Finite Element Method (FEM)

Lecture 1: What is FEM? (Basic concept)

Problem (P): Find $u : [a,b] \to \mathbb{R}$ such that

$$\begin{cases} -u'' + cu = f \\ u(a) = 0, \quad u(b) = 0 \end{cases}$$

$c = \text{const}$

f - given function

Classical formulation of boundary value problem (BVP)

Take a function $v : [a,b] \to \mathbb{R}$, $v(a) = 0$, $v(b) = 0$ (test function)

$$-u'' + cu = f \;\Big|\; \cdot v, \; \int_a^b$$

$$-\int_a^b u'' \cdot v \, dx + c\int_a^b u \cdot v \, dx = \int_a^b f \cdot v \, dx$$

$$-u' \cdot v \Big|_a^b + \int_a^b u'v' dx + c\int_a^b u \cdot v \, dx = \int_a^b f \cdot v \, dx$$

$$-u'(b)\underbrace{v(b)}_{0} + u'(b) \cdot \underbrace{v(a)}_{0} + \int_a^b u'v' + c\int_a^b u \cdot v \, dx = \int_a^b f \cdot v \, dx$$

$$\int_a^b u'v' dx + c\int_a^b u \cdot v \, dx = \int_a^b f \cdot v \, dx$$

We introduce the space $V = H_0^1(a,b) = \{ w : [a,b] \to \mathbb{R} :$

$w \in L^2(a,b), \quad w' \in L^2(a,b)$

$w(a) = w(b) = 0 \}$

Problem (V): Find $u \in V$ such that

$$\underbrace{\int_a^b u'v' dx + c\int_a^b u \cdot v \, dx}_{:= a(u,v)} = \underbrace{\int_a^b f \cdot v \, dx}_{:= L(v)} \qquad \forall v \in V$$

Let $a: V \times V \to \mathbb{R}$ - bilinear, symmetric form

$L: V \to \mathbb{R}$ - linear form

Problem (V) - variational formulation of Problem (P):

Find $u \in V$ s.th.

$$a(u, v) = L(v) \quad \forall v \in V \qquad (1)$$

<u>Galerkin</u> method of solving problem (V):

Let $V^h \subset V$ - finite dimensional subspace of V ($\dim V^h = n$)

Problem (V^h): Find $u^h \in V^h$ s.th.

$$a(u^h, v^h) = L(v^h) \quad \forall v^h \in V^h \qquad (2)$$

Let $\{e_1, e_2, \dots, e_n\}$ - basis of V^h. Then (2) is equivalent to:

$$a(u^h, e_j) = L(e_j) \quad \forall j = 1, \dots, n \qquad (3)$$

(2) $\Rightarrow$ (3) - obvious.

(3) $\Rightarrow$ (2). Assume that (3) holds. We will show (2).

Let $v^h \in V^h$. Then $v^h = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$

$$a(u^h, v^h) = a(u^h, c_1 e_1 + c_2 e_2 + \dots + c_n e_n) =$$

$$c_1 a(u^h, e_1) + c_2 a(u^h, e_2) + \dots + c_n a(u^h, e_n) \overset{(3)}{=}$$

$$\cancel{c_1 a(u} \; c_1 L(e_1) + c_2 L(e_2) + \dots + c_n L(e_n) =$$

$$L(c_1 e_1 + c_2 e_2 + \dots + c_n e_n) = L(v^h) \quad \square$$

The unknown function u^h has a form:

$$u^h = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n \quad , \quad \alpha_i \in \mathbb{R} \; , \; i=1\dots n$$

Then (3):

$$a(\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n , e_j) = L(e_j) \qquad j=1\dots n$$

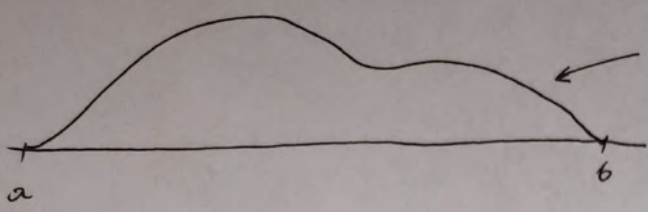
$$\alpha_1 a(e_1, e_j) + \alpha_2 a(e_2, e_j) + \dots + \alpha_n a(e_n, e_j) = L(e_j) \qquad j=1\dots n$$

$$\begin{cases} \alpha_1 a(e_1, e_1) + \alpha_2 a(e_2, e_1) + \dots + \alpha_n a(e_n, e_1) = L(e_1) \\ \alpha_1 a(e_1, e_2) + \alpha_2 a(e_2, e_2) + \dots + \alpha_n a(e_n, e_2) = L(e_2) \\ \vdots \\ \alpha_1 a(e_1, e_n) + \alpha_2 a(e_2, e_n) + \dots + \alpha_n a(e_n, e_n) = L(e_n) \end{cases}$$

$$\begin{bmatrix} a(e_1,e_1) & a(e_2,e_1) & \dots & a(e_n,e_1) \\ a(e_2,e_1) & a(e_2,e_2) & \dots & a(e_n,e_2) \\ & & & \\ a(e_1,e_n) & a(e_2,e_n) & \dots & a(e_n,e_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} L(e_1) \\ L(e_2) \\ \vdots \\ L(e_n) \end{bmatrix}$$

