

# AMATH 584 Assignment 1

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## Exercise 1.1

Let  $B$  be a  $4 \times 4$  matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

- (a) Write the result as a product of eight matrices .
- (b) Write it again as a product  $ABC$  (same  $B$ ) of three matrices.

## Solution

(a) We have,  $O_5 O_3 O_2 B O_1 O_4 O_6$  where,

$$O_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$O_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad O_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad O_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We now simplify the expression from (a) as,  $ABC$  where,

$$A = O_5 O_3 O_2 = \begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad C = O_1 O_4 O_6 O_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We first manually manipulate the inputed matrix. We then define the matrices listed above. Finally, all three methods are compared.

```
import scipy as sp

def exercise_1_1(B):

    M=sp.copy(B)
    M[:,0]=2*M[:,0] # double column 1
    M[2]=1/2*M[2] # halve row 3
    M[0]=M[2]+M[0] # add row 3 to row 1
    M[:, [0,3]]=M[:, [3,0]] # interchange columns 1 and 4
    M[[0,2,3]]=M[[0,2,3]]-M[1] # subtract row 2 from each of the other rows
```

```

M[:,3]=M[:,2] # replace column 4 by column 3
M[:,0]=0 # delete column 1

O1=sp.matrix([[2,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
O2=sp.matrix([[1,0,0,0],[0,1,0,0],[0,0,1/2,0],[0,0,0,1]])
O3=sp.matrix([[1,0,1,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
O4=sp.matrix([[0,0,0,1],[0,1,0,0],[0,0,1,0],[1,0,0,0]])
O5=sp.matrix([[1,-1,0,0],[0,1,0,0],[0,-1,1,0],[0,-1,0,1]])
O6=sp.matrix([[1,0,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,0]])
O7=sp.matrix([[0,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

A=sp.matrix([[1,-1,1/2,0],[0,1,0,0],[0,-1,1/2,0],[0,-1,0,1]])
C=sp.matrix([[0,0,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,0]])

print(M)
print(O5*O3*O2*B*O1*O4*O6*O7)
print(A*B*C)

print(sp.array_equal(M,O5*O3*O2*B*O1*O4*O6*O7) and sp.array_equal(M,A*B*
C))

exercise_1_1(sp.matrix(sp.random.rand(4,4)))

```

Running the function for a few different values of  $B$  always returns True indicating that the three methods are equivalent (at least for the tested matrices). A sample output is displayed below.

```

>> exercise_1_1(sp.matrix(sp.random.rand(4,4)))
>> [[ 0.          -0.07326807  0.33590766  0.33590766]
 [ 0.          0.91030668  0.63417526  0.63417526]
 [ 0.         -0.46052944 -0.4908797  -0.4908797 ]
 [ 0.         -0.28526664 -0.29515107 -0.29515107]]
[[ 0.          -0.07326807  0.33590766  0.33590766]
 [ 0.          0.91030668  0.63417526  0.63417526]
 [ 0.         -0.46052944 -0.4908797  -0.4908797 ]
 [ 0.         -0.28526664 -0.29515107 -0.29515107]]
[[ 0.          -0.07326807  0.33590766  0.33590766]
 [ 0.          0.91030668  0.63417526  0.63417526]
 [ 0.         -0.46052944 -0.4908797  -0.4908797 ]
 [ 0.         -0.28526664 -0.29515107 -0.29515107]]
True

```

### Exercise 2.1

Show that if a matrix  $A$  is both triangular and unitary, then it is diagonal.

### Solution

Suppose a matrix  $A \in \mathbb{C}^{m \times m}$ ,  $m \geq 2$ , is both triangular and unitary. We have  $A^*A = I = AA^*$ , so one of  $A$  or  $A^*$  is upper triangular. Thus, without loss of generality assume  $A$  is upper triangular.

Since  $A$  is upper triangular we have  $A_{ij} = 0$  for  $i > j$ .

