

AMATH 561 Assignment 3

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Exercise 3.1

Let $X \sim \text{Bin}(n, U)$ where $U \sim \mathcal{U}((0, 1))$. What is the probability Generating function $G_X(s)$ of X ? What is $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, \dots, n\}$?

Solution

Using iterated conditioning, since a Binomial random variable is the sum of n iid Bernioully random variables,

$$G_X(s) = \mathbb{E}[s^X] = \mathbb{E}\mathbb{E}[s^X|U] = \mathbb{E}[(1 - U)s^0 + Us^1]^n$$

We calculate this by integrating with Mathematica as,

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Integrate[((1 - x) + x s)^n, {x, 0, 1}, Assumptions -> {s > 0}]
```

This yields,

$$\mathbb{E}[(1 - U) + Us]^n = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}((1 - x) + xs)^n dx = \int_0^1 ((1 - x) + xs)^n dx = \frac{1 - s^{n+1}}{(n+1)(1-s)}$$

This is a finite geometric progression which we simplify so,

$$G_X(s) = \sum_{k=0}^n \frac{s^k}{n+1}$$

Therefore $\mathbb{P}(X = k) = 1/(1 + n)$ for $k = 0, 1, 2, \dots, n$.

Exercise 3.2

Let Z_n be the size of the n -th generation in an ordinary branching process with $Z_0 = 1$, $\mathbb{E}Z_1 = \mu$ and $\mathbb{V}Z_1 > 0$. Show that $\mathbb{E}Z_n Z_m = \mu^{n-m} \mathbb{E}Z_m^2$ for $m \leq n$. Use this to find the correlation coefficient $\rho(Z_m, Z_n)$ in terms of μ, n and m . Consider the case $\mu = 1$ and the case $\mu \neq 1$.

Solution

Let $Y_{m,i}$ denote the number of offspring in the n -th generation that descends from the i -th member of the m -th generation. Then the $(Y_{m,i})$ are iid with distribution Z_{n-m} and $Z_n = Y_{m,1} + Y_{m,2} + \dots + Y_{m,Z_m}$.

Then, since $(Y_{m,i})$ are iid with distribution Z_{n-m} ,

$$\mathbb{E}[Z_n | Z_m] = \mathbb{E}[Y_{m,1} + Y_{m,2} + \dots + Y_{m,Z_m} | Z_m] = Z_m \mathbb{E}[Z_{n-m}] = Z_m \mu^{n-m}$$

Therefore, by taking out what is known,

$$\mathbb{E}[Z_m Z_n] = \mathbb{E}[\mathbb{E}[Z_m Z_n | Z_m]] = \mathbb{E}[Z_m^2 \mathbb{E}[Z_{n-m} | Z_m]] = \mathbb{E}[Z_m^2 \mu^{n-m}] = \mu^{n-m} \mathbb{E}[Z_m^2]$$

Observing that $\mathbb{E}[Z_m Z_n] = \mu^{n-m} \mathbb{E}[Z_m^2] = \mu^{n-m} (\mathbb{V}[Z_m] + \mathbb{E}[Z_m]^2) = \mu^{n-m} (\mathbb{V}[Z_m] + \mu^{2m})$, write,

$$\rho(Z_m, Z_n) = \frac{\text{Cov}(Z_n, Z_m)}{(\mathbb{V}[Z_n] \mathbb{V}[Z_m])^{1/2}} = \frac{\mathbb{E}[Z_n Z_m] - \mathbb{E}[Z_n] \mathbb{E}[Z_m]}{(\mathbb{V}[Z_n] \mathbb{V}[Z_m])^{1/2}} = \frac{\mu^{n-m} (\mathbb{V}[Z_m] + \mu^{2m}) - \mu^{n+m}}{(\mathbb{V}[Z_n] \mathbb{V}[Z_m])^{1/2}}$$

