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Nonlinear Sub-Optimal Control for Polynomial Systems – Design and Stability

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Abstract– *Many real world systems are inherently nonlinear. Therefore, the linear quadratic regulator theory is rarely efficient for these systems. In this paper, we propose the design of an optimal feedback control for polynomial systems in the indeterminate state variables. To deal with the case of a nonlinear infinite-horizon-cost-functional, we investigate the control based on the Lyapunov functions (LF) and by using the Kronecker product (KP) algebra. Then, we analyze the stability of the feedback and its domain of attraction (DA) in form of convex problems based on the linear matrix inequality (LMI) formalism. The practical sub-optimal control is evaluated through simulation results and comparative schemes.*

Keywords: Polynomial Systems, Matrix KP, Nonlinear State-Feedback, Stability, Sub-Optimal Control

1. Introduction

Numerous physical systems are very well known to be nonlinear by nature, but methods for analysing and synthesizing controllers for nonlinear systems are still not as well developed as their counterparts for linear models (Ekman, 2005). The investigation of new techniques for nonlinear problems such as the stability, the estimation and the control design remains a challenge until today (see *e.g.* (Zhu & Khayati, 2012; Zhu & Khayati, 2011; Won & Biswas, 2007; Khayati et *al.*, 2006, Ekman, 2005)). In particular, to deal with the nonlinear optimal control problem, it has been stated in (Khayati, 2013) and references cited therein that a great variety of works shown in the literature used simple techniques, based on the local linearization, and more complex ones, such as (but not limited to) the state-dependent-Riccati (SDR) equation, the nonlinear-matrix-inequality- and frozen-Riccati-equation-based methods (Won & Biswas, 2007; Huang & Lu, 1996; Banks & Mhana, 1992). These methods could work well in some applications but rigorous theoretical proofs were lacking (Won & Biswas, 2007). The related grey area nevertheless covers the stability analysis of these closed loop controllers and also their implementation (complexity of the algorithms) within a large set of plants. These concerns have been discussed in separate works with a lot of compromises to achieve their goals (Won & Biswas, 2007; Ekman, 2005; Banks & Mhana, 1992).

Recently, the KP algebra has shown an important role in research activities dealing with control analysis and design (Mtar et *al.*, 2009; Bouzaouche & Braik, 2006; Rotella & Tanguy, 1988). In these works, polynomial modelling structures represent the nonlinearities using the matrix KP and the vector power algebra (Steeb, 1997; Brewer, 1978). This modelling resembles the classical linearization, but with a difference. In fact, the order of truncation of the decomposition is high enough to represent closely and fairly the actual dynamics of the system.

In this paper, the optimal control for affine input nonlinear systems (*i.e.* linear *w.r.t.* the input but nonlinear in terms of the states (Rotella & Tanguy, 1988)) is considered. Such a large class contains well-known examples in control theory and many physical systems (*e.g.* mass-spring systems with softening/hardening springs, artificial pneumatic muscles, flight engine setups, *etc.*) (Chesi, 2009; Ekman, 2005; Banks & Mhana, 1992). The controller is developed using the well-known optimality conditions (Goh 1993; Borne et *al.*, 1990; Rotella & Tanguy, 1988) by converting the nonlinear SDR equation into a set of algebraic equations using the KP algebra (Steeb, 1997; Rotella & Tanguy, 1988). The proposed method is using the same technique developed in (Rotella & Tanguy, 1988), but with a main difference of considering a given quadratic form for the cost index functional allowing the analysis of the stability of the optimal state-feedback

(Goh, 1993). In fact, this analysis will show cases where the overall system will be globally asymptotically stable (GAS), or will estimate alternatively its DA and how much this domain can be large when the system is locally asymptotically stable (LAS) eventually. The stability and DA estimate features will be cast as convex problems that will be solved using LMI frameworks (Chesi, 2009; Chesi, 2005). Indeed, we will propose a technique that ensures the computation of the largest estimation of the domain of attraction (LEDA) using both the well-known complete square matrix representation (SMR) (Chesi, 2009; Chesi, 2003) and a new formalism of a complete rectangular matrix representation (RMR).

We will proceed as follows. In Section 2, we introduce a set of useful notations, definitions and properties regarding the matrix KP algebra, the vector power series and the SMR/RMR formulations. Section 3 is devoted to the problem statement of the nonlinear dynamics, the nonlinear quadratic cost functional to be optimized and the related optimality conditions. In Section 4, we introduce an LF-based optimal cost index that will be used in the transformation of the polynomial SDR equation. Then, Section 5 deals with the computation of a 'closely' acceptable solution to this nonlinear equation in the unknown constant matrices, while in Section 6, an analytic and practical form of the state-feedback sub-optimal control is developed. Section 7 introduces the stability issue of the designed sub-optimal closed-loop. Moreover, in Section 8, we discuss the computation of the LEDA of this closed loop system. Finally, to illustrate the proposed technique, numerical and comparative results are presented in Section 9, while Section 10 concludes this work.

2. Useful Notations, Definitions and Proprieties

Notations and properties of matrices, vectors, dot product and KP tensors used in this paper are exhaustively discussed in the literature; *e.g.* (Schott, 2001; Steeb, 1997; Brewer, 1978). The proofs of the new lemmas introduced in this Section are based on theorems introduced in these references. Due to lack of space, all these theorems as well as the proofs of the lemmas shown below are omitted.

2. 1. Definitions

Definition 1: For any vector $x \in \mathbb{R}^n$ and any integer j, $x^{[j]} \in \mathbb{R}^{n^j}$ is the j-power of a vector x and $\tilde{x}^{[j]} \in \mathbb{R}^{\tau_j^{(n)}}$ is the non-redundant j-power of the vector x with $\tau_j^{(n)}$ standing for the binomial coefficient. We have $\forall j \in \mathbb{N}$, $\exists! T_j \in \mathbb{R}^{n^j \times \tau_j^{(n)}}$ *s.t.* $x^{[j]} = T_j \tilde{x}^{[j]}$ (Mtar et *al.*, 2009; Brewer, 1978).

