Midterm 2 Exam

15-122 Principles of Imperative Computation

Tuesday 15th November, 2022

Name: ______________________________

Andrew ID: ______________________________

Recitation Section: ______________________________

Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 23 pages (including 2 blank pages at the end).
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
Reducing a Tree	20	
Heterogeneous Data Structures	25	
Tree Sort	40	
Scanning Hash Tables	40	
Total:	125	

1 Reducing a Tree (20 points)

Consider binary trees with integer data and the function `sum` that adds up all the data in its input tree:

```
typedef elem int;

typedef struct tree_node tree;
struct tree_node {
  tree* left;
  elem data;
  tree* right;
};
```

```
1 int sum(tree* T) {
2   if (T == NULL) return 0;
3
4   return    sum(T->left)
5           + T->data
6           + sum(T->right);
7 }
```

Many functions on binary trees work similarly to `sum`:

- they return some value (here `0`) when the tree is empty (line 2), and
- if the tree is not empty, they make two recursive calls to the left and right subtrees (lines 4 and 6), combine the returned values with the data at the root (line 5), and return this as their result.

This pattern is so pervasive that we can capture it in a function that we call `reduce_int`. The function `reduce_int` takes as input a tree, a user-defined *function* that combines the result of the recursive calls and of the root data, and a user-defined value to return in the base case:

```
typedef int combine_fn(int left_result, int root_data, int right_result);

int reduce_int(tree* T, combine_fn* f, int base)  //@requires f != NULL;
{
  if (T == NULL) return base;

  return (*f)(reduce_int(T->left, f, base),
              T->data,
              reduce_int(T->right, f, base));
}
```

Then, we can reimplement `sum` by defining a *combination function*, `plus_cmb`, that adds its three inputs, and invokes `reduce_int` with it:

```
int plus_cmb(int L, int e, int R) { return L + e + R; }

int sum(tree* T) {
  return reduce_int(T, &plus_cmb, 0);
}
```

Study carefully how this works before proceeding.

3pts **Task 1.1** Using `reduce_int`, complete the definition of the function `size_int` that returns the number of elements in a binary tree.

```
int size_cmb_int(int L, int e, int R) { return ____________________; }

int size_int(tree* T) {
  return reduce_int(_______________, _______________, _______________);
}
```

3pts **Task 1.2** Using `reduce_int`, implement the function `height` that returns the height of a binary tree (i.e., the number of nodes in the longest path from the root to a leaf).

```
int height(tree* T) {

  _____________________________________________________________________;
}
```

The above definition of `reduce_int` can be used only for computations that return an integer, and only for trees whose data are integers. We will now write a *generic* version, which we call `reduce`. In the rest of this question, you may assume that `elem` is defined as **`void*`**, and that `reduce` returns a generic pointer.

6pts **Task 1.3** Update the definition of the function type `combine_fn` to support the generic function `reduce`, and complete the implementation of this generic `reduce`.

```
typedef _________________ combine_fn(__________________ left_result,

                                     __________________ root_data,

                                     __________________ right_result);


________________ reduce(tree* T, combine_fn* f, ________________ base)
//@requires f != NULL;
{
  if (T == NULL) return base;

  return (*f)(reduce(T->left, f, base),
              T->data,
              reduce(T->right, f, base));
}
```

5pts **Task 1.4** Implement the combination function `size_cmb` that will allow you to use `reduce` to compute the number of elements in a generic tree (you don't need to know what type elements are). Include all necessary contracts.

```
____________ size_cmb(____________ L, ____________ e, ____________ R)
//@requires L != NULL && e != NULL && R != NULL;

//@requires ________________________________________________________;
//@ensures \result != NULL;

//@ensures  ________________________________________________________;
{










}
```

3pts **Task 1.5** Using `reduce`, complete the definition of the function `size` that returns the number of elements in a generic binary tree. *(You may not need all lines.)*

```
int size(tree* T) {

  ____________________________________________________________________

  ____________________________________________________________________

  ____________________________________________________________________

  ____________________________________________________________________

  ____________________________________________________________________

  ____________________________________________________________________

  return ____________________________________________________________;
}
```

2 Heterogeneous Data Structures (25 points)

In this exercise, we are going to explore *heterogeneous* queues, allowing a client to store elements of *different* types in *one* queue. An immediate thought might be to use **void**∗ as the type for the queue's elements. However, since `\hastag` can only be used in contracts, but not in code in C1, we would lose the ability to process the elements *depending* on their *type*. To make an element's actual type available to C1 code, we introduce the following struct:

```
struct tagged_elem_header {
  int tag; // 0 = int*, 1 = string*, 2 = bool*
  void* value;
};
typedef struct tagged_elem_header tagged_elem;
```

The field `tag` describes the type of the element and the field `value` its value. We use the integer `0` for type **int**∗, the integer `1` for type **string**∗, and the integer `2` for type **bool**∗.

