A SMALL REMARK ON SZEGO THEOREMS.

1. Suppose that we have a real symmetric weight $w(z)$ on the unit circle $\mathbb{T} = \{z : |z| = 1\}$, i.e. $w : \mathbb{T} \to \mathbb{R}_{\geq 0}$ is a non-negative function and $w(\overline{z}) = w(z)$. The space $L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})$ is defined in the usual way. Orthogonal polynomials $\Phi_0, \Phi_1, \Phi_2, \ldots$ with respect to w are defined recursively by

$$\Phi_0(z) = 1,$$
$$\Phi_n(z) = z^n + \ldots, \qquad \langle \Phi_n(z), z^m \rangle_{L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})} = 0,$$

where $\langle f, g \rangle_{L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})} = \int_{\mathbb{T}} f(z) \cdot \overline{g(z)}\, w(z) \frac{|dz|}{2\pi}$. In other words, $\Phi_0, \Phi_1, \ldots$ is the result of the orthogonalization procedure applied to the system $1, z, z^2, \ldots$ in $L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})$.

2. Given a polynomial Φ set $\Phi^*(z) := z^n \Phi(z^{-1})$. It is straightforward to check that Φ_n^* is the only polynomial of degree at most n such that $\Phi_n^*(0) = 1$ and $\Phi_n^* \perp z^m$ if $1 \leq m \leq n$.

3. Given $n \geq 0$ consider the polynomial $z\Phi_n(z)$. Properties of Φ_n immediately implies that $z\Phi_n(z)$ is monic and is orthogonal to z^m if $1 \leq m \leq n$. Thus we should have

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n \Phi_n^*(z). \tag{1}$$

for some constant α_n. Substituting $z = 0$ we find that

$$\alpha_n = -\Phi_{n+1}(0).$$

Coefficients α_n are called *Verblunsky coefficients* and (1) is called *Szego recursive relation.* Note that one can express $\|\Phi_n\|_{L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})}$ via α_n by

$$\beta_n := \|\Phi_n\|^2_{L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})} = \|\Phi_0\|^2_{L^2(\mathbb{T}, \omega \frac{|dz|}{2\pi})} \prod_{j=0}^{n-1} (1 - |\alpha_j|^2).$$

4. Let us introduce another Hilbert space denoted by H^2:

$$H^2 := \{\sum_{j \geq 0} a_j z^j, \ | \ \|\sum_{j \geq 0} a_j z^j\|^2_{H^2} = \sum_{j \geq 0} |a_j|^2 < \infty\}. \tag{2}$$

The space H^2 is called *Hardy space.* Alternatively one can say that H^2 is the (closed) span of $1, z, z^2, \ldots$ in $L^2(\mathbb{T}, \frac{|dz|}{2\pi})$ and one has

$$\|f\|^2_{H^2} = \int_{\mathbb{T}} |f(z)| \frac{|dz|}{2\pi}.$$

Note that $H^2 \subset L^2(\mathbb{T}, \frac{|dz|}{2\pi})$ is exactly the space of square integrable functions that admits a holomorphic continuation to the unit disc. Denote by $P_+ : L^2(\mathbb{T}, \frac{|dz|}{2\pi}) \to H^2$ the orthogonal projection (Riesz projector).

5. Given a non-negative weight on $\mathbb{T}$ one can define the *Toeplitz operator* $T(w) : H^2 \to H^2$ by

$$T(w)f := P_+(w \cdot f).$$

If $w = \sum_{j\in\mathbb{Z}} a_j z^j$ is the Fourier series for w then the matrix of $T(w)$ in the basis $1, z, z^2, \ldots$ is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_{-1} & a_0 & a_1 & \cdots \\ a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

6. Let $Q_n : H^2 \to H^2$ be the orthogonal projection onto the span of $1, z, \ldots, z^n$ and $R_n := \mathbb{1} - Q_n$. Then determinants of *finite size* matrices $Q_n T(w) Q_n$ (thought as operators on the span of $1, z, \ldots, z^n$) can be expressed via the norms β_n of orthogonal polynomials (see the paragraph 3), one has

$$\det_{Q_n H^2} Q_n T(w) Q_n = \beta_n \cdot \beta_{n-1} \cdot \cdots \cdot \beta_0. \tag{3}$$