Definition 2: Let $w(x)$ be any homogenous form of degree $2j$, then the SMR of $w(x)$ in any $x \in \mathbb{R}^n$ is given by $w(x) = \tilde{x}^{[j]T} W \tilde{x}^{[j]}$ (Chesi, 2005; Chesi, 2003). $\tilde{x}^{[j]}$ is considered a base vector of the homogenous function of degree j in x. W is a suitable but non-unique symmetric matrix SMR, also known as Gram matrix. All matrices W can be linearly parameterized as $W(\beta) = W + L(\beta)$, where $\beta \in \mathbb{R}^{\sigma(n,j)}$ is a free vector with $\sigma(n,j) = \frac{1}{2}\tau_j^{(n)} \cdot \left(\tau_j^{(n)} + 1\right) - \tau_{2j}^{(n)}$. $L(\beta) \in \mathbb{R}^{\tau_j^{(n)} \times \tau_j^{(n)}}$ is a linear parameterization of the set $\left\{L = L^T \mid \tilde{x}^{[j]T} L \tilde{x}^{[j]} = 0, \forall x \in \mathbb{R}^n\right\}$. We refer to $W(\beta)$ as the complete SMR of $w(x)$.

Definition 3: Let $w(x)$ any form of degree $2j+1$ in $x \in \mathbb{R}^n$ given by $w(x) = v^T x^{[2j+1]} = x^{[2j+1]T} v$, where $v \in \mathbb{R}^{n^{2j+1}}$. Using theorem T2.13 of (Brewer, 1978), $w(x)$ can be written using a new formulation given by RMR as $w(x) = x^{[j]T} \cdot M \cdot x^{[j+1]} = x^{[j+1]T} \cdot N \cdot x^{[j]}$, with $M = mat^T_{n^j \times n^{j+1}}(v)$ and $N = mat^T_{n^{j+1} \times n^j}(v)$. Then, similarly to the homogenous forms of even order shown above, we propose a complete RMR of $w(x)$ as $\frac{1}{2}\tilde{x}^{[j]T}\left(M + L(\beta)\right)\tilde{x}^{[j+1]} + \frac{1}{2}\tilde{x}^{[j+1]}\left(M + L(\beta)\right)^T \tilde{x}^{[j]T}$, where β is a vector of free parameters. $L(\beta) \in \mathbb{R}^{\tau_j^{(n)} \times \tau_{j+1}^{(n)}}$ is a linear parameterization of the set $\left\{\tilde{x}^{[j]T} L \tilde{x}^{[j+1]} = 0, \forall x \in \mathbb{R}^n\right\}$. We refer to $M(\beta) = M + L(\beta)$ as the complete RMR of $w(x)$. The following two examples illustrate this new formulation.

Example 1: Consider the form of degree 3 in two variables $w(x) = x_1^3 + x_1^2 x_2 + x_2^3$. Noting $\tilde{x}^{[1]} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ and $\tilde{x}^{[2]} = \begin{pmatrix} x_1^2 & x_1 x_2 & x_2^2 \end{pmatrix}^T$, we obtain, for $\beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}^T \in \mathbb{R}^2$, $M + L(\beta) = \begin{pmatrix} 1 & 1+\beta_1 & \beta_2 \\ -\beta_1 & \beta_2 & 1 \end{pmatrix}$.

Example 2: Consider the form of degree 3 in three variables $w(x) = x_1^3 + x_1 x_2 x_3 + x_2^3 + x_2^2 x_3$. Noting $\tilde{x}^{[1]} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ and $\tilde{x}^{[2]} = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 & x_2^2 & x_2 x_3 & x_3^2 \end{pmatrix}$, we obtain, for $\beta = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \end{pmatrix}^T \in \mathbb{R}^7$, $M + L(\beta) =$

$$\begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ -\beta_1 & -\beta_3 & 1-2\beta_4 & 1 & \beta_6 & \beta_7 \\ -\beta_2 & \beta_4 & -\beta_5 & 1-\beta_6 & -\beta_7 & 0 \end{pmatrix}.$$

2. 2. Notations

Notation 1: If V is a vector of dimension $p = n \cdot m$, then $M = mat_{n \times m}(V)$ is the $(n \times m)$-matrix verifying $V = vec(M)$. Therefore it is called the mat notation.

Notation 2: M^+ stands for the Moore-Penrose pseudo-inverse of any full rank matrix M.

Notation 3: Given $x \in \mathbb{R}^n$, for any integer $p \geq 1$, we denote by $X_p = \begin{pmatrix} x^{|1|T} & x^{|2|T} & \cdots & x^{|p|T} \end{pmatrix}^T$ and $\tilde{X}_p = \begin{pmatrix} \tilde{x}^{|1|T} & \tilde{x}^{|2|T} & \cdots & \tilde{x}^{|p|T} \end{pmatrix}$. We have $X_p = \mathbf{T}_p \tilde{X}_p$ where $\mathbf{T}_p \in \mathbb{R}^{N_p \times \tau_p}$ is the direct sum of T_1, T_2, ..., T_p, denoted by $\mathbf{T}_p = \bigoplus_{i=1}^{p} T_p$, with $N_p = n + n^2 + \ldots + n^p$ and $\tau_p = \tau_1^{(n)} + \tau_2^{(n)} + \cdots + \tau_p^{(n)}$ (Halmos, 1974).

Notation 4: For any vector $x \in \mathbb{R}^n$ and integers p and μ, we denote by $\chi_p^{(\mu)} = \begin{pmatrix} 1 & x^{|p|T} & \cdots & x^{|(\mu-1)p|T} \end{pmatrix}^T \in \mathbb{R}^{1+n^p+n^{2p}+\cdots+n^{(\mu-1)p}}$.

2. 3. Lemmata

Lemma 1: $\forall j \in \mathbb{N} \setminus \{0\}$ and $\forall x \in \mathbb{R}^n$ (Khayati & Benabdelkader, 2012a),

$$\frac{\partial x^{|j|}}{\partial x^T} = \mathscr{D}_j^{(n)} \cdot \left(I_n \otimes x^{|j-1|} \right) \tag{1}$$

where $\mathscr{D}_j^{(n)} \in \mathbb{R}^{n^j \times n^j}$ is given by $\mathscr{D}_j^{(n)} = \sum_{i=0}^{j-1} U_{n^i \times n} \otimes I_{n^{j-i-1}}$ and therefore called the j-differential Kronecker matrix. I_n (*resp.* $I_{n^{j-i-1}}$) denotes the identity matrix of $\mathbb{R}^{n \times n}$ (*resp.* $\mathbb{R}^{n^{j-i-1} \times n^{j-i-1}}$), $U_{n^i \times n}$ the permutation matrix of $\mathbb{R}^{n^{i+1} \times n^{i+1}}$ (Rotella & Tanguy, 1988; Brewer, 1978). Equivalently, $\mathscr{D}_j^{(n)}$ can be derived from

$$\mathscr{D}_1^{(n)} = I_n \quad \text{and} \quad \mathscr{D}_{j+1}^{(n)} = \mathscr{D}_j^{(n)} \otimes I_n + U_{n^j \times n}, \forall j \geq 1 \tag{2}$$