7pts **Task 2.1** Complete the function `new_tagged_string`, which creates a new tagged element. Make sure that your implementation satisfies the given contract:

```
tagged_elem* new_tagged_string(string s)
//@ensures \result != NULL;
//@ensures \result->tag == 1;
//@ensures string_equal(*(string*)(\result->value), s);

//@ensures \hastag(________________________________, \result->value);
{
  tagged_elem* new = _______________________________________________;

  __________________________________________________________________;

  __________________________________________________________________;

  __________________________________________________________________;

  __________________________________________________________________;

  return new;
}
```

In addition to the function `new_tagged_string` that you have just implemented, you can assume the existence of analogous functions `new_tagged_int` and `new_tagged_bool`, with the following signatures and with contracts analogous to `new_tagged_string`'s:

```
tagged_elem* new_tagged_int(int i);
tagged_elem* new_tagged_bool(bool b);
```

Here is some C1 code that uses these functions:

```
1  tagged_elem* elem1 = new_tagged_string("Cogito ergo sum.");
2  //@assert \hastag(string*, elem1->value);
3  tagged_elem* elem2 = new_tagged_int(122);
4  //@assert \hastag(bool*, elem2->value);
5  tagged_elem* elem3 = new_tagged_bool(true);
6  //@assert \hastag(tagged_elem*, elem3->value);
7  int i = *(int*)(elem2->value);
8  int t3 = elem3->tag;
```

5pts **Task 2.2** Given the above code, fill in the blanks:

The assert statement on line 2 evaluates to ______________________

The assert statement on line 4 evaluates to ______________________

The assert statement on line 6 evaluates to ______________________

The integer `i` on line 7 evaluates to ______________________

The integer `t3` on line 8 evaluates to ______________________

6pts **Task 2.3** Complete the function `print_elem` that prints the `value` field of input `T`. Use the appropriate print function from the `conio` library (see page 20 for a reference) for each possibility for the field `tag`.

```
void print_elem(tagged_elem* T)
//@requires T != NULL;
{
  if (T->tag == ________________________________)

    __________________________________________________________________;

  else if (T->tag == ___________________________) {

    __________________________________________________________________;

  else if (T->tag == ___________________________) {

    __________________________________________________________________;

  else error("Unknown tag");
}
```

2pts **Task 2.4** The interface of queues is recalled on page 20 of this exam. Complete the below type definition to make the queue store pointers to `tagged_elem` instances:

```
typedef _______________________________________________________ elem;
```

5pts **Task 2.5** Define the type `print_elem_fn` of functions that print values of type `elem`, and use it to implement the function `print_queue(Q, f)` that prints the contents of the queue `Q` using print function `f`. Calling this function destroys the queue.

```
typedef ___________ print_elem_fn _________________________________;

void print_queue(queue_t Q, print_elem_fn* f)
//@requires Q != NULL;
{
  __________________________________________________________________

  __________________________________________________________________

  __________________________________________________________________
}
```

3 Tree Sort (40 points)

Rob learned about binary search trees (BST) this week, and that sparked an idea about a new algorithm to sort an array: insert all elements into a BST and read them off from smallest to biggest, something he was told is called in-order traversal. He proudly calls it *tree sort*.

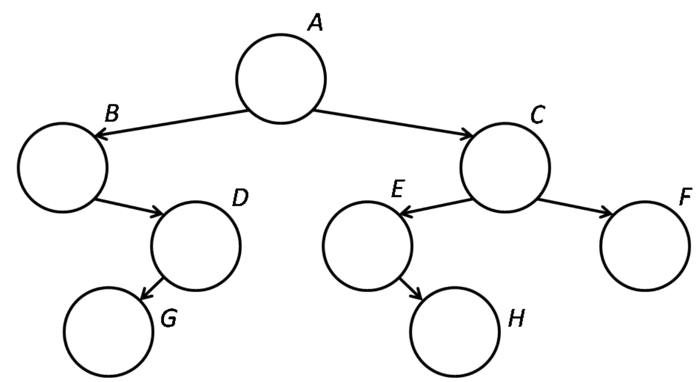
Task 3.1 Before working on the details, he asks for your help getting a good grasp on how BSTs work.