Denote $\mathbb{V}[Z_1]$ by σ .

Suppose $\mu = 1$ so that $\mathbb{V}[Z_m] = m\sigma^2$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
  PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m)) / (Vzn Vzm)^(
    1/2) /. {Vzm -> m \[Sigma]^2, Vzn -> n \[Sigma]^2, \[Mu] -> 1}],
  Assumptions -> {{m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{m}{n}}$$

Now suppose $\mu \neq 1$ so that $\mathbb{V}[Z_m] = \sigma^2(\mu^n - 1)\mu^{n-1}/(\mu - 1)$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
  PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m)) / (Vzn Vzm)^(
    1/2) /. {Vzm -> \[Sigma]^2 (\[Mu]^m - 1) \[Mu]^(m - 1) / (\[Mu] - 1),
    Vzn -> \[Sigma]^2 (\[Mu]^n - 1) \[Mu]^(n - 1) / (\[Mu] - 1) }],
  Assumptions -> {\[Mu] != 1, {m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{\mu^n(\mu^m - 1)}{\mu^m(\mu^n - 1)}}$$

Observe that in the limit $\mu \rightarrow 1$ this coincides with the previous value.

Exercise 3.3

Solution

Exercise 3.4

Consider a branching process with immigration

$$Z_0 = 1 \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n$$

where the $(X_{n,i})$ are iid with common distribution X , the (Y_n) are iid with common distribution Y , and the $(X_{n,i})$ and (Y_n) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_X(s)$, and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_X(s)$ and $G_Y(s)$.

Solution

Define:

$$G_{Z_n}(s) = s^{Z_n} \qquad G_X(s) = \mathbb{E}s^X \qquad G_Y(s) = \mathbb{E}s^Y$$

Write $S_n = \sum_{i=1}^{Z_n} X_{n,i}$ so that, $Z_{n+1} = S_n + Y_n$.

First observe that since the $(X_{n,i})$ are iid with common distribution X ,

$$G_{S_n}(s) = \mathbb{E}[s^{S_n}] = \mathbb{E}[\mathbb{E}[s^{S_n} | Z_n]] = \mathbb{E}[\mathbb{E}[s^X]^{Z_n}] = \mathbb{E}[G_X(s)^{Z_n}] = G_{Z_n}(G_X(s))$$

Since the $(X_{n,i})$ and (Y_n) are independent, S_n and Y_n are independent. Therefore,

$$G_{Z_{n+1}}(s) = G_{S_n + Y_n}(s) = G_{S_n}(s)G_Y(s) = G_{Z_n}(G_X(s))G_Y(s)$$

We calculate,

$$G_{Z_0}(s) = \mathbb{E}[s^{Z_0}] = \mathbb{E}[s] = s$$

Similarly,

$$G_{Z_1}(s) = G_{Z_0}(G_X(s))G_Y(s) = G_X(s)G_Y(s)$$

Therefore,

$$G_{Z_2}(s) = G_{Z_1}(G_X(s))G_Y(s) = G_X(G_X(s))G_Y(G_X(s))G_Y(s)$$

Exercise 3.5

Find $\phi_{X^2}(t) := \mathbb{E} \exp(itX^2)$ where $X \sim \mathcal{N}(\mu, \sigma)$.

Solution

We have,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Thus,

$$\phi_{X^2}(t) = \mathbb{E} \exp(itX^2) = \int_{-\infty}^{\infty} e^{itx^2} f_X(x) dx$$

We evaluate with Mathematica as,