Lemma 2: For x and y column-vectors of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively and for any matrix $A \in \mathbb{R}^{(nk) \times l}$, we have (Khayati & Benabdelkader, 2012a)

$$\left(I_n \otimes x^T \right) Ay = \left(I_n \otimes vec^T \left(A^T \right) \right) \left(vec(I_n) \otimes I_{kl} \right) \left(x \otimes y \right) \tag{3}$$

Lemma 3: Consider a matrix $A \in \mathbb{R}^{p \times nq}$. Let $\begin{bmatrix} A_1 & \ldots & A_n \end{bmatrix}$ be a partition of A, *i.e.* $\forall i = 1, \ldots, n$, $A_i \in \mathbb{R}^{p \times q}$. We have (Khayati & Benabdelkader, 2012a)

$$\left(I_n \otimes vec^T (A) \right) \left(vec(I_n) \otimes I_{pq} \right) = mat_{pq \times n}^T \left(vec(A) \right) \tag{4}$$

3. Problem Statement

Consider the nonlinear system given by

$$\dot{x}(t) = F(x) + G(x) \cdot u(t) = F(x) + \sum_{k=1}^{m} G_k(x) \cdot u_k(t) \tag{5}$$

where $t \in \mathbb{R}$ designates the time, $x(t) \in \mathbb{R}^n$ the state vector, $u(t) = \begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T \in \mathbb{R}^m$ the input vector. $F(\cdot)$ and $G_k(\cdot)$ for $k = 1, \ldots, m$ are analytic vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$ expressed as polynomials in x. Note that $G(x) = \begin{bmatrix} G_1(x) & \cdots & G_m(x) \end{bmatrix} \in \mathbb{R}^{n \times m}$. By using the KP tensor, we write $F(x) = \sum_{j=1}^{f} F_j \cdot x^{|j|}$, $\forall k = 1, \ldots, m$ $G_k(x) = \sum_{j=0}^{g} G_{kj} \cdot x^{|j|}$ and then, $G(x) = \sum_{j=0}^{g} G_j \left(I_m \otimes x^{|j|} \right)$, with $F_j \in \mathbb{R}^{n \times n^j}$, $G_{kj} \in \mathbb{R}^{n \times n^j}$ $\forall k = 1, \ldots, m$ and $G_j = \begin{bmatrix} G_{1j} & | \cdots | & G_{mj} \end{bmatrix} \in \mathbb{R}^{n \times mn^j}$. Let $z(t) = H(x) \in \mathbb{R}^q$ be a vector field in the state vector x given by $H(x) = \sum_{j=1}^{h} H_j \cdot x^{|j|}$ with $H_j \in \mathbb{R}^{q \times n^j}$ (Khayati & Benabdelkader, 2012a; Rotella & Tanguy, 1988).

For Q a symmetric non-negative definite matrix of $\mathbb{R}^{q \times q}$ and R a symmetric positive definite (SPD) matrix of $\mathbb{R}^{m \times m}$, we propose the design of a state feedback which minimizes the continuous-time cost functional

$$J = \frac{1}{2} \int_0^\infty \left[z(t)^T Qz(t) + u(t)^T Ru(t) \right] dt \tag{6}$$

We denote by $V(x)$ the optimal cost with an initial condition x at t (Goh, 1993; Borne et *al.*, 1990)

$$V(x) = \frac{1}{2} \int_t^\infty \left[z(\tau)^T Qz(\tau) + u^*(\tau)^T Ru^*(\tau) \right] d\tau \tag{7}$$

where $u^* = \arg\left(\min_u J \right)$ is the optimal control. The optimality conditions, corresponding to the problem (5) and (6), are given by (Borne et *al.*, 1990)

$$u^*(x) = -R^{-1} G(x)^T V_x(x) \tag{8}$$

$$H(x)^T QH(x) + V_x(x)^T F(x) + F(x)^T V_x(x) - V_x(x)^T G(x) \cdot V_x(x)^T G(x) R^{-1} G(x)^T V_x(x) = 0 \tag{9}$$

where $V_x(x)$ denotes the derivative of $V(x)$ *w.r.t.* the state vector x; *i.e.* $V_x(x) = \frac{\partial V}{\partial x}$.

4. Quadratic Cost Function Representation

Based on the optimality conditions discussed in (Borne et *al.*, 1990; Rotella & Tanguy, 1988), we build the following procedure to obtain a suboptimal state feedback in a polynomial form using the KP tensor, *vec* and *mat* notations (Khayati & Benabdelkader, 2012a). Such a design is based on the determination of the cost function $V(x)$ in a quadratic form. In fact, this function would be expected to satisfy the conditions of any Lyapunov candidate function (Goh, 1993). We propose (Khayati & Benabdelkader, 2012a)

$$V(x)=\frac{1}{2}\begin{pmatrix} x^T & \sum_{j=2}^{\bar{p}} x^{|j|T}\cdot P_j^T \end{pmatrix}\begin{pmatrix} P & \alpha I_n \\ \alpha I_n & I_n \end{pmatrix}\begin{pmatrix} x \\ \sum_{j=2}^{\bar{p}} P_j \cdot x^{|j|} \end{pmatrix} \tag{10}$$

with $\alpha\in\mathbb{R}$, P is an SPD constant matrix of $\mathbb{R}^{n\times n}$ and P_j constant matrices of $\mathbb{R}^{n\times n^j}$. Note that $V(x)$ can be expressed in a compact form

$$V(x)=\frac{1}{2}X_{\bar{p}}^T\mathbf{P}X_{\bar{p}} \tag{11}$$

where

$$\mathbf{P}=\begin{pmatrix} P & \alpha P_2 & \cdots & \alpha P_{\bar{p}} \\ \alpha P_2^T & P_2^T P_2 & \cdots & P_2^T P_{\bar{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha P_{\bar{p}}^T & P_{\bar{p}}^T P_2 & \cdots & P_{\bar{p}}^T P_{\bar{p}} \end{pmatrix} \tag{12}$$

