AMATH 514 Assignment 1

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Exercise 1.7

Find, both with the Dijkstra-Prim algorithm and with Kruskal's algorithm, a spanning tree of minimum length in the graph in Figure 1.5

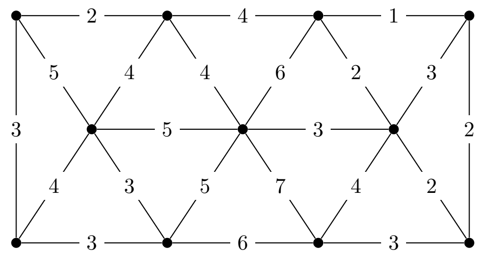
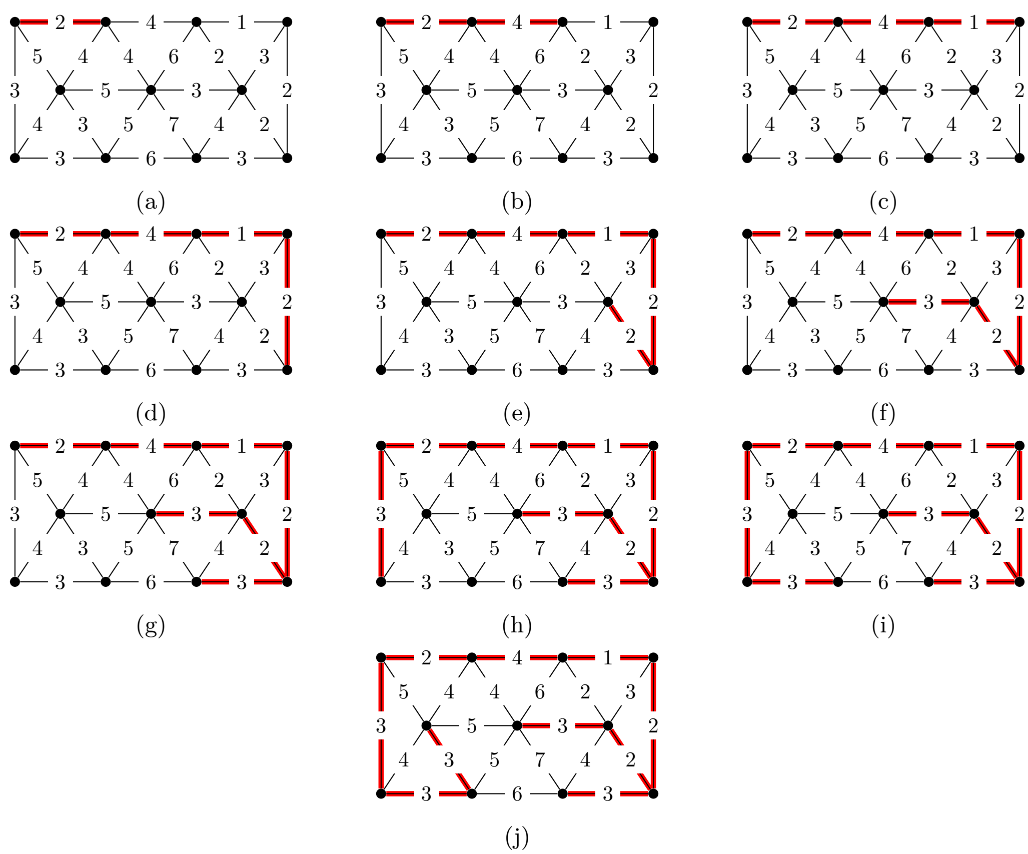


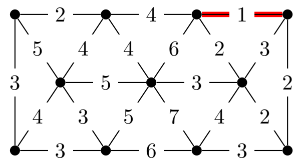
Figure 1

Solution

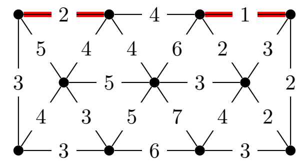
We arbitrarily choose the top left node as the starting node and proceed adding the lowest weight edge in the cut of the existing tree which maintains the tree structure.



