# **AMATH 561** Assignment 3

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**Exercise 3.1**

Let $X \sim \mathrm{Bin}(n, U)$ where $U \sim \mathcal{U}((0,1))$. What is the probability Generating function $G_X(s)$ of $X$? What is $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, ..., n\}$?

**Solution**

Using iterated conditioning, since a Binomial random variable is the sum of $n$ iid Bernioulli random variables,

$$G_X(s) = \mathbb{E}[s^X] = \mathbb{E}\mathbb{E}[s^X|U] = \mathbb{E}[((1-U)s^0 + Us^1)^n]$$

We calculate this by integrating with Mathematica as,

```
Integrate[((1 - x) + x s)^n, {x, 0, 1}, Assumptions -> {s > 0}]
```

This yields,

$$\mathbb{E}[((1-U) + Us)^n] = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}((1-x) + xs)^n \mathrm{d}x = \int_0^1 ((1-x) + xs)^n \mathrm{d}x = \frac{1 - s^{n+1}}{(n+1)(1-s)}$$

This is a finite geometric progression which we simplify so,

$$G_X(s) = \sum_{k=0}^{n} \frac{s^k}{n+1}$$

Therefore $\mathbb{P}(X = k) = 1/(1+n)$ for $k = 0, 1, 2, ..., n$.

### Exercise 3.2

Let $Z_n$ be the size of the $n$-th generation in an ordinary branching process with $Z_0 = 1, \mathbb{E}Z_1 = \mu$ and $\mathbb{V}Z_1 > 0$. Show that $\mathbb{E}Z_nZ_m = \mu^{n-m}\mathbb{E}Z_m^2$ for $m \leq n$ . Use this to find the correlation coefficient $\rho(Z_m, Z_n)$ in terms of $\mu, n$ and $m$. Consider the case $\mu = 1$ and the case $\mu \neq 1$.

### Solution

Let $Y_{m,i}$ denote the number of offspring in the $n$-th generation that descends from the $i$-th member of the $m$-th generation. Then the $(Y_{m,i})$ are iid with distribution $Z_{n-m}$ and $Z_n = Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}$.

Then, since $(Y_{m,i})$ are iid with distribution $Z_{n-m}$,

$$\mathbb{E}\left[Z_n|Z_m\right] = \mathbb{E}\left[Y_{m,1} + Y_{m,2} + ... + Y_{m,Z_m}|Z_m\right] = Z_m\mathbb{E}[Z_{m-n}] = Z_m\mu^{n-m}$$

Therefore, by taking out what is known,

$$\mathbb{E}\left[Z_mZ_n\right] = \mathbb{E}\left[\mathbb{E}\left[Z_mZ_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mathbb{E}\left[Z_n|Z_m\right]\right] = \mathbb{E}\left[Z_m^2\mu^{n-m}\right] = \mu^{n-m}\mathbb{E}\left[Z_m^2\right]$$

Observing that $\mathbb{E}[Z_mZ_n] = \mu^{n-m}\mathbb{E}[Z_m^2] = \mu^{n-m}(\mathbb{V}[Z_m] + \mathbb{E}[Z_m]^2) = \mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m})$, write,

$$\rho(Z_m, Z_n) = \frac{\mathrm{Cov}(Z_n, Z_m)}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mathbb{E}[Z_nZ_m] - \mathbb{E}[Z_n]\mathbb{E}[Z_m]}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}} = \frac{\mu^{n-m}(\mathbb{V}[Z_m] + \mu^{2m}) - \mu^{n+m}}{(\mathbb{V}[Z_n]\mathbb{V}[Z_m])^{1/2}}$$

Denote $\mathbb{V}[Z_1]$ by $\sigma$.

Suppose $\mu = 1$ so that $\mathbb{V}[Z_m] = m\sigma^2$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> m \[Sigma]^2, Vzn -> n \[Sigma]^2, \[Mu] -> 1}],
 Assumptions -> {{m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{m}{n}}$$

Now suppose $\mu \neq 1$ so that $\mathbb{V}[Z_m] = \sigma^2(\mu^n - 1)\mu^{n-1}/(\mu - 1)$. We use Mathematica to simplify the above expression as,

```
FullSimplify[
 PowerExpand[(\[Mu]^(n - m) (Vzm + \[Mu]^(2 m)) - \[Mu]^(
    n + m))/(Vzn Vzm)^(
   1/2) /. {Vzm -> \[Sigma]^2 (\[Mu]^m - 1) \[Mu]^(m - 1)/(\[Mu] - 1),
     Vzn -> \[Sigma]^2 (\[Mu]^n - 1) \[Mu]^(n - 1)/(\[Mu] - 1) }],
 Assumptions -> {\[Mu] != 1, {m, n, \[Sigma], \[Mu]} > 0}]
```

This yields,

$$\rho(Z_m, Z_n) = \sqrt{\frac{\mu^n(\mu^m - 1)}{\mu^m(\mu^n - 1)}}$$

Observe that in the limit $\mu \to 1$ this coincides with the previous value.

