Introduction to Robotics

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This is a direct concatenation and reformatting of all lecture slides and exercises from the Robotics *course (winter term 2014/15, U Stuttgart), including a bullet point list to help prepare for exams.*

Contents

1 Introduction

Why Robotics?

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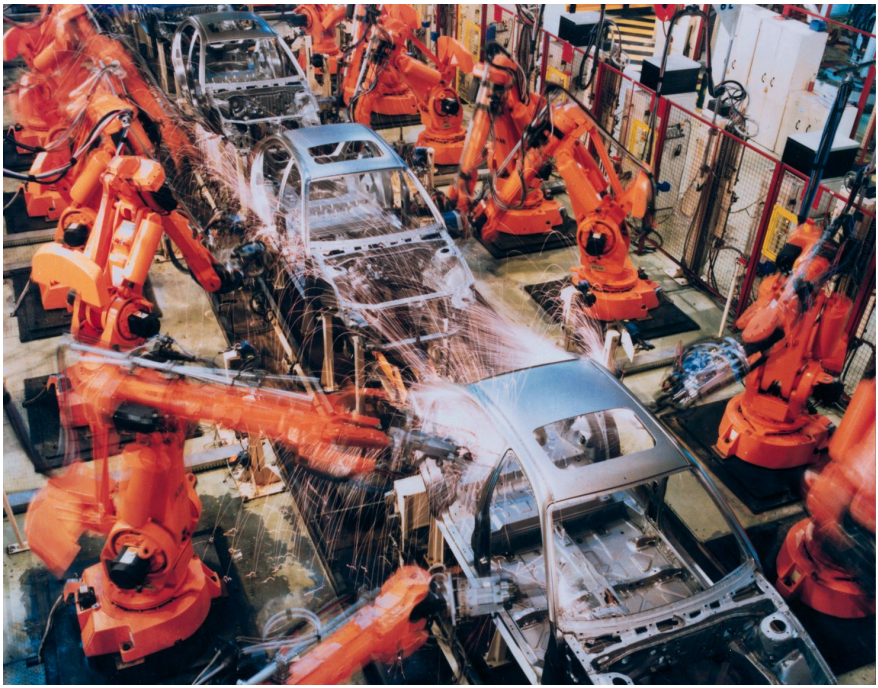

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http://www.saintpatrick.org/index.aspx/Health_Services/da_Vinci_Surgery

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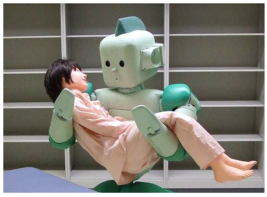


(robot “wife” aico)

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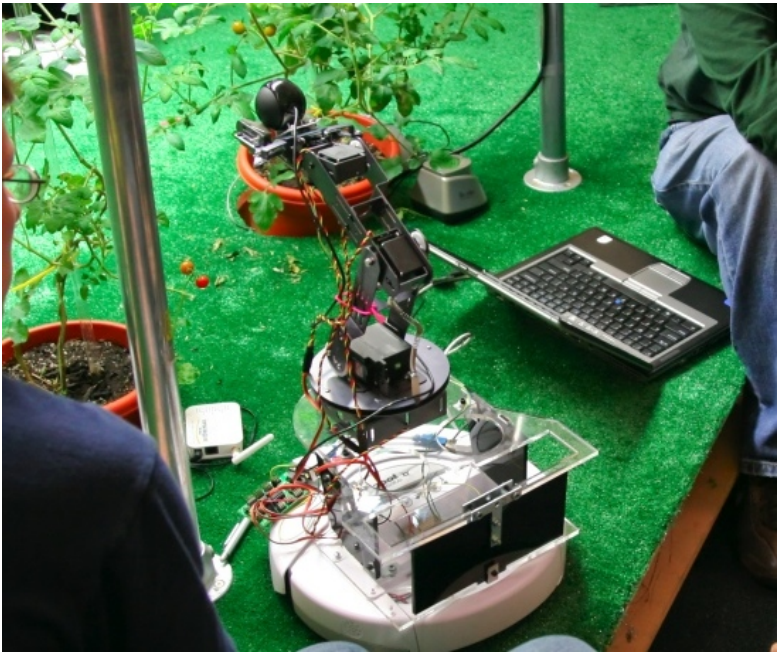


http://people.csail.mit.edu/nikolaus/drg/

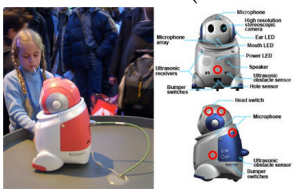
1:5

Why Robotics?

- Commercial:
  Industrial, health care, entertainment, agriculture, surgery, etc

- Critical view:
  – International Committee for Robot Arms Control
  http://icrac.net/

  – Noel Sharkey's articles on robot ethics (Child care robots PePeRo...)



http://www.nec.co.jp/products/robot/en/

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Robotics as intelligence research

AI in the real world

AI: Machine Learning, probabilistic reasoning, optimization
Real World: Interaction, manipulation, perception, navigation, etc

Why AI needs to go real world

Tunicates digest their brain once they settled!

- **Motion** was *the* driving force to develop intelligence
  - motion needs control & decision making $\leftrightarrow$ fast information processing
  - motion needs anticipation & planning
  - motion needs perception
  - motion needs spatial representations
- **Manipulation** requires to acknowledge the structure (geometry, physics, objects) of the real world. Classical AI does not

Robotics as intelligence research

- Machine Learning and AI are **computational** disciplines, which had great success with statistical modelling, analysis of data sets, symbolic reasoning. But they have not solved *autonomous learning, acting & reasoning in real worlds.*
- Neurosciences and psychology are **descriptive** sciences, either on the biological or cognitive level, e.g. with geat sucesses to describe and cure certain deceases. But they are not sufficient to create intelligent systems.
- Robotics is the only **synthetic** discipline to understand intelligent behavior in natural worlds. Robotics tells us what the actual problems are when trying to

organize behavior in natural worlds.

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History

- little movie...

(http://www.csail.mit.edu/videoarchive/history/aifilms http://www.ai.sri.com/shakey/)

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Four chapters

- **Kinematics & Dynamics**
  goal: orchestrate joint movements for desired movement in task spaces

  Kinematic map, Jacobian, optimality principle of inverse kinematics, singularities, configuration/operational/null space, multiple simultaneous tasks, special task variables, trajectory interpolation, motion profiles; 1D point mass, damping & oscillation, PID, general dynamic systems, Newton-Euler, joint space control, reference trajectory following, optimal operational space control

- **Planning & optimization**
  goal: planning around obstacles, optimizing trajectories

  Path finding vs. trajectory optimization, local vs. global, Dijkstra, Probabilistic Roadmaps, Rapidly Exploring Random Trees, differential constraints, metrics; trajectory optimization, general cost function, task variables, transition costs, gradient methods, 2nd order methods, Dynamic Programming

- **Control Theory**
  theory on designing optimal controllers

  Topics in control theory, optimal control, HJB equation, infinite horizon case, Linear-Quadratic optimal control, Riccati equations (differential, algebraic, discrete-time), controllability, stability, eigenvalue analysis, Lyapunov function

- **Mobile robots**
  goal: localize and map yourself

  State estimation, Bayes filter, odometry, particle filter, Kalman filter, Bayes smoothing, SLAM, joint Bayes filter, EKF SLAM, particle SLAM, graph-based SLAM

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Last Year's script

see https://ipvs.informatik.uni-stuttgart.de/mlr/marc/teaching/13-Robotics/13-Robotics-script.pdf

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Prerequisites

- Is this a practical or theoretical course?

  "There is nothing more practical than a good theory."
  (Vapnik, others...)

- Essentially, the whole course is about

  reducing real-world problems to mathematical problems

  that can be solved efficiently
- Required: Basics in
  - Linear Algebra
  - Probability theory
  - Optimization

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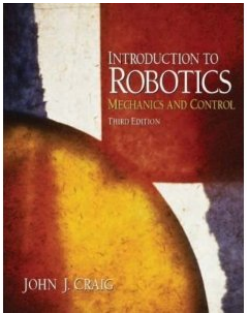
Books

There is no reference book for this lecture. But a basic well-known standard text book is:



Craig, J.J.: *Introduction to robotics: mechanics and control*. Addison-Wesley New York, 1989. (3rd edition 2006)

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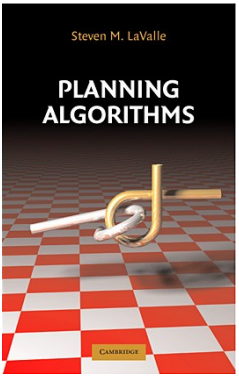
Books

An advanced text book on planning is this:



Steven M. LaValle: *Planning Algorithms*. Cambridge University Press, 2006.
online: http://planning.cs.uiuc.edu/

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Online resources

- VideoLecture by Oussama Khatib: http://academicearth.org/courses/introduction-to-robo
  http://www.virtualprofessors.com/introduction-to-robotics-stanford-cs223a-khat
  (focus on kinematics, dynamics, control)
- Oliver Brock's lecture http://courses.robotics.tu-berlin.de/mediawiki/index.php/Robotics:_Schedule_WT09
- Stefan Schaal's lecture Introduction to Robotics: http://www-clmc.usc.edu/Teaching/TeachingIn
  (focus on control, useful: Basic Linear Control Theory (analytic solution to simple dynamic model → PID), chapter on dynamics)
- Chris Atkeson's "Kinematics, Dynamic Systems, and Control" http://www.cs.cmu.edu/~cga/kdc/
  (uses Schaal's slides and LaValle's book, useful: slides on 3d kinematics http://www.cs.cmu.edu/~cga/kdc/ewhitman1.pptx)
- CMU lecture "introduction to robotics" http://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/16311/www/current/syllabus.html
  (useful: PID control, simple BUGs algorithms for motion planning, non-holonomic constraints)

- *Handbook of Robotics* (partially online at Google books) http://tiny.cc/u6tzl
- LaValle's *Planning Algorithms* http://planning.cs.uiuc.edu/

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Organization

- Course Webpage:
  http://ipvs.informatik.uni-stuttgart.de/mlr/marc/teaching/
  - Slides, Exercises & Software (C++)
  - Links to books and other resources
- Admin things, please first ask:
  Carola Stahl, Carola.Stahl@ipvs.uni-stuttgart.de, Raum 2.217
- Tutorials: Wednesdays, 15:45-17:15 & 17:30-19:00 (0.108)
- Rules for the tutorials:

- Doing the exercises is crucial!
- At the beginning of each tutorial:
  - sign into a list
  - mark which exercises you have (successfully) worked on
- Students are randomly selected to present their solutions
- You need 50% of completed exercises to be allowed to the exam
- Please check 2 weeks before the end of the term, if you can take the exam

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2 Kinematics

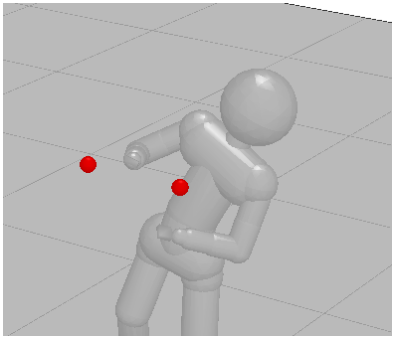
- Two "types of robotics":
  1) Mobile robotics – is all about localization & mapping
  2) Manipulation – is all about interacting with the world
  0) Kinematic/Dynamic Motion Control: same as 2) without ever making it to interaction..

- Typical manipulation robots (and animals) are kinematic trees
  Their pose/state is described by all joint angles

2:1

Basic motion generation problem

- Move all joints in a coordinated way so that the endeffector makes a desired movement



`01-kinematics: ./x.exe -mode 2/3/4`

2:2

Outline

- Basic 3D geometry and notation

- Kinematics: $\phi : q \mapsto y$
- Inverse Kinematics: $y^* \mapsto q^* = \text{argmin}_q \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$
- Basic motion heuristics: Motion profiles

- Additional things to know
  - Many simultaneous task variables

– Singularities, null space,

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2.1 Basic 3D geometry & notation

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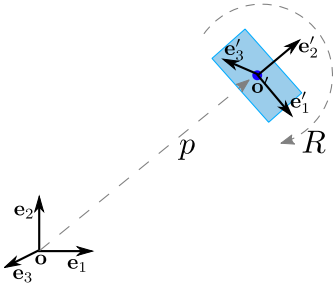
Pose (position & orientation)



- A *pose* is described by a translation $p \in \mathbb{R}^3$ and a rotation $R \in SO(3)$
  - R is an *orthonormal* matrix (orthogonal vectors stay orthogonal, unit vectors stay unit)
  - $R^{-1} = R^\top$
  - columns and rows are orthogonal unit vectors
  - $\det(R) = 1$
  - $R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$

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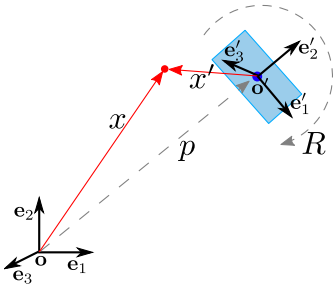
Frame and coordinate transforms



- Let $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$ be the world frame, $(\boldsymbol{o}', \boldsymbol{e}'_{1:3})$ be the body's frame.
  The new basis vectors are the *columns* in R, that is, $\boldsymbol{e}'_1 = R_{11}\boldsymbol{e}_1 + R_{21}\boldsymbol{e}_2 + R_{31}\boldsymbol{e}_3$, etc,
- x = coordinates in world frame $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$
  x' = coordinates in body frame $(\boldsymbol{o}', \boldsymbol{e}'_{1:3})$

p = coordinates of $\boldsymbol{o}'$ in world frame $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$

$$x = p + Rx'$$

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Briefly: Alternative Rotation Representations

(See the "geometry notes" for more details!)

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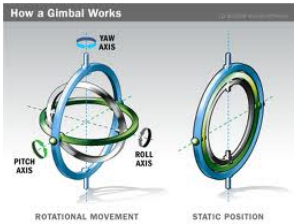
Euler angles

- Describe rotation by consecutive rotation about different axis:
  - 3-1-3 or 3-1-2 conventions, yaw-pitch-roll (3-2-1) in air flight
  - first rotate ϕ about $\boldsymbol{e}_3$, then θ about the new $\boldsymbol{e}_1'$, then ψ about the new $\boldsymbol{e}_3''$
- Gimbal Lock



- Euler angles have severe problem:
  - if two axes align: blocks 1 DoF of rotation!!
  - "singularity" of Euler angles
  - Example: 3-1-3 and second rotation 0 or π

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Rotation vector

- vector $w \in \mathbb{R}^3$
  - length $|w| = \theta$ is rotation angle (in radians)
  - direction of w = rotation axis ($\underline{w} = w/\theta$)
- Application on a vector v (Rodrigues' formula):

$$w \cdot v = \cos\theta\, v + \sin\theta\, (\underline{w} \times v) + (1 - \cos\theta)\, \underline{w}(\underline{w}^\top v)$$

- Conversion to matrix:

$$\begin{aligned} R(w) &= \exp(\hat{w}) \\ &= \cos\theta\, I + \sin\theta\, \hat{w}/\theta + (1 - \cos\theta)\, ww^\top/\theta^2 \end{aligned}$$

$$\hat{w} := \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

($\hat{w}$ is called skew matrix, with property $\hat{w}v = w \times v$; $\exp(\cdot)$ is called exponential map)

- Composition: convert to matrix first
- Drawback: singularity for small rotations

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Quaternion

- A quaternion is $r \in \mathbb{R}^4$ with unit length $|r| = r_0^2 + r_1^2 + r_2^2 + r_3^2 = 1$

$$r = \begin{pmatrix} r_0 \\ \bar{r} \end{pmatrix}, \quad r_0 = \cos(\theta/2)\,, \quad \bar{r} = \sin(\theta/2)\ \underline{w}$$

  where $\underline{w}$ is the unit length rotation axis
- Conversion to matrix

$$R(r) = \begin{pmatrix} 1 - r_{22} - r_{33} & r_{12} - r_{03} & r_{13} + r_{02} \\ r_{12} + r_{03} & 1 - r_{11} - r_{33} & r_{23} - r_{01} \\ r_{13} - r_{02} & r_{23} + r_{01} & 1 - r_{11} - r_{22} \end{pmatrix}$$

$$r_{ij} = 2 r_i r_j\,, \quad r_0 = \frac{1}{2}\sqrt{1 + \text{tr}R}$$

$$r_3 = (R_{21} - R_{12})/(4r_0)\,, \quad r_2 = (R_{13} - R_{31})/(4r_0)\,, \quad r_1 = (R_{32} - R_{23})/(4r_0)$$

- Composition

$$r \circ r' = \begin{pmatrix} r_0 r'_0 - \bar{r}^\top \bar{r}' \\ r_0 \bar{r}' + r'_0 \bar{r} + \bar{r}' \times \bar{r} \end{pmatrix}$$

- Application to vector v: convert to matrix first
- Benefits: fast composition. No $\sin/\cos$ computations. **Use this!**

2:10

Homogeneous transformations

- x^A = coordinates of a point in frame A
  x^B = coordinates of a point in frame B
- Translation and rotation: $x^A = t + Rx^B$
- Homogeneous transform $T \in \mathbb{R}^{4\times 4}$:

$$T_{A\to B} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$$

$$x^A = T_{A\to B}\, x^B = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^B \\ 1 \end{pmatrix} = \begin{pmatrix} Rx^B + t \\ 1 \end{pmatrix}$$

  in homogeneous coordinates, we append a 1 to all coordinate vectors

2:11

Is $T_{A\to B}$ forward or backward?

- $T_{A\to B}$ describes the translation and rotation of *frame* B relative to A
  That is, it describes the forward FRAME transformation (from A to B)

- $T_{A\to B}$ describes the coordinate transformation from x^B to x^A
  That is, it describes the backward COORDINATE transformation

- Confused? Vectors (and frames) transform *covariant*, coordinates *contra-variant*.
  See "geometry notes" or Wikipedia for more details, if you like.

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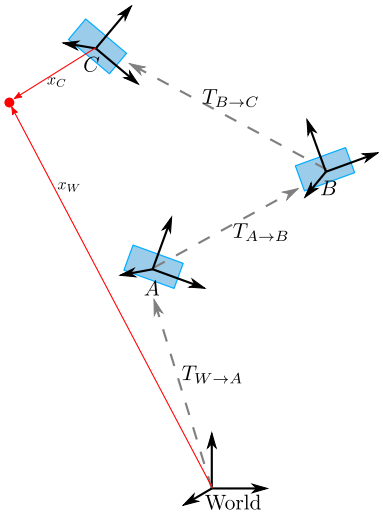
Composition of transforms



$$T_{W\to C} = T_{W\to A}\, T_{A\to B}\, T_{B\to C}$$
$$x^W = T_{W\to A}\, T_{A\to B}\, T_{B\to C}\, x^C$$

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2.2 Kinematics

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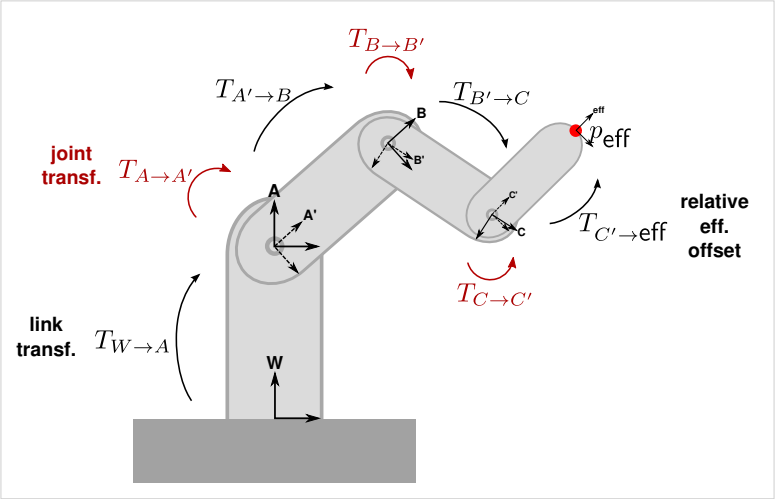
Kinematics



- A *kinematic structure* is a graph (usually tree or chain) of rigid **links** and **joints**

$$T_{W\to\text{eff}}(q) = T_{W\to A}\, T_{A\to A'}(q)\, T_{A'\to B}\, T_{B\to B'}(q)\, T_{B'\to C}\, T_{C\to C'}(q)\, T_{C'\to\text{eff}}$$

2:15

Joint types

- Joint transformations: $T_{A\to A'}(q)$ depends on $q \in \mathbb{R}^n$

  revolute joint: joint angle $q \in \mathbb{R}$ determines rotation about x-axis:

$$T_{A\to A'}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q) & -\sin(q) & 0 \\ 0 & \sin(q) & \cos(q) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  prismatic joint: offset $q \in \mathbb{R}$ determines translation along x-axis:

$$T_{A\to A'}(q) = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  others: screw (1dof), cylindrical (2dof), spherical (3dof), universal (2dof)

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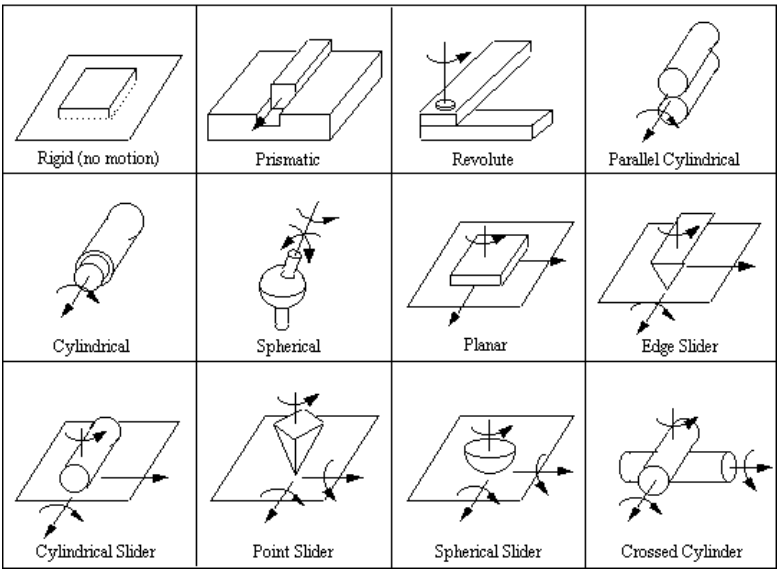

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Kinematic Map

- For any joint angle vector $q \in \mathbb{R}^n$ we can compute $T_{W\to\text{eff}}(q)$ by *forward chaining* of transformations

  $T_{W\to\text{eff}}(q)$ gives us the *pose* of the endeffector in the world frame

- In general, a kinematic map is *any* (differentiable) mapping

  $$\phi:\ q \mapsto y$$

  that maps to *some arbitrary feature* $y \in \mathbb{R}^d$ of the pose $q \in \mathbb{R}^n$

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Kinematic Map

- The three most important examples for a *kinematic map* ϕ are
  - A position v on the endeffector transformed to world coordinates:

    $$\phi^{\text{pos}}_{\text{eff},v}(q) = T_{W\to\text{eff}}(q)\ v \quad \in \mathbb{R}^3$$

  - A direction $v \in \mathbb{R}^3$ attached to the endeffector in world coordinates:

    $$\phi^{\text{vec}}_{\text{eff},v}(q) = R_{W\to\text{eff}}(q)\ v \quad \in \mathbb{R}^3$$

    Where $R_{A\to B}$ is the rotation in $T_{A\to B}$.

– The (quaternion) orientation $q \in \mathbb{R}^4$ of the endeffector:

$$\phi^{\text{quat}}_{\text{eff}}(q) = R_{W\to\text{eff}}(q) \quad \in \mathbb{R}^4$$

- See the technical reference later for more kinematic maps, especially *relative* position, direction and quaternion maps.

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Jacobian

- When we change the joint angles, δq, how does the effector position change, δy?

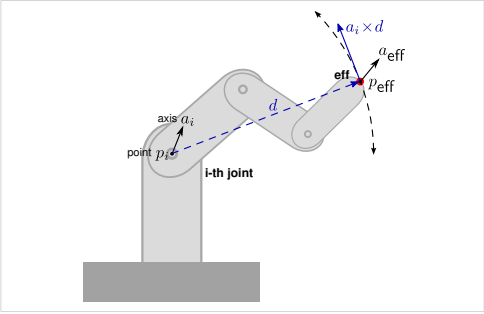
- Given the kinematic map $y = \phi(q)$ and its Jacobian $J(q) = \frac{\partial}{\partial q}\phi(q)$, we have:

$$\delta y = J(q)\, \delta q$$

$$J(q) = \frac{\partial}{\partial q}\phi(q) = \begin{pmatrix} \frac{\partial \phi_1(q)}{\partial q_1} & \frac{\partial \phi_1(q)}{\partial q_2} & \cdots & \frac{\partial \phi_1(q)}{\partial q_n} \\ \frac{\partial \phi_2(q)}{\partial q_1} & \frac{\partial \phi_2(q)}{\partial q_2} & \cdots & \frac{\partial \phi_2(q)}{\partial q_n} \\ \vdots & & & \vdots \\ \frac{\partial \phi_d(q)}{\partial q_1} & \frac{\partial \phi_d(q)}{\partial q_2} & \cdots & \frac{\partial \phi_d(q)}{\partial q_n} \end{pmatrix} \quad \in \mathbb{R}^{d\times n}$$

2:20

Jacobian for a rotational degree of freedom



- Assume the i-th joint is located at $p_i = t_{W\to i}(q)$ and has rotation axis $a_i = R_{W\to i}(q)\begin{pmatrix}1\\0\\0\end{pmatrix}$
- We consider an infinitesimal variation $\delta q_i \in \mathbb{R}$ of the ith joint and see how an endeffector position $p_{\text{eff}} = \phi^{\text{pos}}_{\text{eff},v}(q)$ and attached vector $a_{\text{eff}} = \phi^{\text{vec}}_{\text{eff},v}(q)$ change.

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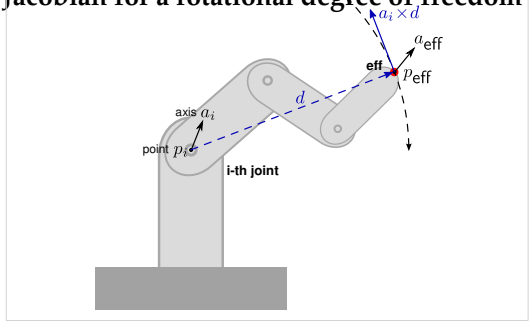
Jacobian for a rotational degree of freedom



Consider a variation δq_i
$\rightarrow$ the whole sub-tree rotates

$$\delta p_{\text{eff}} = [a_i \times (p_{\text{eff}} - p_i)]\, \delta q_i$$
$$\delta a_{\text{eff}} = [a_i \times a_{\text{eff}}]\, \delta q_i$$

$\Rightarrow$ Position Jacobian:

$$J^{\text{pos}}_{\text{eff},v}(q) = \begin{pmatrix} [a_1 \times (p_{\text{eff}} - p_1)] & [a_2 \times (p_{\text{eff}} - p_2)] & \cdots & [a_n \times (p_{\text{eff}} - p_n)] \end{pmatrix} \in \mathbb{R}^{3\times n}$$

$\Rightarrow$ Vector Jacobian:

$$J^{\text{vec}}_{\text{eff},v}(q) = \begin{pmatrix} [a_1 \times a_{\text{eff}}] & [a_2 \times a_{\text{eff}}] & \cdots & [a_n \times a_{\text{eff}}] \end{pmatrix} \in \mathbb{R}^{3\times n}$$

2:22

Jacobian for general degrees of freedom

- Every degree of freedom q_i generates (infinitesimally, at a given q)
  - a rotation around axis a_i at point p_i
  - *and/or* a translation along the axis b_i

  For instance:
  - the DOF of a hinge joint just creates a rotation around a_i at p_i
  - the DOF of a prismatic joint creates a translation along b_i
  - the DOF of a rolling cylinder creates rotation *and* translation
  - the first DOF of a cylindrical joint generates a translation, its second DOF a translation

- We can compute all Jacobians from knowing a_i, p_i and b_i for all DOFs (in the current configuration $q \in \mathbb{R}^n$)

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2.3 Inverse Kinematics

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Inverse Kinematics problem

- Generally, the aim is to find a robot configuration q such that $\phi(q) = y^*$

- Iff ϕ is invertible

$$q^* = \phi^{-1}(y^*)$$

- But in general, ϕ will not be invertible:

  1) The pre-image $\phi^{-1}(y^*) =$ may be empty: No configuration can generate the desired y^*

  2) The pre-image $\phi^{-1}(y^*)$ may be large: many configurations can generate the desired y^*

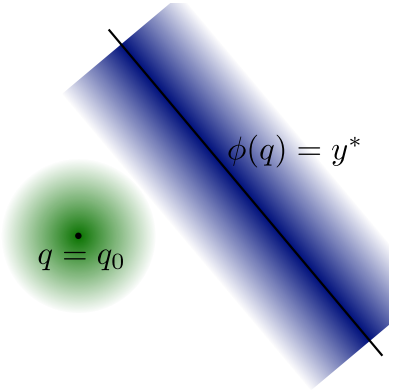
2:25

Inverse Kinematics as optimization problem

- We formalize the inverse kinematics problem as an optimization problem

$$q^* = \underset{q}{\operatorname{argmin}} \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$$

- The 1st term ensures that we find a configuration even if y^* is not exactly reachable
  The 2nd term disambiguates the configurations if there are many $\phi^{-1}(y^*)$



2:26

Inverse Kinematics as optimization problem

$$q^* = \underset{q}{\operatorname{argmin}} \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$$

- The formulation of IK as an optimization problem is very powerful and has many nice properties
- We will be able to take the limit $C \to \infty$, enforcing exact $\phi(q) = y^*$ if possible
- Non-zero C^{-1} and W corresponds to a regularization that ensures numeric stability
- Classical concepts can be derived as special cases:
  - Null-space motion
  - regularization; singularity robutness
  - multiple tasks
  - hierarchical tasks

2:27

Solving Inverse Kinematics

- The obvious choice of optimization method for this problem is Gauss-Newton, using the Jacobian of ϕ
- We first describe just one step of this, which leads to the classical equations for inverse kinematics using the local Jacobian...

2:28

Solution using the local linearization

- When using the local linearization of ϕ at q_0,

$$\phi(q) \approx y_0 + J\,(q - q_0)\,, \quad y_0 = \phi(q_0)$$

- We can derive the optimum as

$$\begin{aligned} f(q) &= \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2 \\ &= \|y_0 - y^* + J\,(q - q_0)\|_C^2 + \|q - q_0\|_W^2 \\ \frac{\partial}{\partial q} f(q) &= 0^\top = 2(y_0 - y^* + J\,(q - q_0))^\top C J + 2(q - q_0)^T W \\ J^\top C\,(y^* - y_0) &= (J^\top C J + W)\,(q - q_0) \end{aligned}$$

$$q^* = q_0 + J^\sharp (y^* - y_0)$$

with $J^\sharp = (J^\top C J + W)^{-1} J^\top C = W^{-1} J^\top (J W^{-1} J^\top + C^{-1})^{-1}$ (*Woodbury identity*)

  - For $C \to \infty$ and $W = \mathbf{I}$, $J^\sharp = J^\top (J J^\top)^{-1}$ is called *pseudo-inverse*
  - W generalizes the metric in q-space
  - C regularizes this pseudo-inverse (see later section on singularities)

2:29

"Small step" application

- This approximate solution to IK makes sense
  - if the local linearization of ϕ at q_0 is "good"
  - if q_0 and q^* are close
- This equation is therefore typically used to iteratively compute small steps in configuration space

$$q_{t+1} = q_t + J^\sharp(y^*_{t+1} - \phi(q_t))$$

where the target y^*_{t+1} moves smoothly with t

2:30

Example: Iterating IK to follow a trajectory

- Assume initial posture q_0. We want to reach a desired endeff position y^* in T steps:

Input: initial state q_0, desired y^*, methods ϕ^{pos} and J^{pos}
Output: trajectory $q_{0:T}$

Set $y_0 = \phi^{\text{pos}}(q_0)$ *// starting endeff position*
for $t = 1 : T$ **do**
  $y \leftarrow \phi^{\text{pos}}(q_{t\text{-}1})$ *// current endeff position*
  $J \leftarrow J^{\text{pos}}(q_{t\text{-}1})$ *// current endeff Jacobian*
  $\hat{y} \leftarrow y_0 + (t/T)(y^* - y_0)$ *// interpolated endeff target*
  $q_t = q_{t\text{-}1} + J^\sharp(\hat{y} - y)$ *// new joint positions*
  Command q_t to all robot motors and compute all $T_{W\to i}(q_t)$
end for

`01-kinematics: ./x.exe -mode 2/3`

- Why does this not follow the interpolated trajectory $\hat{y}_{0:T}$ exactly?
  – What happens if $T = 1$ and y^* is far?

2:31

Two additional notes

- What if we linearize at some arbitrary q' instead of q_0?

$$\begin{aligned}
\phi(q) &\approx y' + J\,(q - q')\,, \quad y' = \phi(q') \\
q^* &= \operatorname*{argmin}_q \|\phi(q) - y^*\|^2_C + \|q - q' + (q' - q_0)\|^2_W \\
&= q' + J^\sharp\,(y^* - y') + (I - J^\sharp J)\,h\,, \quad h = q_0 - q'
\end{aligned} \tag{1}$$

- What if we want to find the *exact* (local) optimum? E.g. what if we want to compute a big step (where q^* will be remote from q) and we cannot not rely only on the local linearization approximation?

- Iterate equation (1) (optionally with a step size < 1 to ensure convergence) by setting the point y' of linearization to the current q^*
- This is equivalent to the Gauss-Newton algorithm

2:32

Where are we?

- We've derived a basic motion generation principle in robotics from
  - an understanding of robot geometry & kinematics
  - a basic notion of optimality

- In the remainder:
  A. Discussion of classical concepts
  B. Heuristic motion profiles for simple trajectory generation
  C. Extension to multiple task variables

2:33

Discussion of classical concepts

- Singularity and singularity-robustness
- Nullspace, task/operational space, joint space
- "inverse kinematics" $\leftrightarrow$ "motion rate control"

2:34

Singularity

- In general: A matrix J **singular** $\iff$ $\text{rank}(J) < d$
  - rows of J are linearly dependent
  - dimension of image is $< d$
  - $\delta y = J\delta q \;\Rightarrow$ dimensions of δy limited
  - Intuition: arm fully stretched

- Implications:
  $\det(JJ^\top) = 0$
  $\rightarrow$ pseudo-inverse $J^\top(JJ^\top)^{-1}$ is ill-defined!
  $\rightarrow$ inverse kinematics $\delta q = J^\top(JJ^\top)^{-1}\delta y$ computes "infinite" steps!

- **Singularity robust pseudo inverse** $J^\top(JJ^\top + \epsilon\mathbf{I})^{-1}$
  The term $\epsilon\mathbf{I}$ is called **regularization**
- Recall our general solution (for $W = \mathbf{I}$)
  $$J^\sharp = J^\top(JJ^\top + C^{-1})^{-1}$$

is already singularity robust

2:35

Null/task/operational/joint/configuration spaces

- The space of all $q \in \mathbb{R}^n$ is called **joint/configuration space**
  The space of all $y \in \mathbb{R}^d$ is called **task/operational space**
  Usually $d < n$, which is called **redundancy**

- For a desired endeffector state y^* there exists a whole manifold (assuming ϕ is smooth) of joint configurations q:

$$\textbf{nullspace}(y^*) = \{q \mid \phi(q) = y^*\}$$

- We have

$$\begin{aligned}\delta q &= \underset{q}{\operatorname{argmin}} \|q - a\|_W^2 + \|Jq - \delta y\|_C^2 \\ &= J^{\#}\delta y + (\mathbf{I} - J^{\#}J)a \,, \quad J^{\#} = W^{-1}J^\top(JW^{-1}J^\top + C^{-1})^{-1}\end{aligned}$$

  In the limit $C \to \infty$ it is guaranteed that $J\delta q = \delta y$ (we are exacty on the manifold). The term a introduces additional "nullspace motion".

2:36

Inverse Kinematics and Motion Rate Control

Some clarification of concepts:

- The notion "kinematics" describes the mapping $\phi:\ q \mapsto y$, which usually is a many-to-one function.
- The notion "inverse kinematics" in the strict sense describes some mapping $g:\ y \mapsto q$ such that $\phi(g(y)) = y$, which usually is non-unique or ill-defined.
- In practice, one often refers to $\delta q = J^\sharp \delta y$ as **inverse kinematics**.

- When iterating $\delta q = J^\sharp \delta y$ in a control cycle with time step τ (typically $\tau \approx 1{-}10$ msec), then $\dot y = \delta y/\tau$ and $\dot q = \delta q/\tau$ and $\dot q = J^\sharp \dot y$. Therefore the control cycle effectively controls the endeffector velocity—this is why it is called **motion rate control**.

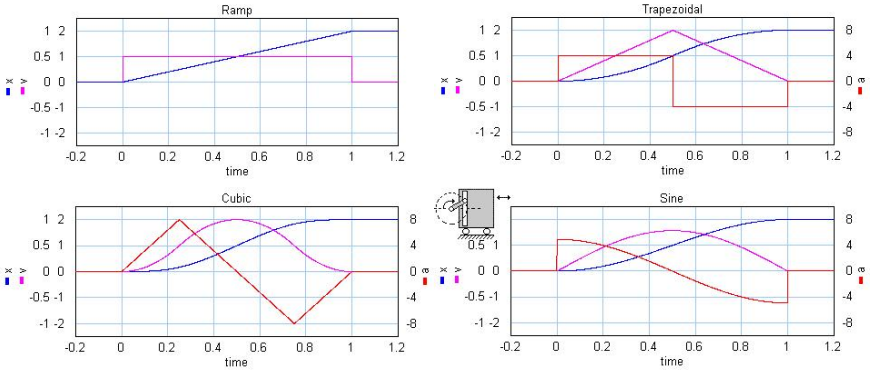
2:37

Heuristic motion profiles

2:38

Heuristic motion profiles

- Assume initially $x = 0, \dot{x} = 0$. After 1 second you want $x = 1, \dot{x} = 0$.
  How do you move from $x = 0$ to $x = 1$ in one second?



The sine profile $x_t = x_0 + \frac{1}{2}[1 - \cos(\pi t/T)](x_T - x_0)$ is a compromise for low max-acceleration and max-velocity

Taken from http://www.20sim.com/webhelp/toolboxes/mechatronics_toolbox/motion_profile_wizard/motionprofiles.htm

2:39

Motion profiles

- Generally, let's define a motion profile as a mapping

$$\text{MP} : [0, 1] \mapsto [0, 1]$$

with $\text{MP}(0) = 0$ and $\text{MP}(1) = 1$ such that the interpolation is given as

$$x_t = x_0 + \text{MP}(t/T)\ (x_T - x_0)$$

- For example

$$\text{MP}_{\text{ramp}}(s) = s$$
$$\text{MP}_{\text{sin}}(s) = \frac{1}{2}[1 - \cos(\pi s)]$$

2:40

Joint space interpolation

1) Optimize a desired final configuration q_T:
Given a desired final task value y_T, optimize a final joint state q_T to minimize the function

$$f(q_T) = \|q_T - q_0\|^2_{W/T} + \|y_T - \phi(q_T)\|^2_C$$

– The metric $\frac{1}{T}W$ is consistent with T cost terms with step metric W.
– In this optimization, q_T will end up remote from q_0. So we need to iterate Gauss-Newton, as described on slide 2.3.

