AMATH 562 Assignment 6

Tyler Chen

Exercise 6.2

Consider the sample space $S = [0, 1]$ with uniform probability distribution, i.e.,

$$\mathbb{P}([a, b]) = b - a, \ \forall 0 \leq a \leq b \leq 1$$

Define the sequence $\{X_n\}_{n \in \mathbb{N}_0}$ as $X_n(s) = \frac{n}{n+1}s + (1-s)^n$. Also, define the random variable X on this sample space as $X(s) = s$. Show that $X_n \to_{a.s.} X$.

Observe that for all $s \in (0, 1]$, $0 \leq (1 - s) < 1$ so,

$$\lim_{n \to \infty} \left[\frac{n}{n+1} s + (1-s)^n \right] = s + 0 = s$$

In particular, this means that,

$$[0, 1) \subseteq \left\{ s \in S : \lim_{n \to \infty} |X_n - X| = 0 \right\}$$

Thus, since $\mathbb{P}[1, 1] = 0$ and $[0, 1) \cap [1, 1] = \emptyset$,

$$\mathbb{P}\left(\lim_{n \to \infty} |X_n - X| = 0 \right) \geq \mathbb{P}([0, 1)) = \mathbb{P}([0, 1)) + \mathbb{P}([1, 1]) = \mathbb{P}([0, 1) \cup [1, 1]) = \mathbb{P}([0, 1]) = 1$$

Probabilities are at most 1, implying $X_n \to_{a.s.} X$. $\square$

Exercise 6.3

Let $\{X_n\}_{n\in\mathbb{N}_0}$ and $\{Y_n\}_{n\in\mathbb{N}_0}$ be two sequences of random variables, defined on the sample space S. Suppose that we know,

$$X_n \to_{a.s.} X \qquad\qquad Y_n \to_{a.s.} Y$$

Prove that $X_n + Y_n \to_{a.s.} X + Y$.

By hypothesis,

$$\mathbb{P}\left(\lim_{n\to\infty} |X_n - X| = 0\right) = 1 \qquad\qquad \mathbb{P}\left(\lim_{n\to\infty} |Y_n - Y| = 0\right) = 1$$

The intersection of sets of measure 1 is still a set of measure 1. Thus,

$$\begin{aligned} 1 &= \mathbb{P}\left(\lim_{n\to\infty} |X_n - X| = 0 \wedge \lim_{n\to\infty} |Y_n - Y| = 0\right) \\ &= \mathbb{P}\left(\lim_{n\to\infty} |X_n - X| + \lim_{n\to\infty} |Y_n - Y| = 0\right) \\ &= \mathbb{P}\left(\lim_{n\to\infty} |X_n - X| + |Y_n - Y| = 0\right) \end{aligned}$$

By the triangle inequality,

$$|(X_n + Y_n) - (X + Y)| = |(X_n - X) + (Y_n - Y)| \leq |X_n - X| + |Y_n - Y|$$

So, $|X_n - X| + |Y_n - Y| = 0$ implies $|(X_n + Y_n) - (X + Y)| = 0$. Thus,

$$\left\{\omega : \lim_{n\to\infty} |X_n(\omega) - X(\omega)| + |Y_n(\omega) - Y(\omega)| = 0\right\} \subseteq \left\{\omega : \lim_{n\to\infty} |(X_n(\omega) + Y_n(\omega) - (X(\omega) + Y(\omega))| = 0\right\}$$

Finally,

$$\mathbb{P}\left(\lim_{n\to\infty} |(X_n + Y_n) - (X + Y)| = 0\right) \geq 1$$

Probabilities are at most 1, implying $X_n + Y_n \to_{a.s.} X + Y$. $\square$

Exercise 6.6

Let $X_1, X_2, ...,$ be independent with $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$. Show that,

(a) $X_n \to_p 0$ if and only if $p_n \to 0$.

(b) $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n < \infty$

(a) Fix $\varepsilon \in (0, 1)$ and consider $\mathbb{P}(|X_n| > \varepsilon)$. For any $\omega \in \Omega$, $|X_n(w)| > \varepsilon$ if $X_n(\omega) = 1$, and $|X_n(\omega)| \leq \varepsilon$ if $X_n(\omega) = 0$. In particular, this means that regardless of the value of ε, $\mathbb{P}(|X_n| > \varepsilon) \geq \mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(|X_n| \leq \varepsilon) \geq \mathbb{P}(X_n = 0) = 1 - p_n$ so that $\mathbb{P}(|X_n| > \varepsilon) \leq p_n$.

