AMATH 561 Assignment 4

Tyler Chen

Exercise 4.1

A six-sided die is rolled repeatedly. Which of the following a Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.
(b) X_n is the number of sixes rolled in the first n rolls.
(c) At time n, X_n is the time since the last six was rolled.
(d) At time n, X_n is the time until the next six is rolled.

Solution

(a) Yes.

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ & & 36 & 1/6 & 1/6 & 1/6 \\ & & & 4/6 & 1/6 & 1/6 \\ & & & & 5/6 & 1/6 \\ & & & & & 1 \end{bmatrix}$$

(b) Yes.

$$P = \begin{bmatrix} 5/6 & 1/6 & & \\ & 5/6 & 1/6 & \\ & & \ddots & \ddots \end{bmatrix}$$

(c) Yes. Suppose $X_n = i$. The next roll is either a 6, in which case $X_{n+1} = 0$. Otherwise $X_{n+1} = i + 1$.

$$P = \begin{bmatrix} 1/6 & 5/6 & & & \\ 1/6 & & 5/6 & & \\ 1/6 & & & 5/6 & \\ \vdots & & & & \ddots \end{bmatrix}$$

(d) Yes. Suppose $X_n = 0$. The probability of $X_{n+1} = j$ is $(1/6)(5/6)^j$ as you must not roll a 6 for j turns, and then must roll a 6 on the j-th. Suppose $X_n = i > 0$. Then the next step you will be on turn closer to rolling a 6. That is, $X_{n+1} = i - 1$.

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6}\left(\frac{5}{6}\right) & \frac{1}{6}\left(\frac{5}{6}^2\right) & \frac{1}{6}\left(\frac{5}{6}\right)^3 & \cdots \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

Exercise 4.2

Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q)

(b) X is a branching process where G is the generating function of the number of offspring from each individual

Solution

(a) In each step we can go down with probability q and then down again with probability q or up with probability p. Alternatively we can go up with probability p and then down with probability q or up again with probability p.

Therefore we will end up two spaces down with probability q^2, in the same position with probability $qp + pq = 2pq$, or up two spaces with probability p^2. Thus,

$$p(i,j) = \begin{cases} p^2 & j = i+2 \\ 2pq & i = j \\ q^2 & j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

(b) We can obtain the exponents of a generating function $G(s) = a_0 + a_1 s + a_2 s^2 + \ldots$ by,

$$a_n = \frac{1}{n!}\frac{d^n}{ds^n}\Big[G(s)\Big]_{s=0}$$

The coefficient of the s^k term is the value of the probability mass function of X evaluated at k.

The generating function of Y is $G(G(s)) = G_2(s)$ from the notes.

For a branching process with current population k, the population of the next generation will be $X_1 + X_2 + \ldots + X_k$, where each X_i is iid with distribution X. Therefore,

$$p(i,j) = \frac{1}{j!}\frac{d^n}{ds^n}\Big[G_2(s)^i\Big]_{s=0}$$

Exercise 4.3

Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k, j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution

Fix a state $j \in S$. By definition of $j \to k$, $\exists N \geq 0 : p_N(j, k) \geq 0$. Since $\{X_N = k | X_0 = j\} \subseteq \{\forall n, X_n \neq j | X_0 = j\}$ we have,

$$0 < p_N(j, k) = \mathbb{P}(X_N = k | X_0 = j) \leq \mathbb{P}(\forall n, X_n \neq j | X_0 = j)$$

Therefore,

$$\mathbb{P}(\exists n \geq 0 : X_n = j | X_0 = j) = 1 - \mathbb{P}(\forall n, X_n \neq j | X_0 = j) < 1$$

This proves state j istransient. $\square$

Exercise 4.4

Suppose two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

where $\tau_j = \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected value of visits to j prior to returning to i is one.

Solution

Write

$$p = \mathbb{P}(\tau_j < \tau_i | X_0 = i) = \mathbb{P}(\tau_i < \tau_j | X_0 = j)$$

That is, p is the probability that we go to state j before state i give we are in state i, and p is also the probability that we go to state i before state j given we are in state j.

Then $1 - p$ is the probability that we do not go to state i before returning state j,0 given we start in state j.

So $(1-p)^k$ is the probability that we return to state j exactly k times before moving to state i, given we start in state j.

Let N be the number of visits to j prior to returning to i given we start in state i.