And equivalently, by using the Cholesky decomposition, P_1 exists *s.t.* $P=P_1^TP_1$, then the cost function $V(x)$ can be rewritten in a summation form as

$$V(x)=\frac{1}{2}\sum_{i,j=1}^{\bar{p}} x^{|i|T}P_{i(j)}^T P_{j(i)}x^{|j|} \tag{13}$$

with

$$P_{i(j)}=\begin{cases} P_1 & \text{for} \quad i=j=1 \\ \alpha I_n & \text{for} \quad i=1 \text{ and } j\geq 2 \\ P_i & \text{for} \quad i\geq 2 \text{ and } j\geq 1 \end{cases} \tag{14}$$

The expression of $V(x)$ given by (13) and (14) will be advantageous to solve the nonlinear SDR (9). Using theorems T2.3 and T4.3 in (Brewer, 1978) and applying lemmas 1, 2 and 3 and the *mat* notation, introduced in Section 2, we obtain the derivative of (13) *w.r.t.* x

$$V_x(x)=\sum_{i,j=1}^{\bar{p}}\frac{\partial x^{|j|T}}{\partial x}P_{j(i)}^T P_{i(j)}x^{|i|}=\sum_{i,j=1}^{\bar{p}}V_{ij}x^{|i+j-1|} \tag{15}$$

with

$$V_{ij}=mat_{n^{i+j-1}\times n}^T\left[vec\left(P_{i(j)}^T P_{j(i)}\mathscr{D}_j^{(n)}\right)\right] \tag{16}$$

where $\mathscr{D}_j^{(n)}$ is the square j-differential Kronecker matrix of $\mathbb{R}^{n^j\times n^j}$ introduced in lemma 1 (see Section 2). Using the KP tensor, the theorem T2.13 of (Brewer, 1978), the lemmas 2 and 3, and the *mat* notation, introduced in Section 2, we obtain from the nonlinear SDR equation (9)

$$\begin{aligned}&\sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f}vec^T\left(V_{ij}^TF_k\right)x^{|i+j+k-1|}+\sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f}vec^T\left(F_k^{\,T}V_{ij}\right)x^{|i+j+k-1|}+\\&\sum_{i,j=1}^{h}vec^T\left(H_i^TQH_j\right)x^{|i+j|}-\sum_{i,j,b,c=1}^{\bar{p}}\sum_{k,d=0}^{g}vec^T\left(W_{ijk}^TR^{-1}W_{bcd}\right)\cdot\\&\qquad x^{|i+j+k+b+c+d-2|}=0\end{aligned} \tag{17}$$

where

$$W_{ijk}=mat_{n^{i+j+k-1}\times m}^T\left[vec\left(V_{ij}^TG_k\right)\right] \tag{18}$$

5. Determination of P_p

In this Section, the matrices P_p, for $p=1,\ldots,\bar{p}$, will be computed from (17) by cancelling the coefficients of $x^{|p+1|}$. The details of such steps, based on the KP notations and theorems introduced in (Steeb, 1997; Brewer, 1978) as well as the lemmas 1, 2 and 3 shown in Section 2, are omitted due to lack of space.

First, the matrix P_1 is obtained by cancelling the terms of $x^{|2|}$, in (17). The operator $vec(\cdot)$ is linear on matrices of the same dimensions. Noting that the first differential Kronecker matrix is given by $\mathscr{D}_1^{(n)}=I_n$ and that $P=P_1^TP_1$ is SPD, we use (14), (16), (18) and the *mat* notation to obtain the classical algebraic Riccati equation (ARE)

$$PF_1+F_1^TP+H_1^TQH_1-PG_0R^{-1}G_0^TP=0 \tag{19}$$

And thence, for a given $\alpha\in\mathbb{R}$, the calculation of P_p, $p=2,\ldots,\bar{p}$, is obtained from (17) by cancelling the coefficients of $x^{|p+1|}$. Using *vec* and *mat* notations, theorems T1.5, T1.6, T3.2, T3.4 of (Brewer, 1978) and the iterative form of the differential Kronecker matrix (2), we combine (14), (16) and (18) to obtain

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{T}_p^{\;T}\mathscr{D}_{p+1}^{(n)T}\alpha\cdot vec\left(P_p\right)=\mathscr{Y}_p \tag{20}$$

where $\mathscr{T}_p=\left(F_1-G_0R^{-1}G_0^TP\right)\otimes I_{n^p}$ and $\mathscr{H}_p=\sum_{\substack{i,j,b,c=1\\ i+j+k+b+c+d=p+3}}^{p-1}\sum_{k,d=0}^{p-1} vec\left(W_{ijk}^TR^{-1}W_{bcd}\right)-\sum_{\substack{i,j=1\\ i+j+k=p+2}}^{p-1}\sum_{k=1}^{p}\left[vec\left(V_{ij}^TF_k\right)+vec\left(F_k^TV_{ij}\right)\right]-\sum_{\substack{i,j=1\\ i+j=p+1}}^{p} vec\left(H_i^TQH_j\right)$. Note that $\left(F_1-G_0R^{-1}G_0^TP\right)$ is a Hurwitz matrix, then $\mathscr{T}_p$ is regular for all $p\in\mathbb{N}$. $\mathscr{D}_{p+1}^{(n)}$ is a singular matrix for all nonzero integers p and $\left(I_{n^{p+1}}+U_{n^p\times n}\right)$ is regular for p even and singular for p odd (Khayati & Benabdelkader, 2012a; Rotella & Tanguy, 1988). Using the non-redundant vector power notation (Bouzaouche & Braiek, 2006), and the theorem T3.4 of (Brewer, 1978), we write $\tilde{P}_p=P_p\cdot T_p$ where $T_p\in\mathbb{R}^{n^p\times\tau_p^{(n)}}$ is the transformation matrix defined in Section 2 (Bouzaouche & Braiek, 2006). Two cases arise depending on p:

Case I – p is even: Let $\mathscr{T}_p=\left(T_p^+\otimes I_n\right)\mathscr{D}_{p+1}^{(n)}$ be a full rank rectangular $\left(\left(n\cdot\tau_p^{(n)}\right)\times n^{p+1}\right)$ matrix. We obtain

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{T}_p^{-T}\mathscr{T}_p^{T}\alpha\cdot vec\left(\tilde{P}_p\right)=\mathscr{H}_p \tag{21}$$

