AMATH 584 Assignment 1

Tyler Chen

Exercise 1.1

Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices .

(b) Write it again as a product ABC (same B) of three matrices.

Solution

(a) We have, $O_5O_3O_2BO_1O_4O_6$ where,

$$O_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$O_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad O_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad O_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We now simply the expression from (a) as, ABC where,

$$A = O_5O_3O_2 = \begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad C = O_1O_4O_6O_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We first manually manipulate the inputed matrix. We then define the matrices listed above. Finally, all three methods are compared.

```
import scipy as sp

def exercise_1_1(B):

    M=sp.copy(B)
    M[:,0]=2*M[:,0] # double column 1
    M[2]=1/2*M[2] # halve row 3
    M[0]=M[2]+M[0] # add row 3 to row 1
    M[:,[0,3]]=M[:,[3,0]] # interchange columns 1 and 4
    M[[0,2,3]]=M[[0,2,3]]-M[1] # subtract row 2 from each of the other rows
    M[:,3]=M[:,2] # replace column 4 by column 3
    M[:,0]=0 # delete column 1

    O1=sp.matrix([[2,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
```

```
    O2=sp.matrix([[1,0,0,0],[0,1,0,0],[0,0,1/2,0],[0,0,0,1]])
    O3=sp.matrix([[1,0,1,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
    O4=sp.matrix([[0,0,0,1],[0,1,0,0],[0,0,1,0],[1,0,0,0]])
    O5=sp.matrix([[1,-1,0,0],[0,1,0,0],[0,-1,1,0],[0,-1,0,1]])
    O6=sp.matrix([[1,0,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,0]])
    O7=sp.matrix([[0,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

    A=sp.matrix([[1,-1,1/2,0],[0,1,0,0],[0,-1,1/2,0],[0,-1,0,1]])
    C=sp.matrix([[0,0,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,0]])

    print(M)
    print(O5*O3*O2*B*O1*O4*O6*O7)
    print(A*B*C)

    print(sp.array_equal(M,O5*O3*O2*B*O1*O4*O6*O7) and sp.array_equal(M,A*B*C))

exercise_1_1(sp.matrix(sp.random.rand(4,4)))
```

Running the function for a few different values of B always returns `True` indicating that the three methods are equivalent (at least for the tested matrices). A sample output is displayed below.

```
>> exercise_1_1(sp.matrix(sp.random.rand(4,4)))
>> [[ 0.         -0.07326807  0.33590766  0.33590766]
[ 0.          0.91030668  0.63417526  0.63417526]
[ 0.         -0.46052944 -0.4908797  -0.4908797 ]
[ 0.         -0.28526664 -0.29515107 -0.29515107]]
[[ 0.         -0.07326807  0.33590766  0.33590766]
[ 0.          0.91030668  0.63417526  0.63417526]
[ 0.         -0.46052944 -0.4908797  -0.4908797 ]
[ 0.         -0.28526664 -0.29515107 -0.29515107]]
[[ 0.         -0.07326807  0.33590766  0.33590766]
[ 0.          0.91030668  0.63417526  0.63417526]
[ 0.         -0.46052944 -0.4908797  -0.4908797 ]
[ 0.         -0.28526664 -0.29515107 -0.29515107]]
True
```

Exercise 2.1

Show that if a matrix A is both triangular and unitary, then it is diagonal.

Solution

Suppose a matrix $A \in \mathbb{C}^{m\times m}$, $m \geq 2$, is both triangular and unitary. We have $A^*A = I = AA^*$, so one of A or A^* is upper triangular. Thus, without loss of generality assume A is upper triangular.

Since A is upper triangular we have $A_{ij} = 0$ for $i > j$.

Consider the product $AA^* = I$. We then have,

$$1 = I_{mm} = \sum_{i=1}^{m} A_{mi}A^*_{im} = A_{mm}A^*_{mm} + \sum_{i=1}^{m-1} A_{mi}A^*_{im} = A_{mm}A^*_{mm}$$

Note that this condition implies $A_{mm} \neq 0$.

Now observe for any index $1 \leq j \leq m-1$,

$$0 = I_{jm} = \sum_{i=1}^{m} A_{mi}A^*_{ij} = A_{mm}A^*_{mj} + \sum_{i=1}^{m-1} A_{mi}A^*_{ij} = A_{mm}A^*_{mj}$$

Since $A_{mm} \neq 0$ we have $A^*_{mj} = 0$. Therefore $A_{jm} = \overline{A^*_{mj}} = 0$.

