AMATH 514 Assignment 7

Tyler Chen

Problem 5.1

(i) Show that a tree has at most one perfect matching

(ii) Show (not using Tutte's 1-factor theorem) that a tree $G = (V, E)$ has a perfect matching if and only if the subgraph $G - v$ has exactly one odd component, for each $v \in V$.

(i) Let $G = (V, E)$ be a tree with a perfect matching M.

Small forests with zero, one, and two vertices clearly have at most one perfect matching.

Suppose $G = (V, E)$ is a forest with $|V| > 1$ and that all forests with fewer than $|V|$ nodes have at most one perfect matching.

Then, since G is a forest, there is at least one vertex u of degree 1.

If there is no perfect matching we are done. Otherwise, since u has degree one, there is unique edge $e = \{u, v\}$ in E. Therefore e must be in the matching on G.

Let $G' = G \setminus \{u, v\}$. Then G' is a subgraph of G and therefore a forest. By the inductive hypothesis, since $|V'| < |V|$, we have G' having at most on perfect matching. Therefore, there is at most one perfect matching on G'.

(ii) Suppose $G = (V, E)$ is a tree with a perfect matching and let $v \in V$. Since G has a perfect matching it must have an even number of vertices. Therefore $G - v$ has an odd number of vertices. This means at least one component must be odd.

Suppose, for the sake of contradiction, that $G - v$ has more than one odd component. Then at least one is not attached to v by an edge in the matching.

Denote one such component by C. Let u be the vertex in C such that $\{v, u\} \in E$. Since $\{u, v\}$ is not in the perfect matching, u must be covered by a matching edge in C. Therefore C must have a perfect matching.

This is a contradiction as C has an odd number of vertices and cannot contain a perfect matching.

Therefore $G - v$ has exactly one odd component.

Conversely, suppose $G-v$ has exactly one odd component for each $v \in V$. We provide an algorithm to find a perfect matching.

Indeed, for each vertex $v \in V$ add the edge of G connecting v to the odd component of $G - v$ to the output (ignore duplicates).

Clearly this will produce a set of edges which cover every vertex. It remains to show that the set of edges output is a matching.

Suppose we are on vertex v and that the edge $\{v, u\}$ is added to the matching. This means the components C_i of $G - v$ not containing u are all even.

Consider $G - u$. We know $\{v\} \cup (\cup_i C_i)$ is a component of $G - u$. Since each C_i is even it must be the unique odd component of $G - u$. Therefore the algorithm will add the edge $\{u, v\} = \{v, u\}$ to the output. That is, the edges the algorithm outputs are a matching on G.

This proves a tree G has a perfect matching if and only if the subgraph $G - v$ has exactly one odd component, for each $v \in V$. $\square$

Problem 5.2

Let G be a 3-regular graph without any bridge. Show that G has a perfect matching. (A bridge is an edge e not contained in any circuit; equivalently, deleting e increases the number of components; equivalently, $\{e\}$ is a cut.)

Write $G = (V, E)$. Let $U \subseteq V$ and consider $G - U$. Let $C = (W, F)$ be an odd component of $G - U$.

Since G is 3-regular the sum of the degrees (in G) of the vertices in W, $\sum_{v \in W} \deg_G(v) = 3|W|$.

However, C is also a graph. The sum of the degrees of vertices in a graph is even, $\sum_{v \in W} \deg_C(v)$ is even.

Therefore there are an odd number of edges between C and U.

Suppose C were connected to U by a single edge. Then deleting this edge in G would mean C again becomes a component. That is, this edge is a bridge.

Therefore there are at least three edges between C to U.

Then there can be at most $|U|$ odd components in $G - U$. Therefore, by Tutte's 1 factor theorem G has a perfect matching. $\square$

I got a hint form here: https://math.stackexchange.com/questions/81257/3-regular-graphs-with-no-bridges. I don't have any graph theory background so I hadn't thought of some of theses facts about the degrees of a graph.

Problem 5.4

Let $G = (V, E)$ be a graph and let T be a subset of V. Show G has a matching covering T if and only if the number of odd components of $G - W$ contained in T is at most $|W|$, for each $W \subseteq V$.

Construct a new graph G_T by reflecting the graph G and connecting each point not in T to its image in the mirror graph. That is, define $G_T = (V \cup V', E \cup E' \cup L)$ where:

- For each $v \in V$ define a new vertex v'. Let V' denote all such vertices.
- For each $e = \{u, v\} \in E$ define a new edge $e' = \{u', v'\} \in E'$.
- For each $v \in V \setminus T$ define a new edge $\{v, v'\} \in L$.

For convenience, for every $W \subseteq V$ denote the set of vertices in the mirror graph by W'. That is, define $W' = \{w' \in V' : w \in W\}$. Similarly, for each $F \subseteq E$ denote the set of edges in the mirror graph by F'. That is, define $F' = \{f' \in E' : f \in F\}$.

Suppose G has a matching M covering T. Each point in G not in T it part of an edge in L. Thus, $L \cup M \cup M'$ is a perfect matching in G_T.

Now, suppose G_T has a perfect matching M_T. Then all the edges $M \subseteq M_T$ contained in E are a matching in G covering T.

Therefore G has a matching covering T if and only if G_T has a perfect matching.

Let $W \subseteq V$ and suppose the number of odd components of $G_T - W_T$ is at most $|W_T|$ for all $W_T \subseteq V \cup V'$.

Let C be an odd component of $G - W$ contained in T. Then C' is an odd component of $G' - W'$ contained in T'. Both C and C' are odd components of $G_T - (W \cup W')$ since being contained in T means they do not touch an edge in L.

By hypothesis the number of odd components of $G_T - (W \cup W')$ is at most $|W \cup W'| = 2|W|$. Therefore the number of odd components of $G - W$ contained in T is less than $|W|$.

Let $W_T \subseteq V \cup V'$ and suppose the number of odd components in $G - W$ contained in T is at most $|W|$ for all $W \subseteq V$.

Partition W_T into W_1, W_2 where $W_1 = \{w \in V : w \in W_T\}$ and $W_2 = \{w \in V : w' \in W_T\}$.

By hypothesis the number of odd components of $G - W_1$ contained in T is at most $|W_1|$, and the number of odd components of $G' - W_2$ contained in T' is at most $|W_2|$.

Edges from L might connect components of $G - W_1$ and $G' - W_2$, however the new component will be odd only if one of the original components were odd. That is, the number of odd components in $G_T - W_T$ is at most $|W_1| + |W_2| = |W_T|$.

Therefore, for all $W_T \subseteq V \cup V'$ is at most $|W_T|$ if and only if for all $W \subseteq V$, the number of odd components in $G - W$ contained in T is at most $|W|$.

Using Tutte's 1 factor theorem we now have: G has a matching covering T if and only if G_T has a perfect matching if and only if for all $W_T \subseteq V \cup V'$ the number of odd components is at most $|W_T|$ if and only if for all $W \subseteq V$, the number of odd components in $G - W$ contained in T is at most $|W|$. $\square$