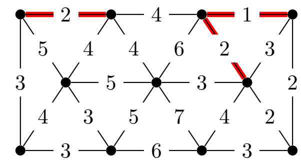
We order the edges. Since the choice of edge in a tie is arbitrary we work from top to bottom left to right. We add edges in this order if they maintain a forest structure (no cycles).


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

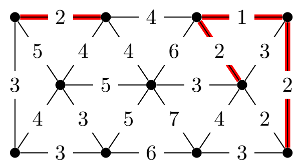
(a)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

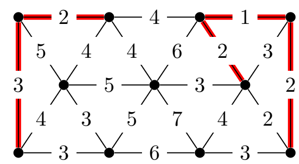
(b)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

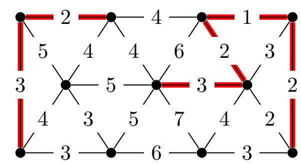
(c)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

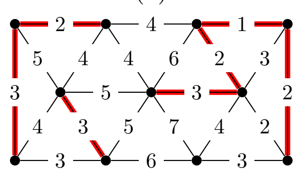
(d)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

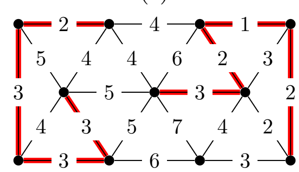
(e)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

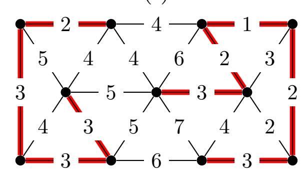
(f)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

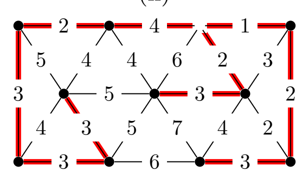
(g)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

(h)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

(i)


2 4 1
5 4 4 6 2 3
3 5 3 2
4 3 5 7 4 2
3 6 3

(j)

Exercise 1.9

Let $G = (V, E)$ be a graph and let $l : E \to \mathbb{R}$ be a 'length' function. Call a forest F good if $l(F') \geq l(F)$ for each forest F' satisfying $|F'| = |F|$.

Let F be a good forest and e be an edge not in F such that $F \cup \{e\}$ is a forest and such that (among all such e) $l(e)$ is as small as possible. Show that $F \cup \{e\}$ is good again.

Solution

We first prove the following:

Let $G = (V, E)$ be a graph, and $(V, F), (V, X)$ be forests in G with $|F| < |X| < \infty$. Then there is an edge $x \in X$ such that $(V, F \cup \{x\})$ is a forest in G.

Denote the connected components of F by $(V_1, F_1), (V_2, F_2), ..., (V_k, F_k)$.

Define $Y = \cup_{j=1}^{k} \{(u, v) \in X | u \in V_j \text{ and } v \in V_j\}$. That is, Y is the set of edges of X not contained in any cut $\delta(V_j)$.

Observe that since X is acyclic, Y is also acyclic. Thus, each set $\{(u, v) \in X | u, v \in V_j\}$ has size at most $|V_j| - 1 = |F_j|$. This means Y has size at most $\sum_{j=1}^{k} |F_j| = |F|$.

Therefore, since X, Y are finite and $Y \subseteq X$, $|X \setminus Y| = |X| - |Y| > |X| - |F| > 0$. This means there is some edge $x = (u, v) \in X \setminus Y$. That is, there is some $x = (u, v) \in X$ satisfying,

$$\forall j \in \{1, 2, ..., k\}, (u, v) \notin \{(u, v) \in X | u \in V_j \text{ and } v \in V_j\}$$

Equivalently, there is some edge $x = (u, v) \in X$ satisfying,

$$\forall j \in \{1, 2, ..., k\}, u \notin V_j \text{ or } v \notin V_j$$

That is both edges of x cannot be in the same set V_j for any j. In other words, $x \in \delta(V_j)$ for some j.