**Exercise 3.3**

**Solution**

**Exercise 3.4**

Consider a branching process with immigration

$$Z_0 = 1 \qquad\qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n$$

where the $(X_{n,i})$ are iid with common distribution $X$, the $(Y_n)$ are iid with common distribution $Y$, and the $(X_{n,i})$ and $(Y_n)$ are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_X(s)$, and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_X(s)$ and $G_Y(s)$.

**Solution**

Define:

$$G_{Z_n}(s) = s^{Z_n} \qquad\qquad G_X(s) = \mathbb{E}s^X \qquad\qquad G_Y(s) = \mathbb{E}s^Y$$

Write $S_n = \sum_{i=1}^{Z_n} X_{n,i}$ so that, $Z_{n+1} = S_n + Y_n$.

First observe that since the $(X_{n,i})$ are iid with common distribution $X$,

$$G_{S_n}(s) = \mathbb{E}\left[s^{S_n}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{S_n}|Zn\right]\right] = \mathbb{E}\left[\mathbb{E}[s^X]^{Z_n}\right] = \mathbb{E}\left[G_X(s)^{Z_n}\right] = G_{Z_n}(G_X(s))$$

Since the $(X_{n,i})$ and $(Y_n)$ are independent, $S_n$ and $Y_n$ are independent. Therefore,

$$G_{Z_{n+1}}(s) = G_{S_n+Y_n}(s) = G_{S_n}(s)G_Y(s) = G_{Z_n}(G_X(s))G_Y(s)$$

We calculate,

$$G_{Z_0}(s) = \mathbb{E}\left[s^{Z_0}\right] = \mathbb{E}[s] = s$$

Similarly,

$$G_{Z_1}(s) = G_{Z_0}(G_X(s))G_Y(s) = G_X(s)G_Y(s)$$

Therefore,

$$G_{Z_2}(s) = G_{Z_1}(G_X(s))G_Y(s) = G_X(G_X(s))G_Y(G_X(s))G_Y(s)$$

**Exercise 3.5**

Find $\phi_{X^2}(t) := \mathbb{E}\exp(itX^2)$ where $X \sim \mathcal{N}(\mu, \sigma)$.

**Solution**

We have,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Thus,

$$\phi_{X^2}(t) = \mathbb{E}\exp(itX^2) = \int_{-\infty}^{\infty} e^{itx^2} f_X(x)\mathrm{d}x$$

We evaluate with Mathematica as,

```
Integrate[Exp[I t x^2] PDF[NormalDistribution[\[Mu], \[Sigma]], x], {x, -\[
    Infinity], \[Infinity]},
 Assumptions -> {\[Mu] \[Element] Reals, t \[Element] Reals, \[Sigma] > 0}]
```

This yields,

$$\phi_{X^2}(t) = \frac{\exp(it\mu^2/(1-2it\sigma^2))}{\sqrt{1-2it\sigma^2}}$$

**Exercise 3.6**

Let $X_n$ have cumulative distribution function

$$F_{X_n}(x) = \left(x - \frac{\sin(2n\pi x)}{2n\pi}\right)\mathbb{1}_{0\le x\le 1} + \mathbb{1}_{x>1}$$

(a) Show that $F_{X_n}$ is a distribution function and find the corresponding density function $f_{X_n}$.

(b) Show that $F_{X_n}$ converges to the uniform distribution function $F_U$ as $n \to \infty$, but that the density function $f_{X_n}$ does NOT converge to $f_U$. Here, $U \sim \mathcal{U}((0,1))$.

**Solution**

(a) Clearly $F_{X_n}(x) = 0$ for $x \le 0$ and $F_{X_n}(x) = 1$ for $x \ge 1$. Observe, $x - \sin(2n\pi x)/2n\pi$ is non-decreasing and continuous on $(0,1)$, since the derivative, calculated below is non-negative on this interval. Moreover, $x - \sin(2n\pi x)/2n\pi$ is equal to zero at $x = 0$, and equal to one at $x = 1$.

