AMATH 561 Assignment 2

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Exercise 2.1

Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = 2^\Omega$ (the set of all subsets of Ω). We define a probability measure $\mathbb{P}$ as follows

$$\mathbb{P}(a) = 1/6, \qquad \mathbb{P}(b) = 1/3, \qquad \mathbb{P}(c) = 1/4, \qquad \mathbb{P}(d) = 1/4$$

Next, define three random variables,

$$\begin{aligned} X(a) &= 1, & X(b) &= 1, & X(c) &= -1, & X(d) &= -1 \\ Y(a) &= 1, & Y(b) &= -1, & Y(c) &= 1, & Y(d) &= -1, \end{aligned}$$

and $Z = X + Y$.

(a) List the sets in $\sigma(X)$.

(b) What are the values of $\mathbb{E}[Y|X]$ for $\{a, b, c, d\}$? Verify the partial averaging property: $\mathbb{E}[\mathbb{1}_A \mathbb{E}[Y|X]] = \mathbb{E}[\mathbb{1}_A Y]$.

(c) What are the values of $\mathbb{E}[Z|X]$ for $\{a, b, c, d\}$? Verify the partial averaging property.

Solution

(a) Recall that $\sigma(X) = \{\{X \in A\} \subseteq \Omega : A \in \mathcal{B}(\mathbb{R})\} = \{\{w : X(w) \in A\} : A \in \mathcal{B}(\mathbb{R})\}$. Therefore,

$$\sigma(X) = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}$$

(b) We compute,

$$\begin{aligned} \mathbb{E}[Y|X](a) = \mathbb{E}[Y|X = X(a)] = \mathbb{E}[Y|X = 1] &= \frac{1\mathbb{P}(a) - 1\mathbb{P}(b)}{\mathbb{P}(a) + \mathbb{P}(b)} = -\frac{1}{3} \\ \mathbb{E}[Y|X](b) = \mathbb{E}[Y|X = X(b)] = \mathbb{E}[Y|X = 1] &= \frac{1\mathbb{P}(a) - 1\mathbb{P}(b)}{\mathbb{P}(a) + \mathbb{P}(b)} = -\frac{1}{3} \\ \mathbb{E}[Y|X](c) = \mathbb{E}[Y|X = X(c)] = \mathbb{E}[Y|X = -1] &= \frac{1\mathbb{P}(c) - 1\mathbb{P}(d)}{\mathbb{P}(c) + \mathbb{P}(d)} = 0 \\ \mathbb{E}[Y|X](d) = \mathbb{E}[Y|X = X(d)] = \mathbb{E}[Y|X = -1] &= \frac{1\mathbb{P}(c) - 1\mathbb{P}(d)}{\mathbb{P}(c) + \mathbb{P}(d)} = 0 \end{aligned}$$

For each set $A \in \sigma(X)$ we verify that $\mathbb{E}[\mathbb{1}_A \mathbb{E}[Y|X]] = \mathbb{E}[\mathbb{1}_A Y]$ as follows,

A	$\mathbb{E}[\mathbb{1}_A \mathbb{E}[Y\|X]]$	$\mathbb{E}[\mathbb{1}_A Y]$
$\emptyset$	0	0
$\{a, b\}$	$-\frac{1}{3}\mathbb{P}(a) - \frac{1}{3}\mathbb{P}(b) = -\frac{1}{6}$	$1\mathbb{P}(a) - 1\mathbb{P}(b) = -\frac{1}{6}$
$\{c, d\}$	$0\mathbb{P}(c) + 0\mathbb{P}(d) = 0$	$1\mathbb{P}(c) - 1\mathbb{P}(d) = 0$
Ω	$-\frac{1}{3}\mathbb{P}(a) - \frac{1}{3}\mathbb{P}(b) + 0\mathbb{P}(c) + 0\mathbb{P}(d) = -\frac{1}{6}$	$1\mathbb{P}(a) - 1\mathbb{P}(b) + 1\mathbb{P}(c) - 1\mathbb{P}(d) = -\frac{1}{6}$