If P, P_2, ..., P_{p-1} are known, $\tilde{P}_p$ can be calculated as a solution of the linear equation (21). Thus, $P_p=\tilde{P}_pT_p^+$ is deduced. In fact, by using $\mathscr{T}_p^{+}=\mathscr{T}_p^{T}\left(\mathscr{T}_p\mathscr{T}_p^{T}\right)^{-1}$ the Moore-Penrose pseudo-inverse of $\mathscr{T}_p$, we obtain

$$vec\left(\tilde{P}_p\right)=\frac{1}{\alpha}\mathscr{T}_p^{+T}\mathscr{T}_p^{-T}\left(I_{n^{p+1}}+U_{n^p\times n}\right)^{-1}\mathscr{H}_p \tag{22}$$

Case II – p is odd: Eq. (17) is rewritten using the non-redundant power series. Then, the coefficients of $\tilde{x}^{|p+1|}$ are given in (20), but multiplied by T_{p+1}^T on the left hand side. Thus, this linear equation becomes

$$\tilde{\mathscr{T}}_p^{T}\cdot vec\left(\tilde{P}_p\right)=\tilde{\mathscr{H}}_p \tag{23}$$

where $\tilde{\mathscr{T}}_p=\alpha\mathscr{T}_p\mathscr{T}_p\left(I_{n^{p+1}}+U_{n\times n^p}\right)T_{p+1}$ is a full rank rectangular $\left(\left(n\cdot\tau_p^{(n)}\right)\times\tau_{p+1}^{(n)}\right)$ matrix and $\tilde{\mathscr{H}}_p=T_{p+1}^T\cdot\mathscr{H}_p$. If P, P_2, ..., P_{p-1} are known, $\tilde{P}_p$ can be calculated as a solution of the linear equation system (23). Thus, $P_p=\tilde{P}_pT_p^+$ is deduced. In fact, by using the Moore Penrose pseudo-inverse of $\tilde{\mathscr{T}}_p$, denoted by $\tilde{\mathscr{T}}_p^{+}=\left(\tilde{\mathscr{T}}_p^{T}\tilde{\mathscr{T}}_p\right)^{-1}\tilde{\mathscr{T}}_p^{T}$, we obtain

$$vec\left(\tilde{P}_p\right)=\tilde{\mathscr{T}}_p^{+T}\cdot\tilde{\mathscr{H}}_p \tag{24}$$

6. Implementation of the State Feedback

Consider the nonlinear dynamics (5). The optimal control minimizing the functional cost (6) is obtained by the optimality conditions (8) and (9). We propose the design of a practical sub-optimal control using the matrices P, P_2, ..., $P_{\bar{p}}$ computed in Section 5. It is based on an approximated optimal cost $V(x)$ given by (10). An analytical form of the state feedback can be obtained by using (8), (15), (16) and (18) (Khayati & Benabdelkader, 2012a)

$$\bar{u}(x)=-\sum_{p=1}^{\bar{p}_g}K_px^{|p|} \tag{25}$$

with $\bar{p}_g=2\bar{p}+g-1$ and

$$K_p=R^{-1}\cdot\sum_{\substack{i,j=1\\ i+j+k=p+1}}^{p}\sum_{k=0}^{g}W_{ijk} \tag{26}$$

The KP tensor is used here to design a systematic computation of a sub-optimal state-feedback. The proposed nonlinear feedback (25) with (26) would not necessarily be implemented with a great number of computed matrices P_p to be so different from the linear control approximation, *a priori*. According to (Rotella & Tanguy, 1988), it can be concluded that the state-feedback obtained with only P (*i.e.*, only the first order of the SDR equation) is more efficient than the solution issued from the linearized system. In fact, by computing only P, we may obtain a polynomial sub-optimal control of order $g+1$ (where g is the order of the term $G(x)$ in (5)), in particular, when g is non-zero. The stability of the proposed closed-loop feedback (5) and (25) will be discussed in the following section.

7. Stability of the Sub-Optimal State Feedback

To investigate the stability of the closed loop system, we consider $V(x)$, given by (10), as a Lyapunov candidate function. $V(x)$ is a radially unbounded continuous function, and its derivative exists and is continuous. From (10), if

$$\begin{pmatrix} P & \alpha I_n\\ \alpha I_n & I_n\end{pmatrix}>0 \tag{27}$$

holds, then the Lyapunov candidate function $V(x)$ is positive definite; that is $V(x)>0$, $\forall x \neq 0$. Note that (27) is equivalent to $P>\alpha^2 I_n$. The time derivative of the LF $V(x)$, along the trajectories of the closed loop system (5) and (25), is given by

$$\dot{V}(x)=V(x)^T \cdot \dot{x}(t)=V(x)^T F(x)-\bar{u}(x)^T R\bar{u}(x) \tag{28}$$

Let us define B_1 and C_1 by $G_0 R^{-1} G_0^T$ and $H_1^T Q H_1$, respectively. We assume the triplet (F_1, B_1, C_1) is stabilizable-detectable. Note that if a solution P of the ARE (19) exists, then it is the unique SPD matrix solution of the optimal control for the linearized system and $(F_1 - B_1 P)$ is a Hurwitz matrix (Rotella & Tunguy, 1988). Thus, the linearized system is asymptotically stable. Moreover, the nonlinear closed loop system (5) and (25) is LAS and $\exists x \neq 0$ *s.t.* $\dfrac{\partial(x^T P x)}{\partial t}<0$.

In the following, we assume $\{x \in \mathbb{R}^n \setminus \{0\} \mid \dot{V}<0\} \neq \varnothing$ and consider the closed ball $\mathcal{B}(\delta)=\{x \in \mathbb{R}^n \mid \|x\| \leq \delta\}$. Given α *s.t.* $P>\alpha^2 I_n$; *i.e.* $V(x)>0$ for all nonzero $x \in \mathbb{R}^n$, $\mathcal{B}(\delta)$ is an estimate of the DA if $\mathcal{B}(\delta) \subset \Delta=\{x \in \mathbb{R}^n \mid \dot{V}<0\} \cup \{0\}$ (Chesi, 2009; Chesi, 2003). The computation of the maximum δ *s.t.* $\mathcal{B}(\delta) \subset \Delta$, *i.e.* (5) and (25) is LAS, corresponds to the LEDA of the closed-loop dynamics and is given by $\mathcal{B}(\gamma)$ where (Chesi, 2009)

$$\gamma=\inf_{x \in \mathbb{R}^n \setminus \{0\} \ s.t.\ \dot{V}(x)=0} \|x\| \tag{29}$$