The following list of integer keys is used to build a BST, not necessarily in the order given:

$$49, \quad 16, \quad 36, \quad 81, \quad 25, \quad 4, \quad 64, \quad 9$$

2pts

a. The shape of the resulting tree is shown below. Fill in each node with one key from the list so that the resulting tree is a BST. *(The letters A–H next to the nodes will be needed in a later task.)*



2pts

b. Give a specific insertion order for the keys above that results in the tree you have just filled in.

2pts

c. Recall that the in-order traversal of a binary tree is the sequence of its entries which places the entries in the left subtree of each node before the entry in the node itself and continues with the entries in its right subtree.

What is the in-order traversal of the tree in task 1a?

For the next few tasks, we will be extending the code for binary search trees discussed in class. Relevant portions are repeated here for your convenience.

```
// typedef ______* entry;  // Type of data in the tree

typedef struct tree_node tree;
struct tree_node {
  entry data;      // != NULL
  tree* left;
  tree* right;
};

bool is_tree(tree* T); // Representation invariant for generic trees
bool is_bst(tree* T);  // Representation invariant for BST

tree* bst_insert(tree* T, entry e)
/*@requires is_bst(T) && e != NULL; @*/
/*@ensures is_bst(\result); @*/ ;
```

For this exercise, you will not need anything more than what is given above.

2pts **Task 3.2** As a warm-up, help Rob write the function `size(T)` which returns the number of nodes in the tree `T`. *Hint: it's very short when done recursively.*

```
int size(tree* T)
//@requires is_tree(T);
//@ensures \result >= 0;
{




}
```

9pts **Task 3.3** Emboldened by this achievement, Rob attempts to implement a recursive function `inorder(T, A, lo, n)` that uses in-order traversal to copy the elements of a tree `T` into a segment of an array `A` starting at index `lo`. The array has size `n`, which is large enough for doing this safely. The function returns the number of elements written into `A`. This is as far as he has gone. Please help him complete his task. *Hint: draw pictures!*

```
int inorder(tree* T, entry[] A, int lo, int n)
//@requires n == \length(A);
//@requires 0 <= lo && lo <= n;
//@requires is_tree(T);
//@requires lo + size(T) <= n;
//@ensures \result == ____________________________________________;
{
  if (T == NULL) return ____________________________________________;

  int s_left = inorder(________________, A, ____________________, n);

  A[________________________________________] = ____________________;

  int s_right = inorder(________________, A, ____________________, n);

  return ____________________________________________________________;
}
```

2pts **Task 3.4** What is the complexity of `inorder` as a function of the size t of the input tree `T`?

$O($______________________$)$

3pts **Task 3.5** With `inorder` done, Rob is ready (for you) to implement his new sorting algorithm. Recall that tree sort sorts an array `A` by inserting each of its `n` elements into a BST and then by doing an in-order traversal to read them off.

```
void tree_sort(entry[] A, int n)
//@requires n == \length(A) &&  ____(SEE NEXT TASK)___;
//@ensures is_sorted(A, 0, n);
{
  ____________________________________________________________________;

  for (int i = 0; i < n; i++) {

    ________________________________________________________________;
  }

  ____________________________________________________________________;
}
```

2pts **Task 3.6** Tree sort, as conceived by Rob and implemented above, has a flaw: it will fail its post-conditions for some arrays that the sorting algorithms you have studied would happily process. Give a 3-element array (using integers for simplicity) for which tree sort will produce an incorrect result. Then, give a precondition on its input that disallows such arrays (either write it in English or use a function seen in a previous homework).

Example array that tree sort will sort incorrectly:

Additional precondition: ______________________________

4pts **Task 3.7** How good is this fixed-up tree sort? Answer the following questions.

