AMATH 586 Assignment 5

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Problem 1

Let A_ϵ be the $m+1$ by $m+1$ matrix

$$A_\epsilon = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & -1 & 0 \end{bmatrix} + \frac{\epsilon}{h^2}\begin{bmatrix} -2 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{bmatrix},$$

as in (10.15). Show that the eigenvalues of A_ϵ are

$$\mu_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\epsilon}{h^2}(1-\cos(2\pi ph)), \quad p = 1, \ldots, m+1,$$

where $h = \frac{1}{m+1}$, and that the corresponding eigenvectors are

$$u_j^p = e^{2\pi ipjh}, \quad j = 1, \ldots, m+1.$$

Solution

Write,

$$A_1 = \begin{bmatrix} 0 & 1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & 0 \end{bmatrix}$$

Recall that for any integer k,

$$\exp\left(\frac{2\pi iz}{m+1}\right) = \exp\left(\frac{2\pi i(z+k(m+1))}{m+1}\right)$$

Let $[z]$ denote all integers equivalent to z modulo $m+1$. When using $[z]$ in an expression, we mean pick any integer from this equivalence class. Then,

$$\begin{aligned}
(A_1u^p)_j &= \left(\exp\left(\frac{2\pi ip[j+1]}{m+1}\right) - \exp\left(\frac{2\pi ip[j-1]}{m+1}\right)\right) \\
&= \left(\exp\left(\frac{2\pi ip}{m+1}\right) - \exp\left(-\frac{2\pi ip}{m+1}\right)\right)\exp\left(\frac{2\pi ipj}{m+1}\right) \\
&= 2i\sin\left(\frac{2\pi p}{m+1}\right)\exp\left(\frac{2\pi ipj}{m+1}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}(A_2u^p)_j &= \left(\exp\left(\frac{2\pi ip[j+1]}{m+1}\right) + \exp\left(\frac{2\pi ip[j-1]}{m+1}\right)\right) \\ &= \left(\exp\left(\frac{2\pi ip}{m+1}\right) + \exp\left(-\frac{2\pi ip}{m+1}\right)\right)\exp\left(\frac{2\pi ipj}{m+1}\right) \\ &= 2\cos\left(\frac{2\pi p}{m+1}\right)\exp\left(\frac{2\pi ipj}{m+1}\right)\end{aligned}$$

This proves that for any integer p, $u^p = \exp(2\pi ipjh)$ is an eigenvector of A_1 and A_2 with eigenvalues $2i\sin(2\pi ph)$ and $2\cos(2\pi ph)$ respectively. Clearly u^p is an eigenvector of $-2I_{m+1}$ with eigenvalue -2.

Finally, observe,

$$A_\epsilon = -\frac{a}{2h}A_1 + \frac{\epsilon}{h^2}\left(A_2 - 2I_{m+1}\right)$$

It follows that u^p is an eigenvector of A_ε with eigenvalue,

$$-\frac{a}{2h}\left(2i\sin\left(\frac{2\pi pj}{m+1}\right)\right) + \frac{\epsilon}{h^2}\left(2\cos\left(\frac{2\pi pj}{m+1}\right) - 2\right) = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\epsilon}{h^2}\left(1-\cos(2\pi ph)\right)$$

$\square$

Problem 2

Suppose $a > 0$ and consider the following *skewed leapfrog* method for solving the advection equation $u_t + au_x = 0$:

$$U_j^{n+1} = U_{j-2}^{n-1} - \left(\frac{ak}{h} - 1\right)(U_j^n - U_{j-2}^n).$$

Note that if $ak/h \approx 1$ then the stencil of this method roughly follows the characteristic of the advection equation ($x - at =$ constant) and might be expected to be more accurate than standard leapfrog. (Like other methods we have studied, if $ak/h = 1$ the method is exact.)

(a) What is the order of accuracy of this method?

(b) For what range of Courant number ak/h does this method satisfy the CFL condition?