Consider the product  $AA^* = I$ . We then have,

$$1 = I_{mm} = \sum_{i=1}^m A_{mi}A_{im}^* = A_{mm}A_{mm}^* + \sum_{i=1}^{m-1} A_{mi}A_{im}^* = A_{mm}A_{mm}^*$$

Note that this condition implies  $A_{mm} \neq 0$ .

Now observe for any index  $1 \leq j \leq m-1$ ,

$$0 = I_{jm} = \sum_{i=1}^m A_{mi}A_{ij}^* = A_{mm}A_{mj}^* + \sum_{i=1}^{m-1} A_{mi}A_{ij}^* = A_{mm}A_{mj}^*$$

Since  $A_{mm} \neq 0$  we have  $A_{mj}^* = 0$ . Therefore  $A_{jm} = \overline{A_{mj}^*} = 0$ .

This proves that the last column of  $A$  is zero, except the diagonal entry.

Consider the  $k$ -th order leading principal sub matrix  $A_k$  formed by deleting the last  $m-k$  rows. That is the sub matrix with entries  $A_{ij}$  for  $1 \leq i, j \leq k$ . This is displayed below as the top left corner of  $A$

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \vdots \\ \vdots & & \vdots & \\ A_{k1} & \cdots & A_{kk} & \vdots \\ \vdots & & \vdots & \end{bmatrix} \quad A_k = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix}$$

Clearly  $A_k$  inherits (upper) triangular from  $A$  as  $A_{ij} = 0$  for  $i > j$ . Moreover, considering block matrix multiplication we see  $A_k A_k^* = I_k$ , where  $I_k$  is the identity matrix in  $\mathbb{C}^{k \times k}$ . That is,  $A_k$  is also unitary (in  $\mathbb{C}^{k \times k}$ ).

Therefore, by the above result,  $A_{jk} = 0$  for any index  $1 \leq j \leq k-1$ . But  $k$  can be any index  $1 \leq k \leq m$  so we see that  $A_{jk} = 0$  for all  $j < k$ . That is,  $A$  is lower triangular. By hypothesis  $A$  is upper triangular as well. This proves  $A$  is diagonal.  $\square$

## Exercise 2.2

The Pythagorean theorem asserts that for a set of  $n$  orthogonal vectors  $\{x_i\}$ ,

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

- (a) Prove this in the case  $n = 2$  by explicit computation of  $\|x_1 + x_2\|^2$ .
- (a) Show that this computation also establishes the general case, by induction

## Solution

We make the assumption that  $x_i \in \mathbb{C}^m$  for  $m \in \mathbb{Z}$ . Suppose the  $x_i$  are orthogonal. That is,  $x_i^* x_j = 0$  for  $i \neq j$ . We denote the  $k$ -th component of  $x_i$  by  $x_{ik}$ .

(a) By orthogonality we have,  $x_1^*x_2 = x_2^*x_1 = 0$ . Thus,

$$\begin{aligned}\|x_1 + x_2\|^2 &= (x_1 + x_2)^*(x_1 + x_2) = (x_1^* + x_2^*)(x_1 + x_2) \\ &= x_1^*x_1 + x_1^*x_2 + x_2^*x_1 + x_2^*x_2 \\ &= \|x_1\|^2 + 0 + 0 + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 \quad \square\end{aligned}$$

(b) Suppose  $\left\|\sum_{i=1}^{n-1} x_i\right\|^2 = \sum_{i=1}^{n-1} \|x_i\|^2$  for some  $n$ . Then, using the above result,

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \left\|x_n + \sum_{i=1}^{n-1} x_i\right\|^2 = \|x_n\|^2 + \left\|\sum_{i=1}^{n-1} x_i\right\|^2 = \|x_n\|^2 + \sum_{i=1}^{n-1} \|x_i\|^2 = \sum_{i=1}^n \|x_i\|^2$$

Thus, using the result from (a) as the base step for induction, for all integer  $n \geq 1$ , we have,

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2 \quad \square$$

### Exercise 2.3

Let  $A \in \mathbb{C}^{m \times m}$  be hermitian. An eigenvector of  $A$  is a nonzero vector  $x \in \mathbb{C}^m$  such that  $Ax = \lambda x$  for some  $\lambda \in \mathbb{C}$ , the corresponding eigenvalue.