Worst-case complexity: $O($______________$)$

The worst-case can occur when ______________________________

______________________________.

Tree sort is an in-place algorithm? *(circle one)* Yes No

A few days later, Rob learns about AVL trees. Since AVL trees are a special form of binary search trees, tree sort will work also if he were to use an implementation of AVL trees!

6pts **Task 3.8** Again, he first needs to wrap his head around AVL trees. Answer the following questions to help him out. *Refer to the nodes of the tree in task 1 using the letters A–H.*

Is the tree in task 1 an AVL tree? *(circle one)* Yes No

If not, it has height violations at node(s) __________

To fix them, we need to do the following rotations: *(you may not need all lines)*

Rotate ________________ at node ________________

Rotate ________________ at node ________________

Rotate ________________ at node ________________

Rotate ________________ at node ________________

Draw the resulting AVL tree here: *(enter numbers in the nodes)*

2pts **Task 3.9** What is the worst-case complexity of tree sort after updating it to use AVL trees?

$O($________________________$)$

Rob mentions tree sort to Frank. Frank shows him the following non-recursive implementation of in-order traversal, which uses a (generic) stack to remember the parts of the tree that still need to be visited. (The stack interface is recalled on page 20 of this exam.)

```
1 void inorder2(tree* T, entry[] A, int n)
2 //@requires is_tree(T) && n == size(T);
3 //@requires n == \length(A);
4 {
5   stack_t S = stack_new();
6   int i = 0;
7
8   while (T != NULL || !stack_empty(S))
9   //@loop_invariant 0 <= i && i <= n;
10  {
11    if (T != NULL) {
12      push(S, (void*)T);
13      T = T->left;
14    } else { // T == NULL
15      T = (tree*)pop(S);
16      A[i] = T->data;          // THIS LINE
17      i++;
18      T = T->right;
19    }
20  }
21 }
```

Rob is not convinced of the safety and termination of this function.

2pts **Task 3.10** Line 9 does *not* support the safety of the array access `A[i]` on line 16. Why? How could you extend the loop guard on line 8 to ensure this access is safe?

Because ____

Change loop guard to `((/*as above */) && ____________________)`

2pts **Task 3.11** In English, describe a loop invariant about the stack `S` that ensures that the dereference `T->data` on line 16 is safe.

____

5 (bonus) **Task 3.12** Why does the loop on lines 8–20 terminate? Frank explains that this is because of a variant of the method seen in class. This new method relies on *two* bounded quantities and goes as follows: at each iteration of the loop,

- *either* the first quantity strictly decreases but cannot go below a certain value (and we don't care how the second quantity changes),
- *or* the first quantity stays the same but the second quantity strictly decreases and is bounded by another value.

In the function above, what are these quantities and what are their bounds?

Quantity 1: ________________________________, which is bounded by __________

Quantity 2: ________________________________, which is bounded by __________

4 Scanning Hash Tables (40 points)

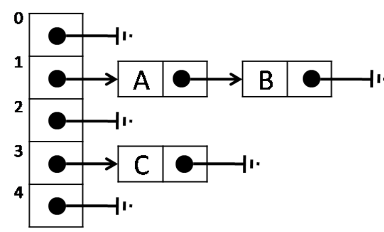
With just creation, lookup and insertion functions, the hash library interface seen in class for hash dictionaries was minimal. It is reproduced on page 21 of this exam. In this exercise, we will equip it with two operations that allow iterating through the entries in a hash dictionary. These operations, together called an *iterator*, are

- `entry hdict_first(hdict_t H) /*@requires H != NULL; @*/;`
  The call `hdict_first(H)` returns the first entry in the hash dictionary `H`, or `NULL` if `H` is empty.
- `entry hdict_next(hdict_t H) /*@requires H != NULL; @*/;`
  Each call to `hdict_next(H)` returns a next entry from `H`, or `NULL` if there are no more entries in `H`.

One can iterate through all the entries in a hash dictionary `H` by first calling `hdict_first(H)` and then repeatedly calling `hdict_next(H)` until `NULL` is returned.

For example, given the operation `print_entry(e)` which prints entry `e` on one line, the following function prints all the entries in hash dictionary `H`.

```
void print_hdict(hdict_t H) {
  for (entry e = hdict_first(H); e != NULL; e = hdict_next(H))
    print_entry(e);
}
```



Applied to the hash table on the right, the initial call to `hdict_first` will return entry A and print it. This will be followed by three calls to `hdict_next`: the first two will returns entries B and C in that order; the last will return `NULL` since the hash table does not contain other entries.

