AMATH 515 Problem Set 4

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Problem 1

Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Solution

Recall that, $\|z\|^2 = \langle z, z\rangle$. Then,

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 &= \langle \alpha x + (1-\alpha)y, \alpha x + (1-\alpha)y\rangle \\ &= \alpha^2 \langle x, x\rangle + 2\alpha(1-\alpha)\langle x, y\rangle + (1-\alpha)^2 \langle y, y\rangle\end{aligned}$$

Similarly,

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x\rangle - 2\langle x, y\rangle + \left\langle y^2\right\rangle$$

Then clearly,

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \qquad \square$$

Problem 2

An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|$ for all (x,y). For any such nonexpansive operator T, define

$$T_\lambda = (1-\lambda)I + \lambda T.$$

(a) Show that T_λ and T have the same fixed points.
(b) Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z-\overline{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2.$$

where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

Solution

Recall that x is a fixed point of f if $f(x) = x$.

(a) If $\lambda = 0$ then all points are fixed points of T_λ but not all points may be fixed points of T. Assuming $\lambda \neq 0$ then,

$$\begin{aligned} & x = T_\lambda x \\ \iff & x = ((1-\lambda)I + \lambda T)x \\ \iff & x = (1-\lambda)x + \lambda Tx \\ \iff & \lambda x = \lambda Tx \\ \iff & x = Tx \end{aligned} \qquad \square$$

(b) Suppose $\overline{z}$ is a fixed point of T so that $\overline{z} = (1-\lambda)\overline{z} + \lambda T\overline{z}$. Then,

$$\|T_\lambda z - \overline{z}\|^2 = \|(1-\lambda)(z-\overline{z}) + \lambda T(z-\overline{z})\|^2$$

By problem 1, and since T is nonexpansive so that $\|T(z-\overline{z})\| \leq \|z-\overline{z}\|$,

$$\begin{aligned} \|T_\lambda z - \overline{z}\|^2 &= \lambda\|z-\overline{z}\|^2 + (1-\lambda)\|T(z-\overline{z})\|^2 - \lambda(1-\lambda)\|(z-\overline{z}) - T(z-\overline{z})\|^2 \\ &\leq \lambda\|z-\overline{z}\|^2 + (1-\lambda)\|z-\overline{z}\|^2 - \lambda(1-\lambda)\|(z-\overline{z}) - T(z-\overline{z})\|^2 \\ &= \|z-\overline{z}\|^2 - \lambda(1-\lambda)\|(z-\overline{z}) - T(z-\overline{z})\|^2 \end{aligned} \qquad \square$$

Problem 3

An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2.$$

(a) Show T is firmly nonexpansive if and only if

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2.$$

(b) Show T is firmly nonexpansive if and only if

$$\langle Tx-Ty, (I-T)x-(I-T)y\rangle \geq 0.$$

(c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 - \|x-y\|^2$$

and let

$$\nu = \|Sx-Sy\|^2 - \|x-y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Solution

(a) Observe that,

$$\begin{aligned}\|(I-T)x-(I-T)y\|^2 &= \|(x-y)-(Tx-Ty)\|^2 \\ &= \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle\end{aligned}$$

Thus,

$$\begin{aligned} & \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 \leq \|x-y\|^2 \\ \iff & \|Tx-Ty\|^2 + \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2 \\ \iff & \|Tx-Ty\|^2 \leq \langle x-y, Tx-Ty\rangle \end{aligned}$$ □

(b) Observe that,

$$\|Tx-Ty\|^2 = \langle Tx-Ty, Tx-Ty\rangle$$

Thus,

$$\begin{aligned} & \langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2 \\ \iff & \langle x-y, Tx-Ty\rangle - \langle Tx-Ty, Tx-Ty\rangle \geq 0 \\ \iff & \langle (x-y)-(Tx-Ty), Tx-Ty\rangle \geq 0 \\ \iff & \langle Tx-Ty, (I-T)x-(I-T)y\rangle \geq 0 \end{aligned}$$ □

(c) Define,

$$u = (I-T)x - (I-T)y, \qquad v = Tx - Ty$$

Then, by problem 1 with $\alpha = 1/2$ we have,

$$\|u/2 + v/2\|^2 + \frac{1}{4}\|u-v\|^2 = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2$$

Equivalently,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

Substituting our expressions for u and v we have,

$$\|x-y\|^2 + \|(I-2T)x - (I-2T)y\|^2 = 2\|(I-T)x - (I-T)y\|^2 + 2\|Tx - Ty\|^2$$