(c) Show that the method is in fact stable for this range of Courant numbers by doing von Neumann analysis. [Hint: Let $\gamma(\xi) = e^{ih\xi}g(\xi)$ and show that $\gamma(\xi)$ satisfies a quadratic equation closely related to the equation (10.34) that arises from a von Neumann analysis of the leapfrog method.]

(d) Produce a plot similar to those in Figure 10.4 using this method with $a = 1$, $h = 0.05$ and $k = 0.8h$.

Solution

(a) We rearrange this to,

$$\frac{1}{2}\left(\frac{U_j^{n+1} - U_j^n}{k} + \frac{U_{j-2}^n - U_{j-2}^{n-1}}{k}\right) + a\left(\frac{U_j^n - U_{j-2}^n}{2h}\right) = 0$$

The local truncation error is then,

$$\tau(x,t) = \frac{1}{2}\left(\frac{u(x,t+k) - u(x,t)}{k} + \frac{u(x-2h,t) - u(x-2h,t-k)}{k}\right) + a\left(\frac{u(x,t) - u(x-2h,t)}{2h}\right)$$

We know that,

$$\begin{aligned}\frac{1}{k}(u(x,t+k) - u(x,t)) &= \frac{1}{k}\left(ku_t(x,t) + \frac{1}{2}k^2u_{tt}(x,t) + \mathcal{O}(k^3)\right)\\ &= u_t(x,t) + \frac{1}{2}ku_{tt}(x,t) + \mathcal{O}(k^2)\end{aligned}$$

$$\begin{aligned}\frac{1}{k}(u(x-2h,t) - u(x-2h,t-k)) &= \frac{1}{k}\left(ku_t(x-2h,t) - \frac{1}{2}k^2u_{tt}(x-2h,t) + \mathcal{O}(k^3)\right)\\ &= u_t(x-2h,k) - \frac{1}{2}ku_{tt}(x-2h,t) + \mathcal{O}(k^2)\end{aligned}$$

$$\begin{aligned}\frac{1}{2h}\left(u(x,t) - u(x-2h,t)\right) &= \frac{1}{2h}\left(2hu_x(x,t) - \frac{1}{2}(2h)^2u_{xx}(x,t) + \mathcal{O}(h^3)\right)\\ &= u_x(x,t) - hu_{xx}(x,t) + \mathcal{O}(h^2)\end{aligned}$$

Putting this together we have,

$$\tau(x,t) = \frac{1}{2}\left(u_t(x,t) + u_t(x-2h,t) + \frac{k}{2}\left(u_{tt}(x,t) - u_{tt}(x-2h,t)\right) + \mathcal{O}(k^2)\right) + a\left(u_x(x,t) - hu_{xx}(x,t) + \mathcal{O}(h^2)\right)$$

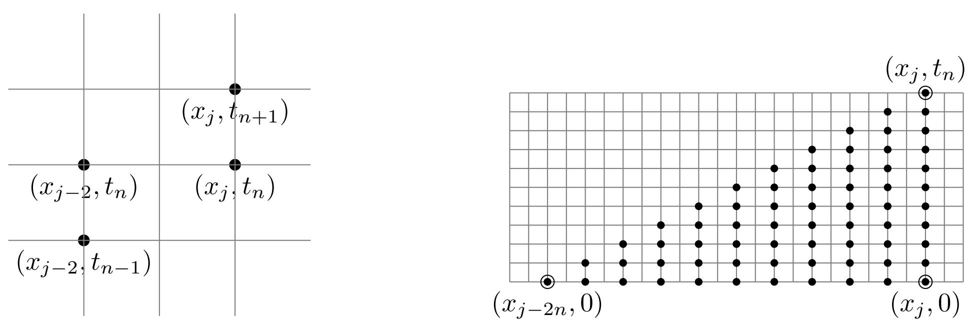
Now observe,

$$u_t(x,t) + u_t(x-2h,t) = 2u_t(x,t) - 2hu_{tx}(x,t) + 2h^2u_{txx}(x,t) + \mathcal{O}(h^2)$$

$$\frac{k}{2}(u_{tt}(x,t) - u_{tt}(x-2h,t)) = \frac{k}{2}(2hu_{ttx}(x,t) + \mathcal{O}(h^2)) = 0 + \mathcal{O}(hk)$$