The probability that $N = k \in \mathbb{Z}_{\geq 0}$ is the probability that starting from state i we go to state j, return to state j $(k-1)$ times without returning to state i, and then return to state i without going to returning to state j.

So $\mathbb{P}(N = k | X_0 = i) = p(1-p)^{k-1}p$. This is the probability mass function for N so,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} np^2(1-p)^{k-1} = p\sum_{n=0}^{\infty} n(1-p)^n = p\frac{p}{(1-(1-p))^2} = 1$$

Exercise 4.5

Let X be a Markov chain with transition matrix,

$$P = \begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}, \qquad\qquad p \in (0,1)$$

Find P^n, the invariant distribution π, and the mean-recurrence times $\overline{\tau}_j$ for $j = 1, 2, 3$.

Solution

Note that P has eigendecomposition $P = V\Lambda V^{-1}$ where,

$$\Lambda = \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix}, \qquad\qquad V = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, $P^n = V\Lambda^n V^{-1}$. Explicitly,

$$P^n = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1-4p & \\ & & 1-2p \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Invariant distributions are linear combinations of left eigenvectors corresponding to eigenvalues of 1. In this case that is the first row of V^{-1}. That is,

$$\pi = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \\ \frac{1}{4} & & \end{bmatrix}$$

Finally, since the invariant distribution is unique, by Theorem we have,

$$\overline{\tau}_i = \frac{1}{\pi(i)}$$

Exercise 4.6

Let X_n be the number of mistakes in the n-th addition of a book. Between the n-th and the $(n+1)$-th addition an editor corrects each mistake independently with probability p and introduces Y_n new mistakes where the(Y_n) are iid and Poisson distributed with parameter λ. Find the invariant distribution π of the number of mistakes in the book.

Solution

Let $M_{n,k}$ be distributed as $\mathrm{Ber}(1-p)$ so that M_k is 0 if this mistake is corrected, and 1 otherwise. Let Y_n be Poisson distributed with parameter λ. Then,

$$X_{n+1} = Y_n + \sum_{k-1}^{X_n} M_k$$

Each $M_{n,k}$ has generating function,

$$G_{M_{n,k}} = p + (1-p)s = 1 - q + qs = 1 - q(1-s)$$

Similarly. Y_n has generating function,

$$G_Y(s) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k!s^k = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)}$$

Therefore X_{n+1} has generating function,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right] \\
&= G_Y(s)\mathbb{E}\left[\mathbb{E}\left[s^{M_{k,1}+M_{k,2}+\ldots+M_{k,X_n}}\right]|X_n\right] \\
&= G_Y(s)\mathbb{E}\left[(1-q(1-s))^{X_n}\right] \\
&= G_Y(s)G_n(1-q(1-s))
\end{aligned}$$

First observe $1 - q^i(1-(1-q(1-s))) = 1 - q^{i+1}(1-s)$. We now use the relation $G_{n+1}(s) = G_Y(s)G_n(1-q(1-s))$ and the fact that $G_0(s) = 1$ to calculate,

$$\begin{aligned}
G_{n+1}(s) &= G_Y(s)G_n(1-q(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_{n-1}(1-q^2(1-s)) \\
&= G_Y(s)G_Y(1-q(1-s))G_Y(1-q^2(1-s))G_{n-2}(1-q^3(1-s)) \\
&\vdots \\
&= \prod_{i=0}^{n} G_Y(1-q^i(1-s))
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} G_n(s) &= \lim_{n\to\infty} G_{n+1}(s) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} G_Y(1-q^i(1-s)) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n} \exp\left(\lambda(-q^i(1-s))\right) \\
&= \exp\left(\sum_{i=0}^{\infty} \lambda(-q^i(1-s)\right) \\
&= \exp\left(\lambda(s-1)\frac{1}{1-q}\right) \\
&= \exp\left(\frac{\lambda}{p}(s-1)\right)
\end{aligned}$$

Thus, $G_n(S)$ converges to the generating function of a Poisson random variable with parameter λ/p.

Then X_n converges to a random variable distributed like a Poisson random variable with parameter λ/p. The random variable for which X_n converges to must be the variable corresponding to the stationary distribution. Therefore, the stationary distribution is distributed like the probability mass function of this random variable. That is,

$$\pi(k) = e^{-\lambda/p}\frac{(\lambda/p)^k}{k!}$$

In the limit $p \to 1$, where we correct all mistakes, the stationary distribution looks like a Poisson distribution with parameter λ. In the limit $\lambda \to 0$ so we do not make any new mistakes, $\pi(0) \to 1$ as expected.