- (a) Prove that all the eigenvalues of  $A$  are real.
- (b) Prove that if  $x$  and  $y$  are eigenvectors corresponding to distinct eigenvalues, then  $x$  and  $y$  are orthogonal.

### Solution

Let  $A \in \mathbb{C}^{m \times m}$  be hermitian. That is,  $A = A^*$ .

- (a) Suppose  $x$  is an eigenvector of  $A$  with corresponding eigenvalue  $\lambda$ . Then  $Ax = \lambda x$ . Recalling that for scalar  $c$ , vectors  $u, v$  and matrices  $A, B$  that  $u^*cv = cu^*v$ , that  $(cA)^* = \bar{c}A^*$ , and that  $(AB)^* = B^*A^*$  we have the following chain of equalities,

$$\lambda \|x\|^2 = \lambda x^*x = x^*\lambda x = x^*Ax = x^*A^*x = (x^*Ax)^* = (x^*\lambda x)^* = x^*\bar{\lambda}x = \bar{\lambda}x^*x = \bar{\lambda} \|x\|^2$$

Since  $x$  is an eigenvector,  $x$  is nonzero. Thus,  $\|x\| > 0$ . In particular, this means that  $\|x\|^2 \neq 0$ . Thus  $\lambda = \bar{\lambda}$  proving  $\lambda$  is real.  $\square$

- (b) Suppose  $y$  is an eigenvector of  $A$  with corresponding eigenvalue  $\gamma \neq \lambda$ . Recall from (a) that  $\lambda = \bar{\lambda}$ . This gives the following chain of equalities,

$$\gamma x^*y = x^*\gamma y = x^*Ay = x^*A^*y = (y^*Ax)^* = (y^*\lambda x)^* = x^*\bar{\lambda}y = x^*\lambda y = \lambda x^*y$$

Therefore,  $\gamma x^*y = \lambda x^*y$  so,

$$0 = \lambda(x^*y) - \gamma(x^*y) = (\lambda - \gamma)(x^*y)$$

However, since  $\lambda \neq \gamma$ , then  $(\lambda - \gamma) \neq 0$ . This proves  $x^*y = 0$ . That is, that  $x$  and  $y$  are orthogonal.  $\square$

### Exercise 3.2

Let  $\|\cdot\|$  denote any norm on  $\mathbb{C}^m$  and also the induced matrix norm on  $\mathbb{C}^{m \times m}$ . Show that  $\rho(A) \leq \|A\|$ , where  $\rho(A)$  is the spectral radius of  $A$ , i.e., the largest absolute value  $|\lambda|$  of an eigenvalue  $\lambda$  of  $A$ .

### Solution

Let  $\|\cdot\|$  denote any norm on  $\mathbb{C}^m$  and also the induced matrix norm on  $\mathbb{C}^{m \times m}$ . Denote the largest absolute value eigenvalue of  $A$  by  $\lambda$  and let  $x$  be the corresponding eigenvector. Then, by definition of supremum,

$$\rho(A) = |\lambda| = \frac{|\lambda| \|x\|}{\|x\|} = \frac{\|\lambda x\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} \leq \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \|A\| \quad \square$$

### Exercise 3.3

Vector and matrix  $p$ -norms are related by various inequalities, often involving the dimensions  $m$  or  $n$ . For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general  $m, n$ ) for which equality is achieved. In this problem  $x$  is an  $m$ -vector and  $A$  is an  $m \times n$  matrix.

- (a)  $\|x\|_\infty \leq \|x\|_2$ ,
- (b)  $\|x\|_2 \leq \sqrt{m} \|x\|_\infty$ ,
- (c)  $\|A\|_\infty \leq \sqrt{n} \|A\|_2$ ,
- (d)  $\|A\|_2 \leq \sqrt{m} \|A\|_\infty$ ,

### Solution

Let  $x \in \mathbb{C}^m$ . Clearly  $|x_i| \leq \max_{1 \leq i \leq m} |x_i| = \|x\|_\infty$  for all  $1 \leq i \leq m$ .