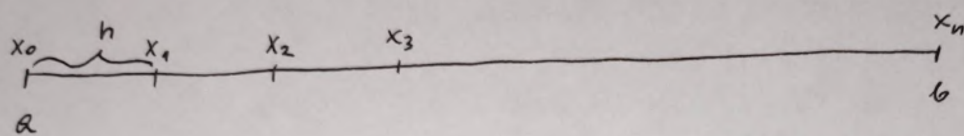
$\hookrightarrow$ SOLVE $\rightarrow$ $\alpha_1, \alpha_2, \dots \alpha_n$ $\rightarrow$ $u^h = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n$

Finite Element Method - a method of constructing particular space V^h.



typical function $\in V = H_0^1(a,b)$

- discretisation of the domain $[a,b]$

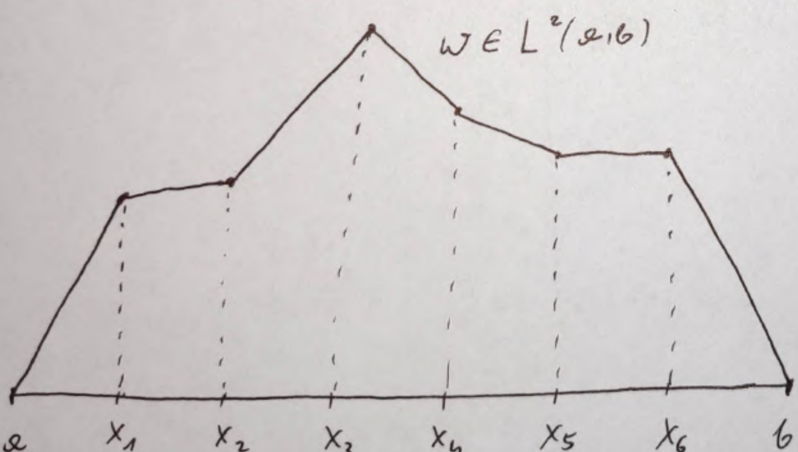


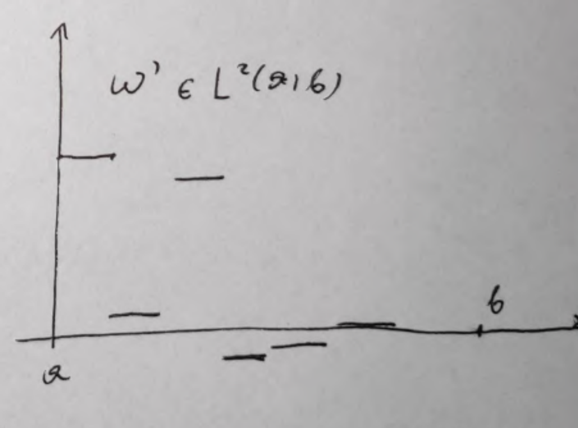
n - given , $n \in \mathbb{N}$

$h = \frac{b-a}{n}$

$x_i = a + ih$, $i = 0, 1, \dots, n$, x_i - nodes

$x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h, \dots,$ $x_n = a + nh = a + n \cdot \frac{b-a}{n} =$
$a + (b-a) = b$





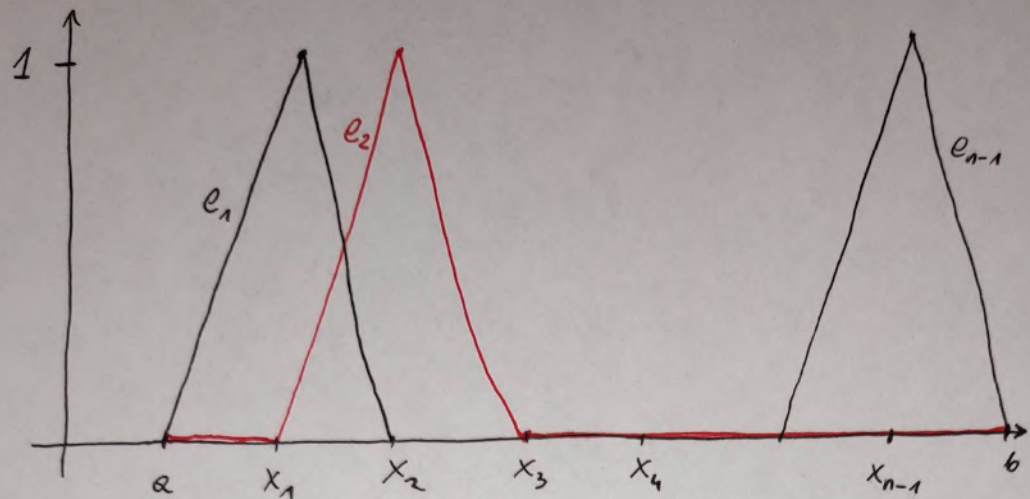
$w \in V = H_0^1(a,b)$

V^h - the subspace of functions, who are :

- continuous
- piecewise linear at $[x_i, x_{i+1}]$ $i = 0, \dots, n-1$

$V^h \subset V$

What about the basis of the space V^h ?



$$e_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We will show, that $\{e_1, e_2, \dots e_{n-1}\}$ is the basis of V^h.