8. LEDA Computation of the Closed Loop System

In this section, we present the mechanism to evaluate the LEDA of the obtained sub-optimal closed-loop system. Let $\mathcal{S}(\delta)=\{x \in \mathbb{R}^n \mid \|x\|=\delta\}$ be a given sphere. The problem (29) turns out that (Chesi, 2003)

$$\gamma=\sup\left\{\bar{\delta} \mid \dot{V}(x)<0, \forall x \in \mathcal{S}(\delta), \forall \delta \in \left(0,\bar{\delta}\right]\right\} \tag{30}$$

We assume that P, P_2, ..., $P_{\bar{p}}$ are obtained from (19), (21) and (23). The terms $V(x)^T F(x)$ and $\bar{u}(x)^T R\bar{u}(x)$ are polynomials in x of degrees $2\bar{p}+f-1$ and $2\bar{p}_g$, respectively. For any $\delta>0$, we have $\forall x \in \mathcal{S}(\delta)$

$$\dot{V}(x)=\sum_{\substack{k=1\\k+l=p}}^{2\bar{p}-1}\sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|}-\sum_{\substack{i,j=1\\i+j=p}}^{\bar{p}_g} x^{|i|T} K_i^T R K_j x^{|j|} \tag{31}$$

with $v_k=\sum_{\substack{i,j=1\\i+j-1=k}}^{\bar{p}} V_{ij}$, where V_{ij} is given by (16). Using the non-redundant vector power series $\tilde{x}^{|p|}$ and the vector notations $X_{\bar{p}}$ introduced in Section 2, without loss of generality, we assume that $\exists \bar{p}_t \in \mathbb{N}$, with $0<\bar{p}_t \leq \bar{p}_g$, and $\exists \Gamma_t=\Gamma_t^T>0$ *s.t.* $\sum_{i,j=1}^{\bar{p}_g} x^{|i|T} K_i^T R K_j x^{|j|}=X_{\bar{p}_t}^T \Gamma_t X_{\bar{p}_t}+w_e(x)+w_o(x)$. The terms $w_e(x)$ and $w_o(x)$ are polynomials in even and odd vector powers in x of orders $2\bar{p}_e$ and $2\bar{p}_o+1$, respectively. $\bar{p}_e$ and $\bar{p}_o$ are integers *s.t.* $0 \leq \bar{p}_e \leq \bar{p}_g$ and $0 \leq \bar{p}_o \leq \bar{p}_g$. Then, we use the SMR and RMR notations introduced in Section 2 to set the time derivative of the LF, $\dot{V}(x)$, in a quadratic form. If we denote by $\bar{p}_q=\max\left(\bar{p}_e, \bar{p}+\dfrac{f-1}{2}\right)$ and $\bar{p}_r=\max\left(\bar{p}_o, \bar{p}+\dfrac{f-3}{2}\right)$ for f odd, and $\bar{p}_q=\max\left(\bar{p}_e, \bar{p}+\dfrac{f}{2}-1\right)$ and $\bar{p}_r=\max\left(\bar{p}_o, \bar{p}+\dfrac{f}{2}-1\right)$ for f even, respectively. We obtain

$$\begin{aligned}\dot{V}(x)=&-\sum_{i=1}^{\bar{p}_q} \tilde{x}^{|i|T} S_{ii}\left(\beta_i^{(e)}\right)\tilde{x}^{|i|}-\sum_{i=1}^{\bar{p}_r}\Big[\tilde{x}^{|i|T} S_{i,i+1}\left(\beta_i^{(o)}\right)\tilde{x}^{|i+1|}+\tilde{x}^{|i+1|T} \cdot \\ &S_{i,i+1}^T\left(\beta_i^{(o)}\right)\tilde{x}^{|i|}\Big]-X_{\bar{p}_t}^T \Gamma_t X_{\bar{p}_t}\end{aligned} \tag{32}$$

where $S_{ii}\left(\beta_i^{(e)}\right) \in \mathbb{R}^{\tau_i^{(n)} \times \tau_i^{(n)}}$ is the SMR matrix of the terms of order $2i$ in $\sum_{k=1}^{2\bar{p}-1}\sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|}+w_e(x)$, $\beta_i^{(e)} \in \mathbb{R}^{\sigma(n,i)}$ a free vector with $\sigma(n,i)=\dfrac{1}{2}\tau_i^{(n)}.\left(\tau_i^{(n)}+1\right)-\tau_{2i}^{(n)}$ and $\tau_i^{(n)}$ stands for binomial coefficients (Mtar et *al.*, 2009), and $S_{i,i+1}\left(\beta_i^{(o)}\right)$ is the RMR of terms of order $2i+1$ in $\dfrac{1}{2}\sum_{k=1}^{2\bar{p}-1}\sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|}+\dfrac{1}{2}w_o(x)$ (see Section 2). (32) can be rewritten as follows

$$\dot{V}(x)=-\tilde{X}_{\bar{p}_s}^T \mathbf{S}(\beta)\tilde{X}_{\bar{p}_s}-X_{\bar{p}_t}^T \Gamma_t X_{\bar{p}_t} \tag{33}$$

where $\bar{p}_s=\max\left(\bar{p}_q, \bar{p}_r\right)$ and

$$\mathbf{S}(\beta)=\begin{pmatrix} S_{11} & S_{12} & 0 & 0 & \ldots & 0 \\ S_{12}^T & S_{22} & S_{23} & 0 & \ldots & 0 \\ 0 & S_{23}^T & S_{33} & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & & 0 & & \end{pmatrix} \tag{34}$$

The decision variables β are set by the concatenation of all free variables $\beta_i^{(e)}$ and $\beta_i^{(o)}$, $\forall i$. Two cases arise depending on the size of the values of $\bar{p}_s$ and $\bar{p}_t$ in (33).

8. 1. Case of $\bar{p}_s = \bar{p}_t$

Using the transformation $X_{\bar{p}_t} = \mathbf{T}_{\bar{p}_t} \tilde{X}_{\bar{p}_t}$ introduced in Section 2, we have $\dot{V}(x) < 0$ if the LMI

$$\mathbf{S}(\beta) + \mathbf{T}_{\bar{p}_t}^T \Gamma_t \mathbf{T}_{\bar{p}_t} > 0 \tag{35}$$

holds in the free decision variable β. Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), if the LMI (35) problem is feasible in β, then the sub-optimal state-feedback (5) and (25) is GAS.