Thus,

$$\begin{aligned}\tau(x,t) &= \frac{1}{2}\left(2u_t(x,t) - 2hu_{tx}(x,t) + \mathcal{O}(h^2) + \mathcal{O}(kh)\right) + a\left(u_x(x,t) - hu_{xx}(x,t) + \mathcal{O}(h^2)\right) \\ &= u_t(x,t) + au_x(x,t) - h(u_{tx}(x,t) + au_{xx}(x,t)) + \mathcal{O}(h^2) + \mathcal{O}(kh) + \mathcal{O}(k^2) \\ &= \mathcal{O}(h^2) + \mathcal{O}(kh) + \mathcal{O}(k^2)\end{aligned}$$



(a) Dependence grid points for leapfrog method.

Figure 1

(b) The solution a a spatial point (x_j, t_{n+1}) depends on the solution at (x_{j-2}, t_n), (x_j, t_n), and (x_{j-2}, t_{n-1}) as shown in Figure 1a. Thus, as we refine the mesh with k/h fixed the dependence of a point (X, T) will be $[X - (2h/k)T, X]$.

The CFL condition requires,

$$X - \frac{2h}{k}T \leq X - aT \leq X$$

This is satisfied if,

$$0 \leq \frac{ak}{h} \leq 2$$

(c) Set $U_j^n = g(\xi)^n e^{i\xi jh}$. Then,

$$g(\xi)^{n+1}e^{i\xi jh} = g(\xi)^{n-1}e^{i\xi(j-2)h} - \left(\frac{ak}{h} - 1\right)\left(g(\xi)^n e^{i\xi jh} - g(\xi)^n e^{i\xi(j-2)h}\right)$$

Dividing by $g(\xi)^{n-1}e^{i\xi(j-2)h}$ yields,

$$g(\xi)^2 e^{2i\xi h} = 1 - \left(\frac{ak}{h} - 1\right)\left(e^{i\xi h} - e^{-i\xi h}\right) g(\xi) e^{i\xi h}$$

Setting $\gamma(\xi) = g(\xi)e^{i\xi h}$ we have,

$$\gamma(\xi)^2 = 1 - \left(\frac{ak}{h} - 1\right) 2i\sin(\xi h)\gamma(\xi)$$

This is of the form of 10.34 so $|\gamma(\xi)| = |g(\xi)| \leq 1$ provided,

$$\left|\frac{ak}{h} - 1\right| \leq 1$$

If $0 \leq ak/h \leq 2$ then clearly the above is satisfied.

(d) We start with initial condition $u(x,0) = \exp\left(-20(x-2)^2\right) + \exp\left(-(x-5)^2\right)$ on $x \in [0,25]$. We assume zero boundary conditions throughout this problem (this is a reasonable assumption since the points nearest to the left and right spatial boundaries have values of `1.38...e-11` and `1.91...e-174` respectively. We find U^1 using forward Euler, then apply the specified skewed leapfrog until time $t = 17$.

Note that we use `convolve1d` rather than difference matrices for convenience.

```
a = 1
h = 0.05
k = 0.8*h
T = 17

x = np.arange(0,25+h,h)
t = np.arange(0,T+k,k)

u = np.zeros((len(t),len(x)))

u[0] = np.exp(-20*(x-2)**2) + np.exp(-(x-5)**2)
u[1] = u0 - k*a*(convolve1d(u0,[1,0,-1],mode='constant'))/(2*h)

for n in range(1,len(t)-1):
    u[n+1] = convolve1d(u[n-1],[0,0,0,0,1],mode='constant') - (a*k/h -
        1) * (u[n] - convolve1d(u[n],[0,0,0,0,1],mode='constant'))
```

The computed solution and actual solution are shown in Figure 2.

