AMATH 561 Assignment 5

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Exercise 5.1

Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Solution

In some small time interval s there is probability $\lambda s+\mathcal{O}(s^2)$ that a patient arrives, probability $1-\lambda s+\mathcal{O}^2$ that a patient does not arrive, and probability $\mathcal{O}(s^2)$ that multiple patients arrive.

If there are patients, in this times there is also probability $\mu s+\mathcal{O}(s^2)$ that a patient is treated, probability $1-\mu s+\mathcal{O}(s^2)$ that a patient is not treated, and probability $\mathcal{O}(s^2)$ that more than one (if possible) patients are treated.

Note that any moves which have more than one transition such as a patient arriving, and a patient being treated are all $\mathcal{O}(s^2)$.

Suppose there are no patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = 0) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = 1 \\ 1 - \lambda s + \mathcal{O}(s^2) & j = 0 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

Now suppose there are $i > 0$ patients at time t. The probability of transitioning to j patients after a short time s is given by,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} (\lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) & j = i+1 \\ (1 - \lambda s + \mathcal{O}(s^2))(1 - \mu s + \mathcal{O}(s^2)) + \mathcal{O}(s^2) & j = i \\ (1 - \lambda s + \mathcal{O}(s^2))(\mu s + \mathcal{O}(s^2)) & j = i-1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This is simplified as,

$$\mathbb{P}(X_{t+s} = j | X_t = i) = \begin{cases} \lambda s + \mathcal{O}(s^2) & j = i+1 \\ 1 - \lambda s - \mu s + \mathcal{O}(s^2) & j = i \\ \mu s + \mathcal{O}(s^2) & j = i-1 \\ \mathcal{O}(s^2) & \text{otherwise} \end{cases}$$

This gives,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We recognize this as a birth-death process (a bit ironic in the context of an emergency room) with $\lambda_i = \lambda$ and $\mu_i = \mu$.

Then if a stationary distribution π exists, for $n \in \mathbb{Z}_{>0}$,

$$\pi(n > 0) = \left(\frac{\lambda}{\mu}\right)^n \pi(0)$$

and

$$\pi(0) = \left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1}$$

This is a geometric series which is convergent exactly when $\lambda/\mu < 1$. That is, when $\lambda < \mu$. In this case,

$$\pi(0) = \left(\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(\frac{\mu}{\mu-\lambda}\right)^{-1} = \frac{\mu-\lambda}{\mu}$$

We condition on knowing the number of people on the queue. Suppose there are n people in the queue when a patient arrives. Then the patient will have to wait a random time distributed as the sum of n exponential random variables with parameter μ to be treated and one more to finish treatment. The expectation of each of each exponential random variable is $1/\mu$, so the patient waits an expected time of $(n+1)/\mu$.

In equilibrium, the probability that there are n people in the queue when a patient arrives is $\pi(n)$.

Therefore, the expected wait time is,

$$\sum_{n=0}^{\infty}\pi(n)\frac{(n+1)}{\mu} = \frac{\mu-\lambda}{\mu^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n(n+1) = \frac{\mu-\lambda}{\mu^2}\left(\frac{\mu\lambda}{(\mu-\lambda)^2} + \frac{\mu}{\mu-\lambda}\right) = \frac{1}{\mu-\lambda}$$

Exercise 5.2

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with stationary distribution π. Let N be an independent Poisson process with intensity λ and denote by τ_n the time of the n-th arrival of N. Define $Y_n := X_{\tau_n+}$ (i.e., Y_n is the value of X immediately after the n-th jump). Show that Y is a discrete time Markov chain with the same stationary distribution as X.

