta^*\sum_{e=(v,v')}J_e^* s(e)\sigma_v\sigma_{v'}) \\
&= \mathcal{Z}^{-1}\sum_{\sigma^*}\prod_{j=1}^{n}\sigma^*_{v_j}\prod_{e=(v,v')}(\cosh(\beta^* J_e^*)+\sinh(\beta^* J_e^*)s(e)\sigma_v\sigma_{v'}) \\
&= \mathcal{Z}^{-1}\left(\prod_e\cosh(\beta^* J_e^*)\right)\sum_{\sigma^*}\prod_{j=1}^{n}\sigma^*_{v_j}\prod_{e=(v,v')}(1+\tanh(\beta^* J_e^*)s(e)\sigma_v\sigma_{v'}) \\
&= Z(G)^{-1}\sum_{C\in\mathcal{E}(G;v_1,\dots,v_n)}x(C)s(C) \\
&= Z(G)^{-1}\sum_{C\in\mathcal{E}(G;v_1,\dots,v_n)}x_{[u_1,\dots,u_m]}(C) \\
&= \mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}].
\end{aligned}$$

Problem 3 (anti-commutativity of variables ψ_c. Recall that the spin-disorder correlations $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ are defined up to a sign which has the same branching structure as $\left[\prod_{p=1}^n\prod_{q=1}^m(v_p-u_q)\right]^{1/2}$. Argue that $\mathbb{E}[\psi_c\psi_d]$ (and, more generally, $\mathbb{E}[\psi_c\psi_d\mathcal{O}_\varpi^{[\mu,\sigma]}]$) is an *anti-commutative* function of two distinct corners $c,d\in\Upsilon(G)$ (resp., $c,d\in\Upsilon_\varpi(G)$).

Solution. The reason why this exercise appears is the we prefer to think of $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ as of a function of u's and v's rather than as just a single expectation. Recall that we need to fix the collection of paths $\gamma^{v_1,\dots,v_n}$ in order to define $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ and the answer *does* depend on this choice as one can see from (1): one should care about the homotopy type of $\gamma^{v_1,\dots,v_n}$ in the "punctured domain" $G\smallsetminus\{u_1,\dots,u_m\}$. In particular, when a face u_j "move across" a path from $\gamma^{v_1,\dots,v_n}$ then some discontinuity may occurs. The expansion of $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ provides a way to define $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ properly. Let us start from some $v_1^0,\dots,v_n^0,u_1^0,\dots,u_m^0$. We fix collection of paths $\gamma^{v_1^0,\dots,v_n^0}$ and $\gamma^{u_1^0,\dots,u_n^0}$ as we did in the solution to Problem 2 and define $\mathbb{E}[\mu_{v_1^0}\dots\mu_{v_n^0}\sigma_{u_1^0}\dots\sigma_{u_m^0}]$ to be *equal* to $Z^{[v_1^0,\dots,v_n^0]}_{[u_1^0,\dots,u_m^0]}(G)/Z(G)$. Now, assume that we want to replace the vertice v_j^0 with the neighbor vertice v_j. Then we add the edge (v_j^0,v_j) to the path from $\gamma^{v_1^0,\dots,v_n^0}$ that is adjacent to v_j^0 and leave all the other pathes from $\gamma^{u_1^0,\dots,u_n^0}$ unchanged. Now we again define $\mathbb{E}[\mu_{v_1^0}\dots\mu_{v_j}\dots\mu_{v_n^0}\sigma_{u_1^0}\dots\sigma_{u_m^0}]$ to be equal to $Z^{[v_1^0,\dots,v_j,\dots,v_n^0]}_{[u_1^0,\dots,u_m^0]}(G)/Z(G)$. In the same way we can replace u_j^0 with a neighboring face. Now, one sees that if u_j makes a full turn arounf v_k o vice versa then the number of intersections between $\gamma^{v_1,\dots,v_n}$ and $\gamma^{u_1,\dots,u_m}$ changes by 1 and thus the sign of $\mathbb{E}[\mu_{v_1}\dots\mu_{v_n}\sigma_{u_1}\dots\sigma_{u_m}]$ also changes due to (1).