(c) Write,

$$Z(a) = 2, \qquad Z(b) = 0, \qquad Z(c) = 0, \qquad Z(d) = -2$$

We compute,

$$
\begin{aligned}
\mathbb{E}[Z|X](a) &= \mathbb{E}[Z|X = X(a)] = \mathbb{E}[Z|X = 1] = \frac{2\mathbb{P}(a) + 0\mathbb{P}(b)}{\mathbb{P}(a) + \mathbb{P}(b)} = \frac{2}{3} \\
\mathbb{E}[Z|X](b) &= \mathbb{E}[Z|X = X(b)] = \mathbb{E}[Z|X = 1] = \frac{2\mathbb{P}(a) + 0\mathbb{P}(b)}{\mathbb{P}(a) + \mathbb{P}(b)} = \frac{2}{3} \\
\mathbb{E}[Z|X](c) &= \mathbb{E}[Z|X = X(c)] = \mathbb{E}[Z|X = -1] = \frac{0\mathbb{P}(c) + 2\mathbb{P}(d)}{\mathbb{P}(c) + \mathbb{P}(d)} = -1 \\
\mathbb{E}[Z|X](d) &= \mathbb{E}[Z|X = X(d)] = \mathbb{E}[Z|X = -1] = \frac{0\mathbb{P}(c) + 2\mathbb{P}(d)}{\mathbb{P}(c) + \mathbb{P}(d)} = -1
\end{aligned}
$$

For each set $A \in \sigma(X)$ we verify that $\mathbb{E}[\mathbb{1}_A \mathbb{E}[Z|X]] = \mathbb{E}[\mathbb{1}_A Z]$ as follows,

A	$\mathbb{E}[\mathbb{1}_A \mathbb{E}[Z\|X]]$	$\mathbb{E}[\mathbb{1}_A Z]$
$\emptyset$	0	0
$\{a, b\}$	$\frac{2}{3}\mathbb{P}(a) + \frac{2}{3}\mathbb{P}(b) = \frac{1}{3}$	$2\mathbb{P}(a) + 0\mathbb{P}(b) = \frac{1}{3}$
$\{c, d\}$	$-1\mathbb{P}(c) + -1\mathbb{P}(d) = -\frac{1}{2}$	$0\mathbb{P}(c) - 2\mathbb{P}(d) = -\frac{1}{2}$
Ω	$\frac{2}{3}\mathbb{P}(a) + \frac{2}{3}\mathbb{P}(b) - 1\mathbb{P}(c) - 1\mathbb{P}(d) = -\frac{1}{6}$	$2\mathbb{P}(a) + 0\mathbb{P}(b) + 0\mathbb{P}(c) - 2\mathbb{P}(d) = -\frac{1}{6}$

Exercise 2.2

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Y be a square integrable random variable: $\mathbb{E}Y^2 < \infty$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Show that

$$\mathbb{V}(Y - \mathbb{E}[Y|\mathcal{G}]) \leq \mathbb{V}(Y - X) \qquad\qquad \forall X \in \mathcal{G}$$

Solution

Suppose further $\mathbb{E}[(Y-X)^2] < \infty$ (we make this assumption so that $\mathbb{V}[Y-X]$ exists).

Clearly $(\mathbb{E}[Y|\mathcal{G}] - X) \in \mathcal{G}$. Then, by partial averaging, $\mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)Y]$. Therefore,

$$\begin{aligned}
\mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)(Y - \mathbb{E}[Y|\mathcal{G}])] &= \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)Y] - \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)\mathbb{E}[Y|\mathcal{G}]] \\
&= \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)Y] - \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)Y] \\
&= 0
\end{aligned}$$

Then, since $\mathbb{E}[(Y-X)^2]$, exists,

$$\begin{aligned}
\mathbb{E}[(Y-X)^2] &= \mathbb{E}[((Y - \mathbb{E}[Y|\mathcal{G}]) + (\mathbb{E}[Y|\mathcal{G}] - X))^2] \\
&= \mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])^2] + \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)^2] + 2\mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])(\mathbb{E}[Y|\mathcal{G}] - X)] \\
&= \mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])^2] + \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)^2]
\end{aligned}$$

Again by partial averaging, $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[\mathbb{1}_\Omega \mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[\mathbb{1}_\Omega Y] = \mathbb{E}[Y]$ so that $\mathbb{E}[Y - \mathbb{E}[Y|G]] = 0$. Then,

$$\begin{aligned}
\mathbb{E}[Y-X]^2 &= \mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}] - X]^2 \\
&= (\mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}]] + \mathbb{E}[\mathbb{E}[Y|\mathcal{G}] - X])^2 \\
&= \mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}]]^2 + \mathbb{E}[\mathbb{E}[Y|\mathcal{G}] - X]^2 + 2\mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}]]\mathbb{E}[\mathbb{E}[Y|\mathcal{G}] - X] \\
&= \mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}]]^2 + \mathbb{E}[\mathbb{E}[Y|\mathcal{G}] - X]^2
\end{aligned}$$