2) Compute $q_{0:T}$ as interpolation between q_0 and q_T:
Given the initial configuration q_0 and the final q_T, interpolate on a straight line with a some motion profile. E.g.,

$$q_t = q_0 + \text{MP}(t/T)\ (q_T - q_0)$$

2:41

Task space interpolation

1) Compute $y_{0:T}$ as interpolation between y_0 and y_T:
Given a initial task value y_0 and a desired final task value y_T, interpolate on a straight line with a some motion profile. E.g,

$$y_t = y_0 + \text{MP}(t/T)\ (y_T - y_0)$$

2) Project $y_{0:T}$ to $q_{0:T}$ using inverse kinematics:
Given the task trajectory $y_{0:T}$, compute a corresponding joint trajectory $q_{0:T}$ using inverse kinematics

$$q_{t+1} = q_t + J^\sharp(y_{t+1} - \phi(q_t))$$

(As steps are small, we should be ok with just using this local linearization.)

2:42

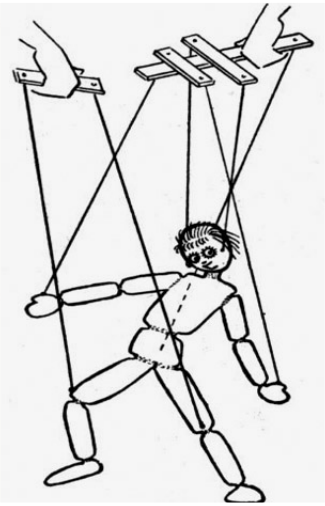
`peg-in-a-hole demo`

2:43

2.4 Multiple tasks

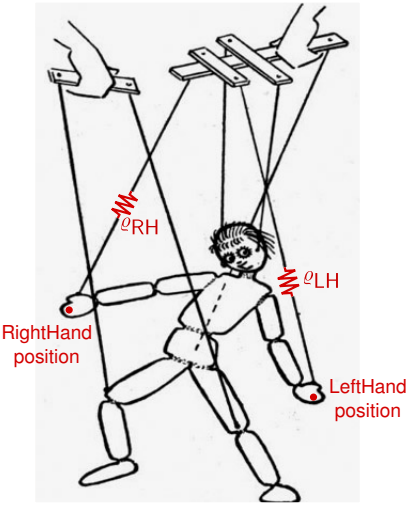
2:44

Multiple tasks



2:45

Multiple tasks



2:46

Multiple tasks

- Assume we have m simultaneous tasks; for each task i we have:
  - a kinematic map $\phi_i : \mathbb{R}^n \to \mathbb{R}^{d_i}$
  - a current value $\phi_i(q_t)$

  - a desired value y_i^*
  - a precision ϱ_i (equiv. to a task cost metric $C_i = \varrho_i \mathbf{I}$)

- Each task contributes a term to the objective function

$$q^* = \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \varrho_1 \|\phi_1(q) - y_1^*\|^2 + \varrho_2 \|\phi_2(q) - y_2^*\|^2 + \cdots$$

  which we can also write as

$$q^* = \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \|\Phi(q)\|^2$$

$$\text{where } \Phi(q) := \begin{pmatrix} \sqrt{\varrho_1}\,(\phi_1(q) - y_1^*) \\ \sqrt{\varrho_2}\,(\phi_2(q) - y_2^*) \\ \vdots \end{pmatrix} \quad \in \mathbb{R}^{\sum_i d_i}$$

2:47

Multiple tasks

- We can "pack" together all tasks in one "big task" Φ.

  Example: We want to control the 3D position of the left hand and of the right hand. Both are "packed" to one 6-dimensional task vector which becomes zero if both tasks are fulfilled.

- The big Φ is scaled/normalized in a way that
  - the desired value is always zero
  - the cost metric is $\mathbf{I}$

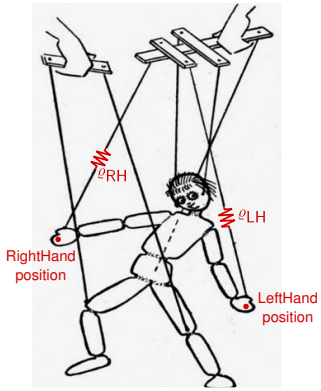
- Using the local linearization of Φ at q_0, $J = \frac{\partial \Phi(q_0)}{\partial q}$, the optimum is

$$q^* = \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \|\Phi(q)\|^2$$

$$\approx q_0 - (J^\top J + W)^{-1} J^\top \Phi(q_0) = q_0 - J^\# \Phi(q_0)$$

2:48

Multiple tasks



- We learnt how to "puppeteer a robot"
- We can handle many task variables (but specifying their precisions ϱ_i becomes cumbersome...)
- In the remainder:
  A. Classical limit of "hierarchical IK" and nullspace motion
  B. What are interesting task variables?

2:49

Hierarchical IK & nullspace motion

- In the classical view, tasks should be executed *exactly*, which means taking the limit $\varrho_i \to \infty$ in some prespecified hierarchical order.
- We can rewrite the solution in a way that allows for such a hierarchical limit:
- One task plus "nullspace motion":

$$\begin{aligned} f(q) &= \|q-a\|_W^2 + \varrho_1\|J_1 q - y_1\|^2 \\ q^* &= [W + \varrho_1 J_1^\top J_1]^{-1}\,[Wa + \varrho_1 J_1^\top y_1] \\ &= J_1^\# y_1 + (\mathbf{I} - J_1^\# J_1)a \\ J_1^\# &= (W/\varrho_1 + J_1^\top J_1)^{-1} J_1^\top = W^{-1}J_1^\top(J_1 W^{-1} J_1^\top + \mathbf{I}/\varrho_1)^{-1} \end{aligned}$$

- Two tasks plus nullspace motion:

$$\begin{aligned} f(q) &= \|q-a\|_W^2 + \varrho_1\|J_1 q - y_1\|^2 + \varrho_2\|J_2 q - y_2\|^2 \\ q^* &= J_1^\# y_1 + (\mathbf{I} - J_1^\# J_1)[J_2^\# y_2 + (\mathbf{I} - J_2^\# J_2)a] \\ J_2^\# &= (W/\varrho_2 + J_2^\top J_2)^{-1} J_2^\top = W^{-1}J_2^\top(J_2 W^{-1} J_2^\top + \mathbf{I}/\varrho_2)^{-1} \end{aligned}$$

- etc...

2:50

Hierarchical IK & nullspace motion

- The previous slide did nothing but rewrite the nice solution $q^* = -J^\# \Phi(q_0)$ (for the "big" Φ) in a strange hierarchical way that allows to "see" nullspace projection
- The benefit of this hierarchical way to write the solution is that one can take the hierarchical limit $\varrho_i \to \infty$ and retrieve classical hierarchical IK

- The drawbacks are:
  - It is somewhat ugly
  - In practise, I would recommend regularization in any case (for numeric stability). Regularization corresponds to NOT taking the full limit $\varrho_i \to \infty$. Then the hierarchical way to write the solution is unnecessary. (However, it points to a "hierarchical regularization", which might be numerically more robust for very small regularization?)
  - The general solution allows for arbitrary blending of tasks

2:51

Reference: interesting task variables

The following slides will define 10 different types of task variables.
This is meant as a reference and to give an idea of possibilities...

2:52

Position

Position of some point attached to link i	
dimension	$d = 3$
parameters	link index i, point offset v
kin. map	$\phi_{iv}^{\text{pos}}(q) = T_{W\to i}\, v$
Jacobian	$J_{iv}^{\text{pos}}(q)_{\cdot k} = [k \prec i]\, a_k \times (\phi_{iv}^{\text{pos}}(q) - p_k)$

Notation:

- a_k, p_k are axis and position of joint k
- $[k \prec i]$ indicates whether joint k is between root and link i
- $J_{\cdot k}$ is the kth column of J

2:53

Vector

Vector attached to link i	
dimension	$d = 3$
parameters	link index i, attached vector v
kin. map	$\phi_{iv}^{\text{vec}}(q) = R_{W\to i}\, v$
Jacobian	$J_{iv}^{\text{vec}}(q) = A_i \times \phi_{iv}^{\text{vec}}(q)$

Notation:

- A_i is a matrix with columns $(A_i)_{\cdot k} = [k \prec i]\, a_k$ containing the joint axes or zeros
- the short notation "$A \times p$" means that each *column* in A takes the cross-product with p.

2:54

Relative position

Position of a point on link i relative to point on link j	
dimension	$d = 3$
parameters	link indices i, j, point offset v in i and w in j
kin. map	$\phi^{\text{pos}}_{iv\|jw}(q) = R_j^{-1}(\phi^{\text{pos}}_{iv} - \phi^{\text{pos}}_{jw})$
Jacobian	$J^{\text{pos}}_{iv\|jw}(q) = R_j^{-1}[J^{\text{pos}}_{iv} - J^{\text{pos}}_{jw} - A_j \times (\phi^{\text{pos}}_{iv} - \phi^{\text{pos}}_{jw})]$

Derivation:

For $y = Rp$ the derivative w.r.t. a rotation around axis a is $y' = Rp' + R'p = Rp' + a \times Rp$. For $y = R^{-1}p$ the derivative is $y' = R^{-1}p' - R^{-1}(R')R^{-1}p = R^{-1}(p' - a \times p)$. (For details see http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/3d-geometry.pdf)

2:55

Relative vector

Vector attached to link i relative to link j	
dimension	$d = 3$
parameters	link indices i, j, attached vector v in i
kin. map	$\phi^{\text{vec}}_{iv\|j}(q) = R_j^{-1}\phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{vec}}_{iv\|j}(q) = R_j^{-1}[J^{\text{vec}}_{iv} - A_j \times \phi^{\text{vec}}_{iv}]$

2:56

Alignment

Alignment of a vector attached to link i with a reference v^*	
dimension	$d = 1$
parameters	link index i, attached vector v, world reference v^*
kin. map	$\phi^{\text{align}}_{iv}(q) = v^{*\top}\,\phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{align}}_{iv}(q) = v^{*\top}\,J^{\text{vec}}_{iv}$

Note: $\phi^{\text{align}} = 1 \leftrightarrow$ align $\quad \phi^{\text{align}} = -1 \leftrightarrow$ anti-align $\quad \phi^{\text{align}} = 0 \leftrightarrow$ orthog.

2:57

Relative Alignment

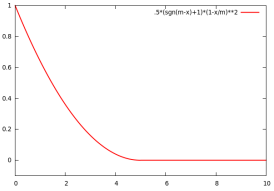
Alignment a vector attached to link i with vector attached to j	
dimension	$d = 1$
parameters	link indices i, j, attached vectors v, w
kin. map	$\phi^{\text{align}}_{iv\|jw}(q) = (\phi^{\text{vec}}_{jw})^\top\,\phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{align}}_{iv\|jw}(q) = (\phi^{\text{vec}}_{jw})^\top\,J^{\text{vec}}_{iv} + \phi^{\text{vec}\top}_{iv}\,J^{\text{vec}}_{jw}$

2:58

Joint limits

Penetration of joint limits	
dimension	$d = 1$
parameters	joint limits $q_{\text{low}}, q_{\text{hi}}$, margin m
kin. map	$\phi_{\text{limits}}(q) = \frac{1}{m}\sum_{i=1}^{n}[m - q_i + q_{\text{low}}]^+ + [m + q_i - q_{\text{hi}}]^+$
Jacobian	$J_{\text{limits}}(q)_{1,i} = -\frac{1}{m}[m - q_i + q_{\text{low}} > 0] + \frac{1}{m}[m + q_i - q_{\text{hi}} > 0]$

$[x]^+ = x > 0?x : 0$ $[\cdots]$: indicator function



2:59

Collision limits

Penetration of collision limits	
dimension	$d = 1$
parameters	margin m
kin. map	$\phi_{\text{col}}(q) = \frac{1}{m}\sum_{k=1}^{K}[m - \lvert p_k^a - p_k^b\rvert]^+$
Jacobian	$J_{\text{col}}(q) = \frac{1}{m}\sum_{k=1}^{K}[m - \lvert p_k^a - p_k^b\rvert > 0]$ $(-J^{\text{pos}}_{p_k^a} + J^{\text{pos}}_{p_k^b})^\top \frac{p_k^a - p_k^b}{\lvert p_k^a - p_k^b\rvert}$

A collision detection engine returns a set $\{(a, b, p^a, p^b)_{k=1}^K\}$ of potential collisions between link a_k and b_k, with nearest points p_k^a on a and p_k^b on b.

2:60

Center of gravity

Center of gravity of the whole kinematic structure	
dimension	$d = 3$
parameters	(none)
kin. map	$\phi^{\text{cog}}(q) = \sum_i \text{mass}_i\ \phi^{\text{pos}}_{ic_i}$
Jacobian	$J^{\text{cog}}(q) = \sum_i \text{mass}_i\ J^{\text{pos}}_{ic_i}$

c_i denotes the center-of-mass of link i (in its own frame)

2:61

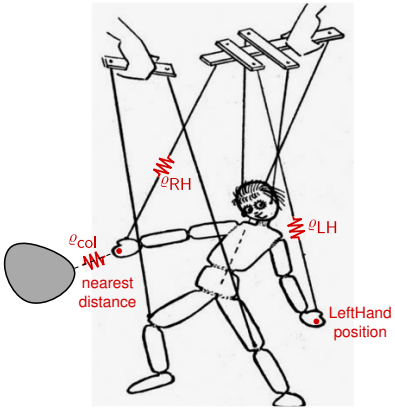
Homing

The joint angles themselves	
dimension	$d = n$
parameters	(none)
kin. map	$\phi_{\text{qitself}}(q) = q$
Jacobian	$J_{\text{qitself}}(q) = \mathbf{I}_n$

Example: Set the target $y^* = 0$ and the precision ϱ very low $\rightarrow$ this task describes posture comfortness in terms of deviation from the joints' zero position. In the classical view, it induces "nullspace motion".

2:62

Task variables – conclusions



- There is much space for creativity in defining task variables! Many are extensions of ϕ^{pos} and ϕ^{vec} and the Jacobians combine the basic Jacobians.
- What the *right* task variables are to design/describe motion is a very hard problem! In what task space do humans control their motion? Possible to learn from data ("task space retrieval") or perhaps via Reinforcement Learning.
- In practice: Robot motion design (including grasping) may require cumbersome hand-tuning of such task variables.

2:63

- Technical Reference: The four rotation axes of a quaternion joint:

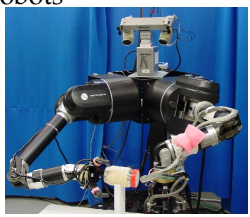
A quaternion joint has four DOFs. If it is currently in configuration $q \in \mathbb{R}^4$, the ith DOFs generates (infinitesimally) a rotation around the axis

$$a_i = \frac{-2}{\sqrt{q^2}}[e_i \circ q^{-1}]_{1:3}$$

where $e_i \in \mathbb{R}^4$ is the ith unit vector, $\circ$ is the concatenation of quaternions, q^{-1} the inverse quaternion, q^2 the quaternion 2-norm (in case it is not normalized), and $[\cdot]_{1:3}$ pics the vector elements of the quaternion (derivation: see geometry notes). As for the hinge joint, these four axes are further transformed to world coordinates, $a_i \leftarrow R_{W\rightarrow j} a_i$, if the quaternion joint is located in the coordinate frame j.

2:64

3 Dynamics

Kinematic	Dynamic
instantly change joint velocities $\dot{q}$: $\delta q_t \overset{!}{=} J^\sharp\,(y^* - \phi(q_t))$	instantly change joint torques u: $u \overset{!}{=} ?$
accounts for kinematic coupling of joints but **ignores inertia, forces, torques**	accounts for dynamic coupling of joints and full Newtonian physics
gears, **stiff**, all of industrial robots 	future robots, **compliant**, few research robots 

3:1

When velocities cannot be changed/set arbitrarily

- Examples:
  - An air plane flying: You cannot command it to hold still in the air, or to move straight up.
  - A car: you cannot command it to move side-wards.
  - Your arm: you cannot command it to throw a ball with arbitrary speed (force limits).
  - A *torque controlled* robot: You cannot command it to instantly change velocity (infinite acceleration/torque).

- What all examples have in comment:
  - One can set **controls** u_t (air plane's control stick, car's steering wheel, your muscles activations, torque/voltage/current send to a robot's motors)
  - But these controls only indirectly influence the **dynamics of state**

$$x_{t+1} = f(x_t, u_t)$$

3:2

Dynamics

- The dynamics of a system describes how the controls u_t influence the change-of-state of the system

$$x_{t+1} = f(x_t, u_t)$$

  - The notation x_t refers to the *dynamic state* of the system: e.g., joint positions *and velocities* $x_t = (q_t, \dot{q}_t)$.
  - f is an arbitrary function, often smooth

3:3

Outline

- We start by discussing a **1D point mass** for 3 reasons:
  - The most basic force-controlled system with inertia
  - We can introduce and understand **PID control**
  - The behavior of a point mass under PID control is a *reference* that we can also follow with arbitrary dynamic robots (if the dynamics are known)

- We discuss computing the dynamics of general robotic systems
  - Euler-Lagrange equations
  - Euler-Newton method
- We derive the dynamic equivalent of inverse kinematics:
  - operational space control

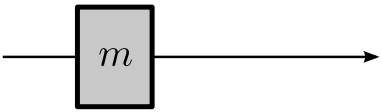
3:4

3.1 PID and a 1D point mass

3:5

The dynamics of a 1D point mass

- Start with simplest possible example: 1D point mass
  (no gravity, no friction, just a single mass)



- The state $x(t) = (q(t), \dot{q}(t))$ is described by:
  - position $q(t) \in \mathbb{R}$
  - velocity $\dot{q}(t) \in \mathbb{R}$

- The controls $u(t)$ is the force we apply on the mass point
- The system dynamics is:

$$\ddot{q}(t) = u(t)/m$$

3:6

1D point mass – proportional feedback

- Assume current position is q.
  The goal is to move it to the position q^*.

  What can we do?
- **Idea 1:**
  "Always pull the mass towards the goal q^*:"

$$u = K_p\,(q^* - q)$$

3:7

1D point mass – proportional feedback

- What's the effect of this control law?

$$m\,\ddot{q} = u = K_p\,(q^* - q)$$

  $q = q(t)$ is a function of time, this is a second order differential equation
- Solution: assume $q(t) = a + be^{\omega t}$
  (a "non-imaginary" alternative would be $q(t) = a + b\,\epsilon^{-\lambda t}\,\cos(\omega t)$)

$$\begin{aligned}
&m\,b\,\omega^2\,e^{\omega t} = K_p\,q^* - K_p\,a - K_p\,b\,e^{\omega t} \\
&(m\,b\,\omega^2 + K_p\,b)\,e^{\omega t} = K_p\,(q^* - a) \\
&\Rightarrow (m\,b\,\omega^2 + K_p\,b) = 0 \;\wedge\; (q^* - a) = 0 \\
&\Rightarrow\; \omega = i\sqrt{K_p/m} \\
&q(t) = q^* + b\,e^{i\sqrt{K_p/m}\,t}
\end{aligned}$$

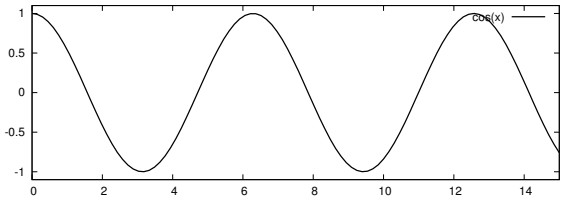
  This is an oscillation around q^* with amplitude $b = q(0) - q^*$ and frequency $\sqrt{K_p/m}$!

3:8

1D point mass – proportional feedback

$$m\,\ddot{q} = u = K_p\,(q^* - q)$$
$$q(t) = q^* + b\,e^{i\sqrt{K_p/m}\,t}$$

Oscillation around q^* with amplitude $b = q(0) - q^*$ and frequency $\sqrt{K_p/m}$



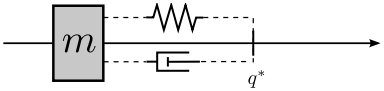
3:9

1D point mass – derivative feedback

- **Idea 2**
  "Pull less, when we're heading the right direction already:"
  "Damp the system:"

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q})$$

$\dot{q}^*$ is a desired goal velocity
For simplicity we set $\dot{q}^* = 0$ in the following.



3:10

1D point mass – derivative feedback

- What's the effect of this control law?

$$m\ddot{q} = u = K_p(q^* - q) + K_d(0 - \dot{q})$$

- Solution: again assume $q(t) = a + be^{\omega t}$

$$m\,b\,\omega^2\,e^{\omega t} = K_p\,q^* - K_p\,a - K_p\,b\,e^{\omega t} - K_d\,b\,\omega e^{\omega t}$$
$$(m\,b\,\omega^2 + K_d\,b\,\omega + K_p\,b)\,e^{\omega t} = K_p\,(q^* - a)$$

$$\Rightarrow (m\,\omega^2 + K_d\,\omega + K_p) = 0 \;\wedge\; (q^* - a) = 0$$
$$\Rightarrow \omega = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m}$$
$$q(t) = q^* + b\,e^{\omega\,t}$$

The term $-\frac{K_d}{2m}$ in ω is real $\leftrightarrow$ exponential decay (damping)

3:11

1D point mass – derivative feedback

$$q(t) = q^* + b\,e^{\omega\,t}\,, \quad \omega = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m}$$

- Effect of the second term $\sqrt{K_d^2 - 4mK_p}/2m$ in ω:

$K_d^2 < 4mK_p \;\Rightarrow$ ω has imaginary part
oscillating with frequency $\sqrt{K_p/m - K_d^2/4m^2}$
$q(t) = q^* + be^{-K_d/2m\;t}\;e^{i\sqrt{K_p/m - K_d^2/4m^2}\;t}$

$K_d^2 > 4mK_p \;\Rightarrow$ ω real
strongly damped

$K_d^2 = 4mK_p \;\Rightarrow$ second term zero
only exponential decay

3:12

1D point mass – derivative feedback

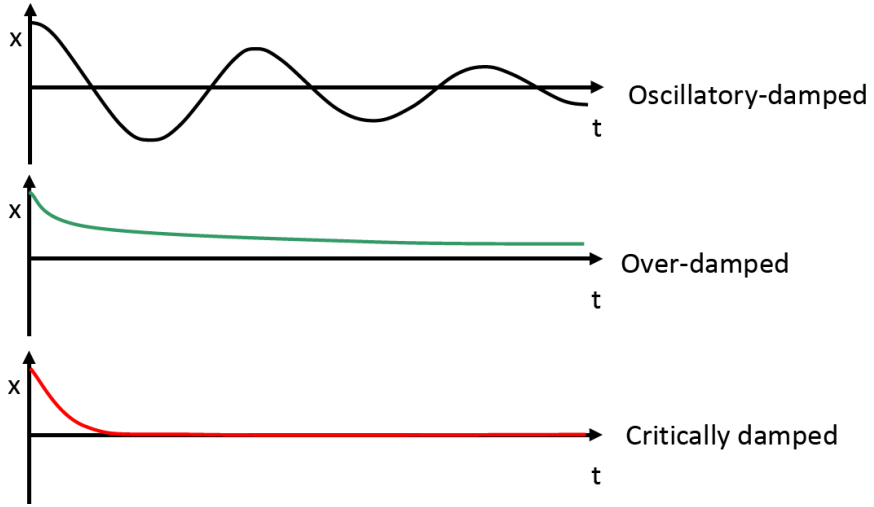


illustration from O. Brock's lecture

3:13

1D point mass – derivative feedback

Alternative parameterization:
Instead of the *gains* K_p and K_d it is sometimes more intuitive to set the

- wave length $\lambda = \frac{1}{\omega_0} = \frac{1}{\sqrt{K_p/m}}$, $\quad K_p = m/\lambda^2$
- damping ratio $\xi = \frac{K_d}{\sqrt{4mK_p}} = \frac{\lambda K_d}{2m}$, $\quad K_d = 2m\xi/\lambda$

$\xi > 1$: over-damped
$\xi = 1$: critically dampled
$\xi < 1$: oscillatory-damped

$$q(t) = q^* + be^{-\xi\ t/\lambda}\ e^{i\sqrt{1-\xi^2}\ t/\lambda}$$

3:14

1D point mass – integral feedback

- **Idea 3**
  "Pull if the position error accumulated large in the past:"

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q}) + K_i \int_{s=0}^{t} (q^*(s) - q(s))\ ds$$

- This is not a linear ODE w.r.t. $x = (q, \dot{q})$.
  However, when we extend the state to $x = (q, \dot{q}, e)$ we have the ODE

$$\begin{aligned}\dot{q} &= \dot{q} \\ \ddot{q} &= u/m = K_p/m(q^* - q) + K_d/m(\dot{q}^* - \dot{q}) + K_i/m\ e \\ \dot{e} &= q^* - q\end{aligned}$$

(no explicit discussion here)

3:15

1D point mass – PID control

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q}) + K_i \int_{s=0}^{t} (q^* - q(s))\ ds$$

- **PID control**
  – Proportional Control ("Position Control")

$u \propto K_p(q^* - q)$

– Derivative Control ("Damping")
$u \propto K_d(\dot{q}^* - \dot{q})$ $(\dot{x}^* = 0 \to \text{damping})$

– Integral Control ("Steady State Error")
$u \propto K_i \int_{s=0}^{t} (q^*(s) - q(s))\, ds$

3:16

Controlling a 1D point mass – lessons learnt

- Proportional and derivative feedback (PD control) are like adding a spring and damper to the point mass

- PD control is a *linear control law*

$$(q, \dot{q}) \mapsto u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q})$$

(linear in the *dynamic system state* $x = (q, \dot{q})$)

- With such linear control laws we can design approach trajectories (by tuning the gains)
– but no optimality principle behind such motions

3:17

3.2 Dynamics of mechanical systems

3:18

Two ways to derive dynamics equations for mechanical systems

- The Euler-Lagrange equation, $L = L(t, q(t), \dot{q}(t))$,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u$$

Used when you want to derive analytic equations of motion ("on paper")

- The Newton-Euler recursion (and related algorithms)

$$f_i = m\dot{v}_i\ , \quad u_i = I_i\dot{w} + w \times Iw$$

Algorithms that "propagate" forces through a kinematic tree and numerically compute the *inverse* dynamics $u = \text{NE}(q, \dot{q}, \ddot{q})$ or *forward* dynamics $\ddot{q} = f(q, \dot{q}, u)$.

3:19

The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u$$

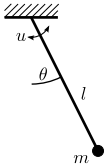
- $L(q, \dot{q})$ is called **Lagrangian** and defined as

$$L = T - U$$

  where T=kinetic energy and U=potential energy.
- q is called generalized coordinate – any coordinates such that $(q, \dot{q})$ describes the state of the system. Joint angles in our case.
- u are external forces

3:20

Example: A pendulum



- Generalized coordinates: angle $q = (\theta)$
- Kinematics:
  - velocity of the mass: $v = (l\dot{\theta}\cos\theta, 0, l\dot{\theta}\sin\theta)$
  - angular velocity of the mass: $w = (0, -\dot{\theta}, 0)$
- Energies:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}w^\top I w = \frac{1}{2}(ml^2 + I_2)\dot{\theta}^2\,, \quad U = -mgl\cos\theta$$

- Euler-Lagrange equation:

$$\begin{aligned} u &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \\ &= \frac{d}{dt}(ml^2 + I_2)\dot{\theta} + mgl\sin\theta = (ml^2 + I_2)\ddot{\theta} + mgl\sin\theta \end{aligned}$$

3:21

The Euler-Lagrange equation

- How is this typically done?
- **First**, describe the *kinematics and Jacobians* for every link i:

$$(q, \dot{q}) \mapsto \{T_{W\to i}(q), v_i, w_i\}$$

  Recall $T_{W\to i}(q) = T_{W\to A}\, T_{A\to A'}(q)\, T_{A'\to B}\, T_{B\to B'}(q)\, \cdots$
  Further, we know that a link's velocity $v_i = J_i\dot{q}$ can be described via its position Jacobian.
  Similarly we can describe the link's *angular velocity* $w_i = J_i^w\dot{q}$ as linear in $\dot{q}$.

- **Second**, formulate the kinetic energy

$$T = \sum_i \frac{1}{2} m_i v_i^2 + \frac{1}{2} w_i^\top I_i w_i = \sum_i \frac{1}{2} \dot{q}^\top M_i \dot{q} \,, \quad M_i = \begin{pmatrix} J_i \\ J_i^w \end{pmatrix}^\top \begin{pmatrix} m_i \mathbf{I}_3 & 0 \\ 0 & I_i \end{pmatrix} \begin{pmatrix} J_i \\ J_i^w \end{pmatrix}$$

  where $I_i = R_i \bar{I}_i R_i^\top$ and $\bar{I}_i$ the inertia tensor in link coordinates
- **Third**, formulate the potential energies (typically independent of $\dot{q}$)

$$U = g m_i \text{height}(i)$$

- **Fourth**, compute the partial derivatives analytically to get something like

$$\underbrace{u}_{\text{control}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \underbrace{M}_{\text{inertia}} \ddot{q} + \underbrace{\dot{M}\dot{q} - \frac{\partial T}{\partial q}}_{\text{Coriolis}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{gravity}}$$

  which relates accelerations $\ddot{q}$ to the forces

3:22

Newton-Euler recursion

- An algorithm that computes the *inverse dynamics*

$$u = \text{NE}(q, \dot{q}, \ddot{q}^*)$$

  by recursively computing force balance at each joint:
  - **Newton's equation** expresses the force acting at the center of mass for an accelerated body:

$$f_i = m \dot{v}_i$$

  - **Euler's equation** expresses the torque (=control) acting on a rigid body given an angular velocity and angular acceleration:

$$u_i = I_i \dot{w} + w \times I w$$

- **Forward recursion:** ($\approx$ kinematics)
  Compute (angular) velocities (v_i, w_i) *and* accelerations $(\dot{v}_i, \dot{w}_i)$ for every link (via forward propagation; see geometry notes for details)

- **Backward recursion:**
  For the leaf links, we now know the desired accelerations q^* and can compute the necessary joint torques. Recurse backward.

3:23

Numeric algorithms for forward and inverse dynamics

- **Newton-Euler recursion**: very fast ($O(n)$) method to compute *inverse* dynamics
$$u = \text{NE}(q, \dot{q}, \ddot{q}^*)$$
Note that we can use this algorithm to also compute
  - gravity terms: $u = \text{NE}(q, 0, 0) = G(q)$
  - Coriolis terms: $u = \text{NE}(q, \dot{q}, 0) = C(q, \dot{q})\,\dot{q} + G(q)$
  - column of Intertia matrix: $u = \text{NE}(q, 0, e_i) = M(q)\,e_i$

- **Articulated-Body-Dynamics**: fast method ($O(n)$) to compute *forward* dynamics $\ddot{q} = f(q, \dot{q}, u)$

3:24

Some last practical comments

- `[demo]`
- Use *energy conservation* to measure dynamic of physical simulation
- Physical simulation engines (developed for games):
  - ODE (Open Dynamics Engine)
  - Bullet (originally focussed on collision only)
  - Physx (Nvidia)

  Differences of these engines to Lagrange, NE or ABD:
  - Game engine can model much more: Contacts, tissues, particles, fog, etc
  - (The way they model contacts looks ok but is somewhat fictional)
  - On kinematic trees, NE or ABD are much more precise than game engines
  - Game engines do not provide *inverse* dynamics, $u = \text{NE}(q, \dot{q}, \ddot{q})$

- Proper modelling of contacts is really really hard

3:25

3.3 Controlling a dynamic robot

3:26

- We previously learnt the effect of PID control on a 1D point mass

- Robots are not a 1D point mass
  - Neither is each joint a 1D point mass
  - Applying separate PD control in each joint neglects force coupling
    (Poor solution: Apply very high gains separately in each joint $\leftrightarrow$ make joints stiff, as with gears.)

- However, knowing the robot dynamics we can transfer our understanding of PID control of a point mass to general systems

3:27

General robot dynamics

- Let $(q, \dot{q})$ be the dynamic state and $u \in \mathbb{R}^n$ the controls (typically joint torques in each motor) of a robot
- Robot dynamics can generally be written as:

$$M(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = u$$

$M(q) \in \mathbb{R}^{n\times n}$	is positive definite intertia matrix (can be inverted $\rightarrow$ forward simulation of dynamics)
$C(q,\dot{q}) \in \mathbb{R}^n$	are the centripetal and coriolis forces
$G(q) \in \mathbb{R}^n$	are the gravitational forces
u	are the joint torques

(cf. to the Euler-Lagrange equation on slide 22)

- We often write more compactly:

$$M(q)\,\ddot{q} + F(q,\dot{q}) = u$$

3:28

Controlling a general robot

- From now on we just assume that we have algorithms to efficiently compute $M(q)$ and $F(q,\dot{q})$ for any $(q,\dot{q})$
- **Inverse dynamics:** If we know the desired $\ddot{q}^*$ for each joint,

$$u = M(q)\,\ddot{q}^* + F(q,\dot{q})$$

gives the necessary torques

- **Forward dynamics:** If we know which torques u we apply, use

$$\ddot{q}^* = M(q)^{-1}(u - F(q,\dot{q}))$$

to simulate the dynamics of the system (e.g., using Runge-Kutta)

3:29

Following a reference trajectory in joint space

- Where could we get the desired $\ddot{q}^*$ from?
  Assume we have a nice smooth **reference trajectory** $q^{\text{ref}}_{0:T}$ (generated with some motion profile or alike), we can at each t read off the desired acceleration as

$$\ddot{q}^{\text{ref}}_t := \frac{1}{\tau}[(q_{t+1} - q_t)/\tau - (q_t - q_{t-1})/\tau] = (q_{t-1} + q_{t+1} - 2q_t)/\tau^2$$

However, tiny errors in acceleration will accumulate greatly over time! This is Instable!!

- Choose a desired acceleration $\ddot{q}_t^*$ that implies a *PD-like behavior around the reference trajectory*!

$$\ddot{q}_t^* = \ddot{q}_t^{\text{ref}} + K_p(q_t^{\text{ref}} - q_t) + K_d(\dot{q}_t^{\text{ref}} - \dot{q}_t)$$

This is a standard and very convenient heuristic to track a reference trajectory when the robot dynamics are known: *All joints will exactly behave like a 1D point particle around the reference trajectory!*

3:30

Following a reference trajectory in task space

- Recall the inverse kinematics problem:
  - We know the desired step δy^* (or velocity $\dot{y}^*$) of the *endeffector*.
  - Which step δq (or velocities $\dot{q}$) should we make in the joints?

- Equivalent dynamic problem:
  - We know how the desired acceleration $\ddot{y}^*$ of the *endeffector*.
  - What controls u should we apply?

3:31

Operational space control

- Inverse kinematics:

$$q^* = \underset{q}{\text{argmin}} \, \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$$

- Operational space control (one might call it "Inverse task space dynamics"):

$$u^* = \underset{u}{\text{argmin}} \, \|\ddot{\phi}(q) - \ddot{y}^*\|_C^2 + \|u\|_H^2$$

3:32

Operational space control

- We can derive the optimum perfectly analogous to inverse kinematics
  We identify the minimum of a locally squared potential, using the local linearization (and approx. $\ddot{J} = 0$)

$$\ddot{\phi}(q) = \frac{d}{dt}\dot{\phi}(q) \approx \frac{d}{dt}(J\dot{q} + \dot{J}q) \approx J\ddot{q} + 2\dot{J}\dot{q} = JM^{-1}(u - F) + 2\dot{J}\dot{q}$$

We get

$$u^* = T^\sharp(\ddot{y}^* - 2\dot{J}\dot{q} + TF)$$
$$\text{with } T = JM^{-1}\,, \quad T^\sharp = (T^\top C T + H)^{-1} T^\top C$$

$$(C \to \infty \;\Rightarrow\; T^\sharp = H^{-1}T^\top(TH^{-1}T^\top)^{-1})$$

3:33

Controlling a robot – operational space approach

- Where could we get the desired $\ddot{y}^*$ from?
  - **Reference trajectory** $y^{\text{ref}}_{0:T}$ in operational space
  - **PD-like behavior** in each operational space:
$$\ddot{y}^*_t = \ddot{y}^{\text{ref}}_t + K_p(y^{\text{ref}}_t - y_t) + K_d(\dot{y}^{\text{ref}}_t - \dot{y}_t)$$

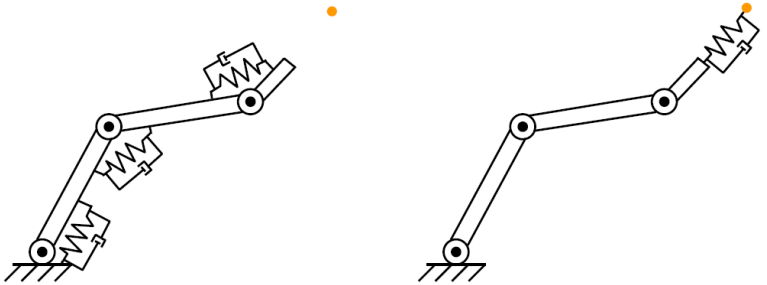


illustration from O. Brock's lecture

- Operational space control: *Let the system behave as if we could directly "apply a 1D point mass behavior" to the endeffector*

3:34

Multiple tasks

- Recall trick last time: we defined a "big kinematic map" $\Phi(q)$ such that

$$q^* = \underset{q}{\text{argmin}} \, \|q - q_0\|^2_W + \|\Phi(q)\|^2$$

- Works analogously in the dynamic case:

$$u^* = \underset{u}{\text{argmin}} \, \|u\|^2_H + \|\Phi(q)\|^2$$

3:35

What have we learned? What not?

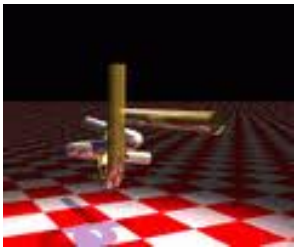
- More theory
  - Contacts → Inequality constraints on the dynamics
  - Switching dynamics (e.g. for walking)
  - Controllling contact forces

- **Hardware limits**
  - I think: the current success stories on highly dynamic robots are all anchored in novel hardware approaches

3:36

4 Path Planning

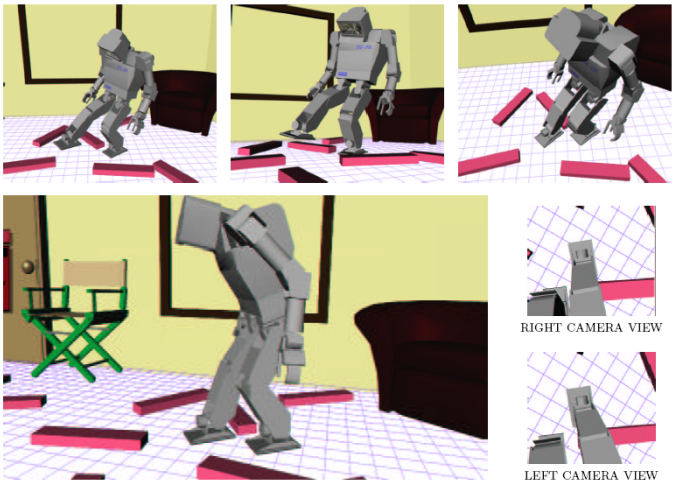
Path finding examples



Alpha-Puzzle, solved with James Kuffner's RRTs

4:1

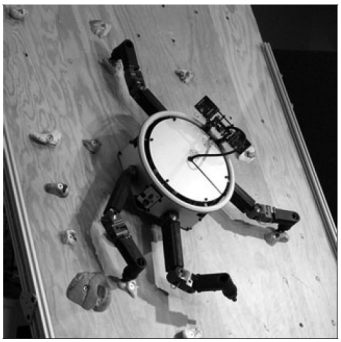
Path finding examples



J. Kuffner, K. Nishiwaki, S. Kagami, M. Inaba, and H. Inoue. Footstep Planning Among Obstacles for Biped Robots. Proc. IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS), 2001.

4:2

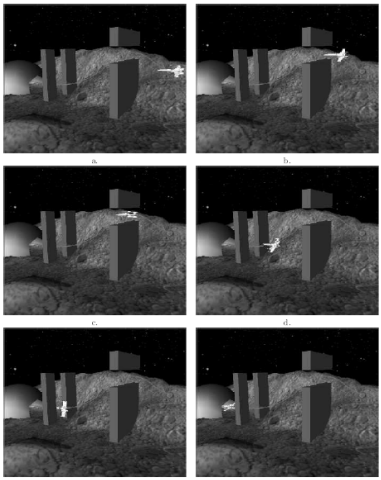
Path finding examples



T. Bretl. Motion Planning of Multi-Limbed Robots Subject to Equilibrium Constraints: The Free-Climbing Robot Problem. International Journal of Robotics Research, 25(4):317-342, Apr 2006.