Take any ~~linear combination~~ $w \in V^h$ and observe, that :

$$\underbrace{w(x)}_{LHS(x)} = \underbrace{w(x_1)e_1(x) + w(x_2)e_2(x) + \dots + w(x_{n-1})e_{n-1}(x)}_{RHS(x)} \qquad \forall x \in [a, b]$$

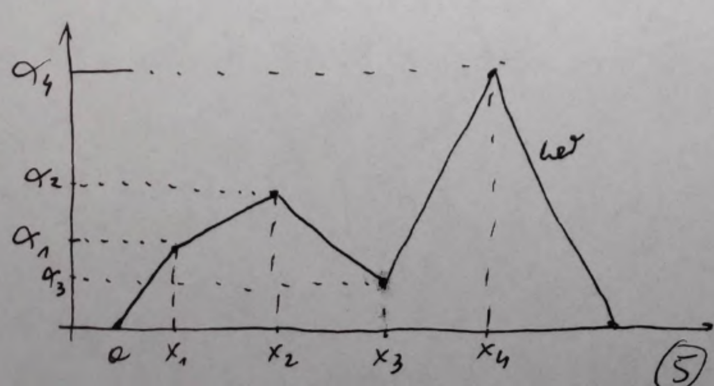
$LHS(x_0) = 0$, $RHS(x_0) = w(x_1)e_1(0) + .. + w(x_{n-1})e_{n-1}(0) = 0$

$LHS(x_1) = w(x_1)$, $RHS(x_1) = w(x_1)e_1(x_1) + w(x_2)e_1(x_2) + .. + w(x_{n-1})e_{n-1}(x_2) =$
$w(x_1) \cdot 1 + w(x_2) \cdot 0 + .. + w(x_{n-1}) \cdot 0 = w(x_1)$

. . . . . .

Inversely: If $w = \alpha_1 e_1 + \alpha_2 e_2 + .. + \alpha_n e_n$, then

$w(x_1) = \alpha_1$
$w(x_2) = \alpha_2$
$\vdots$
$w(x_n) = \alpha_n$



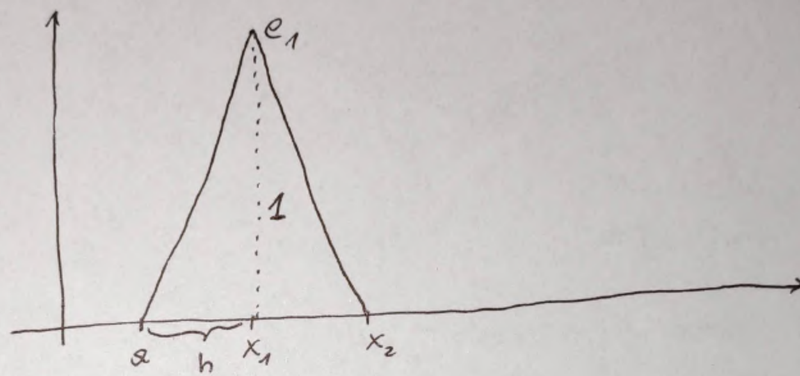
In order to solve the Galerkin problem, we have to evaluate the elements of the matrix

$$a(e_i, e_j) = \int_a^b e_i'(x) e_j'(x)\,dx + c\int_a^b e_i(x) e_j(x)\,dx$$

and elements of RHS $\quad L(e_j) = \int_a^b f(x) e_j(x)\,dx$

The product $e_i e_j$ and $e_i' e_j'$ are not $\equiv 0$ only if $i = j \pm 1$
or $i = j$

1) $e_i = e_j \; (= e_1)$



$$e_1(x) = \begin{cases} \frac{1}{h}(x-a) & \text{for } x \in [a, x_1] \\ -\frac{1}{h}(x-a-2h) & \text{for } x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases} \qquad e_1'(x) = \begin{cases} \frac{1}{h} & x \in [a, x_1] \\ -\frac{1}{h} & x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b e_1(x) e_1(x)\,dx = \int_a^{a+h} \frac{1}{h^2}(x-a)^2\,dx + \int_{a+h}^{a+2h} \frac{1}{h^2}(x-a-2h)^2\,dx =$$

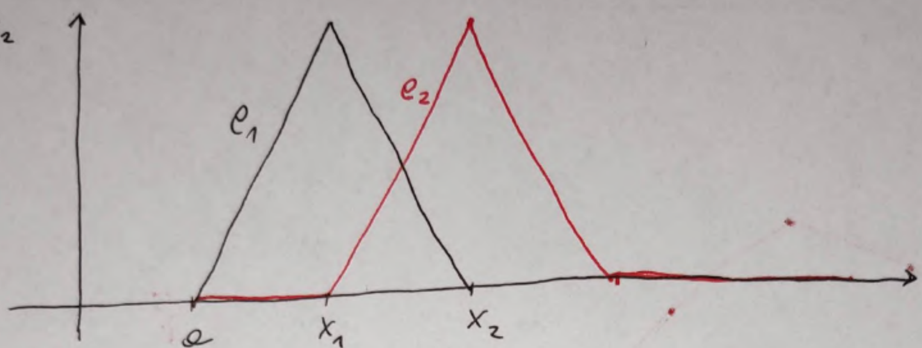
$$\frac{1}{h^2}\left[\int_a^{a+h} (x-a)^2\,dx + \int_{a+h}^{a+2h} (x-a-2h)^2\,dx\right] = \left\{ \begin{array}{l} u = x-a \\ v = x-a-2h \end{array} \right\} = \frac{1}{h^2}\left[\int_0^h u^2\,du + \int_{-h}^0 v^2\,dv\right.$$

$$= \frac{1}{h^2}\left[\frac{1}{3}u^3\Big|_0^h + \frac{1}{3}v^3\Big|_{-h}^0\right] = \frac{1}{3}\frac{1}{h^2}\left[h^3 - 0 + 0 - (-h)^3\right] = \frac{1}{3}\frac{1}{h^2}\cdot 2h^3 = \frac{2}{3}h$$

$$\int_a^b e_1' e_1' \, dx = \int_a^{a+h} \frac{1}{h} \cdot \frac{1}{h} \, dx + \int_{a+h}^{a+2h} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx = \int_a^{a+h} \frac{1}{h^2} \, dx + \int_{a+h}^{a+2h} \frac{1}{h^2} \, dx =$$

$$\frac{1}{h^2} \cdot h + \frac{1}{h^2} \cdot h = \frac{1}{h} + \frac{1}{h} = \frac{2}{h}$$

