AMATH 514 Assignment 3

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Exercise 2.16 (Stiemke's theorem)

Prove that there exists a vector $x > 0$ such that $Ax = 0$ if and only if for each y satisfying $y^TA \geq 0$ one has $y^TA = 0$.

The problem is show exactly one of the following is true,

$$\exists x > 0 : Ax = 0$$
$$\exists y : y^TA \geq 0, y^TA \neq 0$$

Denote the j-th column of A by A_j. Denote the vector of all ones by e. Then,

$$y^TAe = \sum_{j=1}^{n} y^TA_j$$

We first prove a few useful equivalences.

Suppose $y^TA \geq 0$, then $y^jA_k \geq 0$ for all $j = 1, 2, \ldots, n$ so y^TAe is the sum of non-negative terms. If $y^TA \neq 0$ at least one term is nonzero (positive). Thus,

$$y^TA \neq 0 \iff y^T(-Ae) = -(y^TAe) < 0$$

Suppose $\exists x > 0$ with $Ax = 0$. We can scale x so that all entries are at least one. Then $z = x/(\min_i x_i) - e \geq 0$ and $Az = -Ae$.

Suppose $\exists z \geq 0$ with $Az = -Ae$. Then $A(z+e) = 0$ so $x = z + e > 0$ solves $Ax = 0$.

Thus,

$$\exists x > 0 : Ax = 0 \iff \exists z \geq 0 : Az = -Ae$$

We can now apply Farkas Theorem. Indeed, start with,

$$\exists y : y^TA \geq 0, y^TA \neq 0$$

As explained above this is equivalent to,

$$\exists y : y^TA \geq 0, y^T(-Ae) < 0$$

Applying Farkas Theorem, this is equivalent to,

$$\nexists z \geq 0 : Az = -Ae$$

Again, as explained above this is equivalent to,

$$\nexists x > 0 : Ax = 0$$

This is the desired result. $\square$

Exercise 2.26

Give an example of a matrix A and vectors b and c for which both $\{x \mid Ax \leq b\}$ and $\{y \mid y \geq 0; y^T A = c^T\}$ are empty.

Trivially we can pick $A = [0]$, $b = c = [-1]$. Then $Ax = 0 \nleq -1$ and $y^T A = 0 \neq -1$.

We can easily characterize all matrices $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^{2\times 1}$, $c \in \mathbb{R}^{1\times 2}$ such that these sets are empty.

Visually, $\{x \mid Ax \leq b\}$ corresponds to the intersection of two half planes in $\mathbb{R}^2$.

Suppose $a_1x_1 + a_2x_2 \leq b_1$ is one of the half planes. Then we require the other half plane to have the form $a_1x_1 + a_2x_2 \geq b_2$, where $b_2 > b_1$ so that their intersection is empty.

That is, $\{x \mid Ax = b\}$ will be empty if and only if,

$$A = \begin{bmatrix} a_1 & a_2 \\ -a_1 & -a_2 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}, \qquad b_2 < -b_1$$