This proves that the last column of A is zero, except the diagonal entry.

Consider the k-th order leading principal sub matrix A_k formed by deleting the last $m-k$ rows. That is the sub matrix with entries A_{ij} for $1 \leq i, j \leq k$. This is displayed below as the top left corner of A

$$A = \left[\begin{array}{ccc:c} A_{11} & \cdots & A_{1k} & \\ \vdots & & \vdots & \\ A_{k1} & \cdots & A_{kk} & \\ \hdashline & & & \\ & & & \end{array}\right] \qquad A_k = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix}$$

Clearly A_k inherits (upper) triangular from A as $A_{ij} = 0$ for $i > j$. Moreover, considering block matrix multiplication we see $A_kA_k^* = I_k$, where I_k is the identity matrix in $\mathbb{C}^{k\times k}$. That is, A_k is also unitary (in $\mathbb{C}^{k\times k}$).

Therefore, by the above result, $A_{jk} = 0$ for any index $1 \leq j \leq k-1$. But k can be any index $1 \leq k \leq m$ so we see that $A_{jk} = 0$ for all $j < k$. That is, A is lower triangular. By hypothesis A us upper triangular as well. This proves A is diagonal. $\square$

Exercise 2.2

The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

(a) Prove this in the case $n = 2$ by explicit computation of $\|x_1 + x_2\|^2$.

(a) Show that this computation also establishes the general case, by induction

Solution

We make the assumption that $x_i \in \mathbb{C}^m$ for $m \in \mathbb{Z}$. Suppose the x_i are orthogonal. That is, $x_i^* x_j = 0$ for $i \neq j$. We denote the k-th component of x_i by x_{ik}.

(a) By orthogonality we have, $x_1^* x_2 = x_2^* x_1 = 0$. Thus,

$$\begin{aligned}\|x_1 + x_2\|^2 = (x_1 + x_2)^*(x_1 + x_2) &= (x_1^* + x_2^*)(x_1 + x_2) \\ &= x_1^* x_1 + x_1^* x_2 + x_2^* x_1 + x_2^* x_2 \\ &= \|x_1\|^2 + 0 + 0 + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2\end{aligned}$$ $\square$

(b) Suppose $\left\|\sum_{i=1}^{n-1} x_i\right\|^2 = \sum_{i=1}^{n-1} \|x_i\|^2$ for some n. Then, using the above result,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \left\|x_n + \sum_{i=1}^{n-1} x_i\right\|^2 = \|x_n\|^2 + \left\|\sum_{i=1}^{n-1} x_i\right\|^2 = \|x_n\|^2 + \sum_{i=1}^{n-1} \|x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Thus, using the result from (a) as the base step for induction, for all integer $n \geq 1$, we have,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$ $\square$

Exercise 2.3

Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

(a) Prove that all the eigenvalues of A are real.

(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Solution

Let $A \in \mathbb{C}^{m\times m}$ be hermitian. That is, $A = A^*$.

(a) Suppose x is an eigenvector of A with corresponding eigenvalue λ. Then $Ax = \lambda x$. Recalling that for scalar c, vectors u, v and matrices A, B that $u^* cv = cu^* v$, that $(cA)^* = \overline{c}A^*$, and that $(AB)^* = B^* A^*$ we have the following chain of equalities,

$$\lambda \|x\|^2 = \lambda x^* x = x^* \lambda x = x^* Ax = x^* A^* x = (x^* Ax)^* = (x^* \lambda x)^* = x^* \overline{\lambda} x = \overline{\lambda} x^* x = \overline{\lambda} \|x\|^2$$

Since x is an eigenvector, x is nonzero. Thus, $\|x\| > 0$. In particular, this means that $\|x\|^2 \neq 0$. Thus $\lambda = \overline{\lambda}$ proving λ is real. $\square$

(b) Suppose y is an eigenvector of A with corresponding eigenvalue $\gamma \neq \lambda$. Recall from (a) that $\lambda = \overline{\lambda}$. This gives the following chain of equalities,

$$\gamma x^* y = x^* \gamma y = x^* Ay = x^* A^* y = (y^* Ax)^* = (y^* \lambda x)^* = x^* \overline{\lambda} y = x^* \lambda y = \lambda x^* y$$

Therefore, $\gamma x^* y = \lambda x^* y$ so,

$$0 = \lambda(x^* y) - \gamma(x^* y) = (\lambda - \gamma)(x^* y)$$

However, since $\lambda \neq \gamma$, then $(\lambda - \gamma) \neq 0$. This proves $x^* y = 0$. That is, that x and y are orthogonal. $\square$