2) e_1, e_2



The produts $e_1 e_2$ and $e_1' e_2'$ are not $\equiv 0$ only for $x \in [x_1, x_2]$

$e_1(x) = -\frac{1}{h}(x - x_2) = -\frac{1}{h}(x - a - 2h)$ | $e_2(x) = \frac{1}{h}(x - x_1) = \frac{1}{h}(x - a - h)$

$e_1'(x) = -\frac{1}{h}$ | $e_2'(x) = \frac{1}{h}$

$$\int_a^b e_1(x) e_2(x) \, dx = \int_{x_1}^{x_2} e_1(x) e_2(x) \, dx = \int_{a+h}^{a+2h} -\frac{1}{h}(x - a - 2h) \cdot \frac{1}{h}(x - a - h) \, dx =$$

$$-\frac{1}{h^2} \int_{a+h}^{a+2h} (x - a - 2h)(x - a - h) \, dx = \left\{ u = x - a - h \right\} = -\frac{1}{h^2} \int_0^h (u - h) \cdot u \, du =$$

$$-\frac{1}{h^2} \left[\int_0^h u^2 du - h \int_0^h u \, du \right] = -\frac{1}{h^2} \left[\frac{1}{3} h^3 - h \cdot \frac{1}{2} h^2 \right] = -\frac{1}{h^2} \left[-\frac{1}{6} h^3 \right] = \frac{1}{6} h$$

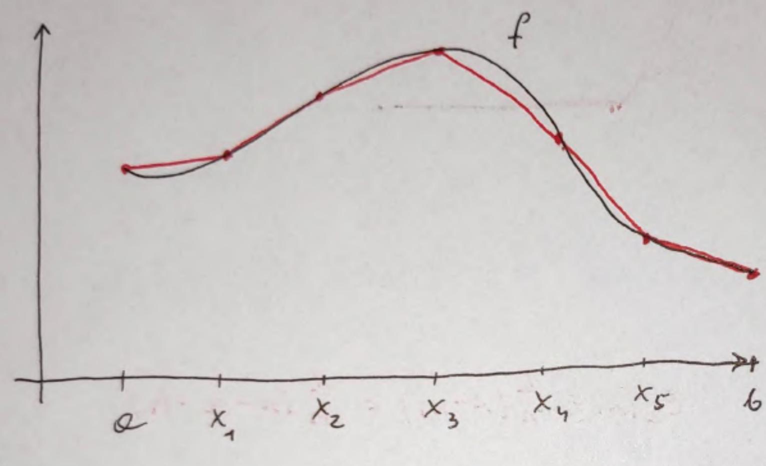
$$\int_a^b e_1'(x) e_2'(x) \, dx = \int_{x_1}^{x_2} \left(-\frac{1}{h}\right) \frac{1}{h} \, dx = -\frac{1}{h^2} \int_{a+h}^{a+2h} dx = -\frac{1}{h^2} \cdot h = -\frac{1}{h}$$

Summarizing :

$$a(e_i, e_i) = c\int_a^b e_i^2\,dx + \int_a^b (e_i')^2\,dx = c\cdot\frac{2}{3}h + \frac{2}{h}$$

$$a(e_i, e_{i\pm1}) = c\int_a^b e_i\cdot e_{i\pm1}\,dx + \int_a^b e_i' e_{i\pm1}' = c\cdot\frac{1}{6}h - \frac{1}{h}$$

How to evaluate $L(e_j) = \int_a^b f(x)e_j(x)\,dx$?



The function f is approximated by the function $\bar{f}$ which shares the values in all nodes with f.

$f(x_0) = \bar{f}(x_0)$

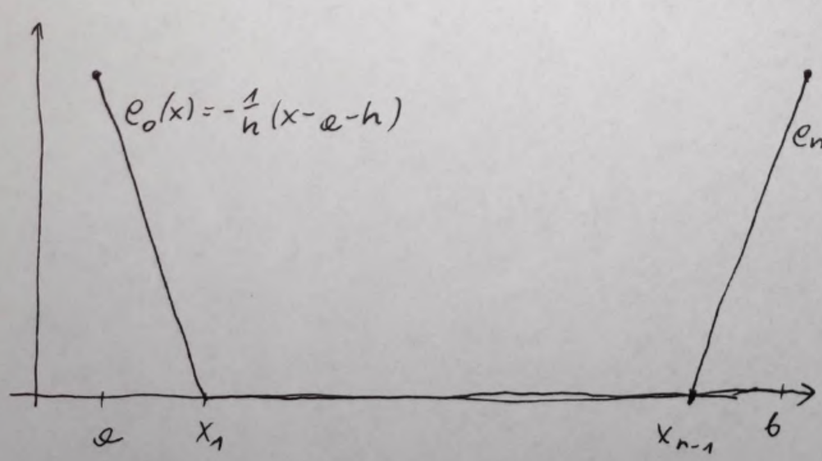
$f(x_1) = \bar{f}(x_1)$

$\vdots$

$f(x_n) = \bar{f}(x_n)$

As the function f doesn't necessary satisfy condition $f(a)=0$, $f(b)=0$, it maybe not possible to obtain $\bar{f}$ only by means of $e_1, \dots e_{n-1}$