4:3

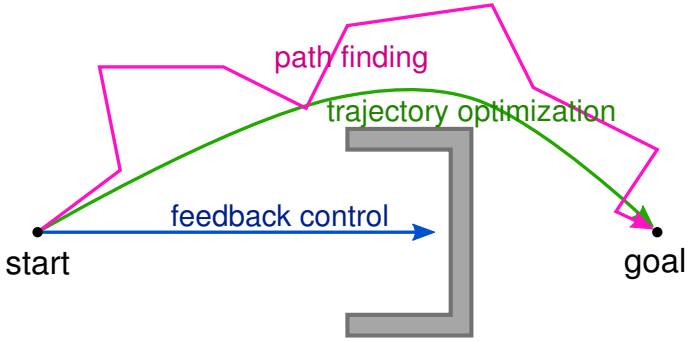
Path finding examples



S. M. LaValle and J. J. Kuffner. Randomized Kinodynamic Planning. International Journal of Robotics Research, 20(5):378–400, May 2001.

4:4

Feedback control, path finding, trajectory optim.



- Feedback Control: E.g., $q_{t+1} = q_t + J^\sharp(y^* - \phi(q_t))$
- Trajectory Optimization: $\text{argmin}_{q_{0:T}} f(q_{0:T})$
- Path Finding: Find some $q_{0:T}$ with only valid configurations

4:5

Outline

- **Really briefly:** Heuristics & Discretization (slides from Howie CHoset's CMU lectures)
  - Bugs algorithm
  - Potentials to guide feedback control
  - Dijkstra
- **Sample-based Path Finding**
  - Probabilistic Roadmaps
  - Rapidly Exploring Random Trees
- **Non-holonomic systems**

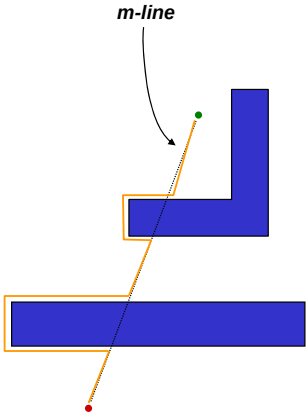
4:6

4.1 Background

4:7

CarnegieMellon

A better bug?



"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

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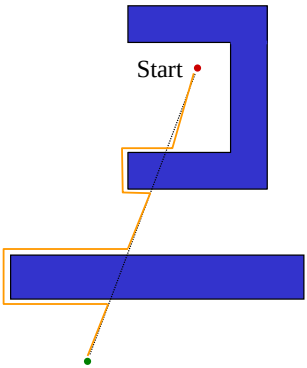
4:8

CarnegieMellon

A better bug?

"Bug 2" Algorithm



1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

Better or worse than Bug1?

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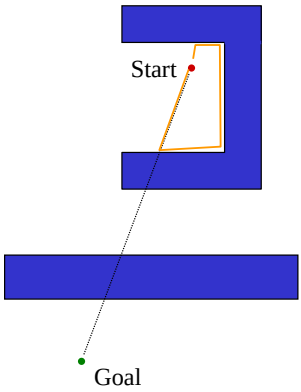
4:9

CarnegieMellon

A better bug?

"Bug 2" Algorithm



1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

NO! How do we fix this?

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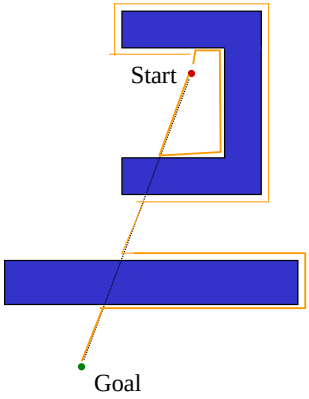
4:10

CarnegieMellon

A better bug?

"Bug 2" Algorithm



1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again ***closer to the goal***.

3) Leave the obstacle and continue toward the goal

Better or worse than Bug1?

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4:11

BUG algorithms – conclusions

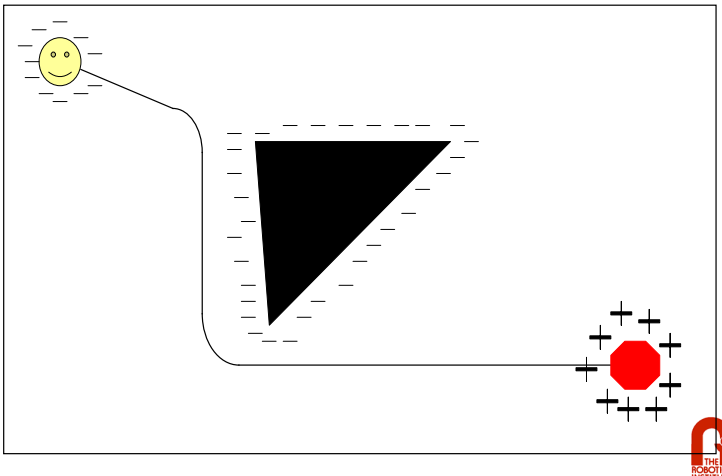
- Other variants: TangentBug, VisBug, RoverBug, WedgeBug, ...
- only 2D! (TangentBug has extension to 3D)
- Guaranteed convergence
- Still active research:
  K. Taylor and S.M. LaValle: *I-Bug: An Intensity-Based Bug Algorithm*

⇒ Useful for minimalistic, robust 2D goal reaching
– not useful for finding paths in joint space

4:12

CarnegieMellon

Start-Goal Algorithm: Potential Functions



4:13

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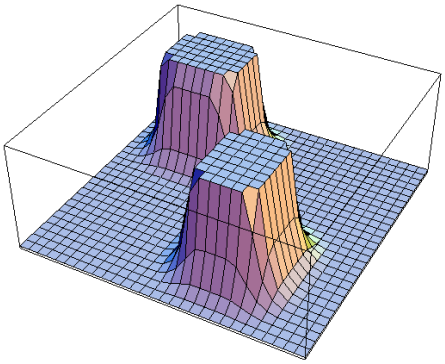
Repulsive Potential



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4:14

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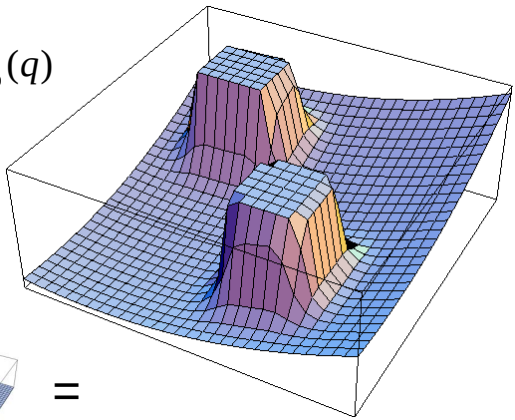
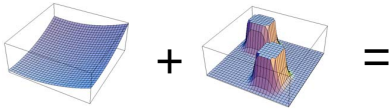
Total Potential Function

$$U(q) = U_{\text{att}}(q) + U_{\text{rep}}(q)$$

$$F(q) = -\nabla U(q)$$





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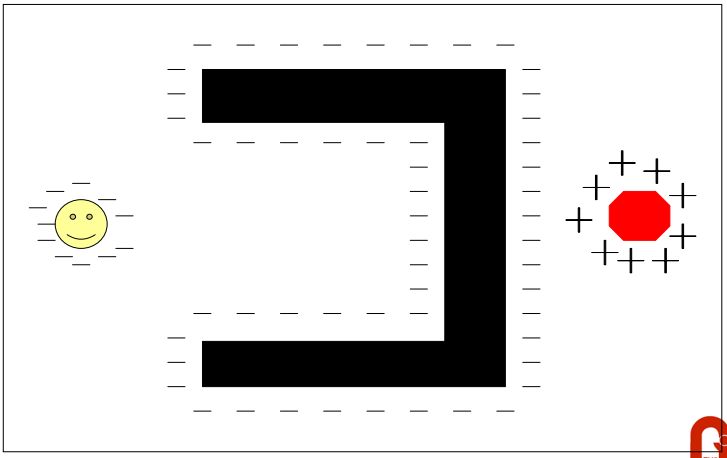
Local Minimum Problem with the Charge Analogy



4:16

Potential fields – conclusions

- Very simple, therefore popular
- In our framework: Combining a goal (endeffector) task variable, with a constraint (collision avoidance) task variable; then using inv. kinematics is *exactly* the same as "Potential Fields"

$\Rightarrow$ Does not solve locality problem of feedback control.

4:17

CarnegieMellon

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0's are adjacent to cells with values >= 2
    - 0's will only remain when regions are unreachable

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

4:18

CarnegieMellon

The Wavefront in Action (Part 1)

- Starting with the goal, set all adjacent cells with "0" to the current cell + 1
  - 4-Point Connectivity or 8-Point Connectivity?
  - Your Choice. We'll use 8-Point Connectivity in our example

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	**2**

4:19

CarnegieMellon

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0's are adjacent to cells with values >= 2
    - 0's will only remain when regions are unreachable

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	**2**

4:20

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The Wavefront in Action (Part 3)

- Repeat again...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

4:21

CarnegieMellon

The Wavefront in Action (Part 4)

- And again...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

4:22

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The Wavefront in Action (Part 5)

- And again until...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	7	7	7	7	7
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	7	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

4:23

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The Wavefront in Action (Done)

- You're done
  - Remember, 0's should only remain if unreachable regions exist

7	**18**	17	16	15	14	13	12	11	10	9	9	9	9	9	9
6	17	17	16	15	14	13	12	11	10	9	8	8	8	8	8
5	17	16	16	15	14	13	12	11	10	9	8	7	7	7	7
4	17	16	15	15	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	17	16	15	14	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	17	16	15	14	13	12	11	10	9	8	7	6	5	4	4	4
1	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	3
0	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

4:24

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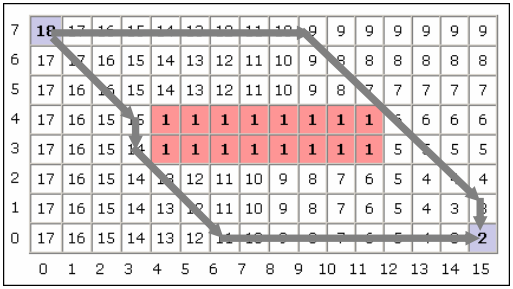
The Wavefront, Now What?

- To find the shortest path, according to your metric, simply always move toward a cell with a lower number
  - The numbers generated by the Wavefront planner are roughly proportional to their distance from the goal

Two possible shortest paths shown



4:25

Dijkstra Algorithm

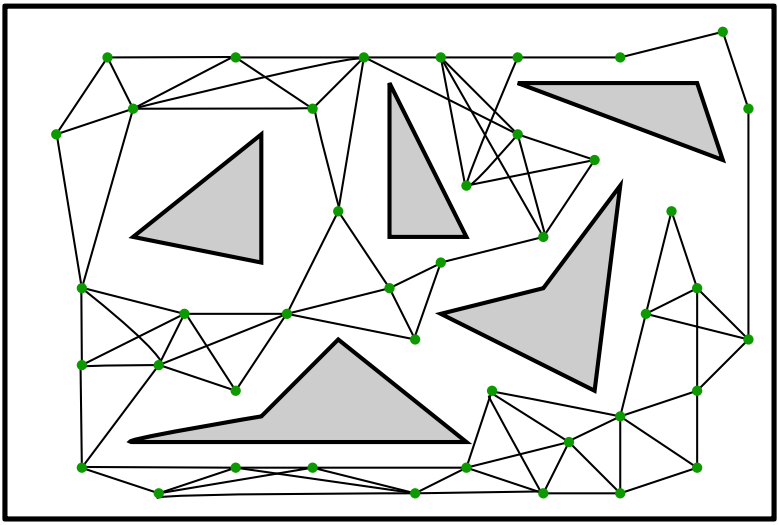
- Is efficient in **discrete domains**
  - Given start and goal node in an arbitrary graph
  - Incrementally label nodes with their distance-from-start
- Produces optimal (shortest) paths
- Applying this to continuous domains requires discretization
  - Grid-like discretization in high-dimensions is daunting! *(curse of dimensionality)*
  - What are other ways to "discretize" space more efficiently?

4:26

4.2 Sample-based Path Finding

4:27

Probabilistic Road Maps

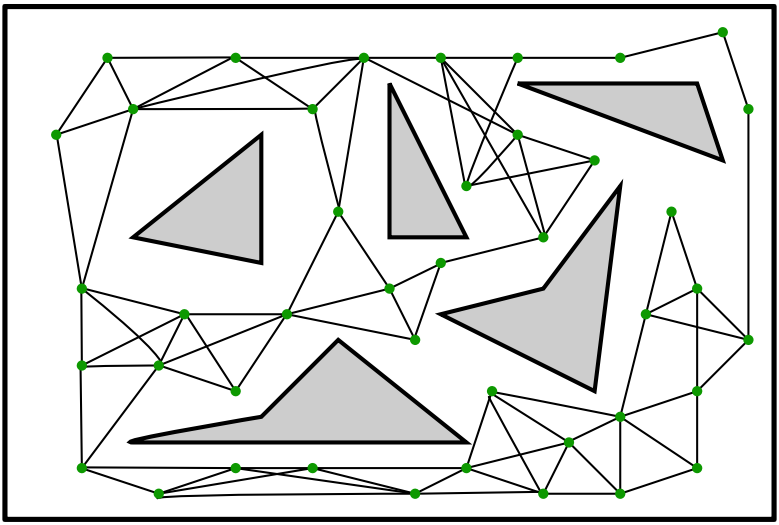


[Kavraki, Svetska, Latombe,Overmars, 95]

$q \in \mathbb{R}^n$ describes configuration
Q_{free} is the set of configurations without collision

4:28

Probabilistic Road Maps



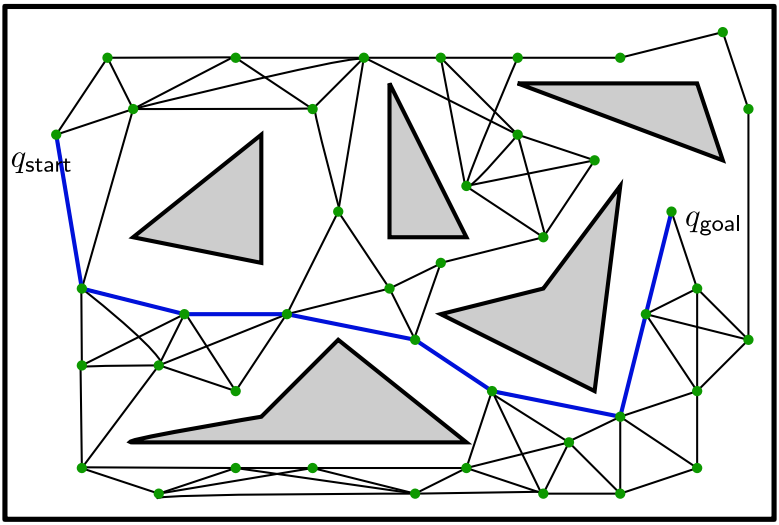
[Kavraki, Svetska, Latombe,Overmars, 95]

Probabilistic Road Map

- generates a graph $G = (V, E)$ of configurations
  - such that configurations along each edges are $\in Q_{\text{free}}$

4:29

Probabilistic Road Maps



Given the graph, use (e.g.) Dijkstra to find path from q_{start} to q_{goal}.

4:30

Probabilistic Road Maps – generation

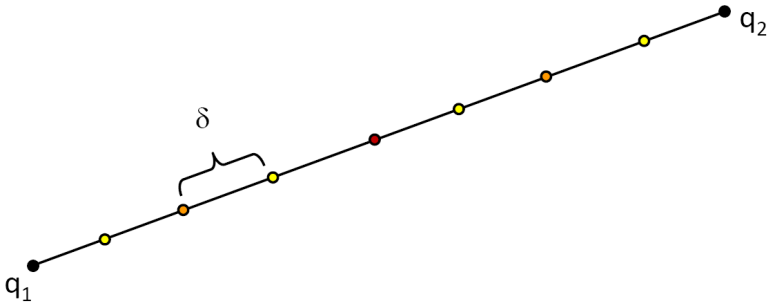
Input: number n of samples, number k number of nearest neighbors
Output: PRM $G = (V, E)$

```
initialize V = ∅, E = ∅
while |V| < n do                          // find n collision free points q_i
    q ← random sample from Q
    if q ∈ Q_free then V ← V ∪ {q}
end while
for all q ∈ V do                          // check if near points can be connected
    N_q ← k nearest neighbors of q in V
    for all q' ∈ N_q do
        if path(q, q') ∈ Q_free then E ← E ∪ {(q, q')}
    end for
end for
```

where path(q, q') is a local planner (easiest: straight line)

4:31

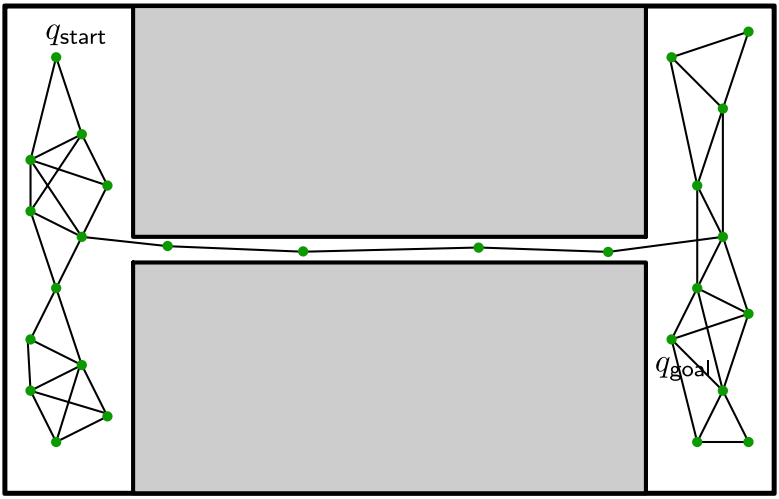
Local Planner



tests collisions up to a specified resolution δ

4:32

Problem: Narrow Passages



The smaller the gap (clearance ϱ) the more unlikely to sample such points.

4:33

PRM theory

(for uniform sampling in d-dim space)

- Let $a, b \in Q_{\text{free}}$ and γ a path in Q_{free} connecting a and b.

Then the probability that PRM found the path after n samples is

$$P(\text{PRM-success} \,|\, n) \geq 1 - \frac{2|\gamma|}{\varrho}\, e^{-\sigma \varrho^d n}$$

$\sigma = \frac{|B_1|}{2^d |Q_{\text{free}}|}$
$\varrho =$ clearance of γ (distance to obstacles)
(roughly: the exponential term are "volume ratios")

- This result is called *probabilistic complete* (one can achieve any probability with high enough n)
- For a given success probability, n needs to be exponential in d

4:34

Other PRM sampling strategies

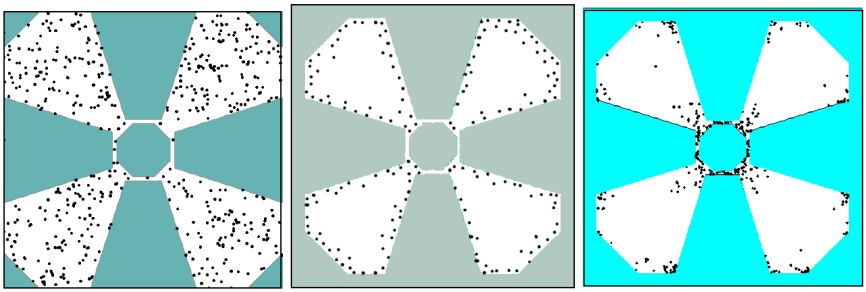


illustration from O. Brock's lecture

Gaussian: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; if $q_1 \in Q_{\text{free}}$ and $q_2 \notin Q_{\text{free}}$, add q_1 (or vice versa).

Bridge: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; $q_3 = (q_1 + q_2)/2$; if $q_1, q_2 \notin Q_{\text{free}}$ and $q_3 \in Q_{\text{free}}$, add q_3.

- Sampling strategy can be made more intelligence: "utility-based sampling"
- Connection sampling
  (once earlier sampling has produced connected components)

4:35

Probabilistic Roadmaps – conclusions

- Pros:
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees
  - Good to answer many queries in an *unchanged* environment

- Cons:
  - Precomputation of exhaustive roadmap takes a long time (but not necessary for "Lazy PRMs")

4:36

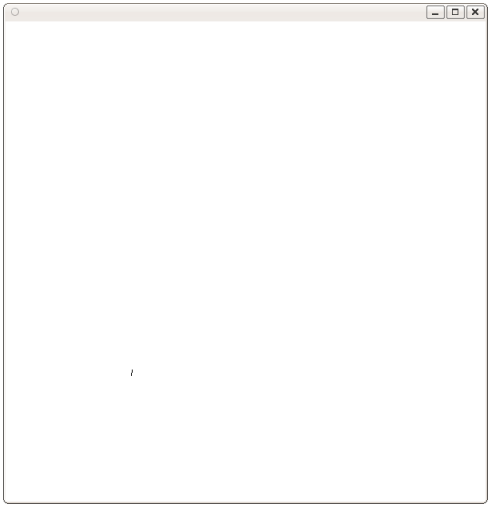
Rapidly Exploring Random Trees

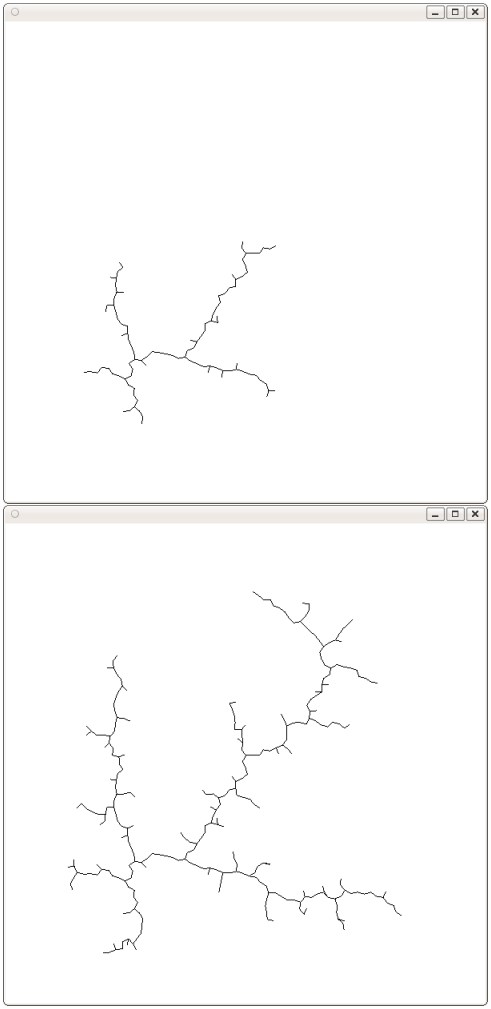
2 motivations:

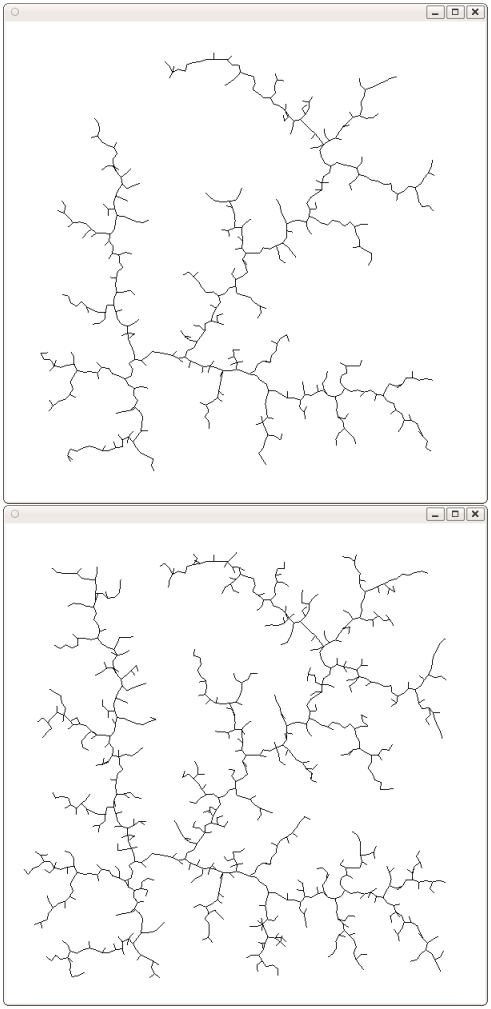
- Single Query path finding: Focus computational efforts on paths for specific $(q_{\text{start}}, q_{\text{goal}})$
- Use actually controllable DoFs to incrementally explore the search space: *control-based* path finding.

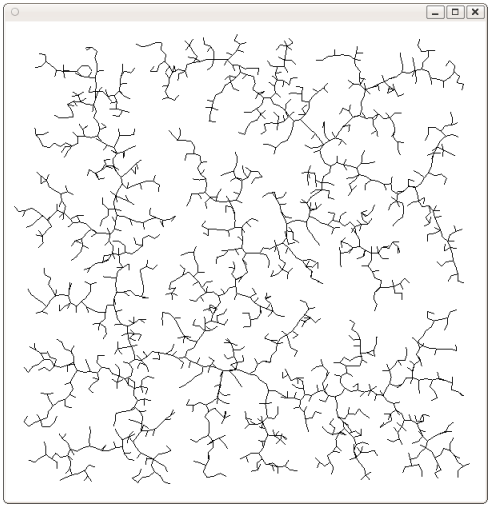
(Ensures that RRTs can be extended to handling differential constraints.)

4:37









$n = 1$ $n = 100$ $n = 300$ $n = 600$ $n = 1000$ $n = 2000$

4:38

Rapidly Exploring Random Trees

Simplest RRT with straight line local planner and step size α

Input: q_{start}, number n of nodes, stepsize α
Output: tree $T = (V, E)$
1: initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
2: **for** $i = 0 : n$ **do**
3: $q_{\text{target}} \leftarrow$ random sample from Q
4: $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
5: $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
6: **if** $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
7: **end for**

4:39

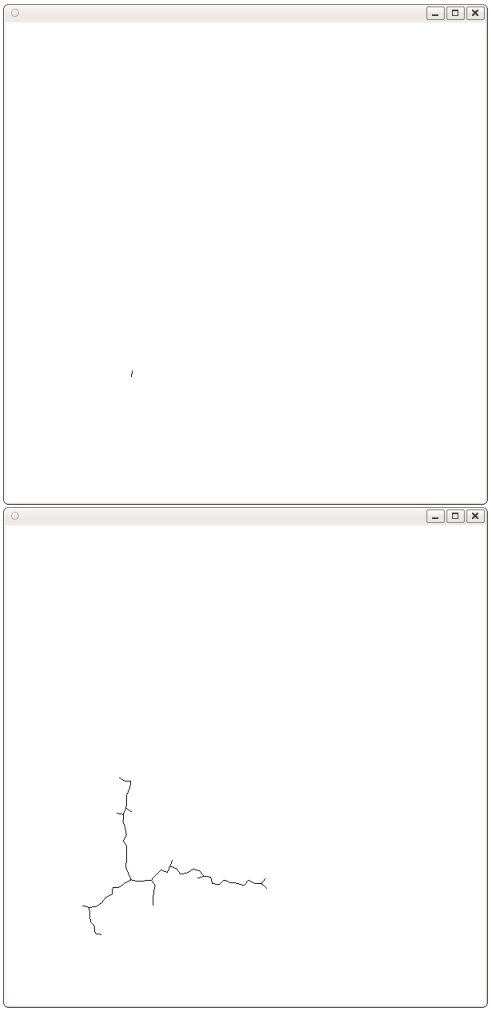
Rapidly Exploring Random Trees

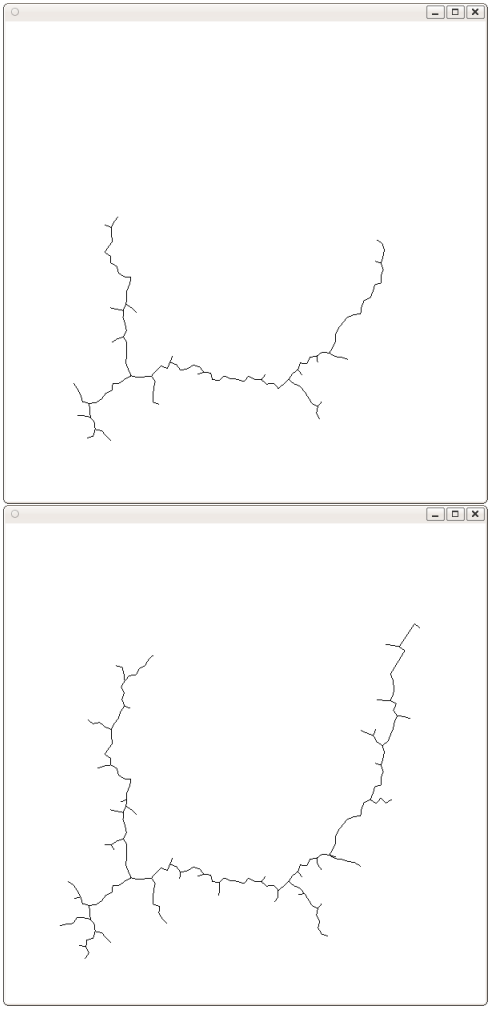
RRT growing directedly towards the goal

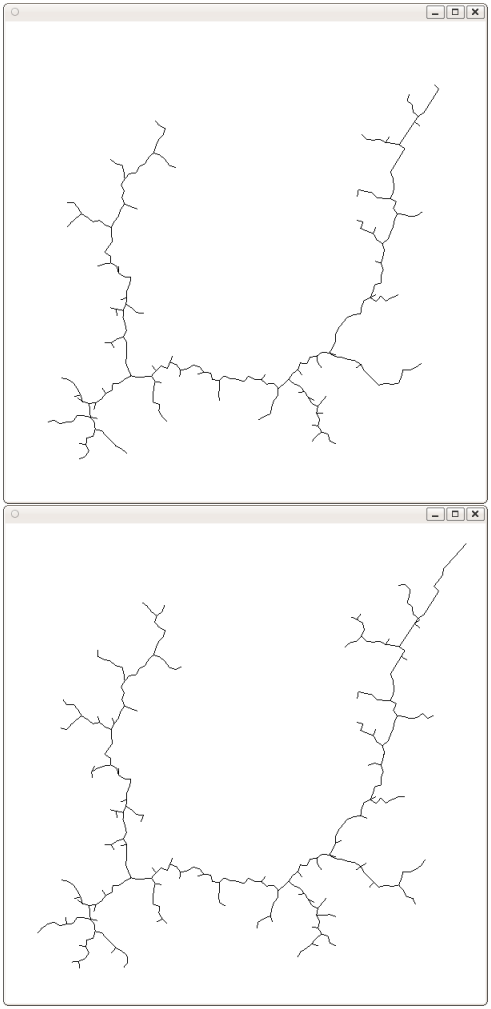
Input: q_{start}, q_{goal}, number n of nodes, stepsize α, β
Output: tree $T = (V, E)$

1: initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
2: **for** $i = 0 : n$ **do**
3: **if** rand$(0, 1) < \beta$ **then** $q_{\text{target}} \leftarrow q_{\text{goal}}$
4: **else** $q_{\text{target}} \leftarrow$ random sample from Q
5: $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
6: $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
7: **if** $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
8: **end for**

4:40





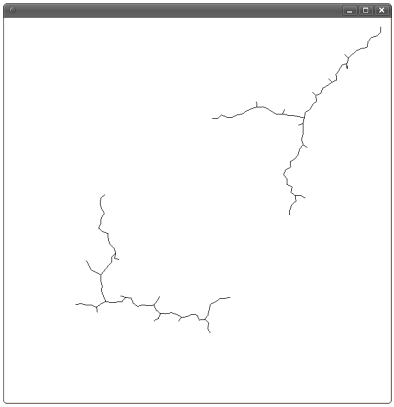


$n = 1$ $n = 100$ $n = 200$ $n = 300$ $n = 400$ $n = 500$

4:41

Bi-directional search

- grow two trees starting from q_{start} and q_{goal}



let one tree grow towards the other
(e.g., "choose q_{new} of T_1 as q_{target} of T_2")

4:42

Summary: RRTs

- Pros (shared with PRMs):
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees
- Pros (beyond PRMs):
  - Focus computation on single query $(q_{\text{start}}, q_{\text{goal}})$ problem
  - Trees from multiple queries can be merged to a roadmap
  - Can be extended to differential constraints (nonholonomic systems)
- To keep in mind (shared with PRMs):
  - The metric (for nearest neighbor selection) is sometimes critical
  - The local planner may be non-trivial

4:43

References

Steven M. LaValle: *Planning Algorithms*, http://planning.cs.uiuc.edu/.

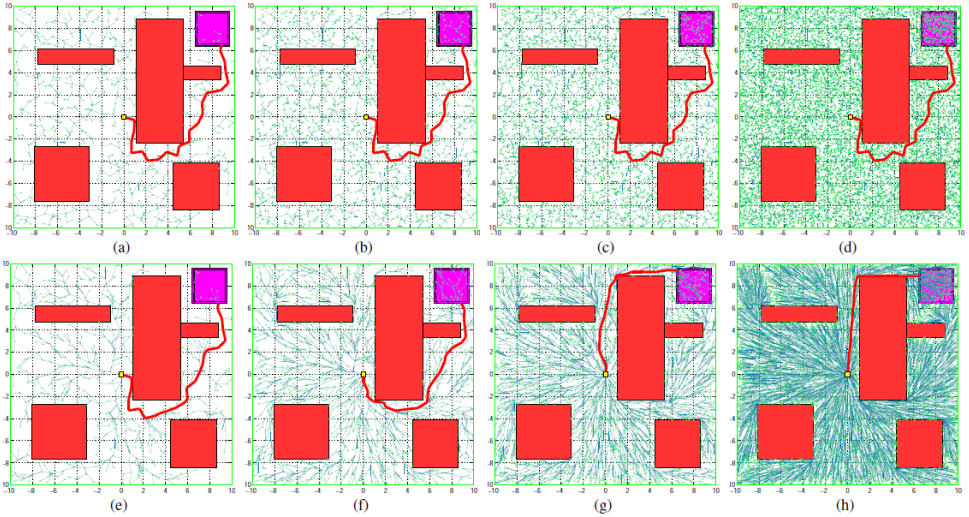
Choset et. al.: *Principles of Motion Planning*, MIT Press.

Latombe's "motion planning" lecture, http://robotics.stanford.edu/~latombe/cs326/2007/schedule.htm

4:44

RRT*

Sertac Karaman & Emilio Frazzoli: Incremental sampling-based algorithms for optimal motion planning, arXiv 1005.0416 (2010).



Algorithm 4: $\texttt{Extend}_{RRT^*}(G, x)$

$V' \leftarrow V$; $E' \leftarrow E$;
$x_{\text{nearest}} \leftarrow \texttt{Nearest}(G, x)$;
$x_{\text{new}} \leftarrow \texttt{Steer}(x_{\text{nearest}}, x)$;
if $\texttt{ObstacleFree}(x_{\text{nearest}}, x_{\text{new}})$ **then**
  $V' \leftarrow V' \cup \{x_{\text{new}}\}$;
  $x_{\min} \leftarrow x_{\text{nearest}}$;
  $X_{\text{near}} \leftarrow \texttt{Near}(G, x_{\text{new}}, |V|)$;
  for *all* $x_{\text{near}} \in X_{\text{near}}$ **do**
    if $\texttt{ObstacleFree}(x_{\text{near}}, x_{\text{new}})$ **then**
      $c' \leftarrow \texttt{Cost}(x_{\text{near}}) + c(\texttt{Line}(x_{\text{near}}, x_{\text{new}}))$;
      if $c' < \texttt{Cost}(x_{\text{new}})$ **then**
        $x_{\min} \leftarrow x_{\text{near}}$;
  $E' \leftarrow E' \cup \{(x_{\min}, x_{\text{new}})\}$;
  for *all* $x_{near} \in X_{\text{near}} \setminus \{x_{\min}\}$ **do**
    if $\texttt{ObstacleFree}(x_{\text{new}}, x_{\text{near}})$ *and* $\texttt{Cost}(x_{\text{near}}) > \texttt{Cost}(x_{\text{new}}) + c(\texttt{Line}(x_{\text{new}}, x_{\text{near}}))$ **then**
      $x_{\text{parent}} \leftarrow \texttt{Parent}(x_{\text{near}})$;
      $E' \leftarrow E' \setminus \{(x_{\text{parent}}, x_{\text{near}})\}$;
      $E' \leftarrow E' \cup \{(x_{\text{new}}, x_{\text{near}})\}$;
return $G' = (V', E')$

4:45

4.3 Non-holonomic systems

4:46

Non-holonomic systems

- We define a **nonholonomic system** as one with **differential constraints**:

$$\dim(u_t) < \dim(x_t)$$

$\Rightarrow$ *Not all degrees of freedom are directly controllable*

- Dynamic systems are an example!

- General definition of a differential constraint:
  For any given state x, let U_x be the tangent space that is generated by controls:

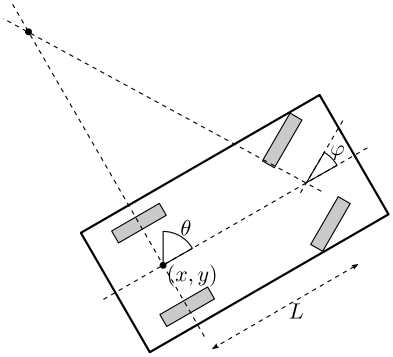
$$U_x = \{\dot{x} \ : \ \dot{x} = f(x,u), \ u \in U\}$$

  (non-holonomic $\iff$ $\dim(U_x) < \dim(x)$)

  The elements of U_x are elements of T_x subject to additional *differential constraints*.

4:47

Car example



$$\begin{aligned}\dot{x} &= v \ \cos\theta \\ \dot{y} &= v \ \sin\theta \\ \dot{\theta} &= (v/L) \ \tan\varphi \\ |\varphi| &< \Phi\end{aligned}$$

State $q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ **Controls** $u = \begin{pmatrix} v \\ \varphi \end{pmatrix}$

Sytem equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v \ \cos\theta \\ v \ \sin\theta \\ (v/L) \ \tan\varphi \end{pmatrix}$$

4:48

Car example

- The car is a *non-holonomic* system: not all DoFs are controlled, $\dim(u) < \dim(q)$
  We have the *differential constraint* $\dot{q}$:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

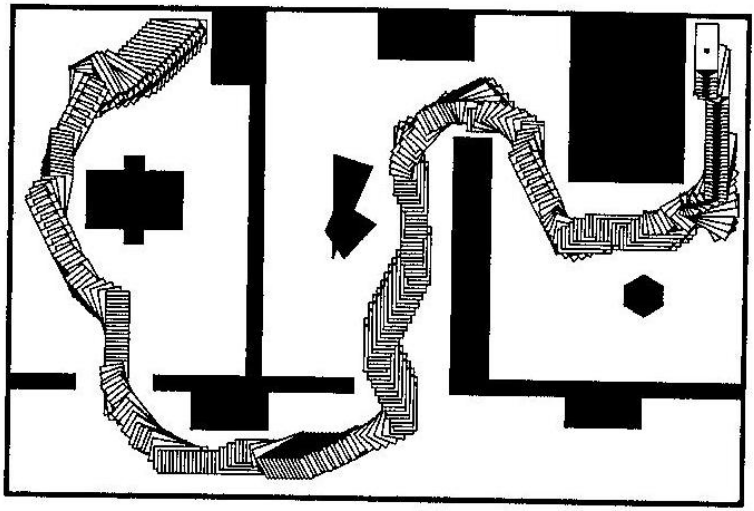
"A car cannot move directly lateral."