Due to some technical reason (see Problem 4) it is convenient to study an observable defined on *corners* c,d by $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$. If we define this quantity using the procedure above then we will see that each time the corner c or the corner d make a full turn the expectation $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$ change the sign. To get rid of this local branching let us consider another function with such a property given by $\eta_c = e^{\frac{\pi i}{4}}\exp(-\frac{i}{2}\mathrm{arg}(v(c)-u(c)))$ where $v(c)-u(c)$ is thought of as a difference of complex numbers (recall that we assume that both G and the dual graph are embedded into $\mathbb{C}$). Since arg is multiply defined the function η_c is multivalued but the product $\eta_c\eta_d\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}] = \mathbb{E}[\psi_c\psi_d]$ then can be defined as a single-valued function. Now let us show that $\mathbb{E}[\psi_c\psi_d] = -\mathbb{E}[\psi_d\psi_c]$. To show this we start moving c and d step by step such that eventually they interchange their position. Let us choose their trajectories to be disjoint from each other, and let us assume that the function $\mathrm{arg}(v(c)-u(c)) - \mathrm{arg}(v(d)-u(d))$ changed its value by $2\pi k$ in the end of this procedure. Then it is easy to observe that in the end of the day $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$ was muliplied by $(-1)^k$. The anticommutativity follows. The case of $\mathbb{E}[\psi_c\psi_d\mathcal{O}_\varpi^{[\mu,\sigma]}]$ can be treated exactly in the same way.

Problem 4 (propagation equation for fermions). (a) Prove the propagation equation $X_\varpi(c_2) = X_\varpi(c_1)\cdot\cos\theta_e + X_\varpi(c_3)\cdot\sin\theta_e$ for Kadanoff-Ceva fermions.

(*Hint:* note that $\exp[-2\beta J_e\sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}]\cdot\cos\theta_e + \sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}\cdot\sin\theta_e = 1$.)

(b) Prove Smirnov's reformulation of the propagation equation for the critical model on isoradial graphs: provided that $z_e = (v^-(e)u^\flat(e)v^+(e)u^\sharp(e))$ is a *rhombus* with the half-angle θ_e and the Ising weights are chosen so that $x_e = \tan\frac{1}{2}\theta_e$, the propagation equation on this rhombus is equivalent to the existence of a value $\Psi_\varpi(z_e)\in\mathbb{C}$ such that

$$\Psi_\varpi(c) \;=\; \tfrac{1}{2}[\Psi_\varpi(z) + \eta_c^2\cdot\overline{\Psi_\varpi(z)}] = \mathrm{Proj}[\Psi_\varpi(z);\eta_c\mathbb{R}].$$

Solution. (a) Recall that $X_\varphi(c) = \mathbb{E}[\mu_{v(c)}\sigma_{u(c)}\mathcal{O}_\varpi^{[\mu,\sigma]}]$ is defined via the procedure described in the solution to Problem 3. In particular, for three consequetive corners c_1, c_2, c_3 we know the particular way (based on the choice of paths $\gamma^v,\ \gamma_u$) to interprete $\mu_{v(c)}\sigma_{u(c)}\mathcal{O}_\varpi^{[\mu,\sigma]}$ as a random variable. The relation in the Hint immediately implies that

$$\mu_{v(c_2)}\sigma_{u(c_2)}\mathcal{O}_\varpi^{[\mu,\sigma]} = \mu_{v(c_1)}\sigma_{u(c_1)}\mathcal{O}_\varpi^{[\mu,\sigma]}\cos\theta_e + \mu_{v(c_3)}\sigma_{u(c_3)}\mathcal{O}_\varpi^{[\mu,\sigma]}\sin\theta_e$$

from which we conclude the propagation equation.

(b) It is clear that the existance of such a function Ψ_ϖ on rombus implies the propagation equation. Vise versa, using the procedure from the solution to the problem 3 we define a complex-valued function Ψ_ϖ on rombus by

$$\Psi_\varpi(z_e) = \mathbb{E}[\psi_{(v^-(e)u^\flat(e))}] + \mathbb{E}[\psi_{(v^+(e)u^\sharp(e))}].$$

It is clear from the definition that if $c = (v^-(e)u^\flat(e))$ or $c = (v^+(e)u^\sharp(e))$ then $\Psi_\varpi(c) = \mathrm{Proj}[\Psi_\varpi(z);\eta_c\mathbb{R}]$. On the other hand, the propagation equation implies that

$$\mathbb{E}[\psi_{(v^-(e)u^\flat(e))}] + \mathbb{E}[\psi_{(v^+(e)u^\sharp(e))}] = \mathbb{E}[\psi_{(u^\sharp(e)v^-(e))}] + \mathbb{E}[\psi_{(u^\flat(e))v^+(e))}]$$

so that the desired relation holds for the other two corners too.