We need additionally e_0 and e_n



$e_n(x) = \frac{1}{h}(x-x_n) = \frac{1}{h}(x-b)$

$$f(x) \approx \bar{f}(x) = \sum_{i=0}^{n} f(x_i)e_i(x)$$

$$= f(x_0)e_0 + f(x_1)e_1 + \dots + f(x_n)e_n$$

$$\int_a^b f(x)e_j(x)\,dx \approx \int_a^b \bar{f}(x)e_j(x)\,dx = \int_a^b \Big(\sum_{i=0}^{n} f_i(x)e_i(x)\Big)e_j(x)\,dx =$$

$$\sum_{i=0}^{n} f(x_i)\int_a^b e_i(x)e_j(x)\,dx$$

1) $j=1$

$$\int_a^b f(x)e_1(x)\,dx \approx \sum_{i=0}^{n} f(x_i)\int_a^b e_i(x)e_1(x)\,dx =$$

$$f(x_0)\int_a^b e_0(x)e_1(x)\,dx + f(x_1)\int_a^b e_1(x)e_1(x)\,dx + f(x_2)\int_a^b e_2(x)e_1(x)\,dx =$$

$$f(x_0)\cdot\tfrac{1}{6}h + f(x_1)\cdot\tfrac{2}{3}h + f(x_2)\cdot\tfrac{1}{6}h$$

2) $j=2$ $\quad \int_a^b f(x)e_2(x)\,dx = f(x_1)\tfrac{1}{6}h + f(x_2)\tfrac{2}{3}h + f(x_3)\tfrac{1}{6}h$

Input : a, b, c, n

$h = (b-a)/n$, $P_1 = a(e_i,e_i) = c\cdot\frac{2}{3}h + \frac{2}{h}$, $P_2 = a(e_i,e_{i\pm1}) = c\cdot\frac{1}{6}h - \frac{1}{h}$