Therefore $F_{X_n}(x)$ is a non-decreasing continuous function with $F_{X_n}(x) \to 0$ as $x \to -\infty$ and $F_{X_n}(x) \to 1$ as $x \to \infty$. So $F_{X_n}(x)$ is a distribution function.

It is straightforward to compute the density function as,

$$f_{X_n}(x) = \frac{d}{dx}F_{X_n}(x) = (1 - \cos(2n\pi x))\mathbb{1}_{0\le x\le 1}$$

(b) The uniform distribution on $(0,1)$ is given by,

$$F_U(x) = x\mathbb{1}_{0\le x\le 1} + \mathbb{1}_{x>1}$$

Obviously outside of $(0,1)$ both $F_U$ and $F_{X_n}$ agree exactly. Consider a point $x \in (0,1)$. Then, since $|\sin(u)| \le 1$ for all $u$,

$$\lim_{n\to\infty}\left[x - \frac{\sin(2n\pi x)}{2n\pi}\right] = x - 0 = x$$

Therefore $F_X$ converges pointwise on to $F_U$ on $(0,1)$, and therefore on all of $\mathbb{R}$.

It is clear that $f_{X_n}(x)$ does not converge to $f_U(x)$ as $f_U(x)$ is constant on $(0,1)$ while $f_{X_n}(x)$ oscillates between zero and two. In particular, fix a rational number $x = p/q$. Then for $n = qk, k \in \mathbb{N}$, $f_{X_n}(x) = 0$.

### Exercise 3.7

A coin is tossed repeatedly, with heads turning up with probability $p$ on each toss. Let $N$ be the minimum number of tosses required to obtain $k$ heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then,

$$f_X(x) = \frac{1}{\Gamma(r)}\lambda^r x^{r-1} e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

### Solution

We have $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\mathrm{d}x$. Thus, making the substitution $u = (\lambda - it)x$,

$$\begin{aligned}
\phi_X(t) &= \mathbb{E}\left[e^{itx} f_X(x)dx\right] \\
&= \int_0^\infty e^{itx}\frac{1}{\Gamma(r)}\lambda^r x^{r-1}e^{-\lambda x}\mathrm{d}x \\
&= \int_0^\infty \frac{\lambda^r}{\Gamma(r)} e^{-u}\frac{u^{r-1}}{(\lambda - it)^{r-1}}\frac{\mathrm{d}u}{(\lambda - it)} \\
&= \frac{\lambda^r}{\Gamma(r)(\lambda - it)^r}\int_0^\infty e^{-u}u^{r-1}\mathrm{d}u \\
&= \frac{\lambda^r}{(\lambda - it)^r}
\end{aligned}$$

Let $(X_i)_{i=1}^k$ be idd with $X, X_i \sim \text{Geo}(p)$. Then $N = \sum_{i=1}^k X_i$ so, since the $X_i$ are iid,

$$\varphi_{2Np}(t) = \mathbb{E}[\exp(it2Np)] = \mathbb{E}[\exp(2itp(X_1 + ... + X_k))] = \mathbb{E}[\exp(2itpX)]^k$$

Therefore, since $|e^{2itp}(1-p)| < 1$ if $p \in (0,1)$,

$$\mathbb{E}[\exp(2itpX)]^k = \left[\sum_{m=1}^\infty e^{2itpm}p(1-p)^{m-1}\right]^k = \left[pe^{2itp}\sum_{m=1}^\infty \left(e^{2itp}(1-p)\right)^{m-1}\right]^k = \left[\frac{pe^{2itp}}{1-(1-p)e^{2itp}}\right]^k$$

With Mathematica we evaluate,

```
Limit[((p Exp[2 I t p])/(1 - (1 - p) Exp[2 I t p]))^k, {p -> 0},
sumptions -> {k \[Element] Integers, k > 0}] // FullSimplify
```

This yields,

$$\lim_{p\to 0}\varphi_{2Np} = \frac{1}{(1-2it)^k} = \frac{(1/2)^k}{(1/2 - it)^k}$$

Thus, for a random variable $X \sim \Gamma(1/2, k)$, by the continuity theorem, $\lim_{p\to 0} f_{2Np}(x) = f_X(x)$