- Analogy to dynamic systems: Just like a car cannot instantly move sidewards, a dynamic system cannot instantly change its position q: the current change in position is *constrained* by the current velocity $\dot{q}$.

4:49

Path finding for a non-holonomic system

Could a car follow this trajectory?

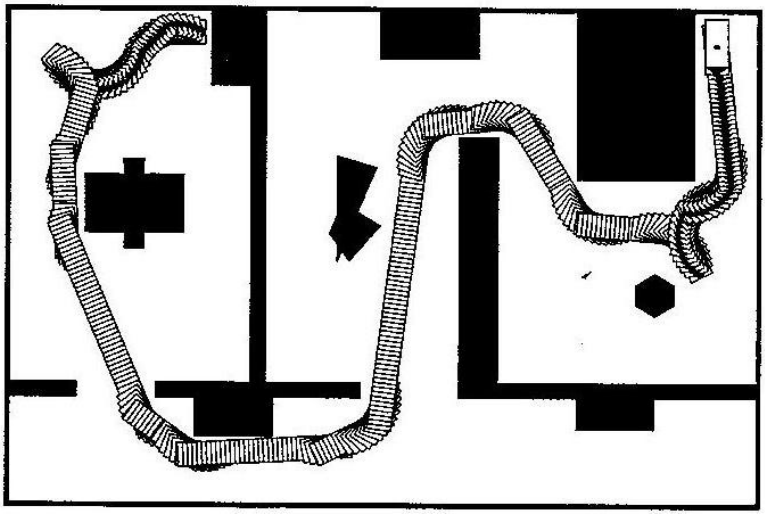


This is generated with a PRM in the state space $q = (x, y, \theta)$ *ignoring the differntial constraint*.

4:50

Path finding with a non-holonomic system

This is a solution we would like to have:



The path respects **differential constraints.**
Each step in the path corresponds to setting certain controls.

4:51

Control-based sampling to grow a tree

- Control-based sampling: fulfils none of the nice exploration properties of RRTs, but fulfils the differential constraints:

  1) Select a $q \in T$ from tree of current configurations

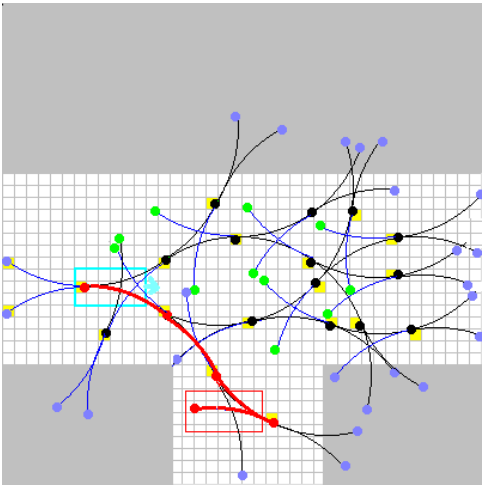
  2) Pick control vector u at random

  3) Integrate equation of motion over short duration (picked at random or not)

  4) If the motion is collision-free, add the endpoint to the tree

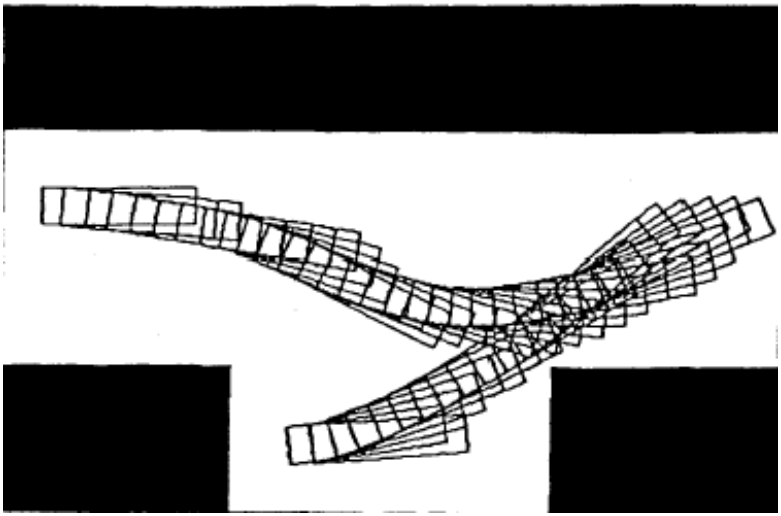
4:52

Control-based sampling for the car



1) Select a $q \in T$
2) Pick v, ϕ, and τ
3) Integrate motion from q
4) Add result if collision-free

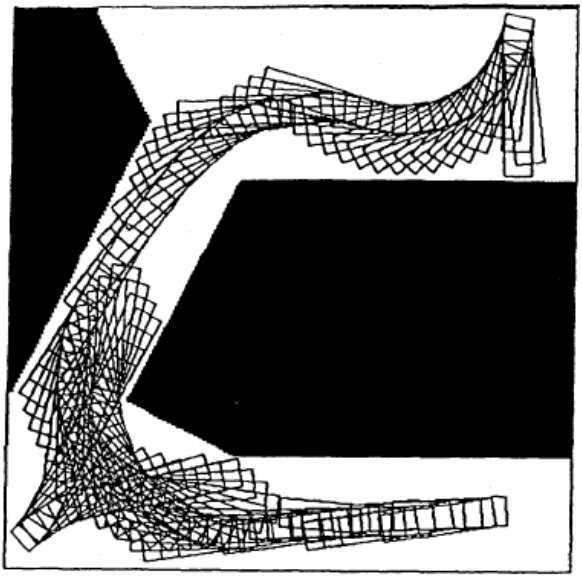
4:53



J. Barraquand and J.C. Latombe. Nonholonomic Multibody Robots: Controllability and Motion Planning in the Presence of Obstacles. Algorithmica, 10:121-155, 1993.

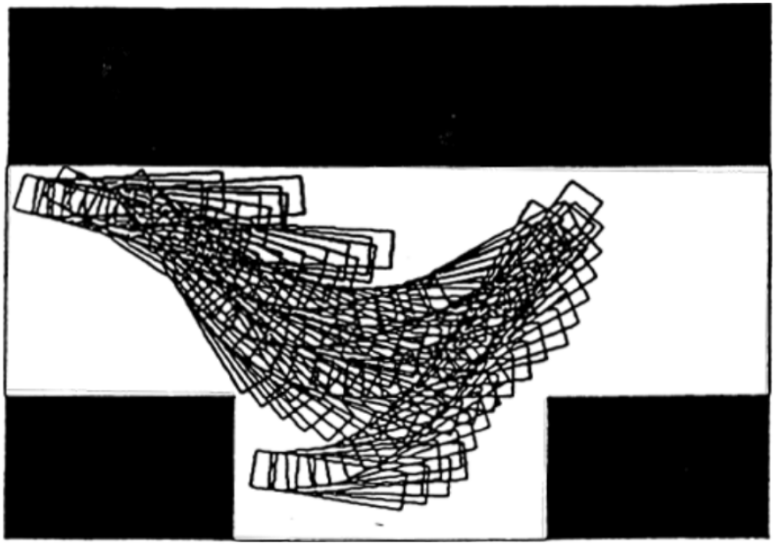
car parking

4:54



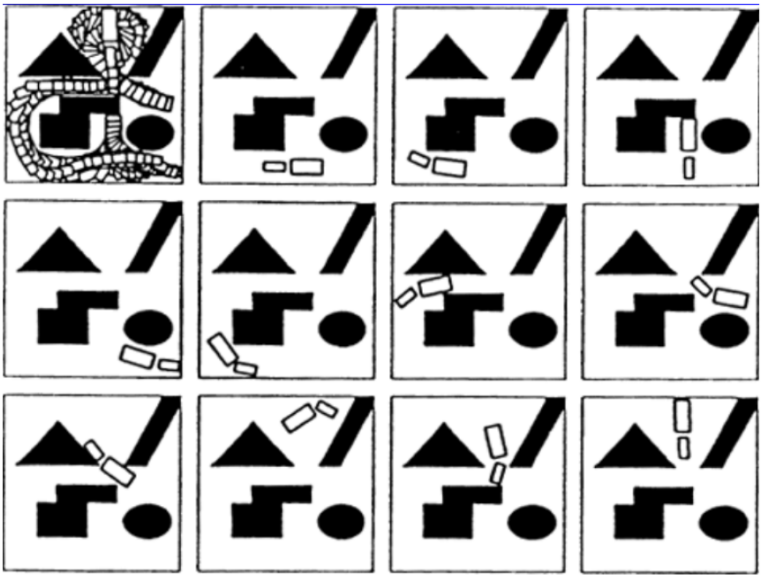
car parking

4:55



parking with only left-steering

4:56



with a trailer

4:57

Better control-based exploration: RRTs revisited

- RRTs with differential constraints:

Input: q_{start}, number k of nodes, time interval τ
Output: tree $T = (V, E)$
1: initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
2: **for** $i = 0 : k$ **do**
3: $q_{\text{target}} \leftarrow$ random sample from Q
4: $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
5: use local planner to compute controls u that steer q_{near} towards q_{target}
6: $q_{\text{new}} \leftarrow q_{\text{near}} + \int_{t=0}^{\tau} \dot{q}(q, u)dt$
7: **if** $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
8: **end for**

- Crucial questions:
- How meassure *near* in nonholonomic systems?
- How find controls u to steer towards target?

4:58

Configuration state metrics

Standard/Naive metrics:

- Comparing two 2D rotations/orientations $\theta_1, \theta_2 \in SO(2)$:
  a) Euclidean metric between $e^{i\theta_1}$ and $e^{i\theta_2}$
  b) $d(\theta_1, \theta_2) = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}$
- Comparing two configurations $(x, y, \theta)_{1,2}$ in $\mathbb{R}^2$:
  Eucledian metric on $(x, y, e^{i\theta})$
- Comparing two 3D rotations/orientations $r_1, r_2 \in SO(3)$:
  Represent both orientations as unit-length quaternions $r_1, r_2 \in \mathbb{R}^4$:
  $$d(r_1, d_2) = \min\{|r_1 - r_2|, |r_1 + r_2|\}$$
  where $|\cdot|$ is the Euclidean metric.
  (Recall that r_1 and $-r_1$ represent exactly the same rotation.)

- **Ideal metric:**
  Optimal cost-to-go between two states x_1 and x_2:
- Use optimal trajectory cost as metric
- This is as hard to compute as the original problem, of course!!
  $\rightarrow$ Approximate, e.g., by neglecting obstacles.

4:59

Side story: Dubins curves

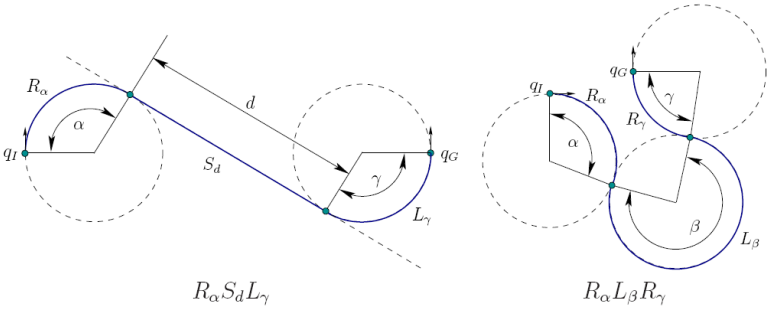
- Dubins car: constant velocity, and steer $\varphi \in [-\Phi, \Phi]$

- Neglecting obstacles, there are only **six** types of trajectories that connect any configuration $\in \mathbb{R}^2 \times \mathbb{S}^1$:
  $$\{LRL, RLR, LSL, LSR, RSL, RSR\}$$

- annotating durations of each phase:
  $$\{L_\alpha R_\beta L_\gamma, , R_\alpha L_\beta R_\gamma, L_\alpha S_d L_\gamma, L_\alpha S_d R_\gamma, R_\alpha S_d L_\gamma, R_\alpha S_d R_\gamma\}$$
  with $\alpha \in [0, 2\pi), \beta \in (\pi, 2\pi), d \geq 0$

4:60

Side story: Dubins curves



$\rightarrow$ By testing all six types of trajectories for (q_1, q_2) we can define a Dubins metric for the RRT – and use the Dubins curves as controls!

- **Reeds-Shepp curves** are an extension for cars which can drive back.
  (includes 46 types of trajectories, good metric for use in RRTs for cars)

5 Path Optimization – briefly

Outline

- These are only some very brief notes on path optimization
- The aim is to explain how to *formulate* the optimization problem. Concerning the optimization algortihm itself, refer to the *Optimization* lecture.

5:1

From inverse kinematics to path costs

- Recall our optimality principle of inverse kinematics

$$\operatorname*{argmin}_q \|q - q_0\|_W^2 + \|\Phi(q)\|^2$$

- A trajectory $q_{0:T}$ is a sequence of robot configurations $q_t \in \mathbb{R}^n$
- Consider the cost function

$$f(q_{0:T}) = \sum_{t=0}^{T} \|q_{t\text{-}1} - q_t\|_W^2 + \sum_{t=0}^{T} \|\Phi_t(q_t)\|^2$$

(where $(q_{\text{-}1}$ is a given prefix)

- $\|q_{t\text{-}1} - q_t\|_W^2$ represents **control costs**
  $\Phi_t(q_t)$ represents **task costs**

5:2

General k-order cost terms

[Notation: x_t instead of q_t represents joint state]

$$\begin{aligned} \min_{x_{0:T}} \quad & \sum_{t=0}^{T} f_t(x_{t-k:t})^\top f_t(x_{t-k:t}) \\ \text{s.t.} \quad & \forall_t : \; g_t(x_{t-k:t}) \le 0 \;, \quad h_t(x_{t-k:t}) = 0 \;. \end{aligned}$$

5:3

Cost terms

- The $f_t(x_{t-k:t})$ terms can penalize various things:

$k=1$	$f_t(x_{t\text{-}1}, x_t) = x_t - x_{t\text{-}1}$	penalize velocity
$k=2$	$f_t(x_{t\text{-}2}, .., x_t) = x_t - 2x_{t\text{-}1} + x_{t\text{-}2}$	penalize acceleration
$k=3$	$f_t(x_{t\text{-}3}, .., x_t) = x_t - 3x_{t\text{-}1} + 3x_{t\text{-}2} - x_{t\text{-}3}$	penalize jerk

or in some arbitrary task spaces

$k=0$	$f_t(x_t) = \phi(x_t) - y^*$	penalize offset in some task space
$k=1$	$f_t(x_{t\text{-}1}, x_t) = \phi x_t - \phi x_{t\text{-}1}$	
$k=2$	$f_t(x_{t\text{-}2}, .., x_t) = \phi x_t - 2\phi x_{t\text{-}1} + \phi x_{t\text{-}2}$	
$k=3$	$f_t(x_{t\text{-}3}, .., x_t) = \phi x_t - 3\phi x_{t\text{-}1} + 3\phi x_{t\text{-}2} - \phi x_{t\text{-}3}$	

- And terms f_t can be stacked arbitrarily

5:4

Choice of optimizer

$$\min_{x_{0:T}} \quad \sum_{t=0}^{T} f_t(x_{t-k:t})^\top f_t(x_{t-k:t})$$
$$\text{s.t.} \quad \forall_t : \; g_t(x_{t-k:t}) \le 0 \;, \quad h_t(x_{t-k:t}) = 0 \;.$$

- Constrained optimization methods:
  - Log-barrier, squared penalties
  - **Augmented Lagrangian**

- Note: also the Lagrangian is the form of the so-called **Gauss-Newton** form. The pseudo Hessian is a banded, symmetric, positive-definite matrix.

5:5

6 Probabilities

Probability Theory

- Why do we need probabilities?

  – Obvious: to express inherent (*objective*) stochasticity of the world

- But beyond this: (also in a "deterministic world"):
  - lack of knowledge!
  - hidden (latent) variables
  - expressing *uncertainty*
  - expressing *information* (and lack of information)
  - **Subjective Probability**

- Probability Theory: an information calculus

6:1

Outline

- Basic definitions
  - Random variables
  - joint, conditional, marginal distribution
  - Bayes' theorem
- Probability distributions:
  - Gauss
  - Dirak & Particles

These are generic slides on probabilities I use throughout my lecture. Only parts are relevant for this course.

6:2

Probability: Frequentist and Bayesian

- Frequentist probabilities are defined in the limit of an infinite number of trials
  Example: "The probability of a particular coin landing heads up is 0.43"

- Bayesian (subjective) probabilities quantify degrees of belief
  Example: "The probability of it raining tomorrow is 0.3"
  – Not possible to repeat "tomorrow"

6:3

6.1 Basic definitions

6:4

Probabilities & Sets

- **Sample Space/domain** Ω, e.g. $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Probability** $P : A \subset \Omega \mapsto [0, 1]$
  e.g., $P(\{1\}) = \frac{1}{6}$, $P(\{4\}) = \frac{1}{6}$, $P(\{2, 5\}) = \frac{1}{3}$,
- Axioms: $\forall A, B \subseteq \Omega$
  – Nonnegativity $P(A) \geq 0$
  – Additivity $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$
  – Normalization $P(\Omega) = 1$
- Implications
  $0 \leq P(A) \leq 1$
  $P(\emptyset) = 0$
  $A \subseteq B \Rightarrow P(A) \leq P(B)$
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  $P(\Omega \setminus A) = 1 - P(A)$

6:5

Probabilities & Random Variables

- For a random variable X with discrete domain $\mathrm{dom}(X) = \Omega$ we write:
  $\forall_{x \in \Omega} : 0 \leq P(X{=}x) \leq 1$
  $\sum_{x \in \Omega} P(X{=}x) = 1$

  Example: A dice can take values $\Omega = \{1, .., 6\}$.
  X is the random variable of a dice throw.
  $P(X{=}1) \in [0, 1]$ is the probability that X takes value 1.
- A bit more formally: a random variable is a map from a measureable space to a domain (sample space) and thereby introduces a probability measure on the domain ("assigns a probability to each possible value")

6:6

Probabilty Distributions

- $P(X{=}1) \in \mathbb{R}$ denotes a specific probability

$P(X)$ denotes the probability distribution (function over Ω)

Example: A dice can take values $\Omega = \{1, 2, 3, 4, 5, 6\}$.
By $P(X)$ we discribe the full distribution over possible values $\{1, .., 6\}$. These are 6 numbers that sum to one, usually stored in a *table*, e.g.: $[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$

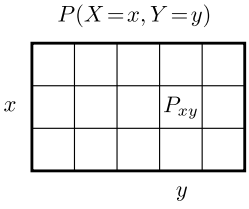
- In implementations we typically represent distributions over discrete random variables as tables (arrays) of numbers

- Notation for summing over a RV:
In equation we often need to sum over RVs. We then write
$$\sum_X P(X) \cdots$$
as shorthand for the explicit notation $\sum_{x \in \text{dom}(X)} P(X=x) \cdots$

6:7

Joint distributions

Assume we have *two* random variables X and Y



- Definitions:
Joint: $P(X, Y)$
Marginal: $P(X) = \sum_Y P(X, Y)$
Conditional: $P(X|Y) = \frac{P(X,Y)}{P(Y)}$

The conditional is normalized: $\forall_Y : \sum_X P(X|Y) = 1$

- X is *independent* of Y iff: $P(X|Y) = P(X)$
(table thinking: all columns of $P(X|Y)$ are equal)

6:8

Joint distributions

joint: $P(X, Y)$
marginal: $P(X) = \sum_Y P(X, Y)$
conditional: $P(X|Y) = \frac{P(X,Y)}{P(Y)}$

- Implications of these definitions:
Product rule: $P(X, Y) = P(X|Y)\, P(Y) = P(Y|X)\, P(X)$

Bayes' Theorem: $P(X|Y) = \frac{P(Y|X)\ P(X)}{P(Y)}$

6:9

Bayes' Theorem

$$P(X|Y) = \frac{P(Y|X)\ P(X)}{P(Y)}$$

$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{normalization}}$$

6:10

Multiple RVs:

- Analogously for n random variables $X_{1:n}$ (stored as a rank n tensor)
  Joint: $P(X_{1:n})$
  Marginal: $P(X_1) = \sum_{X_{2:n}} P(X_{1:n})$,
  Conditional: $P(X_1|X_{2:n}) = \frac{P(X_{1:n})}{P(X_{2:n})}$

- X is *conditionally independent* of Y given Z iff:
  $$P(X|Y,Z) = P(X|Z)$$

- Product rule and Bayes' Theorem:

$$P(X_{1:n}) = \prod_{i=1}^{n} P(X_i|X_{i+1:n})$$

$$P(X_1|X_{2:n}) = \frac{P(X_2|X_1,X_{3:n})\ P(X_1|X_{3:n})}{P(X_2|X_{3:n})}$$

$$P(X,Z,Y) = P(X|Y,Z)\ P(Y|Z)\ P(Z)$$

$$P(X|Y,Z) = \frac{P(Y|X,Z)\ P(X|Z)}{P(Y|Z)}$$

$$P(X,Y|Z) = \frac{P(X,Z|Y)\ P(Y)}{P(Z)}$$

6:11

6.2 Probability distributions

6:12

- We skip Bernoulli & Beta, Multinomial & Dirichlet..

6:13

Bernoulli & Binomial

- We have a binary random variable $x \in \{0,1\}$ (i.e. $\text{dom}(x) = \{0,1\}$)
  The *Bernoulli* distribution is parameterized by a single scalar μ,

$$P(x{=}1 \,|\, \mu) = \mu \,, \quad P(x{=}0 \,|\, \mu) = 1-\mu$$
$$\text{Bern}(x \,|\, \mu) = \mu^x (1-\mu)^{1-x}$$

- We have a data set of random variables $D = \{x_1, .., x_n\}$, each $x_i \in \{0,1\}$. If each $x_i \sim \text{Bern}(x_i \,|\, \mu)$ we have

$$P(D \,|\, \mu) = \textstyle\prod_{i=1}^n \text{Bern}(x_i \,|\, \mu) = \prod_{i=1}^n \mu^{x_i}(1-\mu)^{1-x_i}$$
$$\operatorname*{argmax}_\mu \; \log P(D \,|\, \mu) = \operatorname*{argmax}_\mu \; \sum_{i=1}^n x_i \log \mu + (1-x_i)\log(1-\mu) = \frac{1}{n}\sum_{i=1}^n x_i$$

- The *Binomial distribution* is the distribution over the count $m = \sum_{i=1}^n x_i$

$$\text{Bin}(m \,|\, n, \mu) = \binom{n}{m} \mu^m (1-\mu)^{n-m} \,, \quad \binom{n}{m} = \frac{n!}{(n-m)!\, m!}$$

6:14

Beta

How to express uncertainty over a Bernoulli parameter μ

- The *Beta* distribution is over the interval $[0,1]$, typically the parameter μ of a Bernoulli:

$$\text{Beta}(\mu \,|\, a, b) = \frac{1}{B(a,b)}\, \mu^{a-1}(1-\mu)^{b-1}$$

  with mean $\langle \mu \rangle = \frac{a}{a+b}$ and mode $\mu^* = \frac{a-1}{a+b-2}$ for $a, b > 1$

- The crucial point is:
  - Assume we are in a world with a "Bernoulli source" (e.g., binary bandit), but don't know its parameter μ
  - Assume we have a *prior* distribution $P(\mu) = \text{Beta}(\mu \,|\, a, b)$
  - Assume we collected some data $D = \{x_1, .., x_n\}$, $x_i \in \{0,1\}$, with counts $a_D = \sum_i x_i$ of $[x_i{=}1]$ and $b_D = \sum_i (1-x_i)$ of $[x_i{=}0]$
  - The posterior is

$$\begin{aligned} P(\mu \,|\, D) &= \frac{P(D \,|\, \mu)}{P(D)}\, P(\mu) \propto \text{Bin}(D \,|\, \mu)\, \text{Beta}(\mu \,|\, a, b) \\ &\propto \mu^{a_D}(1-\mu)^{b_D}\, \mu^{a-1}(1-\mu)^{b-1} = \mu^{a-1+a_D}(1-\mu)^{b-1+b_D} \\ &= \text{Beta}(\mu \,|\, a + a_D, b + b_D) \end{aligned}$$

6:15

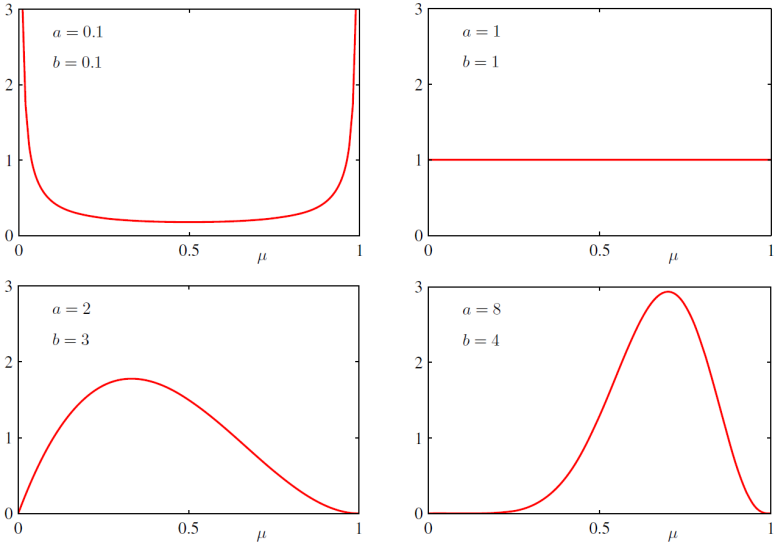
Beta

$$\textit{The prior is } \mathrm{Beta}(\mu \,|\, a, b), \textit{ the posterior is } \mathrm{Beta}(\mu \,|\, a + a_D, b + b_D)$$

- Conclusions:
  - The semantics of a and b are counts of $[x_i = 1]$ and $[x_i = 0]$, respectively
  - The Beta distribution is conjugate to the Bernoulli (explained later)
  - With the Beta distribution we can represent beliefs (state of knowledge) about uncertain $\mu \in [0, 1]$ and know how to update this belief given data

6:16

Beta



from Bishop

6:17

Multinomial

- We have an integer random variable $x \in \{1, .., K\}$
  The probability of a single x can be parameterized by $\mu = (\mu_1, .., \mu_K)$:

$$P(x = k \,|\, \mu) = \mu_k$$

  with the constraint $\sum_{k=1}^{K} \mu_k = 1$ (probabilities need to be normalized)
- We have a data set of random variables $D = \{x_1, .., x_n\}$, each $x_i \in \{1, .., K\}$. If each $x_i \sim P(x_i \,|\, \mu)$ we have

$$P(D \,|\, \mu) = \prod_{i=1}^{n} \mu_{x_i} = \prod_{i=1}^{n} \prod_{k=1}^{K} \mu_k^{[x_i = k]} = \prod_{k=1}^{K} \mu_k^{m_k}$$

where $m_k = \sum_{i=1}^n [x_i = k]$ is the count of $[x_i = k]$. The ML estimator is

$$\underset{\mu}{\text{argmax}} \ \log P(D \,|\, \mu) = \frac{1}{n}(m_1, .., m_K)$$

- The *Multinomial distribution* is this distribution over the counts m_k

$$\text{Mult}(m_1, .., m_K \,|\, n, \mu) \propto \prod_{k=1}^K \mu_k^{m_k}$$

6:18

Dirichlet

How to express uncertainty over a Multinomial parameter μ

- The *Dirichlet* distribution is over the K-simplex, that is, over $\mu_1, .., \mu_K \in [0, 1]$ subject to the constraint $\sum_{k=1}^K \mu_k = 1$:

$$\text{Dir}(\mu \,|\, \alpha) \propto \ \prod_{k=1}^K \mu_k^{\alpha_k - 1}$$

It is parameterized by $\alpha = (\alpha_1, .., \alpha_K)$, has mean $\langle \mu_i \rangle = \frac{\alpha_i}{\sum_j \alpha_j}$ and mode $\mu_i^* = \frac{\alpha_i - 1}{\sum_j \alpha_j - K}$ for $a_i > 1$.

- The crucial point is:
  - Assume we are in a world with a "Multinomial source" (e.g., an integer bandit), but don't know its parameter μ
  - Assume we have a *prior* distribution $P(\mu) = \text{Dir}(\mu \,|\, \alpha)$
  - Assume we collected some data $D = \{x_1, .., x_n\}$, $x_i \in \{1, .., K\}$, with counts $m_k = \sum_i [x_i = k]$
  - The posterior is

$$\begin{aligned} P(\mu \,|\, D) &= \frac{P(D \,|\, \mu)}{P(D)} \ P(\mu) \propto \text{Mult}(D \,|\, \mu) \ \text{Dir}(\mu \,|\, a, b) \\ &\propto \prod_{k=1}^K \mu_k^{m_k} \ \prod_{k=1}^K \mu_k^{\alpha_k - 1} = \prod_{k=1}^K \mu_k^{\alpha_k - 1 + m_k} \\ &= \text{Dir}(\mu \,|\, \alpha + m) \end{aligned}$$

6:19

Dirichlet

The prior is $\text{Dir}(\mu \,|\, \alpha)$*, the posterior is* $\text{Dir}(\mu \,|\, \alpha + m)$

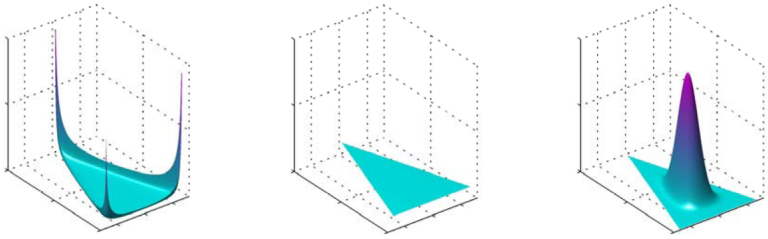
- Conclusions:
  - The semantics of α is the counts of $[x_i = k]$
  - The Dirichlet distribution is conjugate to the Multinomial

– With the Dirichlet distribution we can represent beliefs (state of knowledge) about uncertain μ of an integer random variable and know how to update this belief given data

6:20

Dirichlet

Illustrations for $\alpha = (0.1, 0.1, 0.1)$, $\alpha = (1, 1, 1)$ and $\alpha = (10, 10, 10)$:



from Bishop

6:21

Motivation for Beta & Dirichlet distributions

- Bandits:
  - If we have binary [integer] bandits, the Beta [Dirichlet] distribution is a way to represent and update beliefs
  - The belief space becomes discrete: The parameter α of the prior is continuous, but the posterior updates live on a discrete "grid" (adding counts to α)
  - We can in principle do belief planning using this
- Reinforcement Learning:
  - Assume we know that the world is a finite-state MDP, but do not know its transition probability $P(s' \,|\, s, a)$. For each (s, a), $P(s' \,|\, s, a)$ is a distribution over the integer s'
  - Having a separate Dirichlet distribution for each (s, a) is a way to represent our belief about the world, that is, our belief about $P(s' \,|\, s, a)$
  - We can in principle do belief planning using this $\rightarrow$ *Bayesian Reinforcement Learning*
- Dirichlet distributions are also used to model texts (word distributions in text), images, or mixture distributions in general

6:22

Conjugate priors

- Assume you have data $D = \{x_1, .., x_n\}$ with likelihood

$$P(D \,|\, \theta)$$

that depends on an uncertain parameter θ

Assume you have a prior $P(\theta)$

- The prior $P(\theta)$ is **conjugate** to the likelihood $P(D\,|\,\theta)$ iff the posterior

$$P(\theta\,|\,D) \propto P(D\,|\,\theta)\; P(\theta)$$

is in the *same distribution class* as the prior $P(\theta)$

- Having a conjugate prior is very convenient, because then you know how to update the belief given data

6:23

Conjugate priors

likelihood	conjugate
Binomial $\mathrm{Bin}(D\,|\,\mu)$	Beta $\mathrm{Beta}(\mu\,|\,a,b)$
Multinomial $\mathrm{Mult}(D\,|\,\mu)$	Dirichlet $\mathrm{Dir}(\mu\,|\,\alpha)$
Gauss $\mathcal{N}(x\,|\,\mu,\Sigma)$	Gauss $\mathcal{N}(\mu\,|\,\mu_0,A)$
1D Gauss $\mathcal{N}(x\,|\,\mu,\lambda^{-1})$	Gamma $\mathrm{Gam}(\lambda\,|\,a,b)$
nD Gauss $\mathcal{N}(x\,|\,\mu,\Lambda^{-1})$	Wishart $\mathrm{Wish}(\Lambda\,|\,W,\nu)$
nD Gauss $\mathcal{N}(x\,|\,\mu,\Lambda^{-1})$	Gauss-Wishart $\mathcal{N}(\mu\,|\,\mu_0,(\beta\Lambda)^{-1})\;\mathrm{Wish}(\Lambda\,|\,W,\nu)$

6:24

Distributions over continuous domain

6:25

Distributions over continuous domain

- Let x be a continuous RV. The **probability density function (pdf)** $p(x) \in [0,\infty)$ defines the probability

$$P(a \le x \le b) = \int_a^b p(x)\,dx \;\in [0,1]$$

The **(cumulative) probability distribution** $F(y) = P(x \le y) = \int_{-\infty}^{y} dx\; p(x) \in [0,1]$ is the cumulative integral with $\lim_{y\to\infty} F(y) = 1$

(In discrete domain: *probability distribution* and *probability mass function* $P(x) \in [0,1]$ are used synonymously.)

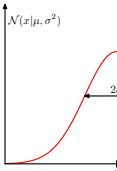
- Two basic examples:

Gaussian: $\mathcal{N}(x\,|\,\mu,\Sigma) = \frac{1}{|\,2\pi\Sigma\,|^{1/2}}\; e^{-\frac{1}{2}(x-\mu)^\top\,\Sigma^{-1}\,(x-\mu)}$

Dirac or δ ("point particle") $\delta(x) = 0$ except at $x = 0$, $\int \delta(x)\, dx = 1$
$\delta(x) = \frac{\partial}{\partial x} H(x)$ where $H(x) = [x \geq 0]$ = Heavyside step function

6:26

Gaussian distribution



- 1-dim: $\mathcal{N}(x\,|\,\mu,\sigma^2) = \frac{1}{|\,2\pi\sigma^2\,|^{1/2}}\, e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$
- n-dim Gaussian in *normal form*:

$$\mathcal{N}(x\,|\,\mu,\Sigma) = \frac{1}{|\,2\pi\Sigma\,|^{1/2}}\ \exp\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}\ (x-\mu)\}$$

with **mean** μ and **covariance** matrix Σ. In *canonical form*:

$$\mathcal{N}[x\,|\,a,A] = \frac{\exp\{-\frac{1}{2}a^\top A^{-1} a\}}{|\,2\pi A^{-1}\,|^{1/2}}\ \exp\{-\frac{1}{2}x^\top\, A\, x + x^\top a\} \tag{2}$$

with **precision** matrix $A = \Sigma^{-1}$ and coefficient $a = \Sigma^{-1}\mu$ (and mean $\mu = A^{-1}a$).

- Gaussian identities: see http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/gaussians.pdf

6:27

Gaussian identities

Symmetry: $\mathcal{N}(x\,|\,a,A) = \mathcal{N}(a\,|\,x,A) = \mathcal{N}(x-a\,|\,0,A)$

Product:
$\mathcal{N}(x\,|\,a,A)\,\mathcal{N}(x\,|\,b,B) = \mathcal{N}[x\,|\,A^{-1}a{+}B^{-1}b, A^{-1}{+}B^{-1}]\,\mathcal{N}(a\,|\,b, A{+}B)\,\mathcal{N}[x\,|\,a,A]\,\mathcal{N}[x\,|\,b,B] = \mathcal{N}[x\,|\,a+b, A+B]\,\mathcal{N}(A^{-1}a\,|\,B^{-1}b, A^{-1}+B^{-1})$

"Propagation":
$\int_y \mathcal{N}(x\,|\,a+Fy,A)\,\mathcal{N}(y\,|\,b,B)\,dy = \mathcal{N}(x\,|\,a+Fb, A+FBF^\top)$

Transformation:
$\mathcal{N}(Fx+f\,|\,a,A) = \frac{1}{|\,F\,|}\,\mathcal{N}(x\,|\ F^{-1}(a-f),\ F^{-1}AF^{-\top})$

Marginal & conditional:
$$\mathcal{N}\left(\begin{array}{c|cc} x & a & A \quad C \\ y & b\,, & C^\top \quad B \end{array}\right) = \mathcal{N}(x\,|\,a,A)\cdot\mathcal{N}(y\,|\,b + C^\top A^{-1}(x\,{-}\,a),\ B - C^\top A^{-1} C)$$

More Gaussian identities: see http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/gaussians.pdf

6:28

Motivation for Gaussian distributions

- Gaussian Bandits
- Control theory, Stochastic Optimal Control
- State estimation, sensor processing, Gaussian filtering (Kalman filtering)
- Machine Learning
- etc

6:29

Particle Approximation of a Distribution

- We approximate a distribution $p(x)$ over a continuous domain $\mathbb{R}^n$

- A particle distribution $q(x)$ is a weighed set $\mathcal{S} = \{(x^i, w^i)\}_{i=1}^N$ of N particles
  - each particle has a "location" $x^i \in \mathbb{R}^n$ and a weight $w^i \in \mathbb{R}$
  - weights are normalized, $\sum_i w^i = 1$

$$q(x) := \sum_{i=1}^{N} w^i \, \delta(x - x^i)$$

  where $\delta(x - x^i)$ is the δ-distribution.
- Given weighted particles, we can estimate for any (smooth) f:

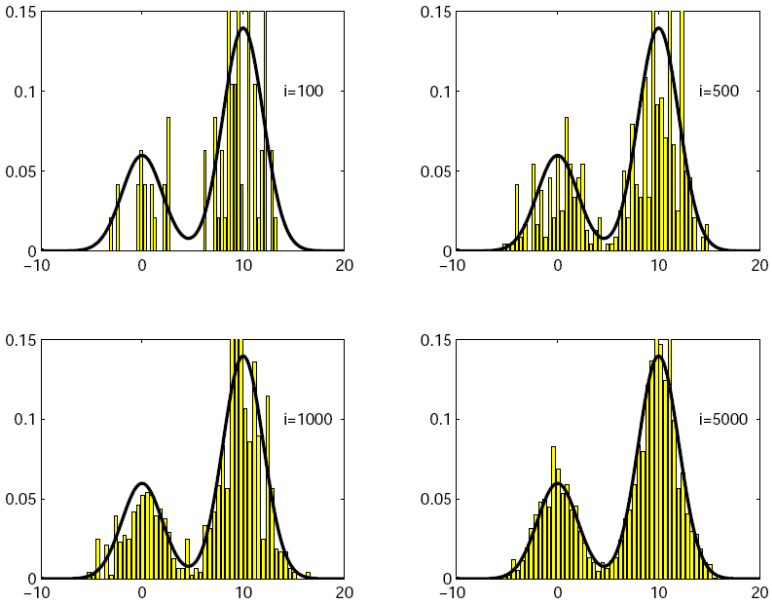
$$\langle f(x) \rangle_p = \int_x f(x) p(x) dx \;\approx\; \sum_{i=1}^N w^i f(x^i)$$

See *An Introduction to MCMC for Machine Learning* www.cs.ubc.ca/~nando/papers/mlintro.pdf

6:30

Particle Approximation of a Distribution

Histogram of a particle representation:



6:31

Motivation for particle distributions

- Numeric representation of "difficult" distributions
  - Very general and versatile
  - But often needs many samples
- Distributions over games (action sequences), sample based planning, MCTS
- State estimation, particle filters
- etc

6:32

Some more continuous distributions*

Gaussian	$\mathcal{N}(x \mid a, A) = \frac{1}{\mid 2\pi A \mid^{1/2}} e^{-\frac{1}{2}(x-a)^\top A^{-1} (x-a)}$
Dirac or δ	$\delta(x) = \frac{\partial}{\partial x} H(x)$
Student's t (=Gaussian for $\nu \to \infty$, otherwise heavy tails)	$p(x;\nu) \propto [1 + \frac{x^2}{\nu}]^{-\frac{\nu+1}{2}}$
Exponential (distribution over single event time)	$p(x;\lambda) = [x \geq 0]\, \lambda e^{-\lambda x}$
Laplace ("double exponential")	$p(x;\mu,b) = \frac{1}{2b} e^{-\mid x-\mu \mid /b}$
Chi-squared	$p(x;k) \propto [x \geq 0]\, x^{k/2-1} e^{-x/2}$
Gamma	$p(x;k,\theta) \propto [x \geq 0]\, x^{k-1} e^{-x/\theta}$

6:33

7 Mobile Robotics

http://www.darpa.mil/grandchallenge05/

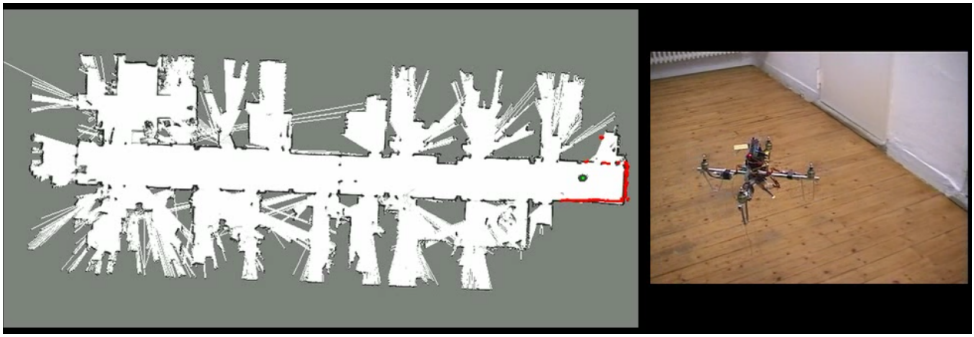
DARPA Grand Challenge 2005

7:1

http://www.darpa.mil/grandchallenge/index.asp

DARPA Grand Urban Challenge 2007

7:2



http://www.slawomir.de/publications/grzonka09icra/grzonka09icra.pdf

Quadcopter Indoor Localization

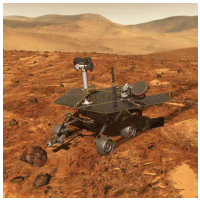
7:3

http://stair.stanford.edu/multimedia.php

STAIR: STanford Artificial Intelligence Robot

7:4

Types of Robot Mobility



7:5

- Each type of robot mobility corresponds to a
  system equation $x_{t+1} = x_t + \tau f(x_t, u_t)$
  or, if the dynamics are stochastic,

$$P(x_{t+1} \,|\, u_t, x_t) = \mathcal{N}(x_{t+1} \,|\, x_t + \tau f(x_t, u_t), \Sigma)$$

- We considered control, path finding, and trajectory optimization

  For this we always assumed to know the state x_t of the robot (e.g., its posture/position)!