Then $(V, F \cup \{x\})$ is still a forest since adding x to F will not not induce a cycle. □

We now prove the main result. Indeed, let F be a good forest and let X be any forest in G satisfying $|X| = |F \cup \{e\}| = |F| + 1$, where e is chosen such that $F \cup \{e\}$ is a forest and $l(e)$ is as small as possible.

By above, there is some $x \in X$ such that $F \cup \{x\}$ is a forest. By our choice of e we have $l(x) \geq l(e)$.

Moreover, $X \setminus \{x\}$ is a forest satisfying $|X \setminus \{x\}| = |F|$. Since F is good, $l(X \setminus \{x\}) \geq l(F)$.

Therefore,

$$l(X) = l((X \setminus \{x\}) \cup \{x\}) = l(X \setminus \{x\}) + l(x) \geq l(F) + l(e) = l(F \cup \{e\})$$

This proves $F \cup \{e\}$ is a good forest. □

Exercise 10.1

Let X be a finite set and $\mathcal{I} \subseteq 2^X$. Suppose (i) $\emptyset \in \mathcal{I}$ and (ii) if $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$.

Show the following statements are equivalent:

(iii) if $Y, Z \in \mathcal{I}$ and $|Y| < |Z|$ then $Y \cup \{x\} \in \mathcal{I}$ for some $x \in Z \setminus Y$.

(3) for any subset Y of X, any two bases of Y have the same cardinality.

Solution

Suppose (iii) and let $Y \subseteq X$ with U, V bases for Y.

Suppose further, for the sake of contradiction, that U and V have different cardinalities. Without loss of generality, assume $|U| < |V| \leq |Y|$. Then, by (iii) there is some $v \in V \setminus U$ such that $U \cup \{v\} \in \mathcal{I}$.

Since U is a proper subset of $U \cup \{v\}$ and U is a basis (inclusionwise maximial) we must have $U \cup \{v\} = Y$. Then $|Y| = |U| + 1$, so since $|V| > |U|$, and $|Y| \geq |V|$, we have $|V| = |Y|$. Moreover, since $V \subseteq Y$, we require $V = Y$. But then $U \subsetneq V$ is not inclusionwise maximial in $Y = V = U \cup \{v\}$, contradicting the hypothesis that U is a basis for Y.

This proves U and V have the same cardinality.

Now, suppose not (iii). That is, let $Y, Z \in \mathcal{I}$ with $|Y| < |Z|$ and suppose $Y \cup \{x\} \notin \mathcal{I}$ for any $x \in Z \setminus Y$.

Let $U = Z \cup Y$. Suppose Y is not a basis for U. That is, that there is some $W \in \mathcal{I}$ with $Y \subsetneq W \subsetneq U$. By size arguments it is obvious that $W \cap Z \neq \emptyset$ so that there is some $x \in Z \cap W \subseteq Z$. But then, $Y, \{x\}$ are subsets of W, so their union $Y \cup \{x\}$ is a subset of W. By (ii), $Y \cup \{x\} \in \mathcal{I}$, contradicting the hypothesis. Therefore Y is a basis for U.

If Z is also inclusionwise maximial in U, then Y, Z are bases for U of different cardinality. If Z is not inclusionwise maximal, then there is some $W \supsetneq Z$ which is inclusionwise maximal (since U is finite). But clearly, $|W| > |Z| > |Y|$ meaning W has a different cardinality than Y.

This proves two bases of a subset of X need not have the same cardinality.

We have now shown (iii) $\Longrightarrow$ (3) and not (iii) $\Longrightarrow$ not (3). This proves (iii) $\Longleftrightarrow$ (3). $\square$