Rearranging, and using the definition $S = 2T - I$ we have,

$$\|Sx - Sy\|^2 - \|x-y\|^2 = 2\|(I-T)x - (I-T)T\|^2 + 2\|Tx - Ty\|^2 - 2\|x-y\|^2$$

This is exactly the statement,

$$2\mu = \nu \qquad \square$$

Problem 4

Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax-b\|^2 \quad \text{s.t.} \quad Cx \leq d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

Solution

We would like to solve $F=0$ where,

$$F = \left[\begin{array}{c} A^T(Ax-b)+C^Tv \\ V(d-Cx)-\mu\cdot 1 \end{array}\right]$$

We compute Jacobian,

$$J_F = \left[\begin{array}{cc} A^TA & C^T \\ -VC & \operatorname{diag}(d-Cx) \end{array}\right]$$

Problem 5

Implement a Chambolle-Pock method to solve

$$\min_x \|Ax - b\|_1 + \|x\|_1.$$

Solution

Suppose we would like to solve,

$$\min_x \hat{h}(Ax) + \hat{k}(x)$$

The Chambolle-Pock algorithm has iterates,

$$\begin{aligned} x^+ &= \operatorname{prox}_{\alpha\hat{k}}(x + \alpha A^T v) \\ v^+ &= \operatorname{prox}_{\alpha\hat{h}^*}(-v - \alpha A(x - 2x^+)) \end{aligned}$$

Now suppose that our functions $\hat{k}$ and $\hat{h}$ have the form,

$$\hat{h}(x) = h(b - x), \qquad \hat{k}(x) = \langle c, x\rangle + k(x)$$

Now observe that,

$$\begin{aligned} \hat{h}^*(x) &= \sup_z \langle x, z\rangle - h(b - z) \\ &= \sup_w \langle x, b - w\rangle - h(w) \\ &= \sup_w \langle -x, w\rangle - h(w) + \langle x, b\rangle \\ &= h^*(-x) + \langle x, b\rangle \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{prox}_{\alpha\hat{h}^*}(y) &= \arg\min_x \frac{1}{2\alpha}\|x - y\|^2 + \hat{h}^*(x) \\ &= \arg\min_x \frac{1}{2\alpha}\|x - y\|^2 + h^*(-x) + \langle x, b\rangle \\ &= \arg\min_x \frac{1}{2\alpha}\|x + (y - \alpha b)\|^2 + h^*(-x) \\ &= \operatorname{prox}_{\alpha h^*}(-(y - \alpha b)) \\ &= \operatorname{prox}_{\alpha h^*}(\alpha b - y) \end{aligned}$$

By completing the square,

$$\begin{aligned} \operatorname{prox}_{\alpha\hat{k}}(y) &= \arg\min_x \frac{1}{2\alpha}\|x - y\|^2 + \hat{k}(x) \\ &= \arg\min_x \frac{1}{2\alpha}\|x - y\|^2 + k(x) + \langle c, x\rangle \\ &= \arg\min_x \frac{1}{2\alpha}\|x - (y - \alpha c)\|^2 + k(x) \\ &= \operatorname{prox}_{\alpha k}(y - \alpha c) \end{aligned}$$

Therefore, in terms of h and k we have iterates,

$$
\begin{aligned}
x^+ &= \operatorname{prox}_{\alpha k}(x + \alpha A^T v - \alpha c) = \operatorname{prox}_{\alpha k}(x + \alpha(A^T v - c)) \\
v^+ &= \operatorname{prox}_{\alpha h^*}(\alpha b + v + \alpha A(x - 2x^+)) = \operatorname{prox}_{\alpha h^*}(v + \alpha(Ax - b + 2b - Ax^+))
\end{aligned}
$$

We now turn to the original problem which we write this as,

$$
\min_x \langle c, x\rangle + h(b - Ax) + k(x)
$$

where,

$$
h(x) = \|x\|_1, \qquad k(x) = \|x\|_1, \qquad c = 0
$$

Therefore,

$$
h^*(z) = \delta_{\mathbb{B}_\infty}(x), \qquad \operatorname{prox}_{\alpha h^*}(z) = \max(\min(z, 1), -1)
$$

and

$$
\operatorname{prox}_{\alpha k}(z) = \begin{cases} z_i + t, & z_i \in (-\infty, t) \\ 0, & z_i \in [-t, t] \\ z_i - t, & z_i \in (t, \infty) \end{cases}
$$