```
Integrate[Exp[I t x^2] PDF[NormalDistribution[[Mu], [Sigma]], x], {x, -[
  Infinity], [Infinity]},
  Assumptions -> {[Mu] [Element] Reals, t [Element] Reals, [Sigma] > 0}]
```

This yields,

$$\phi_{X^2}(t) = \frac{\exp(it\mu^2/(1-2it\sigma^2))}{\sqrt{1-2it\sigma^2}}$$

Exercise 3.6

Let X_n have cumulative distribution function

$$F_{X_n}(x) = \left(x - \frac{\sin(2n\pi x)}{2n\pi} \right) \mathbb{1}_{0 \leq x \leq 1} + \mathbb{1}_{x > 1}$$

- (a) Show that F_{X_n} is a distribution function and find the corresponding density function f_{X_n} .
- (b) Show that F_{X_n} converges to the uniform distribution function F_U as $n \rightarrow \infty$, but that the density function f_{X_n} does NOT converge to f_U . Here, $U \sim \mathcal{U}((0, 1))$.

Solution

- (a) Clearly $F_{X_n}(x) = 0$ for $x \leq 0$ and $F_{X_n}(x) = 1$ for $x \geq 1$. Observe, $x - \sin(2n\pi x)/2n\pi$ is non-decreasing and continuous on $(0, 1)$, since the derivative, calculated below is non-negative on this interval. Moreover, $x - \sin(2n\pi x)/2n\pi$ is equal to zero at $x = 0$, and equal to one at $x = 1$.

Therefore $F_{X_n}(x)$ is a non-decreasing continuous function with $F_{X_n}(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $F_{X_n}(x) \rightarrow 1$ as $x \rightarrow \infty$. So $F_{X_n}(x)$ is a distribution function.

It is straightforward to compute the density function as,

$$f_{X_n}(x) = \frac{d}{dx} F_{X_n}(x) = (1 - \cos(2n\pi x)) \mathbb{1}_{0 \leq x \leq 1}$$

- (b) The uniform distribution on $(0, 1)$ is given by,

$$F_U(x) = x \mathbb{1}_{0 \leq x \leq 1} + \mathbb{1}_{x > 1}$$

Obviously outside of $(0, 1)$ both F_U and F_{X_n} agree exactly. Consider a point $x \in (0, 1)$. Then, since $|\sin(u)| \leq 1$ for all u ,

$$\lim_{n \rightarrow \infty} \left[x - \frac{\sin(2n\pi x)}{2n\pi} \right] = x - 0 = x$$

Therefore F_{X_n} converges pointwise on to F_U on $(0, 1)$, and therefore on all of \mathbb{R} .

It is clear that $f_{X_n}(x)$ does not converge to $f_U(x)$ as $f_U(x)$ is constant on $(0, 1)$ while $f_{X_n}(x)$ oscillates between zero and two. In particular, fix a rational number $x = p/q$. Then for $n = qk, k \in \mathbb{N}$, $f_{X_n}(x) = 0$.

Exercise 3.7

A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \rightarrow 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then,

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

Solution

We have $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$. Thus, making the substitution $u = (\lambda - it)x$,

$$\begin{aligned} \phi_X(t) &= \mathbb{E} [e^{itx} f_X(x) dx] \\ &= \int_0^\infty e^{itx} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} dx \\ &= \int_0^\infty \frac{\lambda^r}{\Gamma(r)} e^{-u} \frac{u^{r-1}}{(\lambda - it)^{r-1}} \frac{du}{(\lambda - it)} \\ &= \frac{\lambda^r}{\Gamma(r)(\lambda - it)^r} \int_0^\infty e^{-u} u^{r-1} du \\ &= \frac{\lambda^r}{(\lambda - it)^r} \end{aligned}$$

Let $(X_i)_{i=1}^k$ be iid with $X, X_i \sim \text{Geo}(p)$. Then $N = \sum_{i=1}^k X_i$ so, since the X_i are iid,

$$\varphi_{2Np}(t) = \mathbb{E}[\exp(it2Np)] = \mathbb{E}[\exp(2itp(X_1 + \dots + X_k))] = \mathbb{E}[\exp(2itpX)]^k$$

Therefore, since $|e^{2itp}(1-p)| < 1$ if $p \in (0, 1)$,

$$\mathbb{E}[\exp(2itpX)]^k = \left[\sum_{m=1}^{\infty} e^{2itpm} p(1-p)^{m-1} \right]^k = \left[pe^{2itp} \sum_{m=1}^{\infty} (e^{2itp}(1-p))^{m-1} \right]^k = \left[\frac{pe^{2itp}}{1 - (1-p)e^{2itp}} \right]^k$$

With Mathematica we evaluate,

```
Limit[(p Exp[2 I t p])/(1 - (1 - p) Exp[2 I t p]))^k, {p -> 0},
sumptions -> {k \[Element] Integers, k > 0}] // FullSimplify
```

This yields,

$$\lim_{p \rightarrow 0} \varphi_{2Np} = \frac{1}{(1 - 2it)^k} = \frac{(1/2)^k}{(1/2 - it)^k}$$

Thus, for a random variable $X \sim \Gamma(1/2, k)$, by the continuity theorem, $\lim_{p \rightarrow 0} f_{2Np}(x) = f_X(x)$