To prove (3) just consider the matrix $Q_n T(w) Q_n$ in the basis $\Phi_0, \ldots, \Phi_n$.
7. From this point we will assume that $w(z) = |D(z)|^2$ where D is holomorphic function in the unit disc and, moreover, it is an *outer function*. The latter means that $2\Im \log D$ is the harmonic conjugate to harmonic extension of $\log w$ to the unit disc; in particular we assume that $\log w$ is integrable on $\mathbb{T}$.
8. There is a natural way to interpret Φ_n^* as a solution to some extremal problem. Namely, let us consider a problem of minimizing

$$\int_{\mathbb{T}} |\Phi(z)|^2\, w(z) \frac{|dz|}{2\pi} \tag{4}$$

over all polynomials Φ such that $\deg \Phi \le n$ and $\Phi(0) = 1$. One can think of this minimum as of the distance between 0 and the affine hyperplane given by the linear condition $\Phi(0) = 1$. Such a distance is achieved on the unique vector in this hyperplane that is orthogonal to it. Using such arguments one sees that the minimum is attained on Φ_n^*.
9. Notice that if we drop the condition that Φ is a polynomial, than the minimum of (4) is attained on $\Phi(z) = D(0)/D(z)$ (recall that we have $w(z) = |D(z)|^2$). The fact that D is an outer function ensures that $D(z)$ do not vanish in the unit disc and thus $D(0)/D(z)$ is a holomorphic function in the unit disc and thus lies in the closed span of $1, z, z^2, \ldots$ in $L^2(\mathbb{T}, w(z)\frac{|dz|}{2\pi})$. A straightforward computation shows that $D(0)/D(z)$ is orthogonal to $z, z^2, \ldots,$ therefore it provides the minimum for (4). This motivates the following theorem called *The first Szego Theorem*:

Theorem 1. *Assume that the weight $w(z)$ is bounded from above and $w(z) = |D(z)|^2$ for some outer function D. Then*

(1) *We have $\beta_n = \|\Phi_n^*\|^2_{L^2(\mathbb{T}, w\frac{|dz|}{2\pi})} \to |D(0)|^2$ as $n \to +\infty$ and*
(2) *$\Phi_n^*(z) \to D(0)/D(z)$ as $n \to +\infty$ for any $|z| < 1$ uniformly on compacts of the unit disc,*

where the notation Φ_n^* was defined above.

10. Assume now that $D(0) = 1$ (this can be achieved just by a renormalization of the weight). Then Theorem 1 provides that $\beta_n \to 1$ and it is reasonable to ask if one

can compute the determinant of the Toeplitz operator $T(w)$, i.e. what is the limit $\lim\limits_{n\to+\infty}\prod\limits_{j=0}^{n}\beta_j$. The answer to this question is provided by *The second (Strong) Szego Theorem*:

Theorem 2. *Assume that $d/dz\log D$ is square integrable on the unit disc and let $\log D=\sum\limits_{n\geq 0}L_nz^n$. Then we have*

$$\prod_{n=0}^{+\infty}\beta_n=\det T(w)=\exp\left[\sum_{n\geq 1}n|L_n|^2\right]=\exp\left[\int\limits_{\{z\ :\ |z|<1\}}\frac{|D'(z)|}{|D(z)|}\frac{|dz|}{\pi}\right].$$

11. The aim of next paragraphs is to sketch the proof of this theorem that is due to Borodin and Okunkov. The $\det T(w)$ is the Fredholm determinant of the operator $T(w)$ (see computations below), so it is equal to $\lim\limits_{n\to+\infty}\det_{Q_nH^2}(Q_nT(w)Q_n)=\prod_{n=0}^{+\infty}\beta_n$. The fact that $\exp\left[\sum\limits_{n\geq 1}n|L_n|^2\right]=\exp\left[\int\limits_{\{z\ :\ |z|<1\}}\frac{|D'(z)|}{|D(z)|}\frac{|dz|}{\pi}\right]$ is a straightforward computation, thus it remains to show that $\det T(w)=\exp\left[\sum\limits_{n\geq 1}n|L_n|^2\right]$. In order to do this we compute $\det T(w)$ using a link between Toepletz and Hankel operators.