It is obvious that Y is Markov, as given the present, the future is independent of the past. We add a bit more rigor below.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By hypothesis X_t is a Markov process. That is, for a filtration $(\mathcal{F}_s)_{s\in[0,T]}$, for $0 \leq s \leq t \leq T$, and for every non-negative Borel measurable function f,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s]$$

Let $\mathcal{F}'_n = \mathcal{F}_{\tau_n+}$ be a sub-σ-algebra of $\mathcal{F}$. Then clearly $(\mathcal{F}'_n)$ is a filtration. Let f be any non-negative Borel measurable function. Then,

$$\mathbb{E}[f(Y_n)|\mathcal{F}'_m] = \mathbb{E}[f(X_{\tau_n+})|\mathcal{F}_{\tau_m+}] = \mathbb{E}[f(X_{\tau_n+})|X_{\tau_m+}] = \mathbb{E}[f(Y_n)|Y_m]$$

This means Y is Markov, and clearly Y is discrete time. Therefore Y is a discrete time Markov chain.

Note we assume X is time homogeneous.

Suppose X has stationary distribution π. Then for all $0 \leq t \leq T$, $\pi P_t = \pi$, where,

$$(P_t)_{i,j} = \mathbb{P}(X_t = j|X_0 = i)$$

Thus, the one step probability transition matrix, denoted $\tilde{P}$, for Y is,

$$\tilde{P}_{i,j} = \mathbb{P}(Y_1 = j|Y_0 = i) = \mathbb{P}(X_{\tau_1+} = j|X_0 = i) = (P_{\tau_1})_{i,j}$$

This means $\pi\tilde{P} = \pi$, so π is a stationary distribution of Y.

Exercise 5.3

Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, ...\}$ and generator G whose i-th row has entries

$$g_{i,i-1} = i\mu \qquad\qquad g_{i,i} = -i\mu - \lambda \qquad\qquad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_t}(s) := \mathbb{E}s^{X_t}$. What is the distribution of X_t as $t \to \infty$?

Solution

We have G in matrix form,

$$G = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\mu+\lambda) & \lambda & & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & & \\ & & 3\mu & -3(\mu+\lambda) & \lambda & \cdots \\ & & & \vdots & \vdots & \ddots \end{bmatrix}$$

We wish to find the transition semi group P_t. We know this can be derived from the Kolmogorov forward equations. That is,

$$\frac{d}{dt}P_t = P_t G$$

With the assumption that $X_0 = i$ (*I am using i rather than j like the problem statement since this is the standard way of doing things*) we have,

$$\begin{aligned} \frac{d}{dt}p_t(i,0) &= \sum_{k=0}^{\infty} p_t(i,k)g(k,0) = -\lambda p_t(i,0) + \mu p_t(i,1) \\ \frac{d}{dt}p_t(i,j) &= \sum_{k=0}^{\infty} p_t(i,k)g_t(k,j) = \lambda p_t(i,j-1) - (j\mu+\lambda)p_t(i,j) + (j+1)\mu p_t(i,j+1) \qquad j \geq 1 \end{aligned}$$

We multiply the j-th equation by s^j. This gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t}p_t(i,j)s^j = \sum_{j=1}^{\infty}\left[\lambda p_t(i,j-1)s^j\right] - \sum_{j=0}^{\infty}\left[(j\mu-\lambda)p_t(i,j)s^j\right] + \sum_{j=0}^{\infty}\left[(j+1)\mu p_t(i,j+1)s^j\right]$$

Summing the left hand sides gives,

$$\sum_{j=0}^{\infty} \frac{\partial}{\partial t}p_t(i,j)s^j = \frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_t(i,j)s^j = \frac{\partial}{\partial t}G_{X_t}(s)$$

The first term of the right hand side gives,

$$\sum_{j=1}^{\infty} \lambda p_t(i,j-1)s^j = \lambda s\sum_{j=1}^{\infty} p_t(i,j-1)s^{j-1} = \lambda s\sum_{j=0}^{\infty} p_t(i,j)s^j = \lambda s G_{X_t}(s)$$