Exercise 4.7

Give an example of a transition matrix P that admits multiple stationary distributions π.

Solution

Define P to be the identity matrix. Then any distribution is a stationary distribution.

Exercise 4.8

A Markov chain on $S = \{0, 1, 2, ..., n\}$ has transition probabilities $p(0, 0) = 1-\lambda_0$, $p(i, i+1) = \lambda_i$ and $p(i + 1, i) = \mu_{i+1}$ for $i = 0, 1, ..., n - 1$, and $p(n, n) = 1 - \mu_n$. Show that the process is reversible in equilibrium.

Solution

We assume all entries not specified are zero. (I heard this is the intent, however I wonder why we are given μ_j when $\mu_j = 1 - \lambda_j$ for all j). We write the matrix P as,

Write $\mu_n = 1-\lambda_n$. Thus, $\mu_i = 1 - \lambda_i$ for $i = 1, ..., n$ as the sum of each row must be 1 (making the assumption that all entries not specified at zero).

$$P = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ \mu_1 & & \lambda_1 & & & \\ & \mu_2 & & \lambda_2 & & \\ & & \mu_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & \mu_n & 1-\mu_n \end{bmatrix} = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & & & & \\ 1-\lambda_1 & & \lambda_1 & & & \\ & 1-\lambda_2 & & \lambda_2 & & \\ & & 1-\lambda_3 & & & \\ & & & & & \lambda_{n-1} \\ & & & & 1-\lambda_n & \lambda_n \end{bmatrix}$$

This chain is irreducible and finite so a unique invariant distribution π exists. Write $\pi = [\pi_0, \pi_1, ..., \pi_n]$. Then $\pi P = \pi$. That is,

$$\pi P = \begin{bmatrix} \pi_0(1-\lambda_0) + \pi_1(1-\lambda_1) \\ \pi_0\lambda_0 + \pi_2(1-\lambda_2) \\ \pi_1\lambda_1 + \pi_3(1-\lambda_3) \\ \vdots \\ \vdots \\ \pi_{n-1}\lambda_{n-1} + \pi_n\lambda_n \end{bmatrix}^T = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_j \\ \vdots \\ \pi_n \end{bmatrix}^T$$

$$\begin{aligned}
\pi_1 &= \lambda_0\pi_0/(1-\lambda_1) & \lambda_0\pi_0 &= \pi_1(1-\lambda_1) \\
\pi_2 &= (\pi_1 - \pi_0\lambda_0)/(1-\lambda_2) = \pi_1\lambda_1/(1-\lambda_2) & \lambda_1\pi_1 &= \pi_2(1-\lambda_2) \\
\pi_3 &= (\pi_2 - \pi_1\lambda_1)/(1-\lambda_3) = \pi_2\lambda_2/(1-\lambda_3) & \lambda_2\pi_2 &= \pi_3(1-\lambda_3) \\
&\vdots \\
\pi_{j+1} &= (\pi_j - \pi_{j-1}\lambda_{j-1})/(1-\lambda_{j+1}) = \pi_j\lambda_j/(1-\lambda_{j+1}) & \lambda_j\pi_j &= \pi_{j+1}(1-\lambda_{j+1}) \\
&\vdots \\
\pi_n &= (\pi_{n-1}\lambda_{n-1})/(1-\lambda_n) & \pi_{n-1}\lambda_{n-1} &= \pi_n(1-\lambda_n)
\end{aligned}$$

Observing the equations on the right hand side we have that for $i = 1, 2, ..., n - 1$,

$$\pi_i p(i, i+p) = \pi_{i+1} p(i+1, i)$$

We now show the detail balance condition. In particular, we must show,

$$\pi_i p(i, j) = \pi_j p(j, i) \qquad \text{for all } i, j$$

However, for $j \notin \{i-1, i+1\}$ we have $p(i,j) = 0$. Therefore, for this matrix the previous condition is equivalent to

$$\pi_i p(i, i+1) = \pi_{i+1} p(i+1, i) \qquad \text{for } i = 1, 2, ..., n-1$$

We have shown that these equations hold for all $i = 1, 2, ..., n-1$.

This proves π is in detailed balance with P, and so this process is reversible in equilibrium. $\square$