7:6

Outline

- PART I:
  A core challenge in mobile robotics is **state estimation**
  → Bayesian filtering & smoothing
  particles, Kalman

- PART II:
  Another challenge is to **build a map** while exploring
  → SLAM (simultaneous localization and mapping)

7:7

7.1 State Estimation

7:8

State Estimation Problem

- Our sensory data does not provide sufficient information to determine our location.

- Given the local sensor readings y_t, the current state x_t (location, position) is *uncertain*.

– which hallway?
– which door exactly?
– which heading direction?

7:9

State Estimation Problem

- What is the probability of being in front of room 154, given we see what is shown in the image?
- What is the probability given that we were just in front of room 156?
- What is the probability given that we were in front of room 156 and moved 15 meters?

7:10

Recall Bayes' theorem

$$P(X|Y) = \frac{P(Y|X)\ P(X)}{P(Y)}$$

$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{(normalization)}}$$

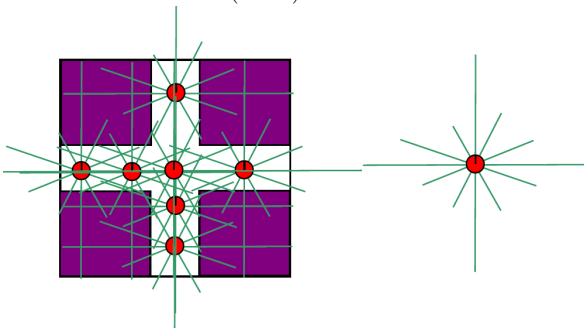
7:11

- How can we apply this to the

State Estimation Problem?

Using Bayes Rule:

$$P(\text{location} \,|\, \text{sensor}) = \frac{P(\text{sensor} \,|\, \text{location}) P(\text{location})}{P(\text{sensor})}$$



7:12

Bayes Filter

x_t = state (location) at time t

y_t = sensor readings at time t

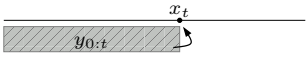
$u_{t\text{-}1}$ = control command (action, steering, velocity) at time t-1

- Given the history $y_{0:t}$ and $u_{0:t\text{-}1}$, we want to compute the probability distribution over the state at time t

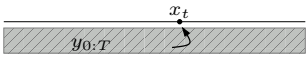
$$p_t(x_t) := P(x_t \,|\, y_{0:t}, u_{0:t\text{-}1})$$

- Generally:

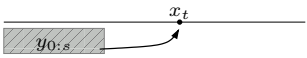
Filtering: $P(x_t|y_{0:t})$



Smoothing: $P(x_t|y_{0:T})$



Prediction: $P(x_t|y_{0:s})$



7:13

Bayes Filter

$$
\begin{aligned}
p_t(x_t) &:= P(x_t \,|\, y_{0:t}, u_{0:t\text{-}1}) \\
&= c_t \; P(y_t \,|\, x_t, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \; P(x_t \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \\
&= c_t \; P(y_t \,|\, x_t) \; P(x_t \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \\
&= c_t \; P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t, x_{t\text{-}1} \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \; dx_{t\text{-}1} \\
&= c_t \; P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, x_{t\text{-}1}, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \; P(x_{t\text{-}1} \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \; dx_{t\text{-}1} \\
&= c_t \; P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, x_{t\text{-}1}, u_{t\text{-}1}) \; P(x_{t\text{-}1} \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \; dx_{t\text{-}1} \\
&= c_t \; P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) \; p_{t\text{-}1}(x_{t\text{-}1}) \; dx_{t\text{-}1}
\end{aligned}
$$

using Bayes rule $P(X|Y,Z) = c\; P(Y|X,Z)\; P(X|Z)$ with some normalization constant c_t

uses conditional independence of the observation on past observations and controls

by definition of the marginal

by definition of a conditional

given $x_{t\text{-}1}$, x_t depends only on the controls $u_{t\text{-}1}$ (Markov Property)

- A Bayes filter updates the posterior belief $p_t(x_t)$ in each time step using the:
  observation model $P(y_t \,|\, x_t)$
  transition model $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$

7:14

Bayes Filter

$$
p_t(x_t) \;\propto\; \underbrace{P(y_t \,|\, x_t)}_{\text{new information}} \; \underbrace{\int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) \; \underbrace{p_{t\text{-}1}(x_{t\text{-}1})}_{\text{old estimate}} \; dx_{t\text{-}1}}_{\text{predictive estimate } \hat{p}_t(x_t)}
$$

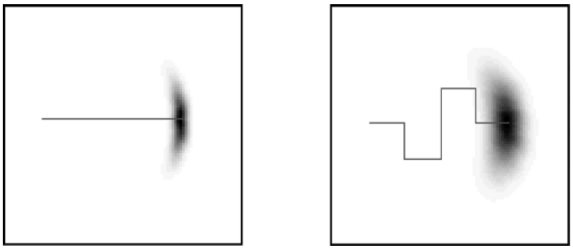
1. We have a belief $p_{t\text{-}1}(x_{t\text{-}1})$ of our previous position

2. We use the motion model to predict the current position
$$\hat{p}_t(x_t) \propto \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\; p_{t\text{-}1}(x_{t\text{-}1})\; dx_{t\text{-}1}$$

3. We integetrate this with the current observation to get a better belief
$$p_t(x_t) \propto P(y_t \,|\, x_t)\; \hat{p}_t(x_t)$$

7:15

- Typical transition model $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$ in robotics:

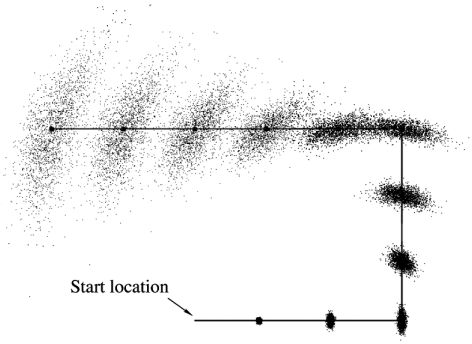


(from *Robust Monte Carlo localization for mobile robots* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

7:16

Odometry ("Dead Reckoning"): Filtering without observations

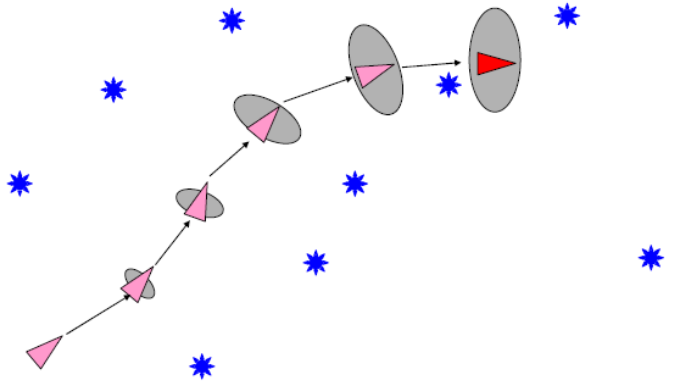
- The predictive distributions $\hat{p}_t(x_t)$ *without integrating observations* (removing the $P(y_t|x_t)$ part from the Bayesian filter)



(from *Robust Monte Carlo localization for mobile robots* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

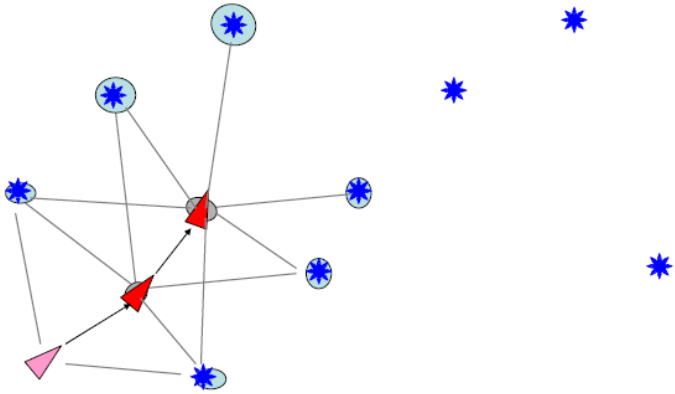
7:17

Again, predictive distributions $\hat{p}_t(x_t)$ *without integrating landmark observations*



7:18

The Bayes-filtered distributions $p_t(x_t)$ integrating landmark observations

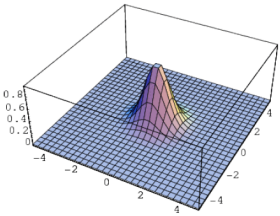


7:19

Bayesian Filters

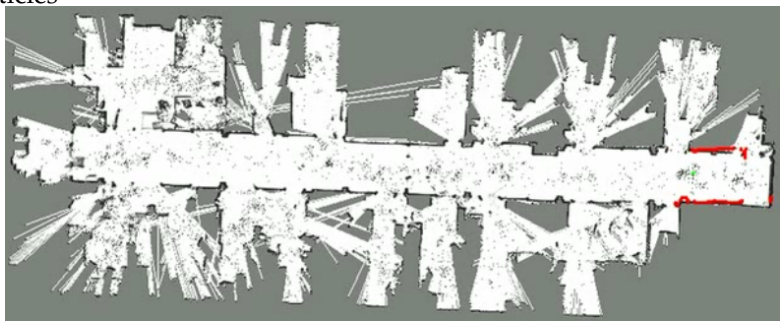
- How to represent the belief $p_t(x_t)$:

- Gaussian



$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Particles

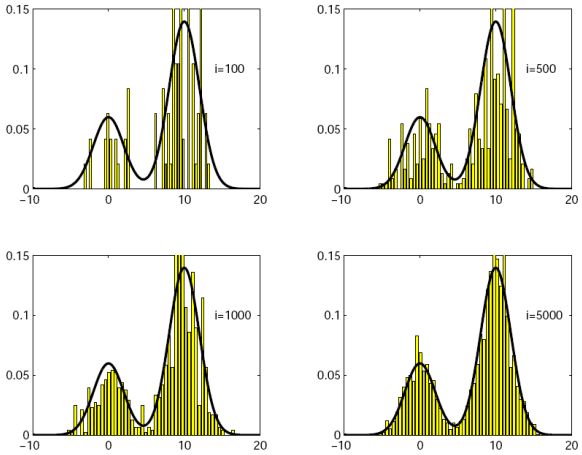


7:20

Recall: Particle Representation of a Distribution

- Weighed set of N particles $\{(x^i, w^i)\}_{i=1}^N$

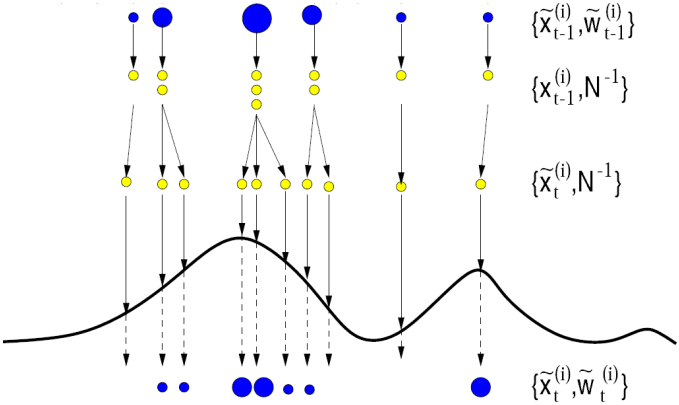
$$p(x) \approx q(x) := \sum_{i=1}^N w^i \delta(x, x^i)$$



7:21

Particle Filter := Bayesian Filtering with Particles

(Bayes Filter: $p_t(x_t) \;\propto\; P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\; p_{t-1}(x_{t-1})\; dx_{t-1}$)



1. Start with N particles $\{(x^i_{t-1}, w^i_{t-1})\}_{i=1}^N$
2. Resample particles to get N weight-1-particles: $\{\hat{x}^i_{t-1}\}_{i=1}^N$
3. Use motion model to get new "predictive" particles $\{x^i_t\}_{i=1}^N$ each $x^i_t \sim P(x_t \,|\, u_{t-1}, \hat{x}^i_{t-1})$
4. Use observation model to assign new weights $w^i_t \propto P(y_t \,|\, x^i_t)$

7:22

- "Particle Filter"

  aka *Monte Carlo Localization* in the mobile robotics community

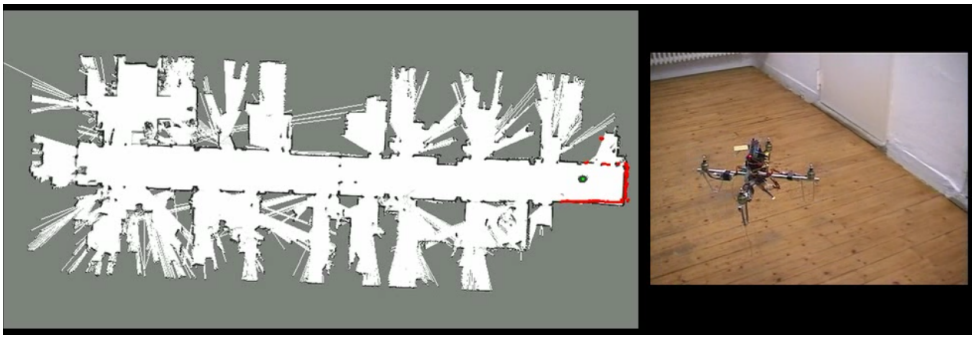
Condensation Algorithm in the vision community

- Efficient resampling is important:
  Typically "Residual Resampling":

  Instead of sampling directly $\hat{n}^i \sim \text{Multi}(\{Nw_i\})$ set $\hat{n}^i = \lfloor Nw_i \rfloor + \bar{n}_i$ with $\bar{n}_i \sim \text{Multi}(\{Nw_i - \lfloor Nw_i \rfloor\})$
  Liu & Chen (1998): Sequential Monte Carlo Methods for Dynamic Systems.
  Douc, Cappé & Moulines: Comparison of Resampling Schemes for Particle Filtering.

7:23

Example: Quadcopter Localization



http://www.slawomir.de/publications/grzonka09icra/grzonka09icra.pdf

Quadcopter Indoor Localization

7:24

Typical Pitfall in Particle Filtering

- Predicted particles $\{x_t^i\}_{i=1}^N$ have very low observation likelihood $P(y_t \,|\, x_t^i) \approx 0$ ("particles die over time")

- Classical solution: generate particles also with other than purely forward proposal $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$:

  – Choose a proposal that depends on the new observation y_t, ideally approximating $P(x_t \,|\, y_t, u_{t\text{-}1}, x_{t\text{-}1})$

  – Or mix particles sampled directly from $P(y_t \,|\, x_t)$ and from $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$.

  (*Robust Monte Carlo localization for mobile robots.* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

7:25

Kalman filter := Bayesian Filtering with Gaussians

Bayes Filter: $p_t(x_t) \;\propto\; P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\; p_{t\text{-}1}(x_{t\text{-}1})\; dx_{t\text{-}1}$

- Can be computed analytically for linear-Gaussian observations and transitions:
  $P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W)$
  $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, A(u_{t\text{-}1})\; x_{t\text{-}1} + a(u_{t\text{-}1}), Q)$

Defition:
$\mathcal{N}(x \,|\, a, A) = \frac{1}{|2\pi A|^{1/2}}\; \exp\{-\frac{1}{2}(x\text{-}a)^\top A^{\text{-}1}\; (x\text{-}a)\}$
Product:
$\mathcal{N}(x \,|\, a, A)\; \mathcal{N}(x \,|\, b, B) = \mathcal{N}(x \,|\, B(A{+}B)^{\text{-}1}a + A(A{+}B)^{\text{-}1}b, A(A{+}B)^{\text{-}1}B)\;\; \mathcal{N}(a \,|\, b, A + B)$
"Propagation":
$\int_y \mathcal{N}(x \,|\, a + Fy, A)\; \mathcal{N}(y \,|\, b, B)\; dy = \mathcal{N}(x \,|\, a + Fb, A + FBF^\top)$
Transformation:
$\mathcal{N}(Fx + f \,|\, a, A) = \frac{1}{|F|}\; \mathcal{N}(x \,|\; F^{\text{-}1}(a - f),\; F^{\text{-}1}AF^{\text{-}\top})$

(more identities: see "Gaussian identities" http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/gaussians.pdf)

7:26

Kalman filter derivation

$$
\begin{aligned}
& p_t(x_t) = \mathcal{N}(x_t \,|\, s_t, S_t) \\
& P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W) \\
& P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, Ax_{t\text{-}1} + a, Q) \\
p_t(x_t) &\propto P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\; p_{t\text{-}1}(x_{t\text{-}1})\; dx_{t\text{-}1} \\
&= \mathcal{N}(y_t \,|\, Cx_t + c, W) \int_{x_{t\text{-}1}} \mathcal{N}(x_t \,|\, Ax_{t\text{-}1} + a, Q)\; \mathcal{N}(x_{t\text{-}1} \,|\, s_{t\text{-}1}, S_{t\text{-}1})\; dx_{t\text{-}1} \\
&= \mathcal{N}(y_t \,|\, Cx_t + c, W)\; \mathcal{N}(x_t \,|\, \underbrace{As_{t\text{-}1} + a}_{=:\hat{s}_t},\; \underbrace{Q + AS_{t\text{-}1}A^\top}_{=:\hat{S}_t}) \\
&= \mathcal{N}(Cx_t + c \,|\, y_t, W)\; \mathcal{N}(x_t \,|\, \hat{s}_t, \hat{S}_t) \\
&= \mathcal{N}[x_t \,|\, C^\top W^{\text{-}1}(y_t - c),\; C^\top W^{\text{-}1}C]\; \mathcal{N}(x_t \,|\, \hat{s}_t, \hat{S}_t) \\
&= \mathcal{N}(x_t \,|\, s_t, S_t) \cdot \langle \text{terms indep. of } x_t \rangle \\
S_t &= (C^\top W^{\text{-}1}C + \hat{S}_t^{\text{-}1})^{\text{-}1} = \hat{S}_t - \underbrace{\hat{S}_t C^\top (W + C\hat{S}_t C^\top)^{\text{-}1}}_{\text{"Kalman gain" } K} C\hat{S}_t \\
s_t &= S_t[C^\top W^{\text{-}1}(y_t - c) + \hat{S}_t^{\text{-}1}\hat{s}_t] = \hat{s}_t + K(y_t - C\hat{s}_t - c)
\end{aligned}
$$

The second to last line uses the general Woodbury identity.
The last line uses $S_t C^\top W^{\text{-}1} = K$ and $S_t \hat{S}_t^{\text{-}1} = \mathbf{I} - KC$

7:27

Extended Kalman filter (EKF) and Unscented Transform

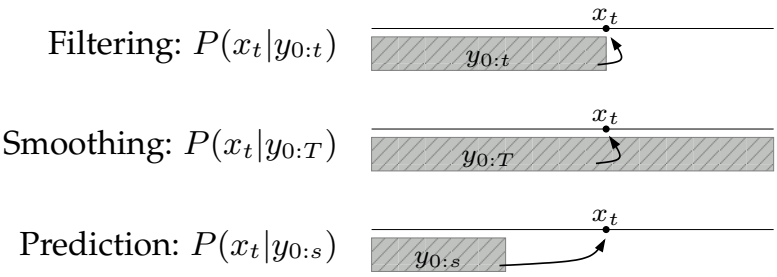
Bayes Filter: $p_t(x_t) \propto P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\, p_{t-1}(x_{t-1})\, dx_{t-1}$

- Can be computed analytically for linear-Gaussian observations and transitions:
  $P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W)$
  $P(x_t \,|\, u_{t-1}, x_{t-1}) = \mathcal{N}(x_t \,|\, A(u_{t-1})x_{t-1} + a(u_{t-1}), Q)$

- If $P(y_t \,|\, x_t)$ or $P(x_t \,|\, u_{t-1}, x_{t-1})$ are not linear:
  $P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, g(x_t), W)$
  $P(x_t \,|\, u_{t-1}, x_{t-1}) = \mathcal{N}(x_t \,|\, f(x_{t-1}, u_{t-1}), Q)$
  – approximate f and g as locally linear (*Extended Kalman Filter*)
  – or sample locally from them and reapproximate as Gaussian (*Unscented Transform*)

7:28

Bayes smoothing



7:29

Bayes smoothing

- Let $\mathcal{P} = y_{0:t}$ past observations, $\mathcal{F} = y_{t+1:T}$ future observations

$$\begin{aligned} P(x_t \,|\, \mathcal{P}, \mathcal{F}, u_{0:T}) &\propto P(\mathcal{F} \,|\, x_t, \mathcal{P}, u_{0:T})\; P(x_t \,|\, \mathcal{P}, u_{0:T}) \\ &= \underbrace{P(\mathcal{F} \,|\, x_t, u_{t:T})}_{=:\beta_t(x_t)}\; \underbrace{P(x_t \,|\, \mathcal{P}, u_{0:t-1})}_{=:p(x_t)} \end{aligned}$$

Bayesian smoothing fuses a forward filter $p_t(x_t)$ with a backward "filter" $\beta_t(x_t)$

- Backward recursion (derivation analogous to the Bayesian filter)

$$\beta_t(x_t) := P(y_{t+1:T} \,|\, x_t, u_{t:T})$$

$$= \int_{x_{t+1}} \beta_{t+1}(x_{t+1})\; P(y_{t+1} \,|\, x_{t+1})\; P(x_{t+1} \,|\, x_t, u_t)\; dx_{t+1}$$

7:30

7.2 Simultaneous Localization and Mapping (SLAM)

7:31

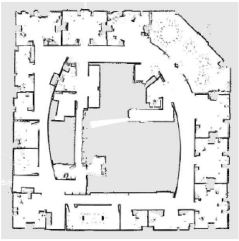
Localization and Mapping

- The Bayesian filter requires an observation model $P(y_t \,|\, x_t)$
- A **map** is something that provides the observation model:
  A map tells us for each x_t what the sensor readings y_t might look like

7:32

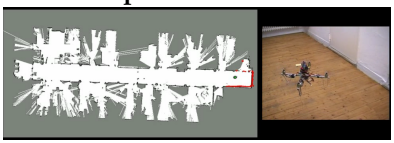
Types of maps

Grid map

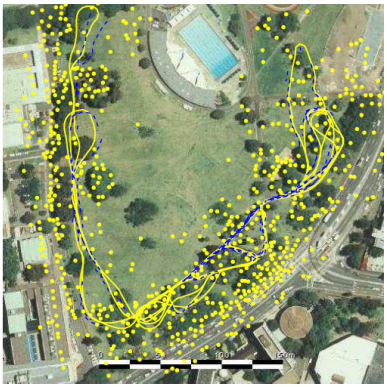


K. Murphy (1999): *Bayesian map learning in dynamic environments.*
Grisetti, Tipaldi, Stachniss, Burgard, Nardi: *Fast and Accurate SLAM with Rao-Blackwellized Particle Filters*

Laser scan map



Landmark map



Victoria Park data set
M. Montemerlo, S. Thrun, D. Koller, & B. Wegbreit (2003): *FastSLAM 2.0: An improved particle filtering algorithm for simultaneous localization and mapping that provably converges.* IJCAI, 1151–1156.

7:33

Simultaneous Localization and Mapping Problem

- Notation:
  x_t = state (location) at time t
  y_t = sensor readings at time t

$u_{t\text{-}1}$ = control command (action, steering, velocity) at time t-1
m = the map; *formally: a map is the parameters that define* $P(y_t \mid x_t, m)$

- Given the history $y_{0:t}$ and $u_{0:t\text{-}1}$, we want to compute the belief over the pose AND THE MAP m at time t

$$p_t(x_t, m) := P(x_t, m \mid y_{0:t}, u_{0:t\text{-}1})$$

- We assume to know the:
  – transition model $P(x_t \mid u_{t\text{-}1}, x_{t\text{-}1})$
  – observation model $P(y_t \mid x_t, m)$ *(defined by the map)*

7:34

SLAM: classical "chicken or egg problem"

- If we knew the state trajectory $x_{0:t}$ we could efficiently compute the belief over the map

$$P(m \mid x_{0:t}, y_{0:t}, u_{0:t\text{-}1})$$

- If we knew the map we could use a Bayes filter to compute the belief over the state

$$P(x_t \mid m, y_{0:t}, u_{0:t\text{-}1})$$

- SLAM requires to tie state estimation and map building together:
  1) Joint inference on x_t and m ($\rightarrow$ Kalman-SLAM)
  2) Tie a state hypothesis (=particle) to a map hypothesis
  ($\rightarrow$ particle SLAM)
  3) Frame everything as a graph optimization problem ($\rightarrow$ graph SLAM)

7:35

Joint Bayesian Filter over x and m

- A (formally) straight-forward approach is the joint Bayesian filter

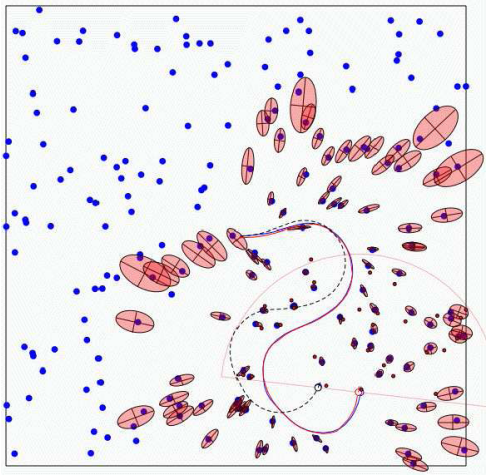
$$p_t(x_t, m) \;\propto\; P(y_t \mid x_t, m) \int_{x_{t\text{-}1}} P(x_t \mid u_{t\text{-}1}, x_{t\text{-}1})\, p_{t\text{-}1}(x_{t\text{-}1}, m)\, dx_{t\text{-}1}$$

But: How represent a belief over high-dimensional x_t, m?

7:36

Map uncertainty

- In the case the map $m = (\theta_1, .., \theta_N)$ is a set of N landmarks, $\theta_j \in \mathbb{R}^2$



- Use Gaussians to represent the uncertainty of landmark positions

7:37

(Extended) Kalman Filter SLAM

- Analogous to Localization with Gaussian for the pose belief $p_t(x_t)$
  – But now: joint belief $p_t(x_t, \theta_{1:N})$ is $3 + 2N$-dimensional Gaussian
  – Assumes the map $m = (\theta_1, .., \theta_N)$ is a set of N landmarks, $\theta_j \in \mathbb{R}^2$
  – Exact update equations (under the Gaussian assumption)
  – Conceptually very simple

- Drawbacks:
  – Scaling (full covariance matrix is $O(N^2)$)
  – Sometimes non-robust (uni-modal, "data association problem")
  – Lacks advantages of Particle Filter
  (multiple hypothesis, more robust to non-linearities)

7:38

SLAM with particles

Core idea: Each particle carries its own map belief

- Use a conditional representation "$p_t(x_t, m) = p_t(x_t)\; p_t(m \,|\, x_t)$"
  (This notation is flaky... the below is more precise)

- As for the Localization Problem use particles to represent the *pose belief* $p_t(x_t)$
  Note: Each particle actually "has a history $x^i_{0:t}$" – a whole trajectory!

- For each particle separately, estimate the *map belief* $p^i_t(m)$ conditioned on the particle history $x^i_{0:t}$.
  The conditional beliefs $p^i_t(m)$ may be factorized over grid points or landmarks of the map

  K. Murphy (1999): *Bayesian map learning in dynamic environments.*
  http://www.cs.ubc.ca/~murphyk/Papers/map_nips99.pdf

7:39

Map estimation for a *given* particle history

- Given $x_{0:t}$ (e.g. a trajectory of a particle), what is the posterior over the map m?

  $\rightarrow$ simplified Bayes Filter:

  $$p_t(m) := P(m \,|\, x_{0:t}, y_{0:t}) \propto P(y_t \,|\, m, x_t)\; p_{t\text{-}1}(m)$$

  (no transtion model: assumption that map is constant)

- In the case of landmarks (FastSLAM):
  $m = (\theta_1, .., \theta_N)$
  θ_j = position of the jth landmark, $j \in \{1, .., N\}$
  n_t = which landmark we observe at time t, $n_t \in \{1, .., N\}$

  We can use a separate (Gaussian) Bayes Filter for each θ_j
  conditioned on $x_{0:t}$, each θ_j is independent from each θ_k:

  $$P(\theta_{1:N} \,|\, x_{0:t}, y_{0:n}, n_{0:t}) = \prod_j P(\theta_j \,|\, x_{0:t}, y_{0:n}, n_{0:t})$$

7:40

Particle likelihood in SLAM

- Particle likelihood for Localization Problem:
  $$w^i_t = P(y_t \,|\, x^i_t)$$
  (determins the new importance weight w^i_t

- In SLAM the map is uncertain $\rightarrow$ each particle is weighted with the *expected* likelihood:
$$w_t^i = \int P(y_t \,|\, x_t^i, m)\; p_{t-1}(m)\; dm$$

- In case of landmarks (FastSLAM):
$$w_t^i = \int P(y_t \,|\, x_t^i, \theta_{n_t}, n_t)\; p_{t-1}(\theta_{n_t})\; d\theta_{n_t}$$

- Data association problem (actually we don't know n_t):
For each particle separately choose $n_t^i = \text{argmax}_{n_t}\; w_t^i(n_t)$

7:41

Particle-based SLAM summary

- We have a set of N particles $\{(x^i, w^i)\}_{i=1}^N$ to represent the pose belief $p_t(x_t)$

- For each particle we have a separate map belief $p_t^i(m)$; in the case of landmarks, this factorizes in N separate 2D-Gaussians

- Iterate
1. Resample particles to get N weight-1-particles: $\{\hat{x}_{t\text{-}1}^i\}_{i=1}^N$
2. Use motion model to get new "predictive" particles $\{x_t^i\}_{i=1}^N$
3. Update the map belief $p_m^i(m) \propto P(y_t \,|\, m, x_t)\; p_{t\text{-}1}^i(m)$ for each particle
4. Compute new importance weights $w_t^i \propto \int P(y_t \,|\, x_t^i, m)\; p_{t-1}(m)\; dm$ using the observation model and the map belief

7:42

Demo: Visual SLAM

- Map building from a freely moving camera

7:43

Demo: Visual SLAM

- Map building from a freely moving camera

  – SLAM has become a big topic in the vision community..
  – features are typically landmarks $\theta_{1:N}$ with SURF/SIFT features
  – PTAM (Parallel Tracking and Mapping) parallelizes computations...

  PTAM1 PTAM2 DTAM KinectFusion LSD-SLAM

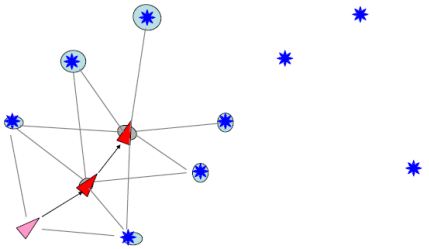
  Klein & Murray: *Parallel Tracking and Mapping for Small AR Workspaces* http://www.robots.ox.ac.uk/~gk/PTAM/

  Newcombe, Lovegrove & Davison: *DTAM: Dense Tracking and Mapping in Real-Time* ICCV 2011. http://www.doc.ic.ac.uk/~rnewcomb/

  Engel, Schöps & Cremers: LSD-SLAM: Large-Scale Direct Monocular SLAM, ECCV 2014. http://vision.in.tum.de/_media/spezial/bib/engel14eccv.pdf

7:44

Alternative SLAM approach: Graph-based

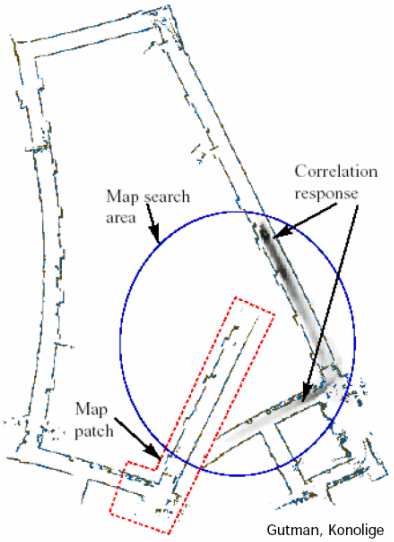


- Represent the previous trajectory as a graph
  – nodes = estimated positions & observations
  – edges = transition & step estimation based on scan matching
- Loop Closing: check if some nodes might coincide → new edges
- Classical Optimization:
  The whole graph defines an optimization problem: Find poses that minimize sum of edge & node errors

7:45

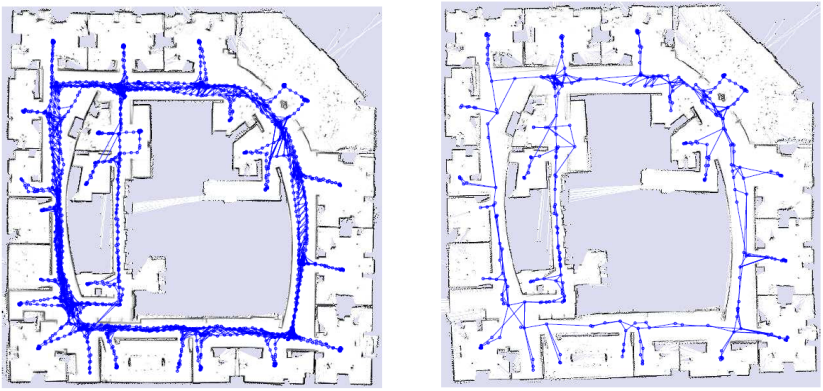
Loop Closing Problem

(Doesn't explicitly exist in Particle Filter methods: If particles cover the belief, then "data association" solves the "loop closing problem")



7:46

Graph-based SLAM



Life-long Map Learning for Graph-based SLAM Approaches in Static Environments Kretzschmar, Grisetti, Stachniss

7:47

SLAM code

- Graph-based and grid map methods:
  http://openslam.org/

- Visual SLAM
  see previous links
  e.g. http://ewokrampage.wordpress.com/

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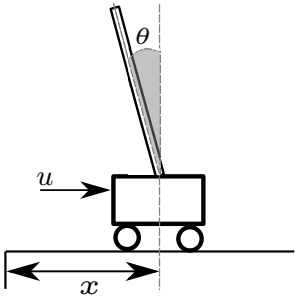
8 Control Theory

Cart pole example



$$\text{state } \boldsymbol{x} = (x, \dot{x}, \theta, \dot{\theta})$$

$$\ddot{\theta} = \frac{g\sin(\theta) + \cos(\theta)\left[-c_1 u - c_2 \dot{\theta}^2 \sin(\theta)\right]}{\frac{4}{3}l - c_2\cos^2(\theta)}$$

$$\ddot{x} = c_1 u + c_2\left[\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\right]$$

8:1

Control Theory

- Concerns controlled systems of the form

$$\dot{x} = f(x, u) + \text{noise}(x, u)$$

  and a controller of the form

$$\pi : (x, t) \mapsto u$$

- We'll neglect stochasticity here

- When analyzing a given controller π, one analyzes **closed-loop system** as described by the differential equation

$$\dot{x} = f(x, \pi(x, t))$$

  (E.g., analysis for convergence & stability)

8:2

Core topics in Control Theory

- **Stability***
  Analyze the stability of a closed-loop system
  → Eigenvalue analysis or Lyapunov function method
- **Controllability***
  Analyze which dimensions (DoFs) of the system can actually in principle be controlled
- **Transfer function**
  Analyze the closed-loop transfer function, i.e., "how frequencies are transmitted through the system". (→ Laplace transformation)
- **Controller design**
  Find a controller with desired stability and/or transfer function properties
- **Optimal control***
  Define a cost function on the system behavior. Optimize a controller to minimize costs

8:3

Control Theory references

- Robert F. Stengel: *Optimal control and estimation*
  Online lectures: http://www.princeton.edu/~stengel/MAE546Lectures.html (esp. lectures 3,4 and 7-9)

- From robotics lectures:
  Stefan Schaal's lecture Introduction to Robotics: http://www-clmc.usc.edu/Teaching/TeachingIntroductionToRoboticsSyllabus
  Drew Bagnell's lecture on Adaptive Control and Reinforcement Learning http://robotwhisperer.org/acrls11/

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Outline

- We'll first consider *optimal control*
  Goal: understand Algebraic Riccati equation
  significance for local neighborhood control

- Then controllability & stability

8:5

8.1 Optimal control

8:6

Optimal control (discrete time)

Given a controlled dynamic system

$$x_{t+1} = f(x_t, u_t)$$

we define a cost function

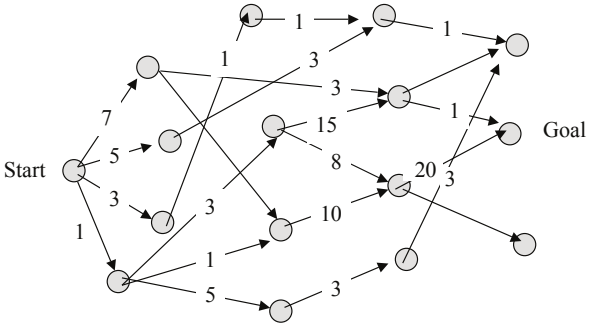
$$J^\pi = \sum_{t=0}^{T} c(x_t, u_t) + \phi(x_T)$$

where x_0 and the controller $\pi : (x, t) \mapsto u$ are given, which determines $x_{1:T}$ and $u_{0:T}$

8:7

Dynamic Programming & Bellman principle

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.