8. 2. Case of $\bar{p}_s \neq \bar{p}_t$

Let ν be the least common multiple of $\bar{p}_s$ and $\bar{p}_t$, *i.e.* $\exists(\nu_s, \nu_t) \in \mathbb{N}^2 \setminus \{(0,0)\}$ *s.t.* $\nu = \nu_s p_s = \nu_t p_t$. Consider the well-posed vectors $\chi_{p_s}^{(\nu_s)}$ and $\chi_{p_t}^{(\nu_t)}$ introduced in Section 2. Noting $\forall x \in \mathbb{S}(\delta)$, $\|x\| = \delta$, then we have $\left\|\chi_{\bar{p}_s}^{(\nu_s)}\right\|^2 = \sum_{i=0}^{\nu_s - 1} \delta^{2i\bar{p}_s} = \delta_s^2$, $\left\|\chi_{\bar{p}_t}^{(\nu_t)}\right\|^2 = \sum_{i=0}^{\nu_t - 1} \delta^{2i\bar{p}_t} = \delta_t^2$ and $\chi_{\bar{p}_s}^{(\nu_s)} \otimes X_{\bar{p}_s} = \chi_{\bar{p}_t}^{(\nu_t)} \otimes X_{\bar{p}_t} = X_\nu$. Thus, (33) is equivalent to

$$\dot{V}(x) = -X_\nu^T \left(\frac{1}{\delta_s^2} I_{\xi_s} \otimes \mathbf{S}^+(\beta) + \frac{1}{\delta_t^2} I_{\xi_t} \otimes \Gamma_t \right) X_\nu \tag{36}$$

with $\mathbf{S}^+(\beta) = \mathbf{T}_{\bar{p}_s}^{+T} \mathbf{S}(\beta) \mathbf{T}_{\bar{p}_s}^+$. $\mathbf{T}_{\bar{p}_s}^+$ is the pseudo-inverse of $\mathbf{T}_{\bar{p}_s} \in \mathbb{R}^{N_{\bar{p}_s} \times \tau_{\bar{p}_s}}$ introduced in Section 2, $\xi_s = 1 + n^{\bar{p}_s} + n^{2\bar{p}_s} + \cdots + n^{(\nu_s - 1)\bar{p}_s}$, $\xi_t = 1 + n^{\bar{p}_t} + n^{2\bar{p}_t} + \cdots + n^{(\nu_t - 1)\bar{p}_t}$. Noting that Γ_t is SPD, let $\mathbf{T}_{\bar{p}_t}^{+T} \Gamma_t \mathbf{T}_{\bar{p}_t}^+ = \bar{\Theta}^T \bar{\Theta}$ be the Cholesky decomposition. Then, from (36), $\forall x \in \mathbb{S}(\delta)$, $\dot{V}(x) < 0$ is equivalent to

$$\left(I_{\xi_t} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_s} \otimes \mathbf{S}^+(\beta)\right)\left(I_{\xi_t} \otimes \bar{\Theta}^{-1}\right) + \bar{\delta} \cdot I_{\xi_t \cdot N_{\bar{p}_t}} > 0 \tag{37}$$

where the factor $\bar{\delta} = \delta_s^2 / \delta_t^2$ depends on δ. If $\nu_s > \nu_t \Leftrightarrow \bar{p}_s < \bar{p}_t$, then $\bar{\delta} > 1$ and monotically increasing with δ. If $\nu_s < \nu_t \Leftrightarrow \bar{p}_s > \bar{p}_t$, then $\bar{\delta} < 1$ and monotically decreasing with δ. The following results hold.

Sub-case $\bar{p}_s < \bar{p}_t$*:* $\forall \beta$, $\exists \bar{\delta} > 1$ *s.t.* the LMI (37) holds. Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), the sub-optimal state-feedback system (5) and (25) is GAS.

Sub-case $\bar{p}_s > \bar{p}_t$*:* Given $\upsilon \in \mathbb{R}$, consider the LMI

$$\left(I_{\xi_t} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_s} \otimes \mathbf{S}^+(\beta)\right)\left(I_{\xi_t} \otimes \bar{\Theta}^{-1}\right) - \upsilon I_{\xi_t \cdot N_{\bar{p}_t}} > 0 \tag{38}$$

in the vector β and the scalar υ. If $\exists \upsilon \geq 0$ *s.t.* the LMI (38) holds, then the LMI constraint (37) holds $\forall \bar{\delta} > 0$, then we select $\bar{\delta} < 1$ and we have $\bar{\delta}$ decreasing with δ (*i.e.* $\delta \to \infty$ as $\bar{\delta} \to 0$). Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), if the LMI (38) is feasible in $\upsilon \geq 0$ and β, then the sub-optimal state-feedback system (5) and (9) is GAS.

Sub-case $\bar{p}_s > \bar{p}_t$ *and* $\exists \upsilon$ *s.t.* $-1 < \upsilon < 0$*:* A lower bound $\bar{\gamma}$ of γ, given by (30), is computed by $\bar{\gamma} = \arg_\delta \dfrac{1 + \delta^{2\bar{p}_s} + \cdots + \delta^{2(\nu_s - 1)\bar{p}_s}}{1 + \delta^{2\bar{p}_t} + \cdots + \delta^{2(\nu_t - 1)\bar{p}_t}} = (-\bar{\upsilon})$, where $\bar{\upsilon}$ is a solution of the following eigen-value problem (EVP): $\bar{\upsilon} = \max \upsilon$ subject to $-1 < \upsilon < 0$ and LMI (38). If $\arg\max_\upsilon$ of this EVP is negative, then the linear inequality constraint $-1 < \upsilon < 0$ corresponds to $\bar{\delta} < 1$ as $\bar{p}_s > \bar{p}_t$.

Remark: The results discussed above can be proven using simply the theorem 1 of (Chesi, 2003) and the proposition 2 of (Chesi, 2005).