$$\underbrace{\begin{bmatrix} P_1 & P_2 & 0 & 0 \dots & & 0 \\ P_2 & P_1 & P_2 & 0 \dots & & 0 \\ 0 & P_2 & P_1 & P_2 & \dots & 0 \\ & \dots & & & & \\ 0 & \dots & & P_2 & P_1 & P_2 \\ 0 & \dots & & 0 & P_2 & P_1 \end{bmatrix}}_{M} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} L(e_1) \\ L(e_2) \\ \vdots \\ L(e_{n-1}) \end{bmatrix}}_{B}$$

```
for k = 1 : n-1
B(k) = f(a+(k-1)h)·h/6 +
        f(a+kh)·2/3 h +
        f(a+(k+1)h)·h/6
end
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M(1,1) = P1;   M(1,2) = P2;
  for k = 2 : n-2
      M(k, k-1) = P2;   M(k,k) = P1;   M(k, k+1) = P2
  end
M(n-1, n-2) = P2;   M(n-1, n-1) = P1;
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Example 1

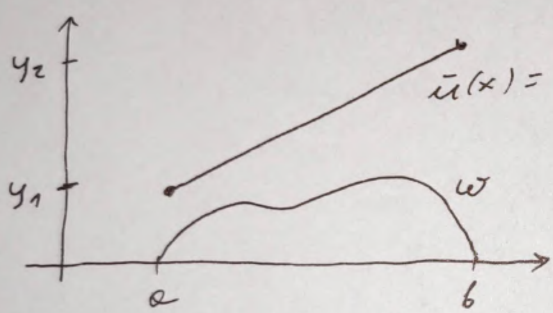
$$\begin{cases} -u'' + 3u = (2x^2 - 12x + 12)e^x \\ u(1) = 0, \quad u(3) = 0 \quad \text{(homogenous Dirichlet boundary conditions)} \end{cases}$$

An exact solution : $u(x) = (x^2 - 4x + 3)e^x$

Matlab $\rightarrow$ example_1.m

How to deal with nonhomogenous Dirichlet boundary conditions?

$$\begin{cases} -u'' + cu = f \\ u(a) = y_1, \quad u(b) = y_2 \end{cases}$$



$$\bar{u}(x) = \frac{y_2 - y_1}{b - a}(x - a) + y_1 \quad \text{- shift function}$$

$$w(a) = 0, \quad w(b) = 0$$

We expect our solution to have the form

$$u = w + \bar{u} \qquad \left(\begin{aligned} u(a) &= w(a) + \bar{u}(a) = 0 + y_1 = y_1 \\ u(b) &= w(b) + \bar{u}(b) = 0 + y_2 = y_2 \end{aligned} \right)$$

$$-u'' + cu = f$$

$$-(w + \bar{u})'' + c(w + \bar{u}) = f$$

$$-w'' - \bar{u}'' + cw + c\bar{u} = f$$

$$-w'' + cw = f - c\bar{u} + \overbrace{\bar{u}''}^{=0}$$

The problem reduces to the following: Find w:

$$\begin{cases} -w'' + cw = f - c\bar{u} \\ w(a) = 0, \quad w(b) = 0 \end{cases}$$

$\rightarrow w \rightarrow u = w + \bar{u}$

Example 2

$$\begin{cases} -u'' + 3u = [-4x^4 + 16x^3 - 15x^2 - 16x + 28]e^{x^2-4x+3} \\ u(1) = -1 \qquad u(3) = 7 \end{cases}$$

Exact solution $u(x) = (x^2-2)e^{x^2-4x+3}$

$\bar{u}(x) = 4x - 5$

Find w:

$$\begin{cases} -w'' + 3w = f - 3\bar{u} \\ w(1) = 0, \quad w(3) = 0 \end{cases} \to w \to u = w + \bar{u}$$

Matlab: example_2.m

Example 3. Can we take a different shift function?

$\bar{u}(x) = (4x-5)(x-2)^2 \qquad \bar{u}(1) = -1 \qquad \bar{u}(3) = 7$

$$u = w + \bar{u}$$

$$-u'' + 3u = f$$

$$-(w+\bar{u})'' + 3(w+\bar{u}) = f$$

$$-w'' + \bar{u}'' + 3w + 3\bar{u} = f$$

$$\begin{cases} -w'' + 3w = f + \bar{u}'' - 3\bar{u} \\ w(1) = -1, \quad w(3) = 7 \end{cases}$$

$$\bar{u}' = 4(x-2)^2 + (4x-5)\cdot 2(x-2) = (x-2)[4(x-2) + 2(4x-5)] =$$

$$(x-2)[4x - 8 + 8x - 10] = (x-2)(12x-18)$$

$$\bar{u}'' = 12x - 18 + (x-2)\cdot 12 = 12x - 18 + 12x - 24 = 24x - 42$$

Matlab: example_3.m

Can we handle the problem directly (without shift funtion)

Example 4

$$\begin{cases} -u'' + cu = f \\ u(a) = y_1 \quad u(b) = y_2 \end{cases}$$

$u \in H^1(a,b) = \{ u : [a,b] \to \mathbb{R} : u \in L^2(a,b),\ u'(a,b),\ \cancel{u(a)=0},\ \cancel{u(b)=0} \}$

$$-u'' + cu = f \quad / \cdot v,\ \int_a^b \qquad (v \in H^1(a,b))$$

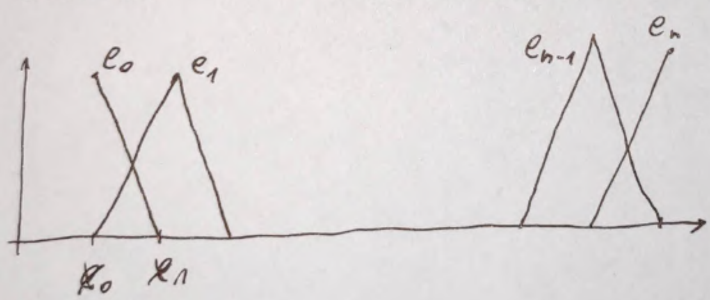
$$-\int_a^b u'' \cdot v\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$-u' \cdot v\big|_a^b + \int_a^b u' \cdot v'\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$-u'(b)v(b) + u'(a)v(a) + \int_a^b u'v'\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$\underbrace{\int_a^b u'v'\,dx + c\int_a^b u \cdot v\,dx}_{a(u,v)} + \underbrace{u'(a)v(a) - u'(b)v(b)}_{\bar{a}(u,v)} = \underbrace{\int_a^b f \cdot v\,dx}_{L(v)}$$