Thus, subtracting this result from the first,

$$\begin{aligned}
\mathbb{E}[(Y-X)^2] - \mathbb{E}[Y-X]^2 &= \mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])^2] - \mathbb{E}[Y - \mathbb{E}[Y|\mathcal{G}]]^2 + \mathbb{E}[(\mathbb{E}[Y|\mathcal{G}] - X)^2] - \mathbb{E}[\mathbb{E}[Y|\mathcal{G}] - X]^2 \\
\mathbb{V}[Y-X] &= \mathbb{V}[Y - \mathbb{E}[Y|\mathcal{G}]] + \mathbb{V}[\mathbb{E}[Y|\mathcal{G}] - X]
\end{aligned}$$

Therefore, since $\mathbb{V}[\mathbb{E}[Y|\mathcal{G} - X] \geq 0$, for any $X \in \mathcal{G}$,

$$\mathbb{V}[Y-X] \geq \mathbb{V}[Y - \mathbb{E}[Y|\mathcal{G}]] \qquad \square$$

Exercise 2.3Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable X and a function f such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but $\sigma(f(X)) \neq \{\emptyset, \Omega\}$. Give a function g such that $\sigma(g(X)) = \{\emptyset, \Omega\}$.

Let $\Omega = \{a, b, c\}$ and $\mathcal{F} = 2^\Omega$. Define $\mathbb{P}(a) = \mathbb{P}(b) = \mathbb{P}(c) = 1/3$.

Define X as $X(a) = 0,\ X(b) = -1,\ X(c) = 1$.

Thus, $\sigma(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}, \Omega\}$

Since $X(\Omega) \subset \mathbb{R}$, define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = 0$. Then $f(X(a)) = 0,\ f(X(b)) = f(X(c)) = 1$ and $g(Y(a)) = g(Y(b)) = g(Y(c)) = 0$.

Therefore $\sigma(f(X)) = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$ so $\sigma(f(X)) \subsetneq \sigma(X)$ so $\sigma(f(X))$ is strictly smaller than $\sigma(X)$.

Similarly, $\sigma(g(X)) = \{\emptyset, \Omega\}$.

Exercise 2.4

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define random variables X and $Y_0, Y_1, Y_2, ...$ and suppose $\mathbb{E}[X] < \infty$. Define $F_n := \sigma(Y_0, Y_1, ..., Y_n)$ and $X_n = \mathbb{E}[X|F_n]$. Show that the sequence $X_0, X_1, X_2, ...$ is a martingale under $\mathbb{P}$ with respect to the filtration $(F_n)_{n\geq 0}$.

Solution

Since $(F_n)_{n\geq 0}$ is a filtration, then F_n is a sub σ-algebra of F_{n+1}. Therefore, by iterated conditioning,

$$\mathbb{E}[X_{n+1}|F_n] = \mathbb{E}[\mathbb{E}[X|F_{n+1}]|F_n] = \mathbb{E}[X|F_n] = X_n$$

This proves the sequence $X_0, X_1, X_2, ...$ is a martingale under $\mathbb{P}$ with respect to the filtration $(F_n)_{n\geq 0}$.
$\square$

Exercise 2.5

Let $X_0, X_1, ...$be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p$). Define $S_n = \sum_{i=1}^n X_i$ where $S_0 = 0$. Define

$$Z_n := \left(\frac{1-p}{p}\right)^{2S_n - n} \qquad n = 0, 1, 2, ...$$

Let $\mathcal{F}_n := \sigma(X_0, X_1, ..., X_n)$. Show that Z_n is a martingale with respect to this filtration.

Solution

Observe,

$$Z_{n+1} = \left(\frac{1-p}{p}\right)^{2S_{n+1}-(n+1)} = \left(\frac{1-p}{p}\right)^{2S_n - n}\left(\frac{1-p}{p}\right)^{2X_{n+1}-1} = Z_n\left(\frac{1-p}{p}\right)^{2X_{n+1}-1}$$

Then, since X_{n+1} is independent of all other X_j, X_{n+1} is is independent of F_n. Thus, using the definition of expectation of a discrete random variable,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{2X_{n+1}-1}\middle| F_n\right] &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{2X_{n+1}-1}\right] \\
&= (p)\left(\frac{1-p}{p}\right)^{2\cdot 1-1} + (1-p)\left(\frac{1-p}{p}\right)^{2\cdot 0-1} \\
&= (1-p) + p \\
&= 1
\end{aligned}$$

Therefore, by taking out what is known,

$$\mathbb{E}[Z_{n+1}|F_n] = \mathbb{E}\left[Z_n((1-p)/p)^{2X_{n+1}-1}|F_n\right] = Z_n\mathbb{E}\left[((1-p)/p)^{2X_{n+1}-1}|F_n\right] = Z_n$$

This proves $(Z_n)_{n\geq 0}$ is a martingale with respect to this filtration. $\square$