- (a) Let  $j$  be an index such that  $|x_j| = \|x\|_\infty$ . Then,

$$\|x\|_\infty = |x_j| = (|x_j|^2)^{1/2} \leq \left( |x_j|^2 + \sum_{i \neq j} |x_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^m |x_i|^2 \right)^{1/2} = \|x\|_2$$

Equality is obtained when  $x$  has exactly one nonzero component  $x_i$ , in which case  $\|x\|_\infty = x_i = (|x_i|^2)^{1/2} = \|x\|_2$ .

(b) Similarly,

$$\begin{aligned}\|x\|_2 &= \left( \sum_{i=1}^m |x_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^m \left( \max_{1 \leq i \leq m} |x_i| \right)^2 \right)^{1/2} \\ &= \left( m \left( \max_{1 \leq i \leq m} |x_i| \right)^2 \right)^{1/2} = \sqrt{m} \max_{1 \leq i \leq m} |x_i| = \sqrt{m} \|x\|_\infty\end{aligned}$$

Equality is obtained when all components of  $x$  are equal, in which case  $\|x\|_2 = \left( \sum_{i=1}^m |x_i|^2 \right)^{1/2} = \left( m|x_i|^2 \right)^{1/2} = \sqrt{m} \|x\|_\infty$ .

We now have  $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{m} \|x\|_\infty$  for  $x \in \mathbb{C}^m$ . Let  $A \in \mathbb{C}^{m \times n}$ . Note that for any vector  $u \in \mathbb{C}^n$ ,  $Au \in \mathbb{C}^m$ .

(c) Denote the  $i$ -th row of  $A$  by  $a_i^*$  and define  $x_0 \in \mathbb{C}^n$  to be the vector with all entries equal to 1. Then observe  $\|a_i^*\|_1 = a_i$

$$\|A\|_\infty = \sup_{u \neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} \leq \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_\infty} \leq \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_2 / \sqrt{n}} = \sqrt{n} \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_2} = \sqrt{n} \|A\|_2$$

Denote the vector with zeros in all components except for a 1 in the  $j$ -th component by  $e_j$ . Denote the vector with all ones by  $1$ .

Now suppose  $e_j$  has length  $m$  and  $1$  has length  $n$ . Let  $A = ae_j 1^*$  for some scalar  $a$ . Then  $A$  is dimension  $m \times n$  and looks like the zero matrix with the  $i$ -th row constant and equal to  $a$ .

Then clearly  $\|A\|_\infty = n|a|$ . Moreover, by our matrix norm rules for outer products,  $\|A\|_2 = |a| \|e_j\|_2 \|1^*\|_2 = |a| 1 \sqrt{n} = \sqrt{m} |a| = \|A\|_\infty / \sqrt{n}$  so equality is obtained. ,

(d)

$$\|A\|_2 = \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_2} \leq \sup_{u \neq 0} \frac{\sqrt{m} \|Au\|_\infty}{\|u\|_2} \leq \sup_{u \neq 0} \frac{\sqrt{m} \|Au\|_\infty}{\|u\|_\infty} = \sqrt{m} \sup_{u \neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} = \sqrt{m} \|A\|_\infty$$

Suppose  $e_j$  has length  $n$  and  $1$  has length  $m$ . Let  $A = a1e_j^*$  for some scalar  $a$ . Then  $A$  is dimension  $m \times n$  and looks like the zero matrix with the  $j$ -th column constant and equal to  $a$ .

Then clearly  $\|A\|_\infty = |a|$ . Moreover, by our matrix norm rules for outer products,  $\|A\|_2 = |a| \|1\|_2 \|e_j^*\|_2 = |a| \sqrt{m} 1 = \sqrt{m} |a| = \sqrt{m} \|A\|_\infty$ , so equality is obtained.