The negative of the first part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} j\mu p_t(i,j)s^j = s\mu \sum_{j=0}^{\infty} jp_t(i,j)s^{j-1} = s\mu \sum_{j=0}^{\infty} \frac{\partial}{\partial s} p_t(i,j)s^j = s\mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j)s^j = s\mu \frac{\partial}{\partial s} G_{X_t}(s)$$

The negative of the second part of the second term of the right hand side gives,

$$\sum_{j=0}^{\infty} \lambda p_t(i,j)s^j = \lambda \sum_{j=0}^{\infty} p_t(i,j)s^j = \lambda G_{X_t}(s)$$

The third term of the right hand side gives,

$$\sum_{j=1}^{\infty} (j+1)\mu p_t(i,j+1)s^j = \mu \sum_{j=1}^{\infty} \frac{\partial}{\partial s} p_t(i,j+1)s^{j+1} = \mu \frac{\partial}{\partial s} \sum_{j=0}^{\infty} p_t(i,j)s^j = \mu \frac{\partial}{\partial s} G_{X_t}(s)$$

Putting these results together we have,

$$\frac{\partial}{\partial t} G_{X_t}(s) = \left[\lambda s - s\mu \frac{\partial}{\partial s} - \lambda + \mu \frac{\partial}{\partial s} \right] G_{X_t}(s)$$

Since $X_0 = j$ we have initial condition,

$$G_{X_0}(s) = s^j$$

We solve with Mathematica by,

```
DSolve[{
    D[G[s,t],t]==\[Lambda] s G[s,t]-s \[Mu] D[G[s,t],s]-\[Lambda] G[s,t]+\[Mu] D[G
        [s,t],s],
    G[s,0]==s^j
    },G[s,t],{s,t}]//FullSimplify
```

This yields,

$$G_{X_t}(s) = \left((s-1)e^{-\mu t} + 1\right)^j \exp\left[\frac{\lambda(s-1)e^{\mu(-t)}\left(e^{\mu t} - 1\right)}{\mu} \right]$$

We find the limit as $t \to \infty$ with Mathematica by,

```
Limit[E^((E^(-t \[Mu]) (-1+E^(t \[Mu])) (-1+s) \[Lambda])/\[Mu]) (1+E^(-t \[Mu])
    (-1+s))^j,{t->\[Infinity]},Assumptions->{\[Lambda]>0,\[Mu]>0}]
```

This yields,

$$G_{X_\infty}(s) = \lim_{t\to\infty} G_{X_t}(s) = e^{\frac{\lambda}{\mu}(s-1)}$$

So $X_\infty = \lim_{t\to\infty} X_t$ is a Poission random variable with parameter λ/μ.

Exercise 5.4

Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $\mathcal{O}(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

Solution

Based on the definition of the generator and the given transition probabilities we have,

$$G(t) = \begin{bmatrix} -\lambda(t) & \lambda(t) & & & \\ & -\lambda(t) & \lambda(t) & & \\ & & -\lambda(t) & \lambda(t) & \cdots \\ & & \vdots & \vdots & \ddots \end{bmatrix}$$

For $t \geq s$ we define,

$$p_{s,t}(i,j) = \mathbb{P}(N_t = j | N_s = i)$$

We first derive the Kolmogorov forward equations. We consider,

$$\begin{aligned} p_{s,t+\Delta t} &= \mathbb{P}(N_{t+\Delta t} = j | N_s = i) \\ &= \sum_k \mathbb{P}(N_{t+\Delta t} = j | N_t = k)\mathbb{P}(N_t = k | N_s = i) \\ &= \begin{cases} \lambda(t)\Delta t p_{s,t}(i,j-1) + (1-\lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ (1-\lambda(t)\Delta t)p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases} \end{aligned}$$

Therefore,

$$\frac{p_{s,t+\Delta t}(i,j) - p_{s,t}(i,j)}{\Delta t} = \begin{cases} \lambda(t)\Delta t p_{s,t}(i,j-1) - \lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j > i \\ -\lambda(t)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta t^2) & j = i \\ 0 & j < i \end{cases}$$