Now observe,

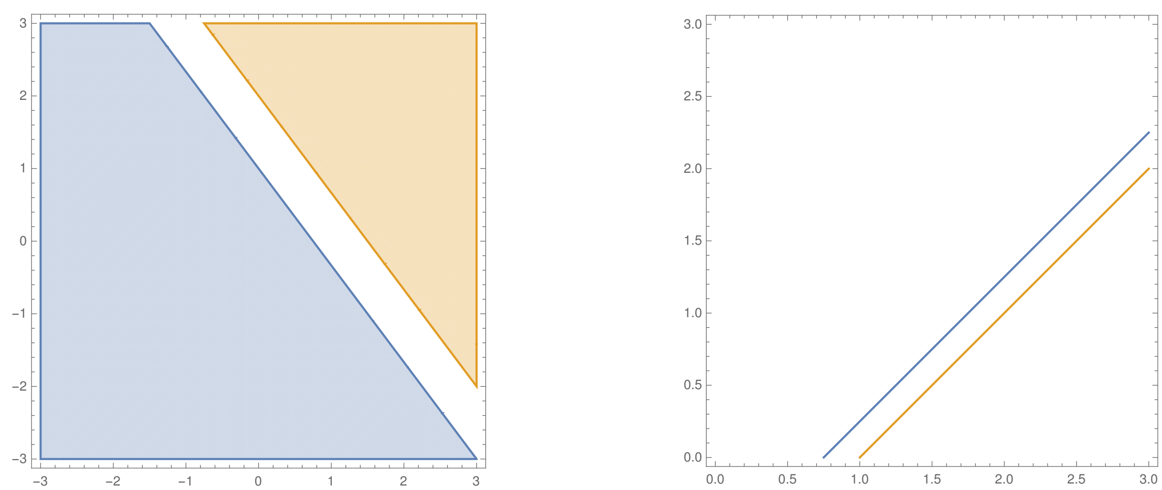
$$y^T A = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} a_1 & a_2 \\ -a_2 & -a_2 \end{bmatrix} = \begin{bmatrix} a_1y_1 - a_1y_2 \\ a_2y_1 - a_2y_2 \end{bmatrix}^T = \begin{bmatrix} a_1(y_1 - y_2) \\ a_2(y_1 - y_2) \end{bmatrix}^T$$

Finally, pick $c = [c_1 \; c_2]$ such that $a_1/a_2 \neq c_1/c_2$ (for instance, pick $c_1 = a_1, c_2 \neq a_2$.

As an example,

$$A = \begin{bmatrix} 4 & 3 \\ -4 & -3 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \qquad c^T = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

The intersection of the regions in Figures 1a and **??** show $\{x \mid Ax \leq b\}$ and $\{y \mid y \geq 0, y^T A = c^T\}$. As we showed above, these intersections are both empty.



(a) $Ax \leq b$

(b) $y^T A = c^T$

Exercise 2.27

Let $\tilde{x}$ be a feasible solution of $\max\{c^T x \mid Ax \le b\}$ and let $\tilde{y}$ be a solution of $\min\{y^T b \mid y \ge 0; y^T A = c^T\}$. Prove that $\tilde{x}$ and $\tilde{y}$ are the optimum solutions of the minimum and maximum, respectively if and only if for each $i = 1, 2, \ldots, m$ one has: $\tilde{y}_i = 0$ or $a_i \tilde{x} = b_i$.

Denote the i-th row of A by a_i.

First, note that if y is feasible, we have $y^T A = c^T$ so that $y^T Ax = c^T x$.

Second, note also that for any $i = 1, 2, \ldots, m$, if x is feasible, we have $a_i x \le b$ so that $a_i x - b \ge 0$ and if y is feasible we have $y_i \ge 0$. Thus, $y_i(a_i x - b) \ge 0$.

We prove both directions at once. Indeed, suppose $\tilde{x}$ and $\tilde{y}$ are feasible.

By duality, $\tilde{x}$ and $\tilde{y}$ are the optimum solutions of the maximum and minimum respectively if and only if $\tilde{y}^T b = c^T \tilde{x}$ which by the first note above is equivalent to,

$$\tilde{y}^T Ax = c^T x = \tilde{y}^T b$$

We can rearrange to find $\tilde{y}^T (A\tilde{x} - b) = 0$. Written in sum notation using the definition of matrix/vector multiplication we have,

$$\sum_{i=1}^{m} \tilde{y}_i \left(a_i \tilde{x} - b \right) = 0$$

Every term in this sum is non-negative by the second note above. Thus, the sum is zero if and only if each term is zero. That is, if and only if,

$$y_i(a_i \tilde{x} - b) = 0 \qquad \forall i = 1, 2, ..., m$$

Equivalently, for each $i = 1, 2, ..., m$ one has: $\tilde{y}_i = 0$ or $a_i \tilde{x} = b_i$. $\square$