Exercise 3.2

Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

Solution

Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Denote the largest absolute value eigenvalue of A by λ and let x be the corresponding eigenvector. Then, by definition of supremum,

$$\rho(A) = |\lambda| = \frac{|\lambda|\,\|x\|}{\|x\|} = \frac{\|\lambda x\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} \leq \sup_{z\neq 0} \frac{\|Az\|}{\|z\|} = \|A\| \qquad \square$$

Exercise 3.3

Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give and example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$,

(b) $\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$,

(c) $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$,

(d) $\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$,

Solution

Let $x \in \mathbb{C}^m$. Clearly $|x_i| \leq \max_{1\leq i\leq m} |x_i| = \|x\|_\infty$ for all $1 \leq i \leq m$.

(a) Let j be an index such that $|x_j| = \|x\|_\infty$. Then,

$$\|x\|_\infty = |x_j| = \left(|x_j|^2\right)^{1/2} \leq \left(|x_j|^2 + \sum_{i\neq j} |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \|x\|_2$$

Equality is obtained when x has exactly one nonzero component x_i, in which case $\|x\|_\infty = x_i = (|x_i|^2)^{1/2} = \|x\|_2$.

(b) Similarly,

$$\begin{aligned}\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} &\leq \left(\sum_{i=1}^{m} \left(\max_{1\leq i\leq m} |x_i|\right)^2\right)^{1/2} \\ &= \left(m\left(\max_{1\leq i\leq m} |x_i|\right)^2\right)^{1/2} = \sqrt{m}\max_{1\leq i\leq m} |x_i| = \sqrt{m}\,\|x\|_\infty\end{aligned}$$

Equality is obtained when all components of x are equal, in which case $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \left(m|x_i|^2\right)^{1/2} = \sqrt{m}\,\|x\|_\infty$.

We now have $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$ for $x \in \mathbb{C}^m$. Let $A \in \mathbb{C}^{m\times n}$. Note that for any vector $u \in \mathbb{C}^n$, $Au \in \mathbb{C}^m$.

(c) Denote the i-th row of A by a_i^* and define $x_0 \in \mathbb{C}^n$ to be the vector with all entries equal to 1. Then observe $\|a_i^*\|_1 = a_i$

$$\|A\|_\infty = \sup_{u\neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} \leq \sup_{u\neq 0} \frac{\|Au\|_2}{\|u\|_\infty} \leq \sup_{u\neq 0} \frac{\|Au\|_2}{\|u\|_2/\sqrt{n}} = \sqrt{n}\sup_{u\neq 0} \frac{\|Au\|_2}{\|u\|_2} = \sqrt{n}\,\|A\|_2$$

Denote the vector with zeros in all components except for a 1 in the j-th component by e_j. Denote the vector with all ones by 1.

Now suppose e_j has length m and 1 has length n. Let $A = ae_j1^*$ for some scalar a. Then A is dimension $m \times n$ and looks like the zero matrix with the i-th row constant and equal to a.

Then clearly $\|A\|_\infty = n|a|$. Moreover, by our matrix norm rules for outer products, $\|A\|_2 = |a|\,\|e_j\|_2\,\|1^*\|_2 = |a|1\sqrt{n} = \sqrt{m}|n| = \|A\|_\infty/\sqrt{n}$ so equality is obtained. ,

(d)

$$\|A\|_2 = \sup_{u\neq 0} \frac{\|Au\|_2}{\|u\|_2} =\leq \sup_{u\neq 0} \frac{\sqrt{m}\,\|Au\|_\infty}{\|u\|_2} \leq \sup_{u\neq 0} \frac{\sqrt{m}\,\|Au\|_\infty}{\|u\|_\infty} = \sqrt{m} \sup_{u\neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} = \sqrt{m}\,\|A\|_\infty$$

Suppose e_j has length n and 1 has length m. Let $A = a1e_j^*$ for some scalar a. Then A is dimension $m \times n$ and looks like the zero matrix with the j-th column constant and equal to a.

Then clearly $\|A\|_\infty = |a|$. Moreover, by our matrix norm rules for outer products, $\|A\|_2 = |a|\,\|1\|_2\,\|e_j^*\|_2 = |a|\sqrt{m}1 = \sqrt{m}|a| = \sqrt{m}\,\|A\|_\infty$, so equality is obtained.