"$V(\text{state}) = \min_{\text{edge}}[c(\text{edge}) + V(\text{next-state})]$"

8:8

Bellman equation (discrete time)

- Define the **value function** or optimal **cost-to-go function**

$$V_t(x) = \min_\pi \Big[\sum_{s=t}^{T} c(x_s, u_s) + \phi(x_T) \Big]_{x_t = x}$$

- Bellman equation

$$V_t(x) = \min_u \Big[c(x, u) + V_{t+1}(f(x, u)) \Big]$$

The argmin gives the optimal control signal: $\pi_t^*(x) = \text{argmin}_u \Big[\cdots \Big]$

Derivation:

$$\begin{aligned} V_t(x) &= \min_\pi \Big[\sum_{s=t}^{T} c(x_s, u_s) + \phi(x_T) \Big] \\ &= \min_{u_t} \Big[c(x, u_t) + \min_\pi [\sum_{s=t+1}^{T} c(x_s, u_s) + \phi(x_T)] \Big] \\ &= \min_{u_t} \Big[c(x, u_t) + V_{t+1}(f(x, u_t)) \Big] \end{aligned}$$

8:9

Optimal Control (continuous time)

Given a controlled dynamic system

$$\dot{x} = f(x, u)$$

we define a cost function with horizon T

$$J^\pi = \int_0^T c(x(t), u(t))\, dt + \phi(x(T))$$

where the start state $x(0)$ and the controller $\pi : (x, t) \mapsto u$ are given, which determine the closed-loop system trajectory $x(t), u(t)$ via $\dot{x} = f(x, \pi(x, t))$ and $u(t) = \pi(x(t), t)$

8:10

Hamilton-Jacobi-Bellman equation (continuous time)

- Define the **value function** or optimal **cost-to-go function**

$$V(x, t) = \min_\pi \Big[\int_t^T c(x(s), u(s))\, ds + \phi(x(T)) \Big]_{x(t)=x}$$

- Hamilton-Jacobi-Bellman equation

$$-\tfrac{\partial}{\partial t} V(x, t) = \min_u \Big[c(x, u) + \tfrac{\partial V}{\partial x} f(x, u) \Big]$$

The argmin gives the optimal control signal: $\pi^*(x) = \operatorname{argmin}_u \Big[\cdots \Big]$

Derivation: Apply the discrete-time Bellman equation for V_t and V_{t+dt}:

$$\begin{aligned} V(x, t) &= \min_u \Big[\int_t^{t+dt} c(x, u)\, dt + V(x(t + dt), t + dt) \Big] \\ &= \min_u \Big[\int_t^{t+dt} c(x, u)\, dt + V(x, t) + \int_t^{t+dt} \frac{dV(x, t)}{dt}\, dt \Big] \\ 0 &= \min_u \Big[\int_t^{t+dt} c(x, u) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x}\, dt \Big] \\ 0 &= \min_u \Big[c(x, u) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u) \Big] \end{aligned}$$

8:11

Infinite horizon case

$$J^\pi = \int_0^\infty c(x(t), u(t))\, dt$$

- This cost function is stationary (time-invariant)!
  $\rightarrow$ the optimal value function is stationary ($V(x,t) = V(x)$)
  $\rightarrow$ the optimal control signal depends on x but not on t
  $\rightarrow$ the optimal controller π^* is stationary

- The HBJ and Bellman equations remain "the same" but with the same (stationary) value function independent of t:

$$0 = \min_u \Big[c(x,u) + \frac{\partial V}{\partial x} f(x,u) \Big] \qquad \text{(cont. time)}$$
$$V(x) = \min_u \Big[c(x,u) + V(f(x,u)) \Big] \qquad \text{(discrete time)}$$

  The argmin gives the optimal control signal: $\pi^*(x) = \mathrm{argmin}_u \Big[\cdots \Big]$

8:12

Infinite horizon examples

- Cart-pole balancing:
  – You always want the pole to be upright ($\theta \approx 0$)
  – You always want the car to be close to zero ($x \approx 0$)
  – You want to spare energy (apply low torques) ($u \approx 0$)
  You might define a cost
$$J^\pi = \int_0^\infty \|\theta\|^2 + \epsilon\|x\|^2 + \rho\|u\|^2$$

- Reference following:
  – You always want to stay close to a reference trajectory $r(t)$
  Define $\tilde{x}(t) = x(t) - r(t)$ with dynamics $\dot{\tilde{x}}(t) = f(\tilde{x}(t) + r(t), u) - \dot{r}(t)$
  Define a cost
$$J^\pi = \int_0^\infty \|\tilde{x}\|^2 + \rho\|u\|^2$$

- Many many problems in control can be framed this way

8:13

Comments

- The Bellman equation is fundamental in optimal control theory, but also Reinforcement Learning

- The HJB eq. is a differential equation for $V(x,t)$ which is in general hard to solve
- The (time-discretized) Bellman equation can be solved by Dynamic Programming starting backward:

$$V_T(x) = \phi(x)\ , \quad V_{T\text{-}1}(x) = \min_u \Big[c(x,u) + V_T(f(x,u))\Big] \qquad \text{etc.}$$

But it might still be hard or infeasible to represent the functions $V_t(x)$ over continuous x!

- Both become significantly simpler under linear dynamics and quadratic costs:
$$\rightarrow \text{Riccati equation}$$

8:14

Linear-Quadratic Optimal Control

linear dynamics

$$\dot{x} = f(x,u) = Ax + Bu$$

quadratic costs

$$c(x,u) = x^\top Q x + u^\top R u\ , \quad \phi(x_T) = x_T^\top F x_T$$

- Note: Dynamics neglects constant term; costs neglect linear and constant terms. This is because
  – constant costs are irrelevant
  – linear cost terms can be made away by redefining x or u
  – constant dynamic term only introduces a constant drift

8:15

Linear-Quadratic Control as Local Approximation

- LQ control is important also to control non-LQ systems in the **neighborhood** of a desired state!

Let x^* be such a desired state (e.g., cart-pole: $x^* = (0,0,0,0)$)
– linearize the dynamics around x^*
– use 2nd order approximation of the costs around x^*
– control the system *locally* as if it was LQ
– pray that the system will never leave this neighborhood!

8:16

Riccati differential equation = HJB eq. in LQ case

- In the Linear-Quadratic (LQ) case, the value function always is a quadratic function of x!

Let $V(x,t) = x^\top P(t) x$, then the HBJ equation becomes

$$\begin{aligned}
-\frac{\partial}{\partial t} V(x,t) &= \min_u \Big[c(x,u) + \frac{\partial V}{\partial x} f(x,u) \Big] \\
-x^\top \dot{P}(t) x &= \min_u \Big[x^\top Q x + u^\top R u + 2x^\top P(t)(Ax + Bu) \Big] \\
0 &= \frac{\partial}{\partial u} \Big[x^\top Q x + u^\top R u + 2x^\top P(t)(Ax + Bu) \Big] \\
&= 2u^\top R + 2x^\top P(t) B \\
u^* &= -R^{-1} B^\top P x
\end{aligned}$$

$\Rightarrow$ **Riccati differential equation**

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q$$

8:17

Riccati differential equation

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q$$

- This is a differential equation for the matrix $P(t)$ describing the quadratic value function. If we solve it with the finite horizon constraint $P(T) = F$ we solved the optimal control problem

- The optimal control $u^* = -R^{-1}B^\top P x$ is called **Linear Quadratic Regulator**

Note: If the state is dynamic (e.g., $x = (q, \dot{q})$) this control is linear in the positions and linear in the velocities and is an instance of **PD control**
The matrix $K = R^{-1}B^\top P$ is therefore also called **gain matrix**
For instance, if $x(t) = (q(t) - r(t), \dot{q}(t) - \dot{r}(t))$ for a reference $r(t)$ and $K = [K_p \quad K_d]$ then

$$u^* = K_p(r(t) - q(t)) + K_d(\dot{r}(t) - \dot{q}(t))$$

8:18

Riccati equations

- Finite horizon continuous time
  Riccati differential equation

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q\,, \quad P(T) = F$$

- Infinite horizon continuous time
  Algebraic Riccati equation (ARE)
  $$0 = A^\top P + PA - PBR^{-1}B^\top P + Q$$

- Finite horizon discrete time $(J^\pi = \sum_{t=0}^{T} \|x_t\|_Q^2 + \|u_t\|_R^2 + \|x_T\|_F^2)$

$$P_{t-1} = Q + A^\top[P_t - P_t B(R + B^\top P_t B)^{-1} B^\top P_t]A\,, \quad P_T = F$$

- Infinite horizon discrete time $(J^\pi = \sum_{t=0}^{\infty} \|x_t\|_Q^2 + \|u_t\|_R^2)$

$$P = Q + A^\top[P - PB(R + B^\top PB)^{-1} B^\top P]A$$

8:19

Example: 1D point mass

- Dynamics:
$$\ddot{q}(t) = u(t)/m$$

$$\begin{aligned} x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} &= \begin{pmatrix} \dot{q} \\ u(t)/m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u \\ &= Ax + Bu\,, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix} \end{aligned}$$

- Costs:
$$c(x,u) = \epsilon\|x\|^2 + \varrho\|u\|^2\,, \quad Q = \epsilon\mathbf{I}\,, \quad R = \varrho\mathbf{I}$$

- Algebraic Riccati equation:

$$\begin{aligned} P &= \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad u^* = -R^{-1}B^\top Px = \frac{-1}{\varrho m}[cq + b\dot{q}] \\ 0 &= A^\top P + PA - PBR^{-1}B^\top P + Q \\ &= \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} - \frac{1}{\varrho m^2}\begin{pmatrix} c^2 & bc \\ bc & b^2 \end{pmatrix} + \epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

8:20

Example: 1D point mass (cont.)

- Algebraic Riccati equation:

$$P = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad u^* = -R^{-1}B^\top Px = \frac{-1}{\varrho m}[cq + b\dot{q}]$$
$$0 = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} - \frac{1}{\varrho m^2}\begin{pmatrix} c^2 & bc \\ bc & b^2 \end{pmatrix} + \epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  First solve for c, then for $b = m\sqrt{\varrho}\sqrt{c+\epsilon}$. Whether the damping ration $\xi = \frac{b}{\sqrt{4mc}}$ depends on the choices of ϱ and ϵ.

- The Algebraic Riccati equation is usually solved numerically. (E.g. `are`, `care` or `dare` in Octave)

8:21

Optimal control comments

- HJB or Bellman equation are very powerful concepts

- Even if we can solve the HJB eq. and have the optimal control, we still don't know how the system really behaves?
  – Will it actually reach a "desired state"?
  – Will it be stable?
  – It is actually "controllable" at all?

- Last note on optimal control:
  Formulate as a constrainted optimization problem with objective function J^π and constraint $\dot{x} = f(x,u)$. $\lambda(t)$ are the Langrange multipliers. It turns out that $\frac{\partial}{\partial x}V(x,t) = \lambda(t)$. (See Stengel.)

8:22

Relation to other topics

- Optimal Control:

$$\min_\pi \; J^\pi = \int_0^T c(x(t), u(t))\, dt + \phi(x(T))$$

- Inverse Kinematics:

$$\min_q \; f(q) = \|q - q_0\|^2_W + \|\phi(q) - y^*\|^2_C$$

- Operational space control:

$$\min_u \; f(u) = \|u\|^2_H + \|\ddot{\phi}(q) - \ddot{y}^*\|^2_C$$

- Trajectory Optimization: (control hard constraints could be included)

$$\min_{q_{0:T}} f(q_{0:T}) = \sum_{t=0}^{T} \|\Psi_t(q_{t\text{-}k}, .., q_t)\|^2 + \sum_{t=0}^{T} \|\Phi_t(q_t)\|^2$$

- Reinforcement Learning:
  – Markov Decision Processes $\leftrightarrow$ discrete time stochastic controlled system $P(x_{t+1} \mid u_t, x$
  – Bellman equation $\rightarrow$ Basic RL methods (Q-learning, etc)

8:23

8.2 Controllability

8:24

Controllability

- As a starting point, consider the claim:
  "Intelligence means to gain maximal controllability over all degrees of freedom over the environment."

  Note:
  – controllability (ability to control) $\neq$ control
  – What does controllability mean exactly?

- I think the general idea of *controllability* is really interesting
  – Linear control theory provides one specific definition of controllability, which we introduce next..

8:25

Controllability

- Consider a linear controlled system

$$\dot{x} = Ax + Bu$$

  How can we tell from the matrices A and B *whether we can control* x *to eventually reach any desired state?*

- Example: x is 2-dim, u is 1-dim:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

  Is x "controllable"?

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$$

Is x "controllable"?

8:26

Controllability

We consider a linear stationary (=time-invariant) controlled system

$$\dot{x} = Ax + Bu$$

- **Complete controllability:** All elements of the state can be brought from arbitrary initial conditions to zero in finite time
- A system is completely controllable iff the **controllability matrix**

$$C := \begin{bmatrix}B & AB & A^2B & \cdots & A^{n\text{-}1}B\end{bmatrix}$$

has full rank $\dim(x)$ (that is, all rows are linearly independent)

- *Meaning of C:*
  The ith row describes how the ith element x_i can be influenced by u
  "B": $\dot{x}_i$ is directly influenced via B
  "AB": $\ddot{x}_i$ is "indirectly" influenced via AB (u directly influences some $\dot{x}_j$ via B; the dynamics A then influence $\ddot{x}_i$ depending on $\dot{x}_j$)
  "A^2B": $\dddot{x}_i$ is "double-indirectly" influenced
  etc...

  Note: $\ddot{x} = A\dot{x} + B\dot{u} = AAx + ABu + B\dot{u}$
  $\dddot{x} = A^3x + A^2Bu + AB\dot{u} + B\ddot{u}$

8:27

Controllability

- When all rows of the controllability matrix are linearly independent $\Rightarrow (u, \dot{u}, \ddot{u}, ...)$ can influence all elements of x independently
- If a row is zero $\rightarrow$ this element of x cannot be controlled at all
- If 2 rows are linearly dependent $\rightarrow$ these two elements of x will remain coupled forever

8:28

Controllability examples

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}u \qquad C = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \text{ rows linearly dependent}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}u \qquad C = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ 2nd row zero}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u \qquad C = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ good!}$$

8:29

Controllability

Why is it important/interesting to analyze controllability?

- The Algebraic Riccati Equation will always return an "optimal" controller – but controllability tells us whether such a controller even has a chance to control x

- *"Intelligence means to gain maximal controllability over all degrees of freedom over the environment."*
  – real environments are non-linear
  – "to have the ability to *gain* controllability over the environment's DoFs"

8:30

8.3 Stability

8:31

Stability

- One of the most central topics in control theory

- Instead of designing a controller by first designing a cost function and then applying Riccati,
  design a controller such that the desired state is provably a stable equilibrium point of the closed loop system

8:32

Stability

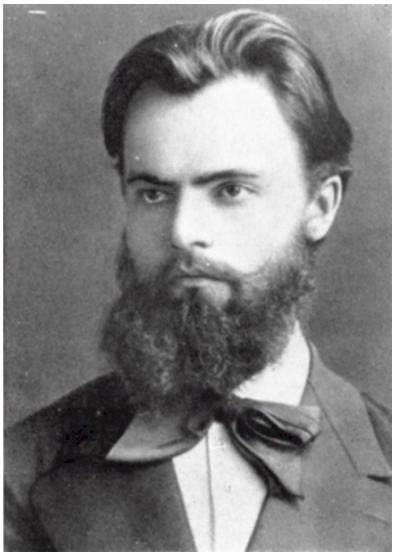
- Stability is an analysis of the *closed loop* system. That is: for this analysis we don't need to distinguish between system and controller explicitly. Both together define the dynamics

$$\dot{x} = f(x, \pi(x,t)) =: f(x)$$

- The following will therefore discuss stability analysis of general differential equations $\dot{x} = f(x)$

- What follows:
  – 3 basic definitions of stability
  – 2 basic methods for analysis by Lyapunov

8:33



Aleksandr Lyapunov (1857–1918)

8:34

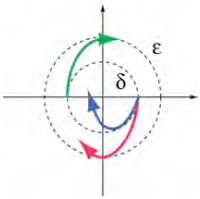
Stability – 3 definitions

$\dot{x} = F(x)$ with equilibrium point $x = 0$

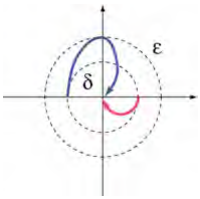
- x_0 is an **equilibrium point** $\iff f(x_0) = 0$

- **Lyapunov stable** or **uniformly stable** $\iff$

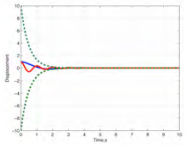
$$\forall \epsilon : \ \exists \delta \text{ s.t. } \|x(0)\| \leq \delta \ \Rightarrow \ \forall t : \ \|x(t)\| \leq \epsilon$$



(when it starts off δ-near to x_0, it will remain ϵ-near forever)



- **asymtotically stable** $\iff$
  Lyapunov stable and $\lim_{t\to\infty} x(t) = 0$



- **exponentially stable** $\iff$
  asymtotically stable and $\exists \alpha, a$ s.t. $\|x(t)\| \leq a e^{-\alpha t}\|x(0)\|$

  ($\to$ the "error" time integral $\int_0^\infty \|x(t)\|dt \leq \frac{a}{\alpha}\|x(0)\|$ is bounded!)

8:35

Linear Stability Analysis

("Linear" $\leftrightarrow$ "local" for a system linearized at the equilibrium point.)

- Given a linear system
$$\dot{x} = Ax$$
Let λ_i be the **eigenvalues** of A
– The system is *asymptotically stable* $\iff \forall_i : \ \text{real}(\lambda_i) < 0$
– The system is *unstable stable* $\iff \exists_i : \ \text{real}(\lambda_i) > 0$
– The system is *marginally stable* $\iff \forall_i : \ \text{real}(\lambda_i) \leq 0$

- Meaning: An eigenvalue describes how the system behaves along one state dimension (along the eigenvector):
$$\dot{x}_i = \lambda_i x_i$$
As for the 1D point mass the solution is $x_i(t) = a e^{\lambda_i t}$ and
– imaginary $\lambda_i \to$ oscillation
– negative $\text{real}(\lambda_i) \to$ exponential decay $\propto e^{-|\lambda_i| t}$
– positive $\text{real}(\lambda_i) \to$ exponential explosion $\propto e^{|\lambda_i| t}$

8:36

Linear Stability Analysis: Example

- Let's take the 1D point mass $\ddot{q} = u/m$ in *closed loop* with a PD $u = -K_p q - K_d \dot{q}$

- Dynamics:

$$\dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + 1/m \begin{pmatrix} 0 & 0 \\ -K_p & -K_d \end{pmatrix} x$$
$$A = \begin{pmatrix} 0 & 1 \\ -K_p/m & -K_d/m \end{pmatrix}$$

- Eigenvalues:

  The equation $\lambda \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K_p/m & -K_d/m \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$ leads to the equation $\lambda \dot{q} = \lambda^2 q = -K_p/m q - K_d/m \lambda q$ or $m\lambda^2 + K_d\lambda + K_p = 0$ with solution (compare slide 05:10)

$$\lambda = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m}$$

  For $K_d^2 - 4mK_p$ negative, the $\text{real}(\lambda) = -K_d/2m$
  $\Rightarrow$ Positive derivative gain K_d makes the system stable.

8:37

Side note: Stability for discrete time systems

- Given a discrete time linear system

$$x_{t+1} = Ax_t$$

  Let λ_i be the **eigenvalues** of A
  – The system is *asymptotically stable* $\iff \forall_i : |\lambda_i| < 1$
  – The system is *unstable stable* $\iff \exists_i : |\lambda_i| > 1$
  – The system is *marginally stable* $\iff \forall_i : |\lambda_i| \leq 1$

8:38

Linear Stability Analysis comments

- The same type of analysis can be done locally for non-linear systems, as we will do for the cart-pole in the exercises

- We can design a controller that minimizes the (negative) eigenvalues of A:
  $\leftrightarrow$ controller with fastest asymtopic convergence

  This is a real alternative to optimal control!

8:39

Lyapunov function method

- A method to analyze/prove stability for general non-linear systems is the famous "Lyapunov's second method"

  Let D be a region around the equilibrium point x_0
- A **Lyaponov function** $V(x)$ for a system dynamics $\dot{x} = f(x)$ is
  – positive, $V(x) > 0$, everywhere in D except...
  at the equilibrium point where $V(x_0) = 0$
  – always decreases, $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} < 0$, in D except...
  at the equilibrium point where $f(x) = 0$ and therefore $\dot{V}(x) = 0$

- If there exists a D and a Lyapunov function $\Rightarrow$ the system is *asymtotically stable*

  If D is the whole state space, the system is *globally stable*

8:40

Lyapunov function method

- The Lyapunov function method is very general. $V(x)$ could be "anything" (energy, cost-to-go, whatever). Whenever one finds some $V(x)$ that decreases, this proves stability

- The problem though is to think of some $V(x)$ given a dynamics!
  (In that sense, the Lyapunov function method is rather a method of proof than a concrete method for stability analysis.)

- In standard cases, a good guess for the Lyapunov function is either the energy or the value function

8:41

Lyapunov function method – Energy Example

- Let's take the 1D point mass $\ddot{q} = u/m$ in *closed loop* with a PD $u = -K_p q - K_d \dot{q}$, which has the solution (slide 05:14):

$$q(t) = be^{-\xi/\lambda\, t}\, e^{i\omega_0\sqrt{1-\xi^2}\, t}$$

- Energy of the 1D point mass: $V(q, \dot{q}) := \frac{1}{2}m\dot{q}^2$

$$\dot{V}(x) = e^{-2\xi/\lambda\, t} V(x(0))$$

  (using that the energy of an undamped oscillator is conserved)
- $V(x) < 0 \iff \xi > 0 \iff K_d > 0$
  Same result as for the eigenvalue analysis

8:42

Lyapunov function method – value function Example

- Consider infinite horizon linear-quadratic optimal control. The solution of the Algebraic Riccati equation gives the optimal controller.
- The value function satisfies

$$\begin{aligned}
V(x) &= x^\top P x \\
\dot V(x) &= [Ax + Bu^*]^\top P x + x^\top P [Ax + Bu^*] \\
u^* &= -R^{-1} B^\top P x = Kx \\
\dot V(x) &= x^\top [(A+BK)^\top P + P(A+BK)] x \\
&= x^\top [A^\top P + PA + (BK)^\top P + P(BK)] x \\
0 &= A^\top P + PA - PBR^{-1}B^\top P + Q \\
\dot V(x) &= x^\top [PBR^{-1}B^\top P - Q + (PBK)^\top + PBK] x \\
&= -x^\top [Q + K^\top R K] x
\end{aligned}$$

(We could have derived this easier! $x^\top Q x$ are the immediate state costs, and $x^\top K^\top R K x = u^\top R u$ are the immediate control costs—and $\dot V(x) = -c(x, u^*)$! See slide 11 bottom.)

- That is: V is a Lyapunov function if $Q + K^\top R K$ is positive definite!

8:43

Observability & Adaptive Control

- When some state dimensions are not directly observable: analyzing higher order derivatives to *infer* them.

  Very closely related to controllability: Just like the controllability matrix tells whether state dimensions can (indirectly) be controlled; an observation matrix tells whether state dimensions can (indirectly) be inferred.

- Adaptive Control: When system dynamics $\dot x = f(x, u, \beta)$ has unknown parameters β.

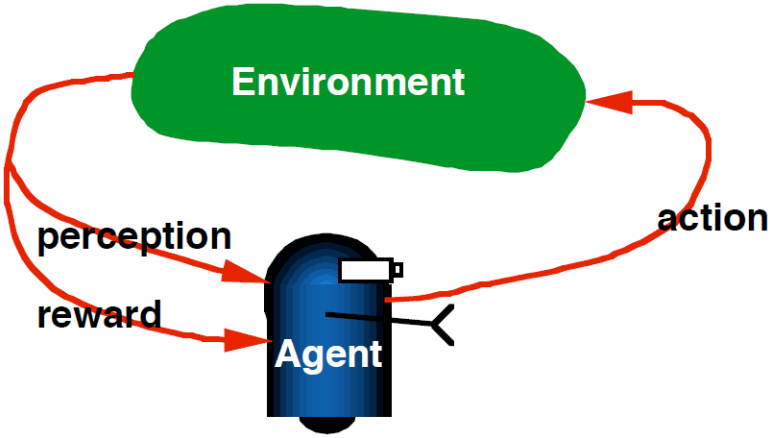
  – One approach is to estimate β from the data so far and use optimal control.

  – Another is to design a controller that has an additional internal update equation for an estimate $\hat\beta$ and is provably stable. (See Schaal's lecture, for instance.)

8:44

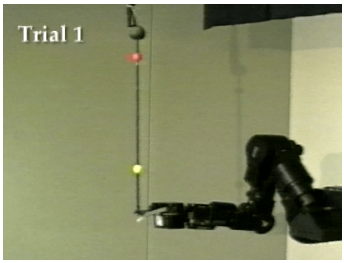
9 Reinforcement Learning in Robotics – briefly

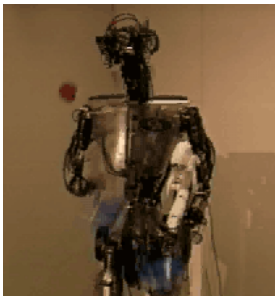
RL = Learning to Act



from Satinder Singh's *Introduction to RL*, videolectures.com

9:1





(around 2000, by Schaal, Atkeson, Vijayakumar)

(2007, Andrew Ng et al.)

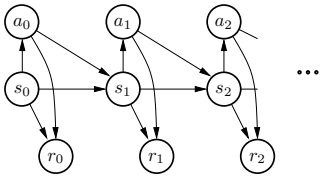
9:2

Applications of RL

- Robotics
  – Navigation, walking, juggling, helicopters, grasping, etc...
- Games
  – Backgammon, Chess, Othello, Tetris, ...
- Control
  – factory processes, resource control in multimedia networks, elevators, ...
- Operations Research
  – Warehousing, transportation, scheduling, ...

9:3

Markov Decision Process



$$P(s_{0:T+1}, a_{0:T}, r_{0:T}; \pi) = P(s_0) \prod_{t=0}^{T} P(a_t|s_t;\pi)\, P(r_t|s_t,a_t)\, P(s_{t+1}|s_t,a_t)$$

– world's initial state distribution $P(s_0)$
– world's transition probabilities $P(s_{t+1} \mid s_t, a_t)$
– world's reward probabilities $P(r_t \mid s_t, a_t)$
– agent's *policy* $\pi(a_t \mid s_t) = P(a_0|s_0;\pi)$ (or deterministic $a_t = \pi(s_t)$)

- **Stationary MDP:**
  – We assume $P(s' \mid s, a)$ and $P(r|s,a)$ independent of time
  – We also define $R(s,a) := \mathrm{E}\{r|s,a\} = \int r\, P(r|s,a)\, dr$

9:4

... in the notation of control theory

- We have a (potentially stochastic) controlled system

$$\dot{x} = f(x,u) + \text{noise}(x,u)$$

- We have costs (neg-rewards), e.g. in the finite horizon case:

$$J^\pi = \int_0^T c(x(t), u(t))\, dt + \phi(x(T))$$

- We want a policy ("controller")

$$\pi : (x, t) \mapsto u$$

9:5

Reinforcement Learning = the dynamics f and costs c are unknown

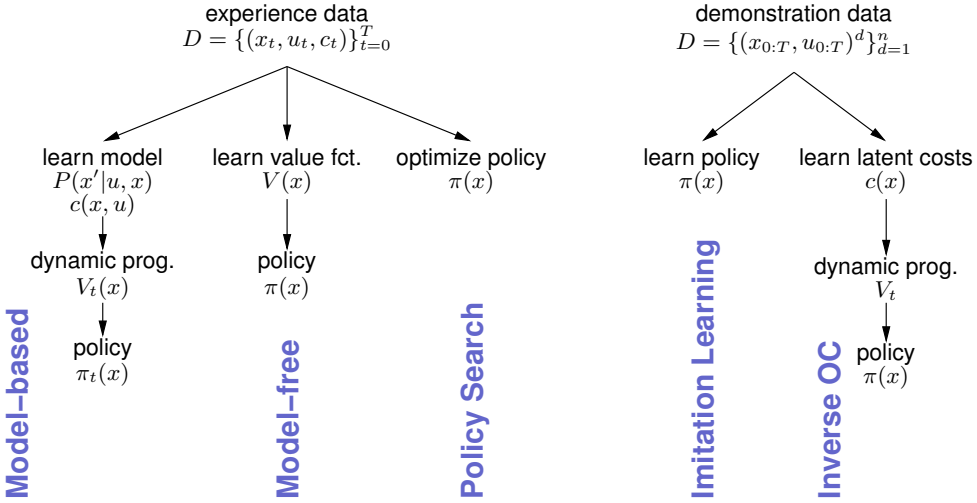
- All the agent can do is collect data

$$D = \{(x_t, u_t, c_t)\}_{t=0}^T$$

What can we do with this data?

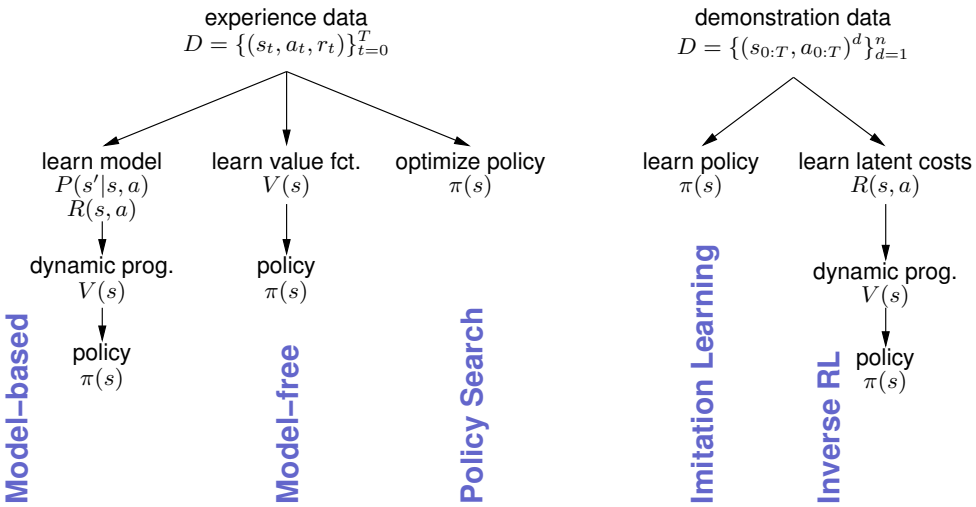
9:6

Five approaches to RL



9:7

Five approaches to RL



9:8

Imitation Learning

$$D = \{(s_{0:T}, a_{0:T})^d\}_{d=1}^n \overset{\text{learn/copy}}{\rightarrow} \pi(s)$$

- Use ML to imitate demonstrated state trajectories $x_{0:T}$

Literature:

Atkeson & Schaal: Robot learning from demonstration (ICML 1997)

Schaal, Ijspeert & Billard: Computational approaches to motor learning by imitation (Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences 2003)

Grimes, Chalodhorn & Rao: Dynamic Imitation in a Humanoid Robot through Nonparametric Probabilistic Inference. (RSS 2006)

Rüdiger Dillmann: Teaching and learning of robot tasks via observation of human performance (Robotics and Autonomous Systems, 2004)

9:9

Imitation Learning

- There a many ways to imitate/copy the oberved policy:

Learn a density model $P(a_t \mid s_t)P(s_t)$ (e.g., with mixture of Gaussians) from the observed data and use it as policy (Billard et al.)

Or trace observed trajectories by minimizing perturbation costs (Atkeson & Schaal 1997)

9:10

Imitation Learning

Atkeson & Schaal

9:11

Inverse RL

$$D = \{(s_{0:T}, a_{0:T})^d\}_{d=1}^n \overset{\text{learn}}{\to} R(s,a) \overset{\text{DP}}{\to} V(s) \to \pi(s)$$

- Use ML to "uncover" the latent reward function in observed behavior

Literature:

Pieter Abbeel & Andrew Ng: Apprenticeship learning via inverse reinforcement learning (ICML 2004)

Andrew Ng & Stuart Russell: Algorithms for Inverse Reinforcement Learning (ICML 2000)

Nikolay Jetchev & Marc Toussaint: Task Space Retrieval Using Inverse Feedback Control (ICML 2011).

9:12

Inverse RL (Apprenticeship Learning)

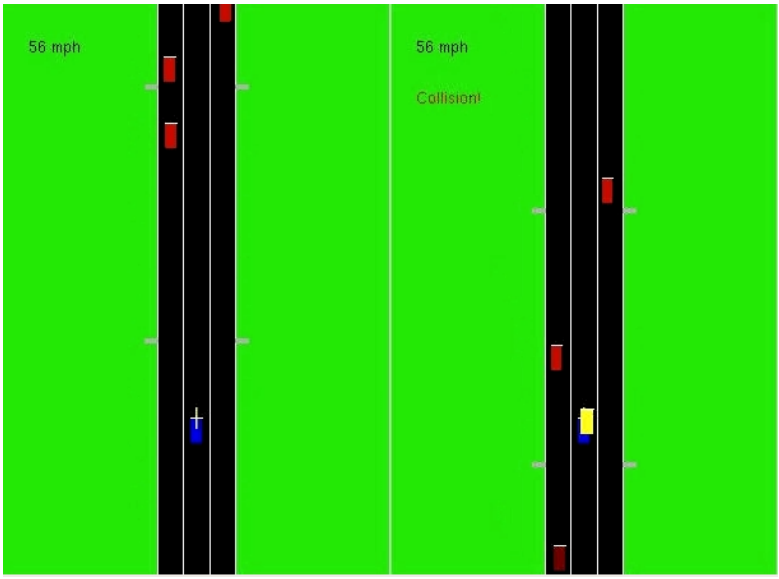
- Given: demonstrations $D = \{x^d_{0:T}\}_{d=1}^n$
- Try to find a reward function that **discriminates demonstrations from other policies**
  - Assume the reward function is linear in some features $R(x) = w^\top \phi(x)$
  - Iterate:
    1. Given a set of candidate policies $\{\pi_0, \pi_1, ..\}$
    2. Find weights w that maximize the value margin between teacher and all other candidates

$$\max_{w,\xi} \ \xi$$
$$\text{s.t. } \forall_{\pi_i}: \quad \underbrace{w^\top \langle \phi \rangle_D}_{\text{value of demonstrations}} \geq \underbrace{w^\top \langle \phi \rangle_{\pi_i}}_{\text{value of } \pi_i} + \xi$$
$$\|w\|^2 \leq 1$$

    3. Compute a new candidate policy π_i that optimizes $R(x) = w^\top \phi(x)$ and add to candidate list.

(Abbeel & Ng, ICML 2004)

9:13



9:14

Policy Search with Policy Gradients

9:15

Policy gradients

- In continuous state/action case, represent the policy as linear in arbitrary state features:

$$\pi(s) = \sum_{j=1}^{k} \phi_j(s)\beta_j = \phi(s)^\top \beta \qquad \text{(deterministic)}$$

$$\pi(a \,|\, s) = \mathcal{N}(a \,|\, \phi(s)^\top \beta, \phi(s)^\top \Sigma \phi(s)) \qquad \text{(stochastic)}$$

  with k features ϕ_j.

- Given an episode $\xi = (s_t, a_t, r_t)_{t=0}^H$, we want to estimate

$$\frac{\partial V(\beta)}{\partial \beta}$$

9:16

Policy Gradients

- One approach is called REINFORCE:

$$\frac{\partial V(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta} \int P(\xi|\beta)\, R(\xi)\, d\xi = \int P(\xi|\beta) \frac{\partial}{\partial \beta} \log P(\xi|\beta) R(\xi) d\xi$$

$$= \mathrm{E}_{\xi|\beta}\{\frac{\partial}{\partial \beta} \log P(\xi|\beta) R(\xi)\} = \mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} \gamma^t \frac{\partial \log \pi(a_t|s_t)}{\partial \beta} \underbrace{\sum_{t'=t}^{H} \gamma^{t'-t} r_{t'}}_{Q^\pi(s_t,a_t,t)}\}$$

- Another is Natural Policy Gradient
  - Estimate the Q-function as linear in the basis functions $\frac{\partial}{\partial \beta} \log \pi(a|s)$:

$$Q(s,a) \approx \Big[\frac{\partial \log \pi(a|s)}{\partial \beta}\Big]^\top w$$

  - Then the natural gradient ($\frac{\partial V(\beta)}{\partial \beta}$ multiplied with inv. Fisher metric) updates

$$\beta^{\text{new}} = \beta + \alpha w$$

- Another is PoWER: Use the stochastic policy $\pi(a \,|\, s)$, let $a_t = \phi(s_t)^\top(\beta + \epsilon_t)$ where $\epsilon_t \sim \mathcal{N}(0, \Sigma)$ is the sampled noise, then $\frac{\partial V(\beta)}{\partial \beta} = 0$ implies

$$\beta \leftarrow \beta + \frac{\mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} \epsilon_t Q^\pi(s_t, a_t, t)\}}{\mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} Q^\pi(s_t, a_t, t)\}}$$

9:17

Kober & Peters: *Policy Search for Motor Primitives in Robotics*, NIPS 2008.

(serious reward shaping!)

9:18

Learning to walk in 20 Minutes

- Policy Gradient method (Reinforcement Learning)
  Stationary policy parameterized as linear in features $u = \sum_i w_i \phi_i(q, \dot{q})$
- Problem: find parameters w_i to minimize expected costs
  cost = mimick $(q, \dot{q})$ of the passive down-hill walker at "certain point in cycle"

Learning To Walk

Tedrake, Zhang & Seung: *Stochastic policy gradient reinforcement learning on a simple 3D biped.* IROS, 2849-2854, 2004. http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake0 pdf

9:19

Policy Gradients – references

Peters & Schaal (2008): *Reinforcement learning of motor skills with policy gradients*, Neural Networks.

Kober & Peters: *Policy Search for Motor Primitives in Robotics*, NIPS 2008.

Vlassis, Toussaint (2009): *Learning Model-free Robot Control by a Monte Carlo EM Algorithm.* Autonomous Robots 27, 123-130.

Rawlik, Toussaint, Vijayakumar(2012): *On Stochastic Optimal Control and Reinforcement Learning by Approximate Inference.* RSS 2012. *(ψ-learning)*

- These methods are sometimes called **white-box optimization**:
  They optimize the policy parameters β for the total reward $R = \sum \gamma^t r_t$ while tying to exploit knowledge of how the process is actually parameterized

9:20

Policy Search with Black-Box Optimization [skipped]

9:21

"Black-Box Optimization"

- The term is not really well defined
  – I use it to express that *only* $f(x)$ can be evaluated
  – $\nabla f(x)$ or $\nabla^2 f(x)$ are not (directly) accessible
  More common terms:

- **Global optimization**
  - This usually emphasizes that methods should not get stuck in local optima
  - Very very interesting domain – close analogies to (active) Machine Learning, bandits, POMDPs, optimal decision making/planning, optimal experimental design
  - Usually mathematically well founded methods

- **Stochastic search** or **Evolutionary Algorithms** or **Local Search**
  - Usually these are local methods (extensions trying to be "more" global)
  - Various interesting heuristics
  - Some of them (implicitly or explicitly) locally approximating gradients or 2nd order models

9:22

Black-Box Optimization

- Problem: Let $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, find

$$\min_x f(x)$$

where we can only evaluate $f(x)$ for any $x \in \mathbb{R}^n$

- A constrained version: Let $x \in \mathbb{R}^n$, $f:\ \mathbb{R}^n \to \mathbb{R}$, $g:\ \mathbb{R}^n \to \{0,1\}$, find

$$\min_x\ f(x) \quad \text{s.t.} \quad g(x) = 1$$

where we can only evaluate $f(x)$ and $g(x)$ for any $x \in \mathbb{R}^n$
I haven't seen much work on this. Would be interesting to consider this more rigorously.