9. Example

As an example, we consider the design of a nonlinear aircraft flight control problem which has been exhaustively treated in literature (see *e.g.* (Banks & Mhana, 1992)) and defined by

$$\begin{aligned}
\dot{x}_1 &= -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 + \\
&\quad -0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u + \\
&\quad 6.265x_1^2u + 46x_1u^2 + 61.4u^3
\end{aligned}$$

where x_1 is the angle of attack in rad, x_2 the pitch angle in rad, x_3 the pitch rate in rad/sec and u the control input provided by the tail deflection angle in rad (Banks & Mhana, 1992). Note that terms involving nonlinearities in u with small effect on the dynamics are eliminated, as the approaches discussed here cannot account for nonlinear control terms, but are taken into consideration in the simulations. The performance index uses $H(x) = x$, $Q =$ $0.25 \cdot I_3$ and $R = 1$. The simulations have been applied for the proposed 'LF'-based technique as well as the linear control 'Lin' where the dynamics is linearized about the origin, the 'KP'-based design introduced in (Rotella & Tanguy, 1988) and the SDR-equation-pointwise-based (referred to as 'PW') technique (Banks & Mhana, 1992). The sub-optimal cost J is evaluated with different initial conditions in terms of angle of attack, $x_1(0)$, and same $x_2(0)=x_3(0)=0$ for the different methods. Table 1 shows the cost performance errors $\varepsilon_J^{\bullet} = \dfrac{J_{\text{pw}} - J_{\bullet}}{J_{\text{pw}}}$ in $\%$. The 'LF'- (of orders 2 and 3), 'KP'- (of orders 2 and 3) and 'Lin'-based design costs are compared to the 'PW'-technique one. A positive value corresponds to an improvement (*i.e.*, a lower cost) with the given method compared to the 'PW' cost, meanwhile a negative value corresponds to a higher cost. Figures 1-3 show the control variable, the angle of attack and the pitch angle, respectively, obtained with the initial condition $x_1(0)=23^{\circ}$. Due to lack of space the pitch rate figure is omitted. Curves of 'LF'-based design, with orders of truncation 2 and 3, overlap almost during all the time showing very similar results in terms of transient behaviour and stability. Furthermore, the proposed design (with both orders 2 and 3 which are relatively small) exhibits a significant added-value in terms of cost estimation and domain of attraction interval performances compared to the other methods.

Table 1. Cost index J^{PW} and cost errors (expressed in $\%$ of J^{PW}) $\varepsilon_{J(p=2)}^{\text{LF}}$, $\varepsilon_{J(p=3)}^{\text{LF}}$, $\varepsilon_{J(p=2)}^{\text{KP}}$, $\varepsilon_{J(p=3)}^{\text{KP}}$, $\varepsilon_{J}^{\text{Lin}}$.

$x_1(0)$	J^{PW}	$\varepsilon_{J(p=2)}^{\text{LF}}$	$\varepsilon_{J(p=3)}^{\text{LF}}$	$\varepsilon_{J(p=2)}^{\text{KP}}$	$\varepsilon_{J(p=3)}^{\text{KP}}$	$\varepsilon_{J}^{\text{Lin}}$
6°	0.0016	**20.2**	**18.6**	-0.6	-0.8	0.0
12°	0.0071	**23.8**	**22.8**	-1.6	-2.6	-0.2
17°	0.0196	**30.9**	**30.3**	-3.7	-6.8	-0.7
23°	0.0519	**46.3**	**45.7**	-13.3	-31.7	-4.3
29°	0.1056	**48.3**	**46.3**	Unstab.	Unstab.	Unstab.
34°	0.4081	**71.4**	**65.6**	Unstab.	Unstab.	Unstab.
40°	1.6170	**58.5**	**50.9**	Unstab.	Unstab.	Unstab.

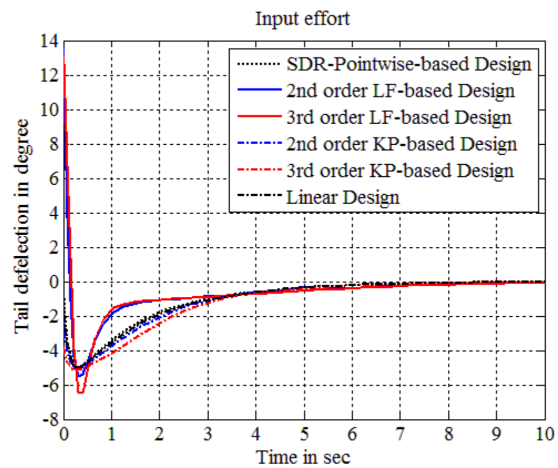

Fig. 1. Input control *vs.* time.

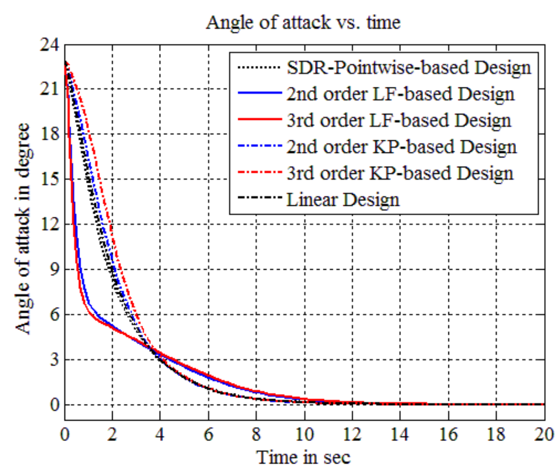

Fig. 2. Angle of Attack *vs.* time

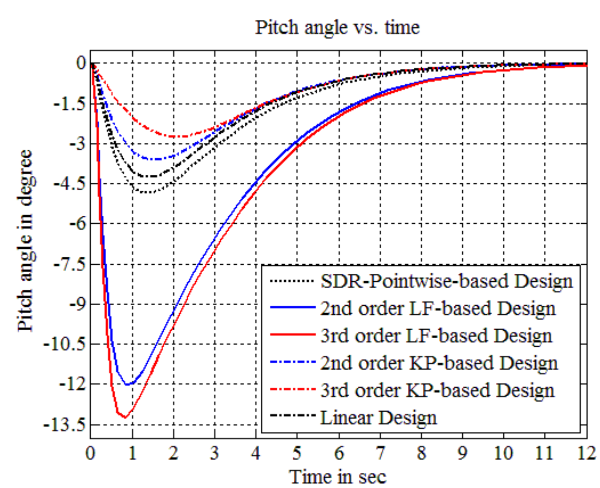

Fig. 3. Pitch Angle *vs.* time

10. Conclusions

A new nonlinear optimal control design for polynomial systems subject to nonlinear cost objectives is proposed. We develop a systematic and practical LF-based sub-optimal control approach using the KP notations. The analysis of the stability of the closed loop system is then discussed using LMI frameworks. The problem of the LEDA computation is cast as a convex EVP design. This method is expected to ensure a best compromise between the feasibility of the implemented scheme and the stability analysis of the overall system. An example showing simulations and comparative results successfully demonstrates the effectiveness of this technique. Furthermore, a modified version of this nonlinear optimal control will be presented to relax the conditions within the computation of the Lyapunov function matrices of high order, and also, improving the formulation of the stability feature (Khayati, 2013). Nevertheless, all those changes will be proposed by following the same overall procedure discussed in this paper.

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