V^h :



Find $u^h = u_0 e_0 + u_1 e_1 + \dots + u_{n-1} e_{n-1} + u_n e_n$ s. th.

$$a(u^h, v^h) + \bar{a}(u^h, v^h) = L(v^h) \qquad \forall v^h \in V^h$$

$$\Leftrightarrow \quad a(u^h, e_i) + \bar{a}(u^h, e_i) = L(e_i) \quad , \ e_0, e_1, \dots, e_{n-1}, e_n$$

$$\Leftrightarrow \begin{bmatrix} a(e_0,e_0) + \bar{a}(e_0,e_0) & \dots & a(e_n,e_0) + \bar{a}(e_n,e_0) \\ & & \\ & & \\ a(e_0,e_n) + \bar{a}(e_0,e_n) & & a(e_n,e_n) + \bar{a}(e_n,e_n) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} L(e_0) \\ L(e_1) \\ \\ \\ L(e_{n-1}) \\ L(e_n) \end{bmatrix}$$

What about y_1, y_2 ?

Bad idea ?

We modify the idea:

$$\begin{cases} -u'' + cu = f \\ u(a) = y_1, \quad u(b) = y_2 \end{cases}$$

$u \in H^1(a,b)$ $\quad$ $\cancel{u(a)=0}$, $\cancel{u(b)=0}$

$v \in H_0^1(a,b)$ $\quad$ $v(a)=0$, $v(b)=0$!

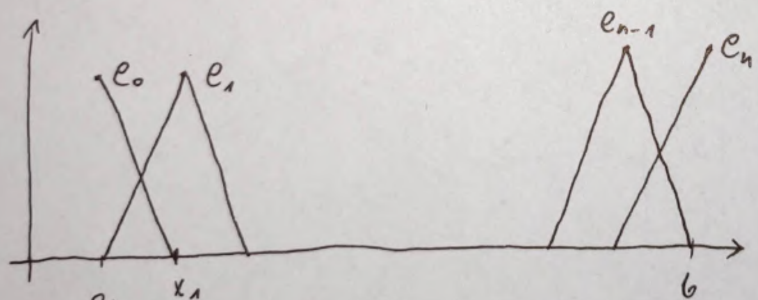
$$-u'' + cu = f \;\Big|\cdot v, \int_a^b$$

$$-\int_a^b u'' \cdot v\,dx + c\int_a^b u\cdot v\,dx = \int_a^b f\cdot v\,dx$$

$$-u'\cdot v\Big|_a^b + \int_a^b u'\cdot v'\,dx + c\int_a^b u\cdot v\,dx = \int_a^b f\cdot v\,dx$$

$$-u'(b)\underbrace{v(b)}_{=0} + u'(b)\underbrace{v(a)}_{=0} + \int_a^b u'v'\,dx + c\int_a^b u\cdot v\,dx = \int_a^b f\cdot v\,dx$$

$$\underbrace{\int_a^b u'v'\,dx + c\int_a^b u\cdot v\,dx}_{a(u,v)} = \underbrace{\int_a^b f\cdot v\,dx}_{L(v)}$$



Find $u^h = u_0 e_0 + u_1 e_1 + \dots + u_{n-1} e_{n-1} + u_n e_n$, s. th.

$$a(u^h, v^h) = L(v^h) \quad \forall v^h \in V^h \subset H_0^1(a,b) \qquad v^h \in \text{lin}\{e_1, \dots, e_{n-1}\}$$

$$\Longleftrightarrow \quad a(u^h, e_i) = L(e_i) \quad \text{for } e_1, e_2, \dots, e_{n-1}$$

$$\Longleftrightarrow \quad a(u_0 e_0 + u_1 e_1 + \dots + u_{n-1} e_{n-1} + u_n e_n, e_i) = L(e_i) \qquad i = 1, 2, \dots, n-1$$

$$\left[\underbrace{\begin{matrix} a(e_0,e_1)\\ a(e_0,e_2)\\ \vdots \\ a(e_0,e_{n-1})\end{matrix}}_{\text{New}} \;\; \underbrace{\begin{matrix} a(e_1,e_1) & \dots & a(e_{n-1},e_1)\\ a(e_1,e_2) & \dots & a(e_{n-1},e_2)\\ \vdots & & \vdots \\ a(e_1,e_{n-1}) & \dots & a(e_{n-1},e_{n-1})\end{matrix}}_{\text{Previous case}} \;\; \underbrace{\begin{matrix} a(e_n,e_1)\\ a(e_n,e_2)\\ \vdots \\ a(e_n,e_{n-1})\end{matrix}}_{\text{New}}\right] \begin{bmatrix}u_0\\u_1\\ \vdots\\ u_{n-1}\\u_n\end{bmatrix} = \underbrace{\begin{bmatrix}L(e_1)\\L(e_2)\\ \vdots \\ L(e_{n-1})\end{bmatrix}}_{\text{Previous case}}$$

$$\begin{bmatrix} P_2 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0\\ 0 & P_2 & P_1 & P_2 & 0 & 0 & 0 & 0\\ 0 & 0 & P_2 & P_1 & P_2 & 0 & 0 & 0\\ 0 & 0 & 0 & P_2 & P_1 & P_2 & 0 & 0\\ 0 & 0 & 0 & 0 & P_2 & P_1 & P_2 & 0\\ 0 & 0 & 0 & 0 & 0 & P_2 & P_1 & P_2 \end{bmatrix} \begin{bmatrix}u_0\\u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\end{bmatrix} = \begin{bmatrix}L(e_1)\\L(e_2)\\L(e_3)\\L(e_4)\\L(e_5)\\L(e_6)\end{bmatrix}$$

+ boundary conditions $u_0 = y_1$, $u_n = y_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ P_2 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0\\ 0 & P_2 & P_1 & P_2 & 0 & 0 & 0 & 0\\ 0 & 0 & P_2 & P_1 & P_2 & 0 & 0 & 0\\ 0 & 0 & 0 & P_2 & P_1 & P_2 & 0 & 0\\ 0 & 0 & 0 & 0 & P_2 & P_1 & P_2 & 0\\ 0 & 0 & 0 & 0 & 0 & P_2 & P_1 & P_2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M:\ (n+1)\times(n+1)} \begin{bmatrix}u_0\\u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\end{bmatrix} = \underbrace{\begin{bmatrix}y_1\\L(e_1)\\L(e_2)\\L(e_3)\\L(e_4)\\L(e_5)\\L(e_6)\\y_2\end{bmatrix}}_{B}$$