Thus, for any $\varepsilon \in (0, 1)$, $\mathbb{P}(|X_n| > \varepsilon) = p_n$, and clearly if $\varepsilon > 1$ then $\mathbb{P}(X_n > \varepsilon) = 0$. We then have,

$$X_n \to_p 0 \iff \forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = 0 \iff \forall \varepsilon > 0, \lim_{n\to\infty} p_n = 0 \iff X_n \to_p 0 \qquad \square$$

(b) Suppose $\sum_n \mathbb{P}(\{\omega : X_n(\omega) = 1\} = \sum_n p_n < \infty$. Then, by Borel-Cantelli Lemma we have,

$$0 = \mathbb{P}(\{\omega : X_n(\omega) = 1, \text{ i.o.}\}) = \mathbb{P}(\{\omega : \lim_{n\to\infty} |X_n(\omega)| \neq 0)$$

Equivalently,

$$1 = \mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\}) \iff X_n \to_{a.s.} 0$$

Now, suppose $\sum_n \mathbb{P}(\{\omega : X_n(\omega) = 1\} = \sum_n p_n = \infty$. Then, by Borel-Cantelli Lemma, since X_n are independent meaning $\{\omega : X_n(\omega) = 1\}$ are independent, we have,

$$1 = \mathbb{P}(\{\omega : X_n(\omega) = 1, \text{ i.o.}\}) = \mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) \neq 0\}) \iff X_n \not\to_{a.s.} 0$$

This proves that $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n = 0$. $\square$

Exercise 6.7

Suppose that $X_1, X_2, ...,$ are independent with $\mathbb{P}(X_n > x) = x^{-5}$ for all $x \geq 1$ and $n = 1, 2, ...,$. Show that $\limsup_{n\to\infty}(\log X_n)/\log n = c$ almost surely for some number c, and find c.

We have,

$$\limsup_{n\to\infty}\{(\log X_n)/\log n = c\} = \{\omega : X_n(\omega)/\log n = c, \text{for infinitely many } n\}$$

Fix $n \in \mathbb{N}, d \in \mathbb{R}$. Consider[1],

$$\mathbb{P}(\log X_n/\log n > d) = \mathbb{P}(\log X_n > d\log n) = \mathbb{P}(X_n > e^{d\log n}) = \mathbb{P}(X_n > n^d) = \left(n^d\right)^{-5} = n^{-5d}$$

Take $c = 1/5$ so that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbb{P}(\log X_n/\log n > c+\varepsilon) = \sum_{n=1}^{\infty} n^{-5(c+\varepsilon)} = \sum_{n=1}^{\infty} n^{-1-5\varepsilon} < \infty$$
$$\sum_{n=1}^{\infty}\mathbb{P}(\log X_n/\log n > c-\varepsilon) = \sum_{n=1}^{\infty} n^{-5(c-\varepsilon)} = \sum_{n=1}^{\infty} n^{-1+5\varepsilon} = \infty$$

By Borel Cantelli, and since $(A_n, \text{i.o.})^c = (A_n^c, \text{a.b.f.m})$,

$$\mathbb{P}(\log X_n/\log n > c+\varepsilon,\ \text{i.o.}) = 0 \iff \mathbb{P}(\log X_n/\log n < c+\varepsilon,\ \text{a.b.f.m}) = 1$$

Since X_n are independent, then $\{\log X_n/\log n > c+\varepsilon\}$ are independent so, by Borel Cantelli,

$$\mathbb{P}(\log X_n/\log n > c-\varepsilon,\ \text{i.o.}) = 1$$

Together these show,

$$\mathbb{P}(\log X_n/\log n = c, \text{for infinitely many } n) = \mathbb{P}\left(\limsup_{n\to\infty}\{(\log X_n)/\log n = 1/5\}\right) = 1 \qquad \square$$

[1] note that when $\log n$ is in the denominator it isn't well defined for $n = 1$. But we interpret it as if the equalities below are actually true