Numerical Methods for solving SDEs

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Why do we want to study SDEs?

- Model physical systems (molecular dynamics, neurodynamics)
- Make $$$$ (sell soul and model financial things)
- solve PDEs (avoid efficiency issues in high dimensions)

Brownian Motion

Brownian motion is characterized by four properties,

1. $W_0 = 0$
2. $(W_d - W_c) \perp\!\!\!\perp (W_b - W_a)$ for $0 \leq a \leq b \leq c \leq d)$
3. $(W_t - W_s) \sim \mathcal{N}(0, t - s)$ for $0 \leq s \leq t$
4. the map $t \to W_t$ is continuous almost surely

What is an SDE?

- A Stochastic Differential Equation (SDE) is an equation of the form,

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t$$

  where W_t denotes a standard Brownian Motion [1].
- This has solution,

$$X_T - X_0 = \int_0^T \mu(t, X_t)\mathrm{d}t + \int_0^T \sigma(t, X_t)\mathrm{d}W_t$$

A Simple Numerical Method

- Approximating each integral in the simplest way gives,

$$\hat{X}_{t+\Delta t} = \hat{X}_t + \mu(t, \hat{X}_t)\Delta t + \sigma(t, \hat{X}_t)\Delta W_t$$

- Given timestep Δt we sample ΔW_t, where $\Delta W_t \sim \mathcal{N}(0, \Delta t)$.
- This method is referred to as the Euler-Maruyama Method [2].

Convergence

- Strong Convergence tells us individual trajectories converge

$$\lim_{\Delta t \to 0} \mathbb{E}\left[\left|X_T - \hat{X}_T\right|\right] = 0$$

- Weak Convergence tells us the statistics (mean and higher moments) converge

$$\lim_{k \to 0} \left|\mathbb{E}\left[p(X_T) - p(\hat{X}_T)\right]\right| = 0$$

Higher Order (Taylor) Methods

- Higher order methods can be derived from stochastic Taylor expansion
- Recall Itô's Lemma

$$f(T, X_T) = f(t, X_t) + \int_t^T \mathcal{M} f(s, X_s) \mathrm{d}s + \int_t^T \mathcal{S} f(s, X_s) \mathrm{d}W_s$$

$$\mathcal{M} := \frac{\partial}{\partial s} + \mu(s, X_s) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(s, X_s) \frac{\partial^2}{\partial x^2}, \qquad \mathcal{S} := \sigma(s, X_s) \frac{\partial}{\partial x}$$

Euler-Maruyama Method

- Apply previous identity to μ and σ to obtain,

$$
\begin{aligned}
X_T - X_0 &= \int_0^T \mu(t, X_t)\mathrm{d}t + \int_0^T \sigma(t, X_t)\mathrm{d}W_t \\
&= X_t + \mu(t, X_t)\int_t^T \mathrm{d}s + \sigma(t, X_t)\int_t^T \mathrm{d}W_s + R
\end{aligned}
$$

where,

$$
\begin{aligned}
R = \int_t^T \int_t^s \mathcal{M}\mu(u, X_u)\mathrm{d}u\mathrm{d}s + \int_t^T \int_t^s \mathcal{S}\mu(u, X_u)\mathrm{d}W_u\mathrm{d}s \\
+ \int_t^T \int_t^s \mathcal{M}\sigma(u, X_u)\mathrm{d}u\mathrm{d}W_s + \int_t^T \int_t^s \mathcal{S}\sigma(u, X_u)\mathrm{d}W_u\mathrm{d}W_s \quad (1)
\end{aligned}
$$

Milstein Method

- using the heursitics the $\mathrm{d}W_u \mathrm{d}W_s$ term is lowest order
- use identity on $\mathcal{S}\sigma$ to obtain,

$$
\begin{aligned}
X_T = X_t + \int_t^T \mu(t, X_t)\mathrm{d}s + \int_t^T \sigma(t, X_t)\mathrm{d}W_s \\
+ \int_t^T \int_t^s \mathcal{S}\sigma(t, X_t)\mathrm{d}W_u \mathrm{d}W_v + R
\end{aligned}
$$

- This gives the Millstein method,

$$
\begin{aligned}
\hat{X}_{t+\Delta t} = \hat{X}_t + \mu(t, \hat{X}_t)\Delta t + \sigma(t, \hat{X}_t)\Delta W_t \\
+ \sigma(t, \hat{X}_t)\partial_x \sigma(t, \hat{X}_t)\left((\Delta W_t)^2 - \Delta t\right)/2
\end{aligned}
$$

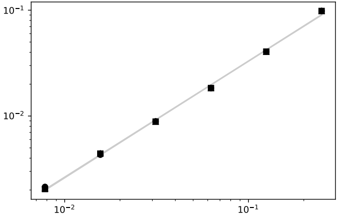
Runge Kutta Method

- Expand integral form of solution using Itô's lemma and drop terms of $\mathcal{O}(\mathrm{d}t)$
- Replace derivatives with finite difference

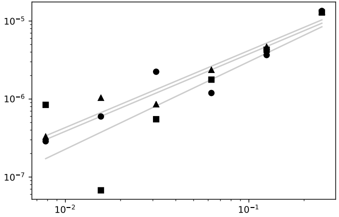
$$
\begin{aligned}
\hat{X}_{t+\Delta t} &= \hat{X}_t + \mu(t, \hat{X}_t)\Delta t + \sigma(t, \hat{X}_t)\Delta W_t \\
&\qquad\qquad + \sigma(t, \hat{X}_t - X^*)\left((\Delta W_t)^2 - \Delta t\right)/(2\sqrt{\Delta t}) \\
X^* &= \hat{X}_t + \sigma(t, \hat{X}_t)\sqrt{\Delta t}
\end{aligned}
$$

Tests of Weak Convergence

OU process with $\theta = 2, \mu = 1, \sigma = 1/100, T = 1$, 5000 trajectories



(a) $|\mathbb{E}[X_T] - \mathbb{E}[\hat{X}_T]|$



(b) $|\mathbb{E}[(X_T - \mathbb{E}[X_T])^2] - \mathbb{E}[(\hat{X}_T - \mathbb{E}[X_T])^2]|$

Figure: Circles: Euler–Maruyama, triangles: Runge-Kutta, squares: Euler–Maruyama with $\Delta W_t \in \{-\Delta t, \Delta t\}$.

Note that all slopes are about 1

Tests of Strong Convergence

Geometric Brownian motion with $\mu = 1, \sigma = 1/2, T = 8$, 300 trajectories

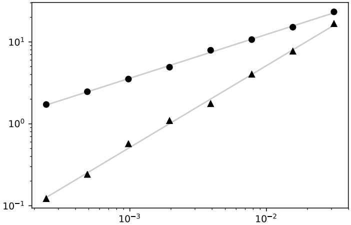


Figure: Values of $\mathbb{E}|X_T - \hat{X}_T|$, circles: Euler–Maruyama, triangles: Runge-Kutta

Note that all slopes are about $1/2$ and 1 for EM and RK respectively.

References

[1] Matthew Lorig, *Introduction to probability and stochastic processes*, 2006.

[2] Timothy Sauer, *Numerical solution of stochastic differential equations in finance*, 2006.