9:23

A zoo of approaches

- People with many different backgrounds drawn into this
  Ranging from heuristics and Evolutionary Algorithms to heavy mathematics
  - Evolutionary Algorithms, esp. Evolution Strategies, Covariance Matrix Adaptation, Estimation of Distribution Algorithms
  - Simulated Annealing, Hill Climing, Downhill Simplex
  - local modelling (gradient/Hessian), global modelling

9:24

Optimizing and Learning

- Black-Box optimization is strongly related to learning:
- When we have local a gradient or Hessian, we can take that local information and run – no need to keep track of the history or learn (exception: BFGS)
- In the black-box case we have no local information directly accessible
  $\to$ one needs to account for the history in some way or another to have an idea where to continue search
- "Accounting for the history" very often means learning: Learning a local or global model of f itself, learning which steps have been successful recently (gradient estimation), or which step directions, or other heuristics

9:25

Stochastic Search

9:26

Stochastic Search

- The general recipe:
  – The algorithm maintains a probability distribution $p_\theta(x)$
  – In each iteration it takes n samples $\{x_i\}_{i=1}^n \sim p_\theta(x)$
  – Each x_i is evaluated $\ \to\ $ data $\{(x_i, f(x_i))\}_{i=1}^n$
  – That data is used to update θ

- Stochastic Search:

Input: initial parameter θ, function $f(x)$, distribution model $p_\theta(x)$, update heuristic $h(\theta, D)$
Output: final θ and best point x
1: **repeat**
2: Sample $\{x_i\}_{i=1}^n \sim p_\theta(x)$
3: Evaluate samples, $D = \{(x_i, f(x_i))\}_{i=1}^n$
4: Update $\theta \leftarrow h(\theta, D)$
5: **until** θ converges

9:27

Stochastic Search

- The parameter θ is the only "knowledge/information" that is being propagated between iterations
  θ encodes what has been learned from the history
  θ defines where to search in the future

- Evolutionary Algorithms: θ is a parent population
  Evolution Strategies: θ defines a Gaussian with mean & variance
  Estimation of Distribution Algorithms: θ are parameters of some distribution model, e.g. Bayesian Network
  Simulated Annealing: θ is the "current point" and a temperature

9:28

Example: Gaussian search distribution (μ, λ)-ES

From 1960s/70s. Rechenberg/Schwefel

- Perhaps the simplest type of distribution model

$$\theta = (\hat{x}) \ , \quad p_t(x) = \mathcal{N}(x|\hat{x}, \sigma^2)$$

  a n-dimenstional isotropic Gaussian with fixed deviation σ

- Update heuristic:
  – Given $D = \{(x_i, f(x_i))\}_{i=1}^\lambda$, select μ best: $D' = \text{bestOf}_\mu(D)$
  – Compute the new mean $\hat{x}$ from D'

- This algorithm is called "Evolution Strategy (μ, λ)-ES"
  – The Gaussian is meant to represent a "species"
  – λ offspring are generated

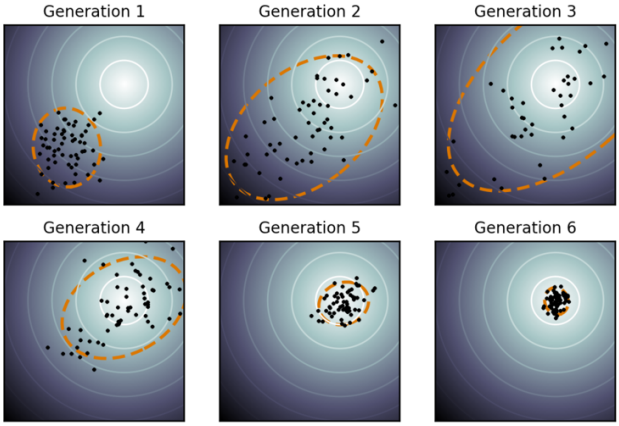
– the best μ selected

9:29

Covariance Matrix Adaptation (CMA-ES)

- An obvious critique of the simple Evolution Strategies:
  – The search distribution $\mathcal{N}(x|\hat{x}, \sigma^2)$ is isotropic
  (no going *forward*, no preferred direction)
  – The variance σ is fixed!

- Covariance Matrix Adaptation Evolution Strategy (CMA-ES)



9:30

Covariance Matrix Adaptation (CMA-ES)

- In Covariance Matrix Adaptation

$$\theta = (\hat{x}, \sigma, C, p_\sigma, p_C) \,, \quad p_\theta(x) = \mathcal{N}(x|\hat{x}, \sigma^2 C)$$

  where C is the covariance matrix of the search distribution
- The θ maintains two more pieces of information: p_σ and p_C capture the "path" (motion) of the mean $\hat{x}$ in recent iterations
- Rough outline of the θ-update:
  – Let $D' = \text{bestOf}_\mu(D)$ be the set of selected points
  – Compute the new mean $\hat{x}$ from D'
  – Update p_σ and p_C proportional to $\hat{x}_{k+1} - \hat{x}_k$
  – Update σ depending on $|p_\sigma|$
  – Update C depending on $p_c p_c^\top$ (rank-1-update) and $\text{Var}(D')$

9:31

CMA references

Hansen, N. (2006), "The CMA evolution strategy: a comparing review"

Hansen et al.: Evaluating the CMA Evolution Strategy on Multimodal Test Functions, PPSN 2004.

Function	f_{stop}	init	n	CMA-ES	DE	RES	LOS
$f_{\text{Ackley}}(x)$	1e-3	$[-30, 30]^n$	20	**2667**	.	.	6.0e4
			30	**3701**	12481	1.1e5	9.3e4
			100	**11900**	36801	.	.
$f_{\text{Griewank}}(x)$	1e-3	$[-600, 600]^n$	20	**3111**	8691	.	.
			30	**4455**	11410 *	*8.5e-3/2e5*	.
			100	**12796**	31796	.	.
$f_{\text{Rastrigin}}(x)$	0.9	$[-5.12, 5.12]^n$	20	68586	**12971**	.	9.2e4
		DE: $[-600, 600]^n$	30	147416	**20150** *	1.0e5	2.3e5
			100	1010989	**73620**	.	.
$f_{\text{Rastrigin}}(\boldsymbol{A}x)$	0.9	$[-5.12, 5.12]^n$	30	**152000**	*171/1.25e6* *	.	.
			100	**1011556**	*944/1.25e6* *	.	.
$f_{\text{Schwefel}}(x)$	1e-3	$[-500, 500]^n$	5	43810	**2567** *	.	7.4e4
			10	240899	**5522** *	.	5.6e5

- For "large enough" populations local minima are avoided

- An interesting variant:

  Igel et al.: A Computational Efficient Covariance Matrix Update and a $(1+1)$-CMA for Evolution Strategies, GECCO 2006.

9:32

CMA conclusions

- It is a good starting point for an off-the-shelf black-box algorithm
- It includes components like estimating the local gradient (p_σ, p_C), the local "Hessian" (Var(D')), smoothing out local minima (large populations)

9:33

Stochastic search conclusions

Input: initial parameter θ, function $f(x)$, distribution model $p_\theta(x)$, update heuristic $h(\theta, D)$
Output: final θ and best point x

1: **repeat**
2: Sample $\{x_i\}_{i=1}^n \sim p_\theta(x)$
3: Evaluate samples, $D = \{(x_i, f(x_i))\}_{i=1}^n$
4: Update $\theta \leftarrow h(\theta, D)$
5: **until** θ converges

- The framework is very general
- The crucial difference between algorithms is their choice of $p_\theta(x)$

9:34

RL under Partial Observability

- Data:
$$D = \{(u_t, c_t, y_t)_t\}_{t=0}^T$$
$\rightarrow$ state x_t not observable

- Model-based RL is dauting: Learning $P(x'|u,x)$ and $P(y|u,x)$ with latent x is very hard

- Model-free: The policy needs to map the **history** to a new control
$$\pi : (y_{t-h,..,t\text{-}1}, u_{t-h,..,t-1}) \mapsto u_t$$
or any **features of the history**
$$u_t = \phi(y_{t-h,..,t\text{-}1}, u_{t-h,..,t-1})^\top w$$

10 SKIPPED THIS TERM – Grasping (brief intro)

Grasping

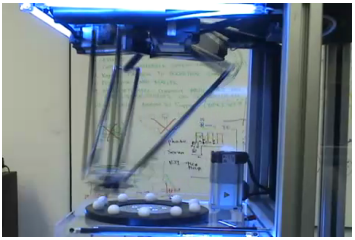
- The most elementary type of interaction with (manipulation of) the environment.
  – Basis for intelligent behavior.

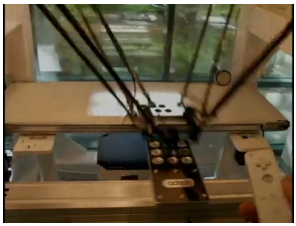
- In industrial settings with high precision sensors and actuators: very fast and precise.

- In general real world with uncertain actuators and perception, still a great research challenge, despite all the theory that has been developed.

10:1

Pick-and-place in industry



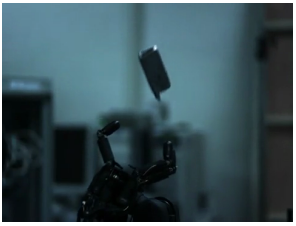


(This type of kinematics is called "Delta Robot", which is a "parallel robot" just as the Stewart platform.)

10:2

Research

- Using ultra high speed and precise cameras and localization: High speed robot hand from the Ishikawa Komuro's "Sensor Fusion" Lab



http://www.k2.t.u-tokyo.ac.jp/fusion/index-e.html

- Asimo's grasping:

10:3

Outline

- Introduce to the basic classical concepts for grasping (force closure)
- Discussion and alternative views
- References:

  Craig's *Introduction to robotics: mechanics and control* – chapter 3.

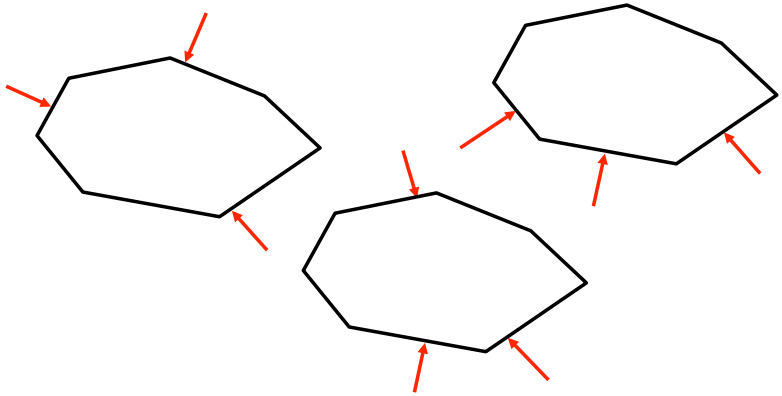
  Matt Mason's lecture: *Static and Quasistatic Manipulation*
  www.cs.cmu.edu/afs/cs/academic/class/16741-s07/www/lecture18.pdf

  Daniela Rus and Seth Teller's lecture: *Grasping and Manipulation*
  courses.csail.mit.edu/6.141/spring2011/pub/lectures/Lec13-Manipulation-I.pdf

10:4

Force Closure

- Which of these objects is in "tight grip"?

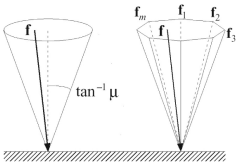


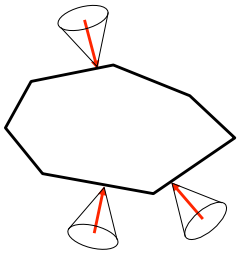
Defining "tight grip": Assume fingers (vectors) have no friction – but can exert arbitrary normal forces. *Can we generate (or counter-act) arbitrary forces on the object?*

10:5

Force Closure – more rigorously

- Assume that each "finger" is a point that can apply forces on the object as decribed by the **friction cone**





- Each finger is described by a point p_i and a force $f_i \in F_i$ in the fingers friction cone. Together they can exert the the force an torque:

$$f^{\text{total}} = \sum_i f_i \ , \quad \tau^{\text{total}} = \sum_i f_i \times (p_i - c)$$

- **Force closure** $\iff$ we can generate (counter-act) arbitrary f^{total} and τ^{total} by choosing $f_i \in F_i$ appropriately.
  $\leftrightarrow$ Check whether the *positive linear span of the fiction cones* covers the whole space.

10:6

Form & Force Closure & Caging

- Force closure: The contacts can apply an arbitrary wrench (=force-torque) to the object.

- Form closure: The object is at an isolated point in configuration space. Note: form closure $\iff$ frictionless force closure

- Caging: The object is not fixated, but cannot escape

- Equilibrium: The contact forces can balance the object's weight and other external forces.

10:7

Traditional research into force closure

- Theorem (Mishra, Schwartz and Sharir, 1987):
  For any bounded shape that is not a surface of revolution, a force closure (or first order form closure) grasp exists.

- Guaranteed synthesis:
  1) put fingers "everywhere"
  2) while redundant finger exists delete any redundant finger
  (A finger is redundant if it can be deleted without reducing the positive linear span.)

  Theorem (Mishra, Schwartz, and Sharir, 1987):
  For any surface not a surface of revolution, [the above method] yields a grasp with at most 6 fingers in the plane, at most 12 fingers in three space.

10:8

Traditional research into force closure

- Force closure turn into a continuous optimization criterion:

  – Constrain the absolute forces each finger can apply
  (cut the friction cones)
  – The friction cones define a *finite* convex polygon in 6D wrench space
  $\rightarrow$ What is the inner radius of this convex wrench polygon?

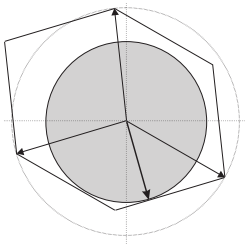


Illustration from Suárez, Roa, Cornellà (2006): *Grasp Quality Measures*

10:9

The "mitten thought experiment"

- From Oliver Brock's research website:

  "Our approach to grasping is motivated by the "mitten thought experiment". This experiment illustrates that a sensory information-deprived subject (blindfolded, wearing a thick mitten to eliminate tactile feedback) is able to grasp a large variety of objects reliably by simply closing the hand, provided that a second experimenter appropriately positioned the object relative to the hand. This thought experiments illustrates that an appropriate perceptual strategy (the experimenter) in conjunction with a simple compliance-based control strategy (the mitten hand) can lead to outstanding grasping performance."

Illustration from O. Brock's page

10:10

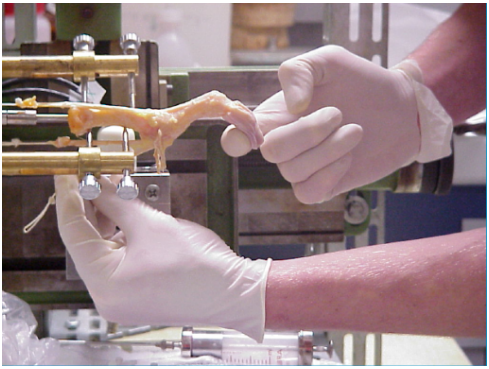
Food for thought

- Are point contact a good model?
- Is the whole idea of "arranging friction cones" the right approach?
- What about biomechanics?

10:11

Biomechanics of the human hand

- Finger tendons:



De Bruijne et al. 1999

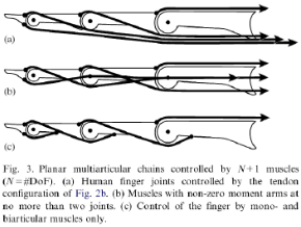


Fig. 3. Planar multiarticular chains controlled by $N+1$ muscles (N=#DoF). (a) Human finger joints controlled by the tendon configuration of Fig. 2b. (b) Muscles with non-zero moment arms at no more than two joints. (c) Control of the finger by mono- and biarticular muscles only.

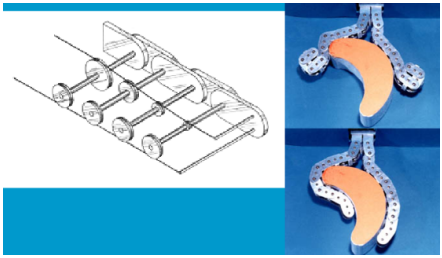
J.N.A.L. Leijnse, 2005

See Just Herder's lecture on "Biograsping"

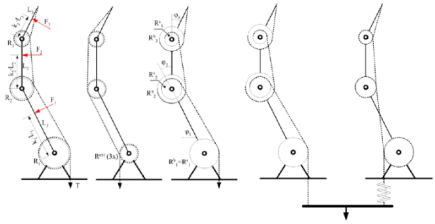
http://www.slideshare.net/DelftOpenEr/bio-inspired-design-lecture6-bio-graspin

10:12

Tendon-based hand mechanisms



Shape Gripper, Shigeo Hirose, TITECH



Jaster Schuurmans, 2004

DLR hand

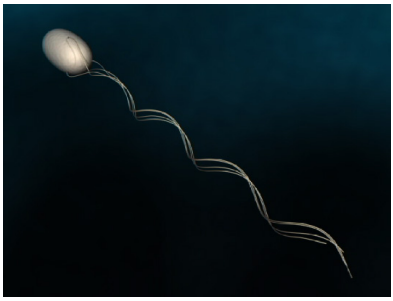
10:13

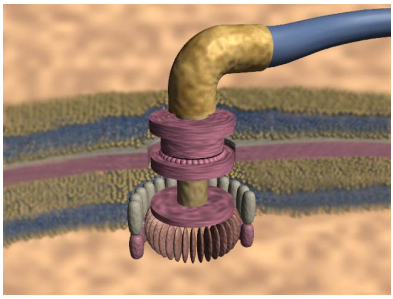
11 SKIPPED THIS TERM – Legged Locomotion (brief intro)

Legged Locomotion

- Why legs?

  Bacterial Flagellum: (rotational "motors" in Biology?)





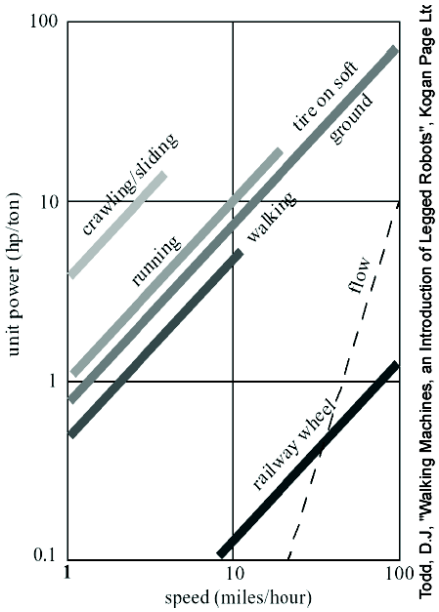
11:1

Legged Locomotion

- Why legs?
  - Human/Animal Locomotion Research
  - Potentially less weight
  - Better handling of rough terrains
    (climbing, isolated footholds, ladders, stairs)

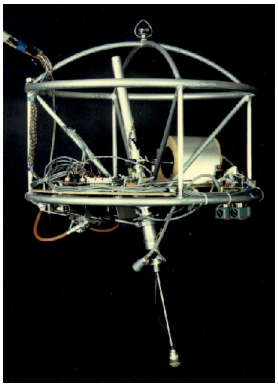
11:2

Rolling vs walking



11:3

One-legged locomotion



- Three separate controllers for:
  - hopping height
  - horizontal velocity (foot placement)
  - attitude (hip torques during stance)
- Each a simple (PD-like) controller

Raibert et al.: *Dynamically Stable Legged Locomotion.* 1985 http://dspace.mit.edu/handle/1721.1/6820

Tedrake: *Applied Optimal Control for Dynamically Stable Legged Locomotion.* PhD thesis (2004). http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake04b.pdf

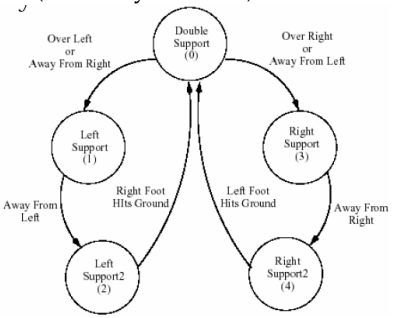
11:4

Biped locomotion

- Walking vs Running

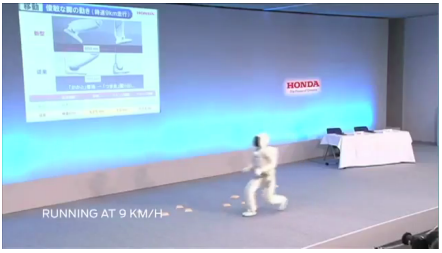
Walking := in all instances at least one foot is on ground
Running := otherwise

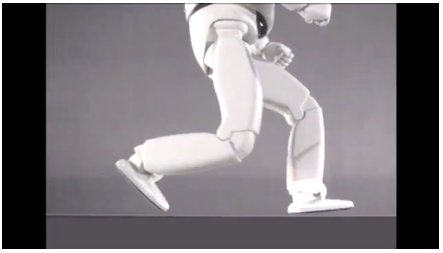
- 2 phases of Walking
  – double-support phase (in Robotics often statically stable)
  – single-support phase (statically instable)



11:5

Asimo





11:6

Statically Stable Walk

- You could rest (hold pose) at any point in time and not fall over

$$\iff \quad \textit{CoG projected on ground is within support polygon}$$

CoG = center of gravity of all body masses
support polygon = hull of foot contact points

- Try yourself: Move as slow as you can but make normal length steps...

11:7

Zero Moment Point

- Vukobratovic's view on walking, leading to ZMP ideas:
  "Basic characteristics of all biped locomotion systems are:

  (i) the possibility of rotation of the overall system about one of the foot edges caused by strong disturbances, which is equivalent to the appearance of an unpowered (passive) DOF,

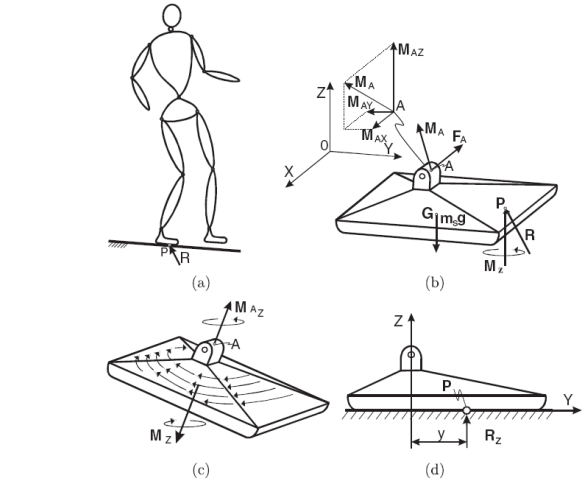
  (ii) gait repeatability (symmetry), which is related to regular gait only

  (iii) regular interchangeability of single- and double-support phases"

Vukobratovic & Borovac: *Zero-moment point—Thirty five years of its life.* International Journal of Humanoid Robotics 1, 157-173, 2004. http://www.cs.cmu.edu/~cga/legs/vukobratovic.pdf

11:8

Zero Moment Point



11:9

Zero Moment Point

- "ZMP is defined as that point on the ground at which the net moment of the inertial forces and the gravity forces has no component along the horizontal axes." (Vukobratovic & Borovac, 2004)

- f_i = force vector acting on body i (gravity plus external)
  w_i = angular vel vector of body i
  I_i = interia tensor ($\in \mathbb{R}^{3\times3}$) of body i
  $r_i = p_i - p_{\text{ZMP}}$ = relative position of body i to ZMP

- Definition: p_{ZMP} is the point on the ground for which
$$\textstyle\sum_i r_i \times f_i + I_i\dot{w}_i + w_i \times I_i w_i \stackrel{!}{=} (0, 0, *)^\top$$

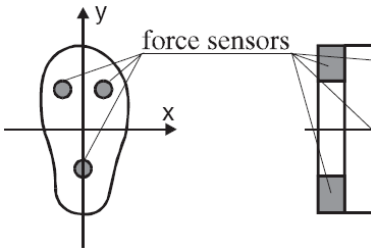
See also: Popovic, Goswami & Herr: *Ground Reference Points in Legged Locomotion: Definitions, Biological Trajectories and Control Implications.* International Journal of Robotics Research 24(12), 2005.
http://www.cs.cmu.edu/~cga/legs/Popovic_Goswami_Herr_IJRR_Dec_2005.pdf

11:10

Zero Moment Point

- If the ZMP is outside the support polygon $\rightarrow$ foot rolls over the edge
  (presumes foot is a rigid body, and non-compliant control)

- Locomotion is *dynamically stable* if the ZMP remains within the foot-print polygons.
  ($\leftrightarrow$ the support can always apply some momentum, if necessary)

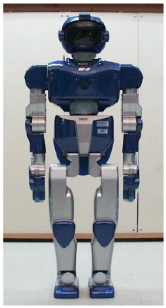
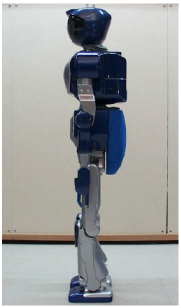
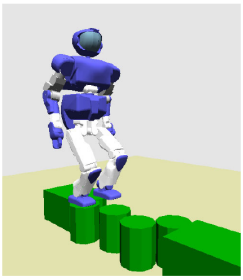


- Computing the ZMP in practise:
  – either exact robot model
  – or foot pressure sensors

11:11

Zero Moment Point – example

- combine ZMP with 3D linear inverted pendulum model and model-predictive control

HRP-2 stair climbing

Kajita et al.: *Biped Walking Pattern Generation by using Preview Control of Zero-Moment Point.* ICRA 2003. http://eref.uqu.edu.sa/files/eref2/folder1/biped_walking_pattern_generatic by_usin_53925.pdf

11:12

ZMP Summary

- ZMP is the "rescue" for conventional methods:
  – ZMP-stability → the robot can be controlled as if foot is "glued" (virtually) to the ground!
  – The whole body behaves like a "conventional arm"
  – Can accellerate $\ddot{q}$ any DoF → conventional dynamic control
  fully actuated system

- Limitations:
  – Humans don't use ZMP stability, we allow our feet to roll (toe-off, heel-strike: ZMP at edge of support polygon)
  – Can't describe robots with point feet (walking on stilts)

11:13

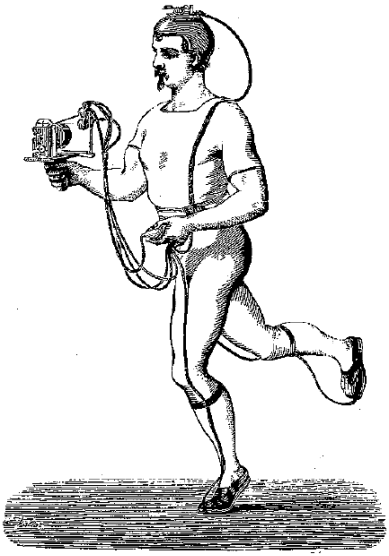
Models of human bipedal locomotion

The following illustrations are from:

McMahon: *Mechanics of Locomotion.* IJRR 3:4-28, 1984
http://www.cs.cmu.edu/~cga/legs/mcmahon1.pdf
http://www.cs.cmu.edu/~cga/legs/mcmahon2.pdf

11:14

Walking research from Marey 1874:

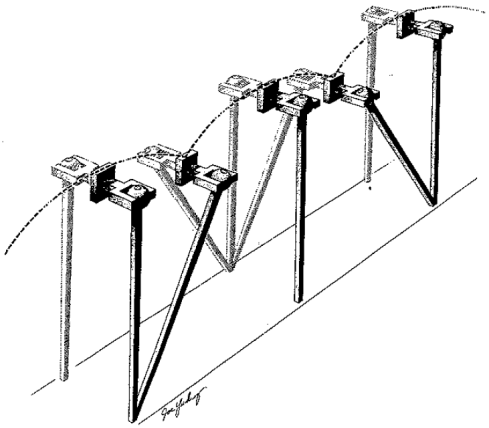


11:15

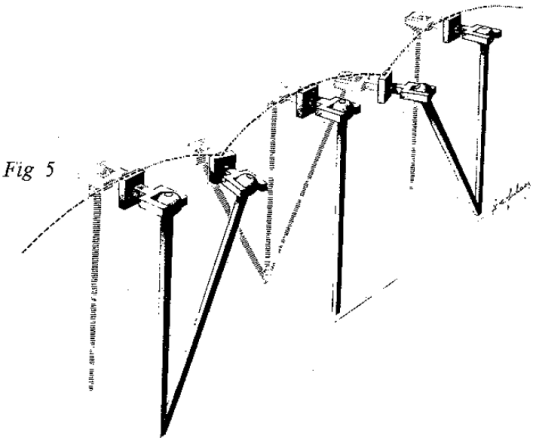
Six determinants of gait

following Saunders, Inman & Eberhart (1953)

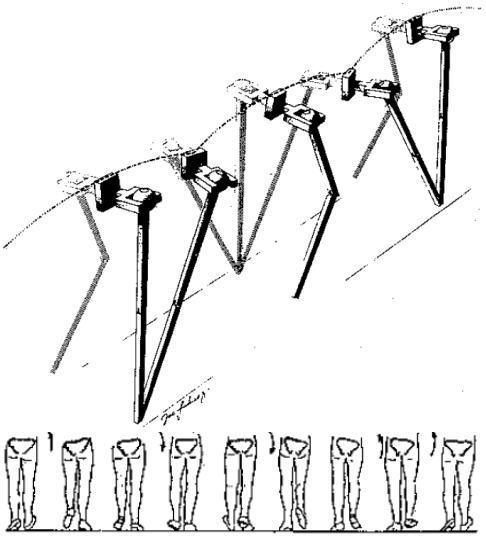
1. Compass Gait:



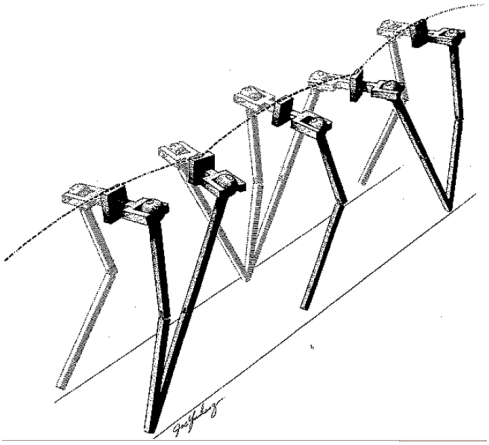
2. Pelvic Rotation:



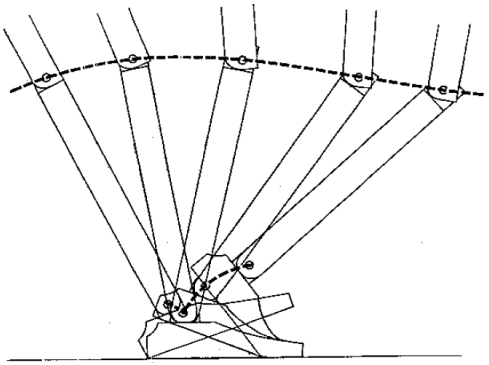
3. Pelvic Tilt:



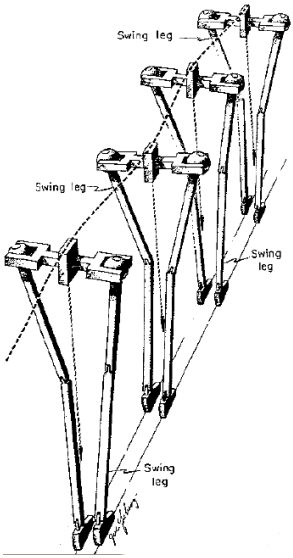
4. Stance Knee Flexion:



5. Stance Ankle Flextion:



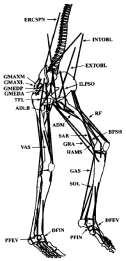
6. Pelvis Lateral Displacement:

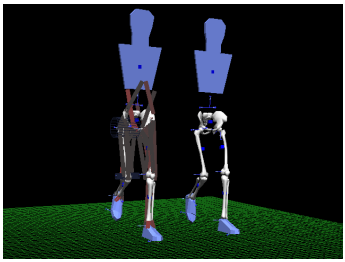


11:16

Models of human bipedal locomotion

- Human model with 23 DoFs, 54 muscles
  – Compare human walking data with model
  – Model: optimize energy-per-distance
  – Energy estimated based on metabolism and muscle heat rate models





Anderson & Pandy: *Dynamic Optimization of Human Walking.* Journal of Biomechanical Engineering 123:381-390, 2001. http://e.guigon.free.fr/rsc/article/AndersonPandy01.pdf

Anderson & Pandy: *Static and dynamic optimization solutions for gait are practically equivalent.* Journal of Biomechanics 34 (2001) 153-161. http://www.bme.utexas.edu/faculty/pandy/StaticOptWalk.pdf

11:17

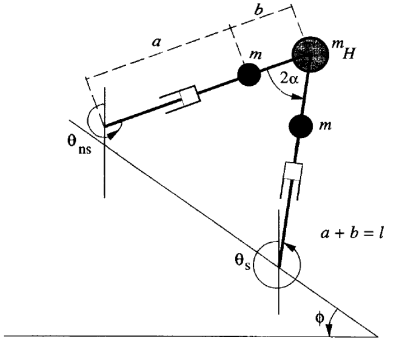
Models of human bipedal locomotion

- Suggest different principles of human motion:
  – passive dynamics (Compass Gait) $\leftrightarrow$ underactuated system
  – modulation of basic passive dynamics
  – Energy minimization

11:18

Passive dynamic walking: Compass Gait

- Basic 2D planar model of the Compass Gait:



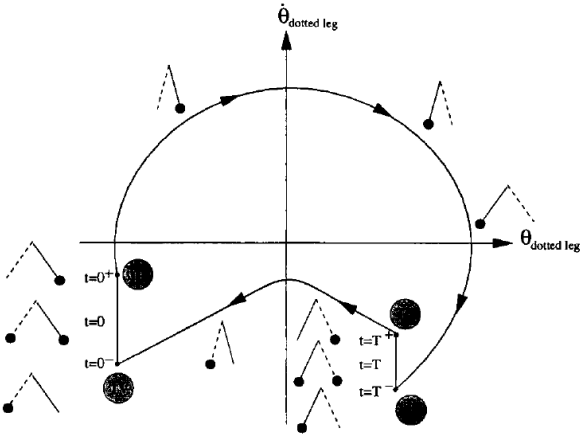
The pose is described by $q = (\theta_s, \theta_{ns})$, the state by $(q, \dot{q})$

Goswami, Thuilot & Espiau: *A study of the passive gait of a compass-like biped robot: symmetry and chaos.* International Journal of Robotics Research 17, 1998.
http://www.ambarish.com/paper/COMPASS_IJRR_Goswami.pdf

11:19

Passive dynamic walking: Compass Gait

- Swing phase has analytic equations of motions
$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = 0$$
but can't be solved analytically...
- Phase space plot of numeric solution:



11:20

Passive walker examples

compass gait simulation
controlled on a circle
passive walker

- Minimally actuated: Minimal Control on rough terrain

11:21

Impact Models in the Compass Gait

- Switch between two consecutive swing phases: depends on slope!
- Typical assumptions made in simulation models:

  – The contact of the swing leg with the ground results in no rebound and no slipping of the swing leg.

  – At the moment of impact, the stance leg lifts from the ground without interaction.

  – The impact is instantaneous.

  – The external forces during the impact can be represented by impulses.

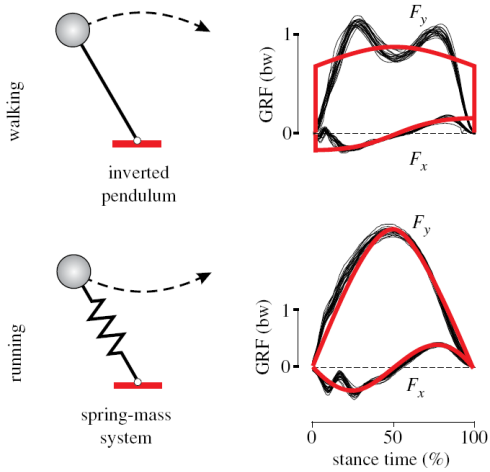
  – The impulsive forces may result in an instantaneous change in the velocities, but there is no instantaneous change in the configuration.

  – The actuators cannot generate impulses and, hence, can be ignored during impact.

Westervelt, Grizzle & Koditschek: *Hybrid Zero Dynamics of Planar Biped Walkers.* IEEE Trans. on Automatic Control 48(1), 2003.
http://repository.upenn.edu/cgi/viewcontent.cgi?article=1124&context=ese_papers

11:22

Implausibility of the stiff Compass Gait leg



Geyer, Seyfarth & Blickhan: *Compliant leg behaviour explains basic dynamics of walking and running.* Proc. Roy. Soc. Lond. B, 273(1603): 2861-2867, 2006. http://www.cs.cmu.edu/~cga/legs/GeyerEA06RoySocBiolSci.pdf

11:23

Learning to walk in 20 Minutes

- Policy Gradient method (Reinforcement Learning)
  Stationary policy parameterized as linear in features $u = \sum_i w_i \phi_i(q, \dot{q})$
- Problem: find parameters w_i to minimize expected costs
  cost = mimick $(q, \dot{q})$ of the passive down-hill walker at "certain point in cycle"

Learning To Walk

Tedrake, Zhang & Seung: *Stochastic policy gradient reinforcement learning on a simple 3D biped.* IROS, 2849-2854, 2004. http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake0 pdf

11:24

Summary

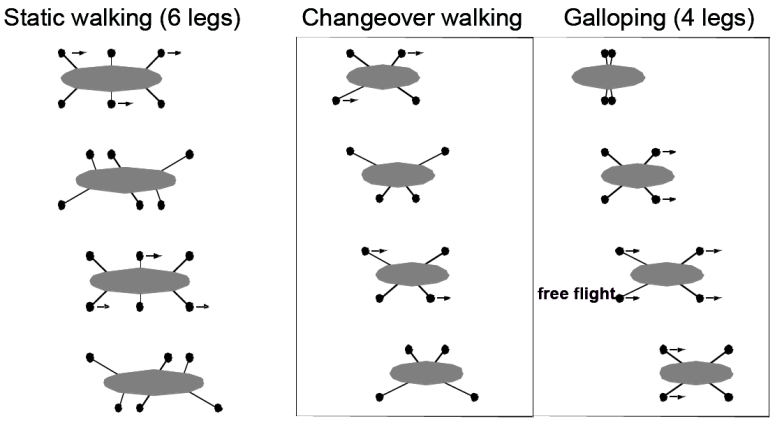
- ZMP type walking was successful (ASIMO, HRP-2, etc), but limited

- Future types of walking:
  – Exploit passive dynamics, cope with *underactuation*
  – Follow some general optimiality principle (but real-time!)
  – Learn (esp. Reinforcement Learning)
  – Compliant hardware! (controllable elasticity & damping)

- Recommended reading: Tedrake: *Underactuated Robotics: Learning, Planning, and Control for Effcient and Agile Machines.* Course Notes for MIT 6.832
  www.cs.berkeley.edu/~pabbeel/cs287-fa09/readings/Tedrake-Aug09.pdf

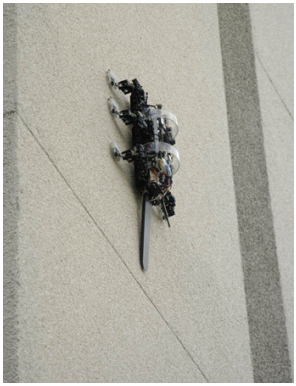
11:25

Finally, multi-legged locomotion



11:26

Finally, multi-legged locomotion



http://www.bostondynamics.com/robot_rise.html

12 Exercises

12.1 Exercise 1

No need to prepare for this first tutorial. We'll do the exercises together on the fly.