```
M(1,1) = 1
for i = 2:n
  M(i,i-1) = P2
  M(i,i+1) = P2
  M(i,i) = P1
end
M(n+1,n+1) = 1

B(1) = y1
for i = 2:n
  B(i) = F(i-1)
end
B(n+1) = y2
```

Example 5

$$\begin{cases} -u'' + cu = f \\ u(a) = y_1, \quad \boxed{\alpha u(b) + \beta u'(b) = y_2} \end{cases}$$

Robin boundary condition

if $\alpha = 0$, then

$u'(b) = \text{const}$ - Neuman b. c

$u \in H^1(a,b) \quad , \quad v(a) = 0$

$$-\int_a^b u'' \cdot v\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$-u' \cdot v\Big|_a^b + \int_a^b u' \cdot v'\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$-u'(b)v(b) + u'(a)\underbrace{v(a)}_{=0} + \int_a^b u'v'\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

from Robin b.c.

$$-\beta u'(b) = \cancel{y_2}\ \alpha u(b) - y_2$$

$$-u'(b) = \frac{\alpha}{\beta} u(b) - \frac{y_2}{\beta}$$

$$\left(\frac{\alpha}{\beta} u(b) - \frac{y_2}{\beta}\right) v(b) + \int_a^b u'v'\,dx + c\int_a^b u \cdot v\,dx = \int_a^b f \cdot v\,dx$$

$$\underbrace{\int_a^b u'v'\,dx + c\int_a^b u \cdot v\,dx}_{a(u,v)} + \underbrace{\frac{\alpha}{\beta} u(b)v(b)}_{\tilde{a}(u,v)\,+} = \underbrace{\int_a^b f \cdot v\,dx}_{L(v)} + \underbrace{\frac{y_2}{\beta} v(b)}_{\bar{L}(v)}$$

$$u^h = u_0 e_0 + u_1 e_1 + \dots + u_{n-1} e_{n-1} + u_n e_n$$

$$a(u^h, v^h) + \tilde{a}(u^h, v^h) = L(v^h) + \bar{L}(v^h) \qquad \forall v^h$$

$$\Leftrightarrow \quad a(u^h, e_i) + \tilde{a}(u^h, e_i) = L(e_i) + \bar{L}(e_i) \qquad \cancel{\forall}\ e_1, e_2, \dots, e_n$$

$$a(u^h, e_i) = a(u_0 e_0 + u_1 e_1 + \dots + u_{n-1} e_{n-1} + u_n e_n, e_i) \qquad e_1, e_2, \dots, e_n$$

$$n\left\{\begin{bmatrix} a(e_0,e_1) & a(e_1,e_1) & \dots & a(e_{n-1},e_1) & a(e_n,e_1) \\ a(e_0,e_2) & a(e_1,e_2) & \dots & a(e_{n-1},e_2) & a(e_n,e_2) \\ & & \text{Previous case} & & \\ & & (n-1)\times(n-1) & & \\ a(e_0,e_{n-1}) & a(e_1,e_{n-1}) & \dots & a(e_{n-1},e_{n-1}) & a(e_n,e_{n-1}) \\ a(e_0,e_n) & a(e_1,e_n) & \dots & a(e_{n-1},e_n) & a(e_n,e_n) \end{bmatrix}\right. \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{n+1}$$

$$\begin{bmatrix} P_2 & P_1 & P_2 & 0 & 0 & 0 & 0 \\ 0 & P_2 & P_1 & P_2 & 0 & 0 & 0 \\ 0 & 0 & P_2 & P_1 & P_2 & 0 & 0 \\ 0 & 0 & 0 & P_2 & P_1 & P_2 & 0 \\ 0 & 0 & 0 & 0 & P_2 & P_1 & P_2 \\ 0 & 0 & 0 & 0 & 0 & P_2 & P_1 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$\tilde{a}(u^h, e_i) = \frac{\alpha}{\beta} u^h(b) e_i(b) = \frac{\alpha}{\beta}\left(u_0 e_0(b) + u_1 e_1(b) + \dots + u_{n-1} e_{n-1}(b) + u_n e_n(b)\right) \cdot e_i(b)$$

$$\frac{\alpha}{\beta} \cdot \begin{bmatrix} e_0(b)e_1(b) & \dots & e_n(b)e_1(b) \\ e_0(b)e_2(b) & \dots & e_n(b)e_2(b) \\ & & \\ e_0(b)e_{n-1}(b) & \dots & e_n(b)e_{n-1}(b) \\ e_0(b)e_n(b) & \dots & \underbrace{e_n(b)e_n(b)}_{\neq 0} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = n\left\{\underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & \frac{\alpha}{\beta} \end{bmatrix}}_{n+1}\right. \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

RHS

$$L(e_i): \begin{bmatrix} L(e_1) \\ L(e_2) \\ \vdots \\ L(e_{n-1}) \\ L(e_n) \end{bmatrix}$$

new

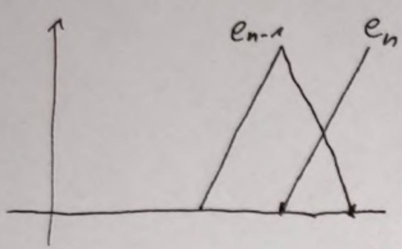
$$L(e_i) = \int_a^b f \cdot e_i$$

$$L(e_n) = \int_a^b f \cdot e_n \, dx \approx \int_a^b \left(\sum_{i=0}^{n} f(x_i) e_i e_n\right) =$$

$$\sum_{i=0}^{n} f(x_i) \int_a^b e_i e_n \, dx =$$

$$f(x_{n-1}) \int_a^b e_{n-1} e_n \, dx + f(x_n) \int_a^b e_n e_n \, dx =$$

$$f(x_{n-1}) \cdot \frac{1}{6} h + f(x_n) \cdot \frac{1}{3} h$$



$$\bar{L}(e_i) = \frac{y_2}{\beta} e_i(b): \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \cdot \frac{y_2}{\beta} \end{bmatrix}$$

+ boundary condition $u(a) = y_1 \Rightarrow u_0 = y_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & P_2 & 0 & 0 & 0 & 0 \\ 0 & P_2 & P_1 & P_2 & 0 & 0 & 0 \\ 0 & 0 & P_2 & P_1 & P_2 & 0 & 0 \\ 0 & 0 & 0 & P_2 & P_1 & P_2 & 0 \\ 0 & 0 & 0 & 0 & P_2 & P_1 & P_2 \\ 0 & 0 & 0 & 0 & 0 & P_2 & P_1 + \frac{\alpha}{\beta} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} y_1 \\ L(e_1) \\ L(e_2) \\ L(e_3) \\ L(e_4) \\ L(e_5) \\ L(e_6) + \frac{y_2}{\beta} \end{bmatrix}$$