Taking the limit as $\Delta t \to 0$ we have,

$$\frac{\partial}{\partial t} p_{s,t}(i,j) = \begin{cases} \lambda(t)p_{s,t}(i,j-1) - \lambda(t)p_{s,t}(i,j) & j > i \\ -\lambda(t)p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_F(x)$ is also a function of s, t and j, we have,

$$G_F(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i)x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j)x^j$$

Thus, multiplying the j-th KFE by x^j and summing, we have,

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{j=i}^{\infty} p_{s,t}(i,j)x^j = \sum_{j=i}^{\infty}\frac{\partial}{\partial t}p_{s,t}(i,j)x^j &= \sum_{j=i+1}^{\infty}\lambda(t)p_{s,t}(i,j-1)x^j + \sum_{j=i}^{\infty}(-\lambda(t))p_{s,t}(i,j)x^j \\ &= \lambda(t)x\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j - \lambda(t)\sum_{j=i}^{\infty}p_{s,t}(i,j)x^j\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}G_F(x) = \lambda(t)xG_F(x) - \lambda(t)G_F(x) = \lambda(t)(x-1)G_F(x)$$

We have initial condition $N_s = i$, so $G_B(x) = x^i$ when $s = t$.

We solve with Mathematica as,

```
DSolve[{D[G[s, t], t] == \[Lambda][t] (x - 1) G[s, t],
   G[s, s] == x^i
   }, G[s, t], {s, t}] // FullSimplify
```

This gives,

$$G_F(x) = x^i \exp\left((x-1)\int_s^t \lambda(z)\mathrm{d}z\right)$$

Write $I = \int_s^t \lambda(z)dz$. Then,

$$G_F(x) = e^{-I}x^ie^{Ix} = e^{-I}x^i\sum_{k=0}^{\infty}\frac{1}{k!}(Ix)^k = e^{-I}\sum_{k=0}^{\infty}\frac{1}{k!}I^kx^{k+i} = e^{-I}\sum_{j=i}^{\infty}\frac{I^{j-i}}{(j-i)!}x^j$$

Therefore, from the definition of the Generating function we have,

$$P_{s,t}(i,j) = \mathbb{P}(N_t = j|N_s = i) = \frac{1}{(j-i)!}\left[\int_s^t \lambda(z)\mathrm{d}z\right]^{j-i}\exp\left(-\int_s^t \lambda(z)\mathrm{d}z\right)$$

We now derive the Kolmogorov Backward equations. We consider,

$$\begin{aligned}p_{s-\Delta s,t} &= \mathbb{P}(N_t = j|N_{s-\Delta s} = i) \\ &= \sum_k \mathbb{P}(N_t = j|N_st = k)\mathbb{P}(N_s = k|N_{s-\Delta s} = i) \\ &= \begin{cases}\lambda(s)\Delta s p_{s,t}(i+1,j) + (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ (1-\lambda(s)\Delta s)p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i\end{cases}\end{aligned}$$

Therefore,

$$\frac{p_{s-\Delta s,t}(i,j) - p_{s,t}(i,j)}{\Delta s} = \begin{cases}\lambda(s)\Delta t p_{s,t}(i+1,j) - \lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j > i \\ -\lambda(s)\Delta t p_{s,t}(i,j) + \mathcal{O}(\Delta s^2) & j = i \\ 0 & j < i\end{cases}$$

Taking the limit as $\Delta s \to 0$ we have,

$$-\frac{\partial}{\partial s} p_{s,t}(i,j) = \begin{cases} \lambda(s) p_{s,t}(i+1,j) - \lambda(s) p_{s,t}(i,j) & j > i \\ -\lambda(s) p_{s,t}(i,j) & j = i \\ 0 & j < i \end{cases}$$