12. Given a function $f\in L^2(\mathbb{T},\frac{|dz|}{2\pi})$ let us introduce the Hankel operator $H(f):H^2\to H^2$ that acts by

$$(H(f)g)(z)=P_+(\overline{z}f\cdot g(\overline{z})). \tag{5}$$

We denote the function $g(\overline{z})$ by $\tilde{g}(z)$ for simplicity. If $f=\sum\limits_{n\in\mathbb{Z}}a_nz^n$ then in the basis $1,z,z^2,\ldots$ the operator $H(f)$ is given by the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots\\ a_2 & a_3 & a_4 & \cdots\\ a_3 & a_4 & a_5 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

Let us note that

$$T(fg)=T(f)T(g)+H(f)H(\tilde{g}). \tag{6}$$

Using this relation we conclude that if f is antiholomorphic *or* g is holomorphic then

$$T(fg)=T(f)T(g). \tag{7}$$

In particular, $T(f^{-1})=T(f)^{-1}$ and $T(g^{-1})=T(g)^{-1}$. We also have

$$H(\overline{\tilde{g}})=H(g)^*. \tag{8}$$

13. *The main trick appears in this paragraph.* Due to (7) we have $T(w)=T(\overline{D}D)=T(\overline{D})T(D)$. It follows that

$$T(D^{-1})T(w)T(\overline{D}^{-1})=\Big(T(D^{-1})T(\overline{D})\Big)\Big(T(D)T(\overline{D}^{-1})\Big)=\Big(T(b)T(\overline{b})\Big)^{-1} \tag{9}$$

where $b = \overline{D}^{-1}D$. Note that $T(D^{-1})$ and $T(\overline{D}^{-1})$ are both triangular in the basis $1, z, z^2, \dots$ with $D(0)^{-1} = \overline{D}^{-1}(0) = 1$ on the diagonal. It follows that

$$\begin{aligned}\det_{Q_nH^2}(Q_nT(w)Q_n) &= \det_{Q_nH^2}(Q_nT(D^{-1})T(w)T(\overline{D}^{-1})Q_n)\\ &= \det_{Q_nH^2}(Q_n\Big(T(b)T(\bar{b})\Big)^{-1}Q_n).\end{aligned} \tag{10}$$

Using that $b\bar{b} = 1$ and (6) we get that

$$T(b)T(\bar{b}) = 1 - H(b)H(b)^* \tag{11}$$

and thus due to the Jacobi identity the (10) can be rewritten as

$$\det_{Q_nH^2}(Q_nT(w)Q_n) = \frac{\det_{R_nH^2}(\mathbb{1} - R_n(H(b)H(b)^*)R_n)}{\det(\mathbb{1} - H(b)H(b)^*)}, \tag{12}$$

recall that $R_n = \mathbb{1} - Q_n$.

14. Recall that

$$\begin{aligned}\mathbb{1} - H(b)H(b)^* = T(b)T(\bar{b}) = T(\overline{D})T(D^{-1})T(\overline{D}^{-1})T(D) =\\ = T(\overline{D})T(D^{-1})T(\overline{D}^{-1})T(D) = [T(\overline{D}^{-1}), T(D)].\end{aligned}$$

Note that (7) implies that $T(D) = e^{T(\log D)}$ and $T(\overline{D}) = e^{T(\log \overline{D})}$. It follows that

$$\mathbb{1} - H(b)H(b)^* = [e^{T(\log D)}, e^{T(\log \overline{D})}]. \tag{13}$$

To compute the determinant of the righ-hand side we use the *Helton-Howe Theorem*:

Theorem 3. *Let A, B be bounded operators on a Hilbert space so that $[A, B]$ is of trace class. Then $[e^A, e^B] - \mathbb{1}$ is of trace class and*

$$\det[e^A, e^B] = \exp(\mathrm{Tr}[A, B]).$$

Using this theorem and (13) we can write

$$\det(\mathbb{1} - H(b)H(b)^*) = \exp(-\mathrm{Tr}[T(\log D), T(\log \overline{D})]).$$

Due to (7) we get

$$[T(\log D), T(\log \overline{D})] = H(\log D)H(\log D)^*$$

and a straightforward computation gives us $\mathrm{Tr}(H(\log D)H(\log D)^*) = \sum_{n\geq 0} n|L_n|^2$. By this compuatation and (11) we conclude that

$$\det(\mathbb{1} - H(b)H(b)^*) = \exp(-\sum_{n\geq 0} n|L_n|^2).$$

and finaly (12) implies that

$$\det_{Q_nH^2}(Q_nT(w)Q_n) = \det_{R_nH^2}(\mathbb{1} - R_n(H(b)H(b)^*)R_n) \cdot \exp(\sum_{n\geq 0} n|L_n|^2).$$

Passing to the limit as $n \to +\infty$ we get the desired relation

$$\det T(w) = \exp(\sum_{n\geq 0} n|L_n|^2).$$