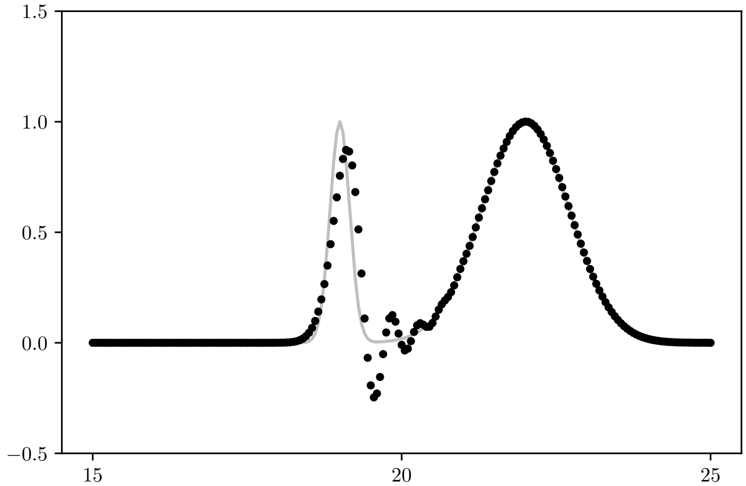


Figure 2: Computed solution (black dots) vs. actual solution (grey)

Problem 3

Derive the modified equation (10.45) for the Lax-Wendroff method.

Solution

We have standard Lax-Wendroff method,

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{2h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)$$

For any sufficiently differentiable function v we have,

$$\begin{aligned} v(x, t+k) &= v + kv_t + \frac{1}{2}k^2 v_{tt} + \frac{1}{6}k^3 v_{ttt} + \mathcal{O}(k^4) \\ v(x \pm h, t) &= v \pm hv_x + \frac{1}{2}h^2 v_{xx} \pm \frac{1}{6}h^3 v_{xxx} + \mathcal{O}(h^4) \end{aligned}$$

Therefore,

$$\frac{1}{k}\left(v(x, t+k) - v(x,t)\right) = v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2 v_{ttt} + \mathcal{O}(k^3)$$

$$\frac{a}{2h}\left(v(x+h,t) - v(x-h,t)\right) = \frac{a}{2h}\left(2hv_x + \frac{1}{3}h^3 v_{xxx} + \mathcal{O}(h^5)\right) = av_x + \frac{1}{6}ah^2 v_{xxx} + \mathcal{O}(h^3)$$

$$\frac{a^2k}{2h^2}\left(v(x-h,t) - 2v(x,t) + v(x+h,t)\right) = \frac{a^2k}{2h^2}\left(h^2 v_{xx} + \mathcal{O}(h^4)\right) = \frac{1}{2}a^2 k v_{xx} + \mathcal{O}(kh^4)$$

Let $v(x,t)$ satisfy the Lax-Wendroff method. That is, Inserting $v(x,t)$ into the difference equation gives,

$$v(x,t+k) = v(x,t) - \frac{ak}{2h}\left(v(x+h,t) - v(x-h,t)\right) + \frac{a^2k^2}{2h^2}\left(v(x-h,t) - 2v(x,t) + v(x+h,t)\right)$$

Substituting our expansions we have,

$$\begin{aligned} 0 &= v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2 v_{ttt} + \mathcal{O}(k^3) + av_x + \frac{1}{6}ah^2 v_{xxx} + \mathcal{O}(h^3) + \frac{1}{2}a^2 k v_{xx} + \mathcal{O}(kh^3) \\ &= v_t + av_x + \frac{1}{6}ah^2 v_{xxx} + \frac{1}{6}k^2 v_{ttt} + \frac{k}{2}\left(v_{tt} + a^2 v_{xx}\right) + \mathcal{O}(kh^3) + \mathcal{O}(k^3) + \mathcal{O}(h^3) \end{aligned}$$

Since the Lax-Wendroff method is a second order accurate discretization of $u_t + au_x = 0$ in space and time, then $v_t + av_x = 0$ to second order. Therefore,

$$v_{tt} = -a^3 v_{xx} + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

Dropping higher order terms,

$$0 = v_t + av_x + \frac{1}{6}ah^2 v_{xxx} - \frac{1}{6}a^3k^2 v_{xxx} = v_t + av_x + \frac{1}{6}ah^2\left(1 - \left(\frac{ak}{h}\right)^2\right) v_{xxx}$$