Fix i. Noting that $G_B(x)$ is also a function of s, t and j, we have,

$$G_B(x) = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j | N_s = i) x^j = \sum_{j=i}^{\infty} p_{s,t}(i,j) x^j$$

Thus, multiplying the j-th KBE by x^j and summing, we have,

$$\begin{aligned} -\frac{\partial}{\partial s} \sum_{j=i}^{\infty} p_{s,t}(i,j) x^j = -\sum_{j=i}^{\infty} \frac{\partial}{\partial s} p_{s,t}(i,j) x^j &= \sum_{j=i+1}^{\infty} \lambda(s) p_{s,t}(i+1,j) x^j + \sum_{j=i}^{\infty} (-\lambda(s)) p_{s,t}(i,j) x^j \\ &= \sum_{j=i+1}^{\infty} \lambda(s) p_{s,t}(i,j-1) x^j + \sum_{j=i}^{\infty} (-\lambda(s)) p_{s,t}(i,j) x^j \\ &= \lambda(s) x \sum_{j=i}^{\infty} p_{s,t}(i,j) x^j - \lambda(s) \sum_{j=i}^{\infty} p_{s,t}(i,j) x^j \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s} G_B(x) = -\lambda(s) x G_B(x) + \lambda(s) G_B(x) = -\lambda(s)(x-1) G_B(x)$$

From the result for $G_F(x)$ we know,

$$G_B(x) = x^i \exp\left(-(x-1) \int_t^s \lambda(z) \mathrm{d}z\right) = x^i \exp\left((x-1) \int_s^t \lambda(z) \mathrm{d}z\right) = G_F(x)$$

We now show that for $\lambda(t) = c/(1+t)$, that $\mathbb{E}\tau_1 < \infty$ if and only if $c < 1$. Indeed,

$$\int_0^t \lambda(z) \mathrm{d}z = \int_0^t \frac{c}{1+z} \mathrm{d}z = c \ln(1+t) - c \ln(1) = c \ln(1+t)$$

Therefore,

$$\mathbb{E}[\tau_1] = \int_0^{\infty} \mathbb{P}(\tau_1 > t) \mathrm{d}t = \int_0^{\infty} \mathbb{P}(N_t = 0 | N_0 = 0) \mathrm{d}t = \int_0^{\infty} \exp(-c \ln(1+t)) \mathrm{d}t = \int_0^{\infty} \frac{\mathrm{d}t}{(1+t)^c}$$

This is convergent if and only if $c > 1$.

Exercise 5.5

Let N_t be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = \mathbb{E}s^{N_t}$. What is the mean and variance of N_t?

Solution

Recall the generating function for a Poisson process with intensity λ is,

$$G(s) = e^{-\lambda t(1-s)}$$

Therefore,

$$G_{N_t}(s) = \mathbb{E}\left[s^{N_t}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{N_t}\right]\middle|\Lambda\right] = \mathbb{E}\left[e^{-\Lambda t(1-s)}\middle|\Lambda\right] = pe^{-\lambda_1 t(1-s)} + (1-p)e^{-\lambda_2(1-s)}$$

We use Mathematica to caluculate moments,

```
GNt[s_]:=p Exp[-\[Lambda]1 t  (1-s)]+(1-p)Exp[-\[Lambda]2 t(1-s)]
D[GNt[s],{s,1}]/.{s->1}
D[GNt[s],{s,2}]-D[GNt[s],{s,1}]^2+D[GNt[s],{s,1}]/.{s->1}
```

This yields,

$$\begin{aligned}
\mu &= G'_{N_t}(1) = p\lambda_1 t + (1-p)\lambda_2 t \\
\sigma^2 &= G''_{N_t}(1) - [G'_{N_t}(1)]^2 + G'_{N_t}(1) = p(\lambda_1 t)^2 + (1-p)(\lambda_2 t)^2 - \mu^2 + \mu
\end{aligned}$$