12.1.1 Prerequisites: Matrix equations

a) Let X, A be arbitrary matrices, A invertible. Solve for X:

$$XA + A^\top = \mathbf{I}$$

b) Let X, A, B be arbitrary matrices, $(C - 2A^\top)$ invertible. Solve for X:

$$X^\top C = [2A(X + B)]^\top$$

c) Let $x \in \mathbb{R}^n, y \in \mathbb{R}^d, A \in \mathbb{R}^{d\times n}$. A obviously *not* invertible, but let $A^\top A$ be invertible. Solve for x:

$$(Ax - y)^\top A = \mathbf{0}_n^\top$$

d) As above, additionally $B \in \mathbb{R}^{n\times n}$, B positive-definite. Solve for x:

$$(Ax - y)^\top A + x^\top B = \mathbf{0}_n^\top$$

12.1.2 Prerequisites: Vector derivatives

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$, $f, g : \mathbb{R}^n \to \mathbb{R}^d$, $A \in \mathbb{R}^{d\times n}$, $C \in \mathbb{R}^{d\times d}$. (Also provide the dimensionality of the results.)

a) What is $\frac{\partial}{\partial x} x$?

b) What is $\frac{\partial}{\partial x}[x^\top x]$?

c) What is $\frac{\partial}{\partial x}[f(x)^\top f(x)]$?

d) What is $\frac{\partial}{\partial x}[f(x)^\top C g(x)]$?

e) Let B and C be symmetric (and pos.def.). What is the minimum of $(Ax - y)^\top C(Ax - y) + x^\top Bx$?

12.1.3 Optimization

Given $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, we want to find $\text{argmin}_x f(x)$. (We assume f is uni-modal.)

a) What 1st-order optimization methods (querying $f(x), \nabla f(x)$ in each iteration) do you know?

b) What 2nd-order optimization methods (querying $f(x), \nabla f(x), \nabla^2 f(x)$ in each iteration) do you know?

c) What is backtracking line search?

12.2 Exercise 2

12.2.1 Geometry

a) Read the notes on basic 3D geometry at http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/3d-geometry.pdf at least until section 2. Prepare questions for the exercises if you have any.

b) You have a book (coordinate frame B) lying on the table (world frame W). Initially B and W are identical. Now you move the book 1 unit to the right, then rotate it by 45° counter-clock-wise around its origin. Given a dot p marked on the book at position $p^B = (1, 1)$ in the book coordinate frame, what are the coordinates p^W of that dot with respect to the world frame?

c) Given a point x with coordinates $x^W = (0, 1)$ in world frame, what are its coordinates x^B in the book frame?

d) What is the *coordinate* transformation from world frame to book frame, and from book frame to world frame?

Please use homogeneous coordinates to derive these answers.

12.2.2 Subscribe to the Email list

Please subscribe here:
https://mailman.informatik.uni-stuttgart.de/mailman/listinfo/lehre14-rob

12.2.3 Simulation software

Future exercises will require to code some examples in C/C++. Test if you can compile and run the lib that accompanies this lecture. Report on problems with installation.

On Ubuntu:

- install the packages
  `liblapack-dev freeglut3-dev libqhull-dev libf2c2-dev libann-dev gnuplot doxygen libglew1.6-dev`
- get the code from
  http://ipvs.informatik.uni-stuttgart.de/mlr/marc/source-code/libRobo14.tgz

- `tar xvzf libRoboticsCourse.14.tgz`
  `cd share/examples/Ors/ors`
  `make`
  `./x.exe`

12.3 Exercise 3

12.3.1 Task spaces and Jacobians

In the lecture we introduced the basic kinematic maps $\phi^{\text{pos}}_{i,v}(q)$ and $\phi^{\text{vec}}_{i,v}(q)$, and their Jacobians, $J^{\text{pos}}_{i,v}(q)$ and $J^{\text{vec}}_{i,v}(q)$, where i may refer to any part of the robot and v is any point or direction on this part. In the following you may assume that we have routines to compute these for any q. The problem is to express other kinematic maps and their Jacobians in terms of these knowns. In the following you're asked to define more involved kinematics maps (a.k.a. task maps) $\phi(q)$ to solve certain motion problems. Please formulate all these maps such that the overall optimization problem is

$$f(q) = \|q - q_0\|^2_W + \|\phi(q)\|^2$$

that is, the motion aims to minimize $\phi(q)$ to zero (absorb the y^* in the definition of phi.)

a) Assume you would like to control the pointing direction of the robot's head (e.g., its eyes) to point to a desired external world point x^W. What task map can you define to achieve this? What is the Jacobian?

b) You would like the two hands or the robot to become parallel (e.g. for clapping). What task map can you define to achieve this? What is the Jacobian?

c) You would like to control a standard endeffector position p_{eff} to be at y^*, as usual. Can you define a 1-dimensional task map $\phi : \mathbb{R}^n \to \mathbb{R}$ to achieve this? What is its Jacobian?

12.3.2 IK in the simulator

Download the simulator code from `http://ipvs.informatik.uni-stuttgart.de/mlr/marc/source-code/libRoboticsCourse.14.tgz`. (See last exercise for instructions. The `libRoboticsCourse.13.msvc.tgz` might include a project file for MSVC.) The header `<src/Ors/roboticsCourse.h>` provides a very simple interface to the simulator—we will use only this header and some generic matrix functionalities. In `share/examples/Core/array` you find many examples on how to use the `arr` for working with vectors and matrices (many conventions are similar to Matlab).

Consider the example in `teaching/RoboticsCourse/01-kinematics` (rename `main.problem.cpp` to `main.cpp`). The goal is to reach the coordinates $y^* = (-0.2, -0.4, 1.1)$ with the right hand of the robot. Assume $W = \mathbf{I}$ and $\sigma = .01$.

a) The example solution generates a motion in one step using inverse kinematics $\delta q = J^\sharp\ \delta y$ with $J^\sharp = (J^\top J + \sigma^2 W)^{-1} J^\top$. Record the task error, that is, the deviation of hand position from y^* after each step. Why is it initially large?

b) Try to do 100 smaller steps $\delta q = \alpha J^\sharp\ \delta y$ with $\alpha = .1$ (each step starting with the outcome of the previous step). How does the task error evolve over time?

c) Generate a nice trajectory composed of $T = 100$ time steps. Interpolate the target linearly $\hat{y} \leftarrow y_0 + (t/T)(y^* - y_0)$ in each time step.

d) Generate a trajectory that moves the right hand in a circle centered at $(-0.2, -0.4, 1.1)$, aligned with the xz-plane, with radius 0.2.

12.4 Exercise 4

12.4.1 Verify some things from the lecture

a) On slide 02:46 it says that

$$\begin{aligned}\operatorname*{argmin}_q \|q - q_0\|_W^2 + \|\Phi(q)\|^2 \\ \approx q_0 - (J^\top J + W)^{-1} J^\top\ \Phi(q_0) = q_0 - J^\# \Phi(q_0)\end{aligned}$$

where the approximation $\approx$ means that we use the local linearization $\Phi(q) = \Phi(q_0) + J(q - q_0)$. Verify this.

b) On slide 02:34 there is a term $(\mathbf{I} - J^\# J)$ called nullspace projection. Verify that for $\varrho \to \infty$ (and $C = \varrho\mathbf{I}$) and any choice of $a \in \mathbb{R}^n$

$$\delta q = (\mathbf{I} - J^\# J)a \ \Rightarrow\ \delta y = 0$$

(assuming linearity of ϕ, i.e., $J\delta q = \delta y$). Note: this means that any choice of a, the motion $(\mathbf{I} - J^\# J)a$ will not change the "endeffector position" y.

c) On slide 02:27 we derived the basic inverse kinematics law. Verify that (assuming linearity of ϕ) for $C \to \infty$ the desired task is fulfilled exactly, i.e., $\phi(q^*) = y^*$. By writing $C \to \infty$ we mean that C is a matrix of the form $C = \varrho C_0$, $\varrho \in \mathbb{R}$, and we take the limit $\varrho \to \infty$.

12.4.2 Multiple task variables

Consider again the last week's exercise where the robot moved his hand in a circle (Exercise 3.2d).

a) We've seen that the solution does track the circle nicely, but the trajectory "gets lost" in joint space, leading to very strange postures. We can fix this by adding more tasks, esp. a task that permanently tries to (moderatly) minimize the distance

of the configuration q to a natural posture q_{home}. Realize this by adding a respective task to the code given on the next page.

b) Make the robot simultaneously point upward with the left hand. Use `kinematicsVec` and `jacobianVec` for this.

```
void multiTask(){
  Simulator S("man.ors");
  arr q,q_home,y_target,y,J,W,Phi,PhiJ;
  uint n = S.getJointDimension();

  S.getJointAngles(q);
  W.setDiag(1.,n); //we define W as identity matrix
  q_home = q; //we store the initial posture as q_home

  for(uint i=0;i<10000;i++){
    Phi.clear();
    PhiJ.clear();

    //1st task: track with right hand
    y_target = ARR(-0.2, -0.4, 1.1);
    y_target += .2 * ARR(cos((double)i/20), 0, sin((double)i/20));
    S.kinematicsPos(y,"handR");
    S.jacobianPos  (J,"handR");

    Phi .append( 1e2 * (y - y_target) );  //rho = 1e4 (cp. slide 02:45)
    PhiJ.append( 1e2 * J );

    //add the "stay close to home" task here

    //add another task for the left hand here, if you like

    //compute joint updates
    q -= inverse(~PhiJ*PhiJ + W)*~PhiJ* Phi; //(cp. slide 02:46)
    S.setJointAngles(q);
  }
}
```

12.5 Exercise 5

In the lecture we discussed PD force control on a 1D point mass, which leads to oscillatory behavior for high K_p and damped behavior for high K_d. Slide 03:15 replaces the parameters K_p, K_d by two other, more intuitive parameters, λ and ξ: λ roughly denotes the time (or time steps) until the goal is reached, and ξ whether it is reached agressively ($\xi > 1$, which overshoots a bit) or by exponential decay ($\xi \leq 1$). Use this to solve the following exercise.

12.5.1 PD force control on a 1D mass point

a) Implement the system equation for a 1D point mass with mass $m = 3.456$. That is, implement the Euler intergration (see below) of the system dynamics that com-

putes x_{t+1} given x_t and u_t in each iteration, where $x_t = (q_t, \dot{q}_t)$.

There is no need for the robot simulator code—implement it directly in plain C++ or another language.

Assume a step time of $\tau = 0.01$sec. Generate a trajectory from the start position $q_0 = 0$ and velocity $\dot{q}_0 = 0$ that approaches the goal position $q^* = 1$ with high precision within about 1 second using PD force control. Find 3 different parameter sets for K_p and K_d to get oscillatory, overdamped and critical damped behaviors. Plot the point trajectory (e.g. using the routine `gnuplot(arr& q); MT::wait();` .)

b) Repeat for time horizon $t = 2$sec and $t = 5$sec. How should the values of K_p and K_d change when we have more time?

c) Implement a PID controller (including the integral (stationary error) term). How does the solution behave with only K_i turned on ($K_p = K_d = 0$); how with K_i and K_d non-zero?

Note on Euler Integration: See also http://en.wikipedia.org/wiki/Euler_method. Given a differential equation $\dot{x} = f(x, u)$, Euler integration numerically "simulates" this equation forward by iterating

$$x_{t+1} = x_t + \tau f(x, u)$$

for small time steps τ.

12.5.2 Following a reference trajectory with PD control

We have the same point mass as above. But now the goal position q^* changes over time: we have a reference trajectory

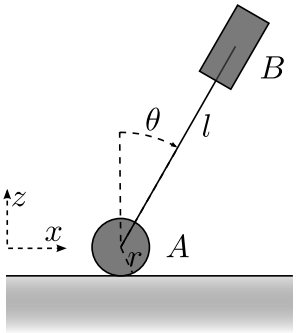
$$q^*(t) = \cos(t)$$

How can one use a PD controller to let the point mass robustly and *precisely* follow this reference trajectory?

Simulate the problem as above, but at every full second apply a randomized impuls (=instantaneous change in velocity) on the point mass. That is, simply set $\dot{q} \leftarrow \dot{q} + \xi$ for $\xi \sim \mathcal{N}(0, .1)$. (In code: `double xi = 0.1 * rnd.gauss();`) Generate oscillatory, overdamped and critical recovery behaviors.

12.6 Exercise 6

12.6.1 Fun with Euler-Lagrange



Consider an inverted pendulum mounted on a wheel in the 2D x-z-plane; similar to a Segway. The exercise is to derive the Euler-Lagrange equation for this system.

Tips:

- Strictly follow the scheme we discussed on slide 03:23.
- Use the generalized coordinates

$$q = (x, \theta) \tag{3}$$

  with x is the position of the wheel and θ the angle of the pendulum relative to the world-vertical.

- The system can be parameterized by
  - m_A, I_A, m_B, I_B: masses and inertias of bodies A (=wheel) and B (=pendulum)
  - r: radius of the wheel
  - l: length of the pendulum (height of its COM)
- Describe the **pose** of every body (depending on q) by the (x, z, ϕ) coordinate: its position in the x-z-plane, and its rotation relative to the world-vertical. Accordingly represent the (linear and angular) velocities as a 3-vector.
- In this 3-dim space, the mass matrix of every body is

$$M_i = \begin{pmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & I_i \end{pmatrix} \tag{4}$$

12.6.2 Optional: PD control to hold an arm steady

In our code, in `03-dynamics` you find an example (rename `main.problem.cpp` to `main.cpp`). Please change `../02-pegInAHole/pegInAHole.ors` to `pegArm.ors`.

You will find an arm with three joints that is swinging freely under gravity (ignoring collisions).

a) Apply direct PD control (*without* using M and F) to each joint separately and try to find parameters K_p and K_d (potentially different for each joint) to hold the arm steady, i.e., $q^* = 0$ and $\dot{q}^* = 0$. Bonus: If you are successful, try the same for the arm in `pegArm2.ors`.

Play with `setDynamicsSimulationNoise` and check stability.

b) As above, try to hold the arm steady at $q^* = 0$ and $\dot{q}^* = 0$. But now use the knowledge of M and F in each time step. For this, decide on a desired wavelength λ and damping behavior ξ and compute the respective K_p and K_d (assuming $m = 1$), the same for each joint. Use the PD equation to determine desired accelerations $\ddot{q}^*$ (slide 03:31) and use inverse dynamics to determine the necessary u.

12.7 Exercise 7

12.7.1 RRTs for path finding

In the code in `teaching/RoboticsCourse/04-rrt` you find an example problem (rename `main.problem.cpp` to `main.cpp`).

a) The code demonstrates an RRT exploration and displays the explored endeffector positions. What is the endeffector's exploration distribution in the limit $n \to \infty$? Specify such a distribution analytically for a planar 2 link arm.

b) First grow an RRT *backward* from the target configuration $q^* = (0.945499, 0.431195, -1.9$ $-0.665206, -1.48356)$. Stop when there exists a node close (<`stepSize`) to the $q = 0$ configuration. Read out the collision free path from the tree and display it. Why would it be more difficult to grow the tree *forward* from $q = 0$ to q^*?

c) Find a collision free path using bi-directional RRTs (that is, 2 RRTs growing together). Use q^* to root the backward tree and $q = 0$ to root the forward tree. Stop when a newly added node is close to the other tree. Read out the collision free path from the tree and display it.

d) (Bonus) Propose a simple method to make the found path smoother (while keeping it collision free). You're free to try anything.

12.7.2 A distance measure in phase space

Consider the 1D point mass with mass $m = 1$ state $x = (q, \dot{q})$. The 2D space of $(q, \dot{q})$ combining position and velocity is also called phase space. In RRT's (in line 4 on slide 04:39) we need to find the nearest configuration q_{near} to a q_{target}. But what does "near" mean in phase space? This exercise explores this question.

Consider a current state $x_0 = (0, 1)$ (at position 0 with velocity 1). Pick *any* random phase state $x_{\text{target}} \in \mathbb{R}^2$. How would you connect x_0 with x_{target} in a way that fulfils the differential constraints of the point mass dynamics? Given this trajectory connecting x_0 with x_{target}, how would you quantify/meassure the distance? (If you defined the connecting trajectory appropriately, you should be able to give an analytic expression for this distance.) Given a set (tree) of states $x_{1:n}$ and you pick the closest to x_{target}, how would you "grow" the tree from this closest point towards x_{target}?

12.8 Exercise 8

12.8.1 Bayes Basics

a) Box 1 contains 8 apples and 4 oranges. Box 2 contains 10 apples and 2 oranges. Boxes are chosen with equal probability. What is the probability of choosing an apple? If an apple is chosen, what is the probability that it came from box 1?

b) The Monty Hall Problem: I have three boxes. In one I put a prize, and two are empty. I then mix up the boxes. You want to pick the box with the prize in it. You choose one box. I then open *another* one of the two remaining boxes and show that it is empty. I then give you the chance to change your choice of boxes—should you do so?

c) Given a joint probability $P(X, Y)$ over 2 binary random variables as the table

	Y=0	Y=1
X=0	.06	.24
X=1	.14	.56

What are $P(X)$ and $P(Y)$? Are X and Y conditionally independent?

12.8.2 Gaussians

On slide 06:17 there is the definition of a multivariate (n-dim) Gaussian distribution. Proof the following using only the definition. (You may ignore terms independent of x.)

a) Proof that:

$$\mathcal{N}(x|a, A) = \mathcal{N}(a|x, A)$$

$$\mathcal{N}(x \,|\, a, A) = |F|\, \mathcal{N}(Fx \,|\, Fa,\ FAF^\top)$$

$$\mathcal{N}(Fx + f \,|\, a, A) = \tfrac{1}{|F|}\, \mathcal{N}(x \,|\ F^{-1}(a - f),\ F^{-1}AF^{-\top})$$

b) Multiplying two Gaussians is essential in many algorithms (typically, a prior and a likelihood, to get a posterior). Prove the general rule

$$\mathcal{N}(x \,|\, a, A)\, \mathcal{N}(x \,|\, b, B)$$

$$\propto \mathcal{N}(x \,|\, (A^{-1} + B^{-1})^{-1}(A^{-1}a + B^{-1}b), (A^{-1} + B^{-1})^{-1}) \,, \tag{5}$$

where the proportionality $\propto$ allows you to drop all terms independent of x.

Note: The so-called canonical form of a Gaussian is defined as $\mathcal{N}[x \,|\, \bar{a}, \bar{A}] = \mathcal{N}(x \,|\, \bar{A}^{-1}\bar{a}, \bar{A}^{1})$; in this convention the product reads much nicher: $\mathcal{N}[x \,|\, \bar{a}, \bar{A}]\, \mathcal{N}[x \,|\, \bar{b}, \bar{B}] \propto \mathcal{N}[x \,|\, \bar{a}+\bar{b}, \bar{A}+\bar{B}]$.

c) The "forward propagation" of a Gaussian belief $\mathcal{N}(y \,|\, b, B)$ along a stochastic linear dynamics $\mathcal{N}(x \,|\, a + Fy, A)$ is given as

$$\int_y \mathcal{N}(x \,|\, a + Fy, A)\; \mathcal{N}(y \,|\, b, B)\; dy = \mathcal{N}(x \,|\, a + Fb, A + FBF^\top) \tag{6}$$

Prove the equation.

12.9 Exercise 9

12.9.1 Particle Filtering the location of a car

Start from the code in `RoboticsCourse/05-car`.

The `CarSimulator` simulates a car exactly as described on slide 04:48 (using Euler integration with step size 1sec). At each time step a control signal $u = (v, \phi)$ moves the car a bit and Gaussian noise with standard deviation $\sigma_{\text{dynamics}} = .03$ is added to x, y and θ. Then, in each step, the car meassures the relative positions of m landmark points, resulting in an observation $y_t \in \mathbb{R}^{m\times 2}$; these observations are Gaussian-noisy with standard deviation $\sigma_{\text{observation}} = .5$. In the current implementation the control signal $u_t = (.1, .2)$ is fixed (roughly driving circles).

a) Odometry (dead reckoning): First write a particle filter (with $N = 100$ particles) that ignores the observations. For this you need to use the cars system dynamics (described on 04:48) to propagate each particle, and add some noise σ_{dynamics} to each particle (step 3 on slide 07:22). Draw the particles (their x, y component) into the display. Expected is that the particle cloud becomes larger and larger.

b) Next implement the likelihood weights $w_i \propto P(y_t|x_t^i) = \mathcal{N}(y_t|y(x_t^i), \sigma) \propto e^{-\frac{1}{2}(y_t - y(x_t^i))^2/}$ where $y(x_t^i)$ is the (ideal) observation the car would have if it were in the particle possition x_t^i. Since $\sum_i w_i = 1$, normalize the weights after this computation.

c) Test the full particle filter including the likelihood weights (step 4) and resampling (step 2). Test using a larger ($10\sigma_{\text{observation}}$) and smaller ($\sigma_{\text{observation}}/10$) variance in the computation of the likelihood.

12.9.2 Kalman filter

We consider the same car example as above, but track the car using a Kalman filter.

a) To apply a Kalman filter (slide 07:27) we need Gaussian models for $P(x_t \mid x_{t\text{-}1}, u_{t\text{-}1})$ as well as $P(y_t|x_t)$. We assume that the dynamics model is given as a local Gaussian of the form

$$P(x_{t+1} \mid x_t, u_t) = \mathcal{N}(x_{t+1}|x_t + B(x_t)u_t, \sigma_{\text{dynamics}})$$

where the matrix $B(x_t) = \frac{\partial x_{t+1}(x_t,u_t)}{\partial u_t}$ gives the local linearization of the deterministic discrete time car dynamics

$$x_{t+1}(x,u) = x + \tau \begin{pmatrix} u_1 \cos x_3 \\ u_1 \sin x_3 \\ (u_1/L)\tan u_2 \end{pmatrix}$$

What is $B(x_t)$ for the car dynamics?

b) Concerning the observation likelihood $P(y_t|x_t)$ we assume

$$P(y_t|x_t, \theta_{1:N}) = \mathcal{N}(y_t \mid C(x_t)x_t + c(x_t), \sigma_{\text{observation}})$$

What is the matrix $C(x_t)$ (the Jacobian of the landmark positions w.r.t. the car state) in our example?

c) Start with the code in `RoboticsCourse/06-kalmanSLAM`.

Write a Kalman filter to track the car. You can use the routine `getObservationJacobian` to access $C(x_t) = \frac{\partial y}{\partial x}$. Note that $c(x_t) = \hat{y}_t - C(x_t)x_t$, where $\hat{y}_t$ is the mean observation in state x_t (there is another routine for this).

12.10 Exercise 10

12.10.1 The value function in Markov Decision Processes

On slide 5 of the RL lecture we defined MDPs as

$$P(s_{0:T+1}, a_{0:T}, r_{0:T}; \pi) = P(s_0) \prod_{t=0}^{T} P(a_t|s_t;\pi)\, P(r_t|s_t,a_t)\, P(s_{t+1}|s_t,a_t)\,, \qquad (7)$$

where $P(a_0|s_0;\pi)$ described the agent's policy. We assume a deterministic agent and write $a_t = \pi(s_t)$. The *value* of a state s is defined as the expected discounted sum of future rewards,

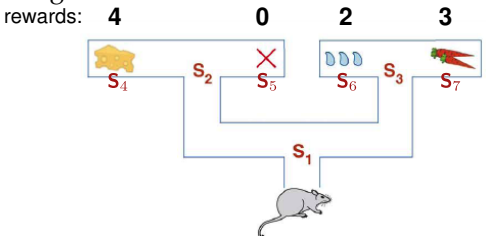
$$V^\pi(s) = \mathrm{E}\{r_0 + \gamma r_1 + \gamma^2 r_2 + \cdots \mid s_0 = s\} \qquad (8)$$

given that the agent starts in state s and from there executes the policy π.

a) Prove

$$V^\pi(s) = \mathrm{E}\{r_0 \mid s, \pi(s)\} \;+\; \gamma \textstyle\sum_{s'} P(s' \mid s, \pi(s))\, V^\pi(s')\,. \qquad (9)$$

b) Consider the following T-maze:



We distinguish 7 states $s_1, .., s_7$ in the maze. The first 3 states are the T-junctions; the last 4 states receive rewards $(4, 0, 2, 3)$. At each T-junction we have two possible actions: left, right. Everything is deterministic. Assume a discounting $\gamma = 0.5$.

Compute (by hand) the value function V^π over all states when π is the *random policy* (50/50 left/right at each junction).

c) Bellman's principle of optimality says that the optimal policy π^* has a value function $V^{\pi^*}(s) = V^*(s)$,

$$V^*(s) = \max_a \Big[\mathrm{E}\{r_0 \,|\, s, a\} \;+\; \gamma \textstyle\sum_{s'} P(s' \,|\, s, a)\, V^\pi(s') \Big] \,. \tag{10}$$

Compute (by hand) the optimal value function V^* over all states for the example above.

d) Now consider continuous state s and action a. Let the policy be stochastic and linear in features $\phi(s) \in \mathbb{R}^k$, that is,

$$\pi(a|s; \beta) = \mathcal{N}(a|\phi(s)^\top \beta, \phi(s)^\top \Sigma \phi(s)) \,. \tag{11}$$

The covariance matrix $\phi(s)^\top \Sigma \phi(s)$ describes that each action $a_t = \phi(s_t)^\top(\beta + \epsilon_t)$ was generated by adding a noise term $\epsilon_t \sim \mathcal{N}(0, \Sigma)$ to the parameter β. We always start in the same state $\hat{s}$ and the value $V^\pi(s_0)$ is

$$V^\pi(\hat{s}) = \mathrm{E}\{\sum_{t=0} H r_t \,|\, s_0 = \hat{s}\} \tag{12}$$

(no discounting, but only a finite horizon H.)

Optimality now requires that $\frac{\partial V^{\pi(\beta)}}{\partial \beta} = 0$. Assume that $a \in \mathcal{R}$ is just 1-dimensional and $\Sigma \in \mathbb{R}$ just a number. Try to prove (see slide 18) that we can derive

$$\beta^* = \beta^{\text{old}} + \Big[\mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} W(s_t) Q^{\pi(\beta)}(s_t, a_t, t)\} \Big]^{-1} \mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} W(s_t) e_t Q^{\pi(\beta)}(s_t, a_t, t)\} \,, \quad W(s) =$$

from $\frac{\partial V^{\pi(\beta^*)}}{\partial \beta} = 0$. (In the scalar case, $W(s) = 1$.) As a first step, derive

$$\frac{\partial}{\partial \beta} \log \pi(a|s)$$

Then insert $a_t = \phi(s_t)^\top(\beta^{\text{old}} + \epsilon_t)$ and solve

$$\mathrm{E}_{\xi|\beta}\{\sum_{t=0}^{H} \frac{\partial}{\partial\beta} \log\pi(a_t|s_t)\, Q(s_t, a_t, t)\} = 0$$

for β. This shows how you can get the optimal policy parameters β^* based on samples generated with β.

12.11 Exercise 11

12.11.1 Local linearization and Algebraic Riccati equation

Slide 08:02 describes the cart pole, which is similar to the Segway-type system of Exercise 6, but a little simpler. We'll solve the cart pole in this exercise.

The state of the cart-pole is given by $x = (p, \dot{p}, \theta, \dot{\theta})$, with $p \in \mathbb{R}$ the position of the cart, $\theta \in \mathbb{R}$ the pendulums angular deviation from the upright position and $\dot{p}, \dot{\theta}$ their respective temporal derivatives. The only control signal $u \in \mathbb{R}$ is the force applied on the cart. The analytic model of the cart pole is

$$\ddot{\theta} = \frac{g\sin(\theta) + \cos(\theta)\left[-c_1 u - c_2\dot{\theta}^2\sin(\theta)\right]}{\frac{4}{3}l - c_2\cos^2(\theta)} \tag{13}$$

$$\ddot{p} = c_1 u + c_2\left[\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\right] \tag{14}$$

with $g = 9.8ms^2$ the gravitational constant, $l = 1m$ the pendulum length and constants $c_1 = (M_p + M_c)^{-1}$ and $c_2 = lM_p(M_p + M_c)^{-1}$ where $M_p = M_c = 1kg$ are the pendulum and cart masses respectively.

a) Derive the local linearization of these dynamics around $x^* = (0, 0, 0, 0)$. The eventual dynamics should be in the form

$$\dot{x} = Ax + Bu$$

Note that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial\ddot{p}}{\partial p} & \frac{\partial\ddot{p}}{\partial\dot{p}} & \frac{\partial\ddot{p}}{\partial\theta} & \frac{\partial\ddot{p}}{\partial\dot{\theta}} \\ 0 & 0 & 0 & 1 \\ \frac{\partial\ddot{\theta}}{\partial p} & \frac{\partial\ddot{\theta}}{\partial\dot{p}} & \frac{\partial\ddot{\theta}}{\partial\theta} & \frac{\partial\ddot{\theta}}{\partial\dot{\theta}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{\partial\ddot{p}}{\partial u} \\ 0 \\ \frac{\partial\ddot{\theta}}{\partial u} \end{pmatrix}$$

where all partial derivatives are taken at the point $p = \dot{p} = \theta = \dot{\theta} = 0$.

The solution (to continue with the other parts) is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g}{\frac{4}{3}l - c_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\frac{4}{3}l - c_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c_1 + \frac{c_1 c_2}{\frac{4}{3}l - c_2} \\ 0 \\ \frac{-c_1}{\frac{4}{3}l - c_2} \end{pmatrix}$$

b) We assume a stationary infinite-horizon cost function of the form

$$J^\pi = \int_0^\infty c(x(t), u(t))\, dt$$
$$c(x, u) = x^\top Q x + u^\top R u$$
$$Q = \text{diag}(c, 0, 1, 0)\,, \quad R = \mathbf{I}\,.$$

That is, we penalize position offset $c\|p\|^2$ and pole angle offset $\|\theta\|^2$. Choose $c = \varrho = 1$ to start with.

Solve the Algebraic Riccati equation

$$0 = A^\top P + P^\top A - PBR^{-1}B^\top P + Q$$

by initializing $P = Q$ and iterating using the following iteration:

$$P_{k+1} = P_k + \epsilon[A^\top P_k + P_k^\top A - P_k B R^{-1} B^\top P_k + Q]$$

Choose $\epsilon = 1/1000$ and iterate until convergence. Output the gains $K = -R^{-1}B^\top P$. (Why should this iteration converge to the solution of the ARE?)

c) Solve the same Algebraic Riccati equation by calling the `are` routine of the octave control package (or a similar method in Matlab). For Octave, install the Ubuntu packages `octave3.2`, `octave-control`, and `qtoctave`, perhaps use `pkg load control` and `help are` in octave to ensure everything is installed, use `P=are(A,B*inverse(R)*B',Q)` to solve the ARE. Output $K = -R^{-1}B^\top P$ and compare to b).

(I found the solution $K = (1.0000, 3.8271, 48.3070, 17.615)$.)

d) Implement the optimal Linear Quadratic Regulator $u^* = -R^{-1}B^\top P x$ on the cart pole simulator. In the Robotics code base, path `07-cartPole` there is some code – but the main.problem.cpp was outdated. Please use the one from the website or email.

Simulate the optimal LQR and test it for various noise levels.

12.12 Exercise 13

On Wed. 29th we meet as usual for the exercise.

On Tue. 28th, 14:00 my office, interested students are invited to try whatever they like on the racer hardware, play around, etc.

12.12.1 Policy Search for the Racer

We consider again the simulation of the racer as given in `09-racer` in your code repo.

In this exercise the goal is to find a policy

$$\pi : \ \phi(y) \mapsto u = \phi(u)^\top w$$

that maps some features of the (history of the) direct sensor signals y to the control policy.

Use black-box optimization to find parameters w that robustly balance the racer.

Some notes:

- Features: In the lecture I suggested that a range of interesting features is:

  $$\Big(y_t, \dot{y}_t, \langle y\rangle_{0.5}, \langle \dot{y}\rangle_{0.5}, \langle y\rangle_{0.9}, \langle \dot{y}\rangle_{0.9}, u_t, u_{t-1}\Big)$$

  However, I noticed that a balancing policy can also be found for the direct sensor signals only, that is:

  $$\phi(y) = (1, y) \in \mathbb{R}^5$$

  (the augmentation by 1 is definitly necessary).

- Cost function: Realistically, the running costs c_t would have to be defined on the sensor signals only. (On the real robot we don't know the real state – if the robot is to reward itself it needs to rely on sensor signals only.) I tried

  ```
  costs += .1*MT::sqr(y(3)) + 1.*MT::sqr(y(2));
  ```

  which combines the wheel encoder `y(3)` and the gyroscope reading `y(2)`. That worked mediocre. For a start, cheat and directly use the state of the simulator to compute costs:

  ```
  costs += 1.*MT::sqr(R.q(0)) + 10.*MT::sqr(R.q(1));
  ```

- Episodes and duration costs: To compute the cost for a given w you need to simulate the racer for a couple of time steps. For the optimization it is really bad if an episode is so long that it includes a complete failure and wrapping around of the inverted pendulum. Therefore, abort an episode if `fabs(R.q(1))` too large and penalize an abortion with an extra cost, e.g., proportional to $T - t$. Try different episode horizons T up to 500; maybe increase this horizon stage-wise.

- Optimizer: You are free to use any optimizer you like. On the webpage you find a reference implementation of CMA by Niko Hansen (with wrapper using our `arr`), which you may use. In that case, add `cmaes.o` and `search_CMA.o` in the `Makefile`. The typical loop for optimization is

```
SearchCMA cma;
cma.init(5, -1, -1, -0.1, 0.1);
arr samples, values;

uint T=500;
for(uint t=0;t<1000;t++){
  cma.step(samples, values);
  for(uint i=0;i<samples.d0;i++) values(i) = evaluateControlParameters(samp
  uint i=values.minIndex();
  cout <<t <<' ' <<values(i) <<' ' <<samples[i] <<endl;
}
```

12.13 Exercise extra

12.13.1 Kalman SLAM

Slide 07:38 outlines how to use a high-dimensional Kalman filter to simultaneously estimate the robot position (localization) and the landmarks position (mapping).

a) Concerning $P(y_t|x_t, \theta_{1:N})$ we assume

$$P(y_t|x_t, \theta_{1:N}) = \mathcal{N}(y_t \,|\, D(x_t)\boldsymbol{\theta} + d(x_t), \sigma_{\text{observation}})$$

where $D(x_t) = \frac{\partial y}{\partial \boldsymbol{\theta}}$ is the observation Jacobian w.r.t. the unknown landmarks, and $\boldsymbol{\theta} \in \mathbb{R}^{2N}$ is the same as $\theta_{1:N}$ written as a $2N$-dim vector.

Write a pseudo-code for Kalman SLAM. (There is much freedom in how to organize the code, choices of notation and variables, etc. Try to write it as concise as possible.)

b) Try to implement Kalman SLAM, which tracks the car simultaneous to the landmarks. You should now access the routines `getMeanObservationAtStateAndLandma` and `getObservationJacobianAtStateAndLandmarks` to retrieve the mean observation and the necessary Jacobians given the current mean estimate $\boldsymbol{\theta}$ of the landmarks.

13 Bullet points to help learning

This is a summary list of core topics in the lecture and intended as a guide for preparation for the exam. Test yourself also on the bullet points in the table of contents. Going through all exercises is equally important.

13.1 Kinematics

- 3D geometry
  - Definition of an object pose, frame, transformations
  - Homogeneous transformations (4×4 matrix)
  - Composition of transformations, notation $x^W = T_{W\to A}\, T_{A\to B}\, T_{B\to C}\, x^C$
- Fwd kinematics & Jacobian
  - Fwd kinematics as composition of transformations
  - Transformations of a rotational joint
  - Kinematic maps $\phi^{\text{pos}}:\ q \mapsto y$ and ϕ^{vec}
  - Definition of a Jacobian
  - Derivation of the position and vector Jacobians J^{pos}, J^{vec}
- Inverse kinematics (IK)
  - Optimality criterion for IK
  - Using the local linearization to find the optimum
  - Pseudo code of following a task trajectory in small steps using Inverse Kinematics
  - Definition & example for a singularity
- Motion profiles & Interpolation
  - Motion profiles (esp. sine profile)
  - Joint space vs. task space interpolation of a trajectory [e.g. using a motion profile in one or the other space]
- Multiple Tasks
  - How to incorporate multiple tasks
  - What are interesting task maps; know at least about pos, vec, align, and limits
- Further
  - Be able to explain the consequences of the local linearization in IK (big steps $\to$ errors)

13.2 Dynamics

- 1D point mass & PID control
  - General form of a dynamic system
  - Dynamics of a 1D point mass
  - Position, derivative and integral feedback to control it to a desired state
  - Solution to the *closed-loop* PD system equations
  - Qualitative behaviors: oscillatory-damped, over-damped, critically damped
- Euler-Lagrange equation
  - Definition
  - Roughly: structure of the Euler-Lagrange equation for systems; understand at least $T = \frac{1}{2}\dot{q}^\top M \dot{q}$
  - Be able to apply on minimalistic system (e.g. pendulum, point mass)
- Robot dynamics & joint/operational space control
  - General form of the dynamics equation
  - Joint space control: given desired $\ddot{q}^*$, choose $u^* = M(q)\, \ddot{q}^* + F(q, \dot{q})$
  - Operational space control: given desired $\ddot{y}^*$, choose $u^* = T^\sharp(\ddot{y}^* - \dot{J}\dot{q} + TF)$ (5:31, 32, 34)
  - Following a reference trajectory $q^{\text{ref}}_{0:T}$ in joint space by imposing PD behavior around it

13.3 Path planning

- Basics
  - Path finding vs. trajectroy optimization vs. feedback control
  - Roughly: BUG algorithms
  - Potential functions, and that they're nothing but IK with special task variables
  - Dijkstra Algorithm
- Probabilistic Road Maps (PRMs)
  - Definition & Generation
  - Importance of local planner
  - Roughly: knowing about probabilistic completeness
  - Roughly: alternative sampling strategies

- Rapidly Exploring Random Trees (RRTs)
  - Algorithm
  - Goal-directed & bi-directional extensions
- Non-holonomic Systems
  - Definition of non-holonomicity. Be able to give example
  - Path finding: control-based sampling
  - RRT extension for control-based exploration
  - Roughly: Intricacies with metrics for non-holonomic systems

13.4 Mobile Robotics

- Probability Basics
  - Definitions of random variable, probability distribution, joint, marginal, conditional distribution, independence
  - Bayes' Theorem
  - Continuous distributions, Gaussian, particles
- State Estimation
  - Formalization of the state estimation problem
  - The Bayes Filter as the general analytic solution
  - Gaussians and particles to approximate the Bayes filter and make it computationally feasible:
  - Details of a Particle Filter
  - Kalman filter (esp. assumptions made, not eq. or derivation)
  - Extended KF (assumptions made)
  - Odometry (dead reckoning) as "Bayes filter without observations"
- SLAM
  - Definition of the SLAM problem
  - In what sense SLAM is a "chicken or egg problem"
  - Joint filtering over x and m: Extended Kalman SLAM
  - Particle-based SLAM (map belief for each particle)
  - Roughly: graph-based SLAM & loop closing

13.5 Control Theory

- Optimal Control
  - Definition of the (continuous time) optimal control problem
  - Concept & definition of the value function
  - HJB equation
  - Infinite horizon $\rightarrow$ stationary solution
  - Awareness that optimal control is not the only approach; it shifts the problem of designing a controller to designing a cost function.
- Linear-Quadratic Optimal Control
  - Definition of problem (esp. assumptions made)
  - Be able to express system dynamics in (locally linearized) standard form $\dot{x} = Ax + Bu$
  - Fact that the value function is quadratic $V(x,t) = x^\top P(t) x$
  - Riccati differential equation
  - How P gives the optimal Linear-Quadratic Regulator
  - Algebraic (infinite horizon) Riccati equation
- Controllability
  - Definition and understanding/interpretation of the controllability matrix C
  - Definition of controllability
  - Be able to apply on simple examples
- Stability
  - Definitions of stability
  - Eigenvalue analysis for linear systems
  - Optimize controllers for negativity of eigenvalues
  - Definition of a Lyapunov function
  - Lyanunov's theorem: $\exists$ Lyapunov fct. $\rightarrow$ stability
  - Energy and value function as candidates for a Lyapunov fct.

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