We begin by implementing the functions `hdict_first` and `hdict_next`. To do so, we extend the struct `hdict_header` seen in class with two fields:

- `last_node` points to the node containing the entry that the iterator reported the last time `hdict_first` or `hdict_next` were called. If the hash dictionary is empty, `last_node` is `NULL`.
- `last_idx` is the hash table index of the chain where `last_node` is found. It can be arbitrary when `last_node` is `NULL`.

In the above example, after returning A, `last_node` points to that entry and `last_idx` contains 1; after returning B, `last_idx` still contains 1 but `last_node` points to B; after returning C, `last_idx` is 3. After the final call to `hdict_next`, `last_node` is `NULL`.

The relevant type declarations are as follows:

```
typedef struct chain_node chain;
struct chain_node {
  entry entry;
  chain* next;
};

typedef hdict* hdict_t;
```

```
typedef struct hdict_header hdict;
struct hdict_header {
  int size;
  chain*[] table;
  int capacity;
  int last_idx;        // NEW
  chain* last_node;    // NEW
};
```

7pts **Task 4.1** Implement the helper function `first_from(H, i)` that returns the entry of the first node in the first non-empty chain of `H` starting at table index `i`, and `NULL` if no such node exists. You will need to update the fields `last_node` and `last_idx` appropriately.

In the previous example, `first_from(H, 1)` returns A's node, `first_from(H, 2)` returns C's node, and `first_from(H, 4)` returns `NULL`.

```
entry first_from(hdict* H, int i)
//@requires is_hdict(H) && 0 <= i && i <= H->capacity;
{
  for (H->last_idx = ___; __________________________; H->last_idx++) {

    chain* bucket = ____________________________________;

    if (____________________________________) { // Found!

      H->last_node = ____________________________________;

      return ____________________________________;
    }
  }
  return ____________________________________; // Not found
}
```

2pts **Task 4.2** Implement `hdict_first` so that it returns the entry of the first node in the first non-empty chain of `H`, and `NULL` if no such node exists. In the previous example, that's A's node.

```
entry hdict_first(hdict* H)  //@requires is_hdict(H);
{
  return ____________________________________;
}
```

7pts **Task 4.3** Implement `hdict_next` so that it returns the entry of the next node in the current chain or the first node in the first non-empty chain thereafter. It returns `NULL` if no such entry exists. In our example, successive calls return B's node, then C's node, and finally `NULL`.

```
entry hdict_next(hdict* H)  //@requires is_hdict(H);
{
  if (H->last_node == NULL) return __________________________;

  if (______________________) { // Next entry in current chain

    H->last_node = ____________________________________;

    return ____________________________________;
  }
  // Look for next entry in later chains

  return ____________________________________;
}
```

Task 4.4 We now consider the cost of iterating through a hash dictionary `H` with n entries and whose table has capacity m. Our **measure of cost** will consist of the number of accesses to the underlying table (e.g., as `H->table[i]`) and to an entry in a chain node (e.g., as `p->entry`).

2pts **a.** Consider the example function `print_hdict` on page 15. To print all n entries in the dictionary, how many times are the functions `hdict_first` and `hdict_next` called?

`hdict_first` is called ________________ time(s)

`hdict_next` is called ________________ time(s).

2pts **b.** What is the worst-case cost of each call *separately*?

`hdict_first` has worst-case cost $O($________________$)$

`hdict_next` has worst-case cost $O($________________$)$

1pt **c.** Assume that printing a single entry has constant cost. What is the worst-case complexity of `print_hdict` based *only* on these figures?

$O($________________$)$

2pts **d.** But is this the real cost of `print_hdict`? Overall, how many accesses (*see above definition*) are effectively carried out when calling this function to print all entries in the dictionary? Give the exact value, not a complexity bound.

Total number of accesses: ________________

6pts **e.** Chances are that your answers to the last two questions are very different. We can use the techniques of amortized analysis to charge a cost (in terms of tokens) to use `hdict_first` and `hdict_next` so that the number of tokens collected during a call to `print_hdict` is at most 1 more than the number of accesses made by this function. Recall that we always need to have enough saved tokens to pay for the true cost of an operation in full.

