AMATH 514 Assignment 9

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Problem 10.5

Let $M = (X, \mathcal{I})$ be a matroid and let k be a natural number. Let $\mathcal{I}' := \{Y \subseteq \mathcal{I} : |Y| \leq k\}$. Show that $(X, \mathcal{I}')$ is again a matroid.

(i) Since $(X, \mathcal{I})$ is a matroid then $\emptyset \in \mathcal{I}$. Clearly $|\emptyset| \leq k$ so $\emptyset \in \mathcal{I}'$.

(ii) Let $Y \in \mathcal{I}'$ and $Z \subseteq Y$. Since $(X, \mathcal{I})$ is a matroid then $Z \in \mathcal{I}$. Moreover, $|Z| \leq |Y| \leq k$ as $Z \in \mathcal{I}'$. Therefore $Z \in \mathcal{I}'$.

(iii) Let $Y, Z \in \mathcal{I}'$ with $|Y| < |Z|$. Since $(X, \mathcal{I})$ is a matroid and $Y, Z \in \mathcal{I}$ then $Y \cup \{x\} \in \mathcal{I}$ for some $x \in Z \setminus Y$. Since $Z \in \mathcal{I}'$, $|Z| \leq k$. Therefore $|Y \cup \{x\}| = |Y| + 1 \leq |Z| \leq k$. Therefore $Y \cup \{x\} \in \mathcal{I}'$.

This proves $(X, \mathcal{I}')$ is a matroid. $\square$

Problem 10.19

Let $M = (X, \mathcal{I})$ be a matroid, let B be a basis of M, and let $w : X \to \mathbb{R}$ be a weight function. Show that B is a basis of maximum weight if and only if $w(B') \leq w(B)$ for every basis B' with $|B' \setminus B| = 1$.

Let B, B' be bases. From theorem we have:

(i) for any $x \in B' \setminus B$, $(B' \setminus \{x\}) \cup \{y\}$ is a basis of M for some $y \in B \setminus B'$.
(ii) for any $x \in B' \setminus B$, $(B \setminus \{y\}) \cup \{x\}$ is a basis of M for some $y \in B \setminus B'$.

Suppose B is a basis of maximum weight and let B' be some basis (with $|B' \setminus B| = 1$). Then $w(B') \leq w(B)$.

Conversely, suppose that $w(B^\dagger) \leq w(B)$ for every basis $B^\dagger$ with $|B^\dagger \setminus B| = 1$.

Let B' be a basis of M.

Suppose $|B' \setminus B| = 0$. Then, since all bases have the same size we have $B' = B$ so $w(B') \leq w(B)$.

Now, suppose $|B' \setminus B| > 0$.

We induct on $k := |B' \setminus B|$, assuming that $w(B^\dagger) \leq w(B)$ for all bases $B^\dagger$ of M with $|B^\dagger \setminus B| < k$. Clearly the original hypothesis is the base case of our induction.

Since $|B' \setminus B| > 0$ there is some $x \in B' \setminus B$. Therefore, by the theorem listed above there is some $y \in B \setminus B'$ such that $B^\dagger = (B' \setminus \{x\}) \cup \{y\}$ is a basis for M.

Observe,

$$|B^\dagger \setminus B| = |((B' \setminus \{x\}) \cup \{y\}) \setminus B| = |(B' \setminus B) \setminus \{x\}| = k - 1$$

Therefore, by our induction hypothesis we have,

$$w(B') - w(x) + w(y) = w(B^\dagger) \leq w(B)$$

Since,

$$|((B \setminus \{y\}) \cup \{x\}) \setminus B| = |\{x\}| = 1$$

by the base case we have,

$$w((B \setminus \{y\}) \cup \{x\}) \leq w(B)$$

Therefore,

$$w(B') \leq w(B) - w(y) + w(x) = w((B \setminus \{y\}) \cup \{x\}) \leq w(B)$$

This proves the result. □

Problem 10.22

Derive König's matching theorem from Edmonds' matroid intersection theorem.

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Let $\mathcal{I}_1$ be the collection of all subsets F of E so that no two edges in F have a vertex in V_1 in common. Similarly, let $\mathcal{I}_w$ be the collection of all subsets F of E so that no two edges in F have a vertex in V_2 in common. Then both $M_1 := (E, \mathcal{I}_1)$ and $M_2 := (E, \mathcal{I}_2)$ are partition matroids.

Now observe that elements of $\mathcal{I}_1 \cap \mathcal{I}_2$ are matchings in G, and matchings in G are elements of $\mathcal{I}_1 \cap \mathcal{I}_2$.

Therefore,

$$\nu(G) = \max_{Y \in \mathcal{I}_1 \cap \mathcal{I}_2} |Y|$$

Let $U \subseteq E$. Define,

$$C_1 = \{v \in V_1 : \exists e \in E \text{ with } v \in e\}$$

Now define,

$$C_2 = \{v \in V_2 : v \text{ reachable from } V_1 \setminus C_1\}$$

Finally, let $C = C_1 \cup C_2$. If $e \in U$ then $e \cap C_1 \neq \emptyset$. Similarly, if $e \in E \setminus U$ then $e \cap C_2 \neq \emptyset$. Therefore C is a vertex cover of G.

Every edge in U touches a vertex in C_1, and every vertex in C_1 touches an edge in U. Similarly, every edge in $E \setminus U$ touches a vertex in C_2, and every vertex in C_2 touches an edge in $E \setminus U$.

Therefore $r_{M_1}(U) = |C_1|$ and $r_{M_2}(E \setminus U) = |C_2|$ so that $r_{M_1}(U) + r_{M_2}(E \setminus U) = |C|$.

In particular, this proves,

$$\min_{U \subseteq E} (r_{M_1}(U) + r_{M_2}(E \setminus U)) \geq \tau(G)$$

Now, let C be a minimum size vertex cover of G. Define,

$$U = \{e \in E : (C \cap V_1) \cap e \neq \emptyset\}$$

Therefore every edge in U is covered by some vertex in $C \cap V_1$ and every vertex in $C \cap V_1$ touches some edge in U. This means that that $r_{M_1}(U) = |C \cap V_1|$.

Similarly, every edge in $E \setminus U$ must be covered by some vertex in C. Moreover, by the minimality of C, every vertex in C must touch an edge in $E \setminus U$ (otherwise C without this edge would be a smaller vertex cover). Therefore $r_{M_2}(E \setminus U) = |C \cap V_2|$

Since points in C are either in V_1 or V_2 we have found a $U \subseteq E$ such that,

$$r_{M_1}(U) + r_{M_2}(E \setminus U) = |C| = \tau(G)$$

This proves,

$$\min_{U \subseteq E} (r_{M_1}(U) + r_{M_2}(E \setminus U)) \leq \tau(G)$$

Using Edmond's matroid intersection theorem we have,

$$\nu(G) = \max_{Y \in \mathcal{I}_1 \cap \mathcal{I}_2} |Y| = \min_{U \subseteq X} (r_{M_1}(U) + r_{M_2}(E \setminus U)) = \tau(G)$$

This is Konig's matching theorem. $\square$