Cost of `hdict_first`: ____________ token(s), to be used as follows:

- ______ token(s), used to ______________________________
- ______ token(s), used to ______________________________

Cost of `hdict_next`: ____________ token(s), to be used as follows

- ______ token(s), used to ______________________________
- ______ token(s), used to ______________________________

Iterators make it easy to implement operations that require scanning all the elements in one or more hash dictionaries. We will examine a couple.

5pts **Task 4.5** Complete the implementation of the function `hdict_inboth`. The call `hdict_inboth(H1, H2)` returns a new dictionary containing the entries of `H1` whose key are also present in `H2`. The initial capacity of the new dictionary should be big enough to hold the contents of the smallest among `H1` and `H2` without collisions, if we are lucky.

```
hdict* hdict_inboth(hdict* H1, hdict* H2)
//@requires is_hdict(H1) && is_hdict(H2);
//@ensures is_hdict(\result);
{
  hdict* H = hdict_new(____________________________________________);

  ___________________________________________________________________

  ___________________________________________________________________

  ___________________________________________________________________

  return H;
}
```

6pts **Task 4.6** Iterators even make it easy to resize a hash dictionary H once its load factor becomes too big: create a temporary hash dictionary with the new capacity, insert all entries from H into it, and finally update the header of H to the values of the header of the temporary dictionary — you do not need to concern yourself with the new iterator fields. Complete the implementation of resize to realize this idea.

```
void resize(hdict* H, int new_capacity)
//@requires is_hdict(H);
//@ensures is_hdict(H);
{
  hdict* tmp = ______________________________________________________;

  // Copy contents of H into tmp

  ___________________________________________________________________

  ___________________________________________________________________

  // Copy header values of tmp into header of H

  ___________________________________________________________________

  ___________________________________________________________________

  ___________________________________________________________________
}
```

The *Queue* Interface *(semi-generic)*

```
/************************/
/*** Client interface ***/
/************************/

// typedef ______* elem;

/************************/
/*** Library interface ***/
/************************/


// typedef ______* queue_t;

bool queue_empty(queue_t Q)
/*@requires Q != NULL; @*/ ;

queue_t queue_new()
/*@ensures \result != NULL; @*/
/*@ensures queue_empty(\result); @*/ ;

void enq(queue_t Q, elem e)
/*@requires Q != NULL; @*/ ;

elem deq(queue_t Q)
/*@requires Q != NULL; @*/
/*@requires !queue_empty(Q); @*/ ;
```

The *stack* Interface *(generic)*

```
/************************/
/*** Library interface ***/
/************************/

typedef void* elem;
// typedef ______* stack_t;

bool stack_empty(stack_t S)
/*@requires S != NULL; @*/ ;

stack_t stack_new()
/*@ensures \result != NULL; @*/
/*@ensures stack_empty(\result); @*/ ;

void push(stack_t S, elem x)
/*@requires S != NULL; @*/ ;

elem pop(stack_t S)
/*@requires S != NULL; @*/
/*@requires !stack_empty(S); @*/ ;
```

Basic Printing Functions

```
void print(string s);      // print string  s to standard output
void printint(int i);      // print integer i to standard output
void printbool(bool b);    // print boolean b to standard output
```

The *Hash Dictionary* Interface *(semi-generic)*

```
/************************/
/*** Client interface ***/
/************************/

// typedef ______* entry;                // Supplied by client
// typedef ______  key;                  // Supplied by client

key  entry_key(entry x)                  // Supplied by client
  /*@requires x != NULL; @*/ ;
int  key_hash(key k);                    // Supplied by client
bool key_equiv(key k1, key k2);          // Supplied by client


/*************************/
/*** Library interface ***/
/*************************/

// typedef ______* hdict_t;

hdict_t hdict_new(int capacity)
/*@requires capacity > 0; @*/
/*@ensures \result != NULL; @*/ ;

entry hdict_lookup(hdict_t H, key k)
/*@requires H != NULL; @*/
/*@ensures \result == NULL || key_equiv(entry_key(\result), k); @*/ ;

void hdict_insert(hdict_t H, entry x)
/*@requires H != NULL && x != NULL; @*/
/*@ensures hdict_lookup(H, entry_key(x)) == x; @*/ ;
```

(This page intentionally left blank.)

(This page intentionally left blank.)