AMATH 561 Assignment 1

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Exercise 1.1

Let $\mathcal{F}$ be a σ-algebra of Ω. Suppose $B \in \mathcal{F}$. Show that $\mathcal{G} := \{A \cap B : A \in \mathcal{F}\}$ is a σ-algebra of B.

Solution

Let $\mathcal{F}$ be a σ-algebra of Ω. Suppose $B \in \mathcal{F}$ and define $\mathcal{G} := \{A \cap B : A \in \mathcal{F}\}$.

(i) Since $\mathcal{F}$ is a σ-algebra, and $B \in \mathcal{F}$ then $B^c \in \mathcal{F}$. Thus, $\emptyset = B^c \cap B \in \mathcal{G}$.

(ii) Suppose $B_1, B_2, B_3, ... \in \mathcal{G}$. Then for each i, $B_i = A_i \cap B$ for some $A_i \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-algebra then $\bigcup_i A_i \in \mathcal{F}$. Thus $\bigcup_i B_i = \bigcup_i (A_i \cap B) = \{x : \forall i, (x \in A_i) \wedge (x \in B)\} = \{x : (\forall i, x \in A_i) \wedge (x \in B)\} = (\bigcup_i A_i) \cap B \in \mathcal{G}$.

(iii) Suppose $B_0 \in \mathcal{G}$. Then $B_0 = A_0 \cap B$ for some $A_0 \in \mathcal{F}$. Consider the compliment of B_0 in B. We have, $B_0^C = (A_0 \cap B)^c = A_0^c \cup B^c = \{x \in B : x \in A_0^c \cup B^c\} = \{x \in B : (x \in A_0^c) \vee (x \in B_0^c)\} = \{x \in B : (x \in A_0^c)\} = \{x \in \Sigma : (x \in A_0^c) \wedge (x \in B))\} = A_0^c \cap B$. Since $\mathcal{F}$ is a σ-algebra then $A_0^c \in \mathcal{F}$ so $B_0^c = A_0^c \cap B \in \mathcal{G}$.

This proves $\mathcal{G}$ is a σ-algebra. $\square$

Exercise 1.2

Let $\mathcal{F}$ and $\mathcal{G}$ be σ algebras of Ω.

(a) Show that $\mathcal{F} \cap \mathcal{G}$ is a σ-algebra of Ω.

(b) Show that $\mathcal{F} \cup \mathcal{G}$ is not necessarily a σ-algebra of Ω.

Solution

Let $\mathcal{F}$ and $\mathcal{G}$ be σ algebras of Ω.

(a) Consider $\mathcal{F} \cap \mathcal{G}$.

(i) Since each $\mathcal{F}$ and $\mathcal{G}$ are σ-algerbas, $\emptyset \in \mathcal{F}$ and $\emptyset \in \mathcal{G}$. Therefore $\emptyset \in \mathcal{F} \cap \mathcal{G}$

(ii) Suppose $A_1, A_2, A_3, ... \in \mathcal{F} \cap \mathcal{G}$. Then $A_1, A_2, A_3, ... \in \mathcal{F}$ and $A_1, A_2, A_3, ... \in \mathcal{G}$. These are each σ-algebras, so $\bigcup_i A_i \in \mathcal{F}$ and $\bigcup_i A_i \in \mathcal{G}$. Therefore $\bigcup_i A_i \in \mathcal{F} \cap \mathcal{G}$.

(iii) Suppose $A \in \mathcal{F} \cap \mathcal{G}$. Then $A \in \mathcal{F}$ and $A \in \mathcal{G}$. These are each σ-algebras, so $A^c \in \mathcal{F}$ and $A^c \in \mathcal{G}$. Therefore $A^c \in \mathcal{F} \cap \mathcal{G}$.

This proves $\mathcal{F} \cap \mathcal{G}$ is a σ-algebra. $\square$

(b) Let $\Omega = (0, 3), A = (0, 1), B = (2, 3)$. Define,

$$\mathcal{F} = \{\emptyset, A, A^c, \Omega\} = \{\emptyset, (0, 1), [1, 3), (0, 3)\}\mathcal{G} = \{\emptyset, B, B^c, \Omega\} = \{\emptyset, (2, 3), (0, 2], (0, 3)\}$$

Then,

$$\mathcal{F} \cup \mathcal{G} = \{\emptyset, A, B, A^c, B^c, \Omega\} = \{\emptyset, (0, 1), (2, 3), [1, 3), (0, 2], (0, 3)\}$$

Observe that $A \cup B = (0, 1) \cup (2, 3) \notin \mathcal{F} \cup \mathcal{G}$.

This proves that for σ-algebras $\mathcal{F}, \mathcal{G}$ of Ω, their union $\mathcal{F} \cup \mathcal{G}$ is not necessarily a σ-algebra of Ω. $\square$

Exercise 1.3

Describe the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for the following three experiments:

(a) a biased coin is tossed three times

(b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls

(c) a biased coin is tossed repeatedly until a head turns up

Solution

(a) $\Omega = \{w_1w_2w_3 : w_i \in \{\mathrm{H}, \mathrm{T}\} = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}$.

$\mathcal{F}$ contains sets which correspond to to questions we could ask about the three coin tosses. For instance, "was the second coin toss a heads?" ({HHH,HHT,THH,THT}), "were there two heads?" ({HHT,HTH,THH}), "were the final two coin tosses tails?" ({HTT,TTT}), etc.

Explicitly, $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω.

Suppose the coin is biased such that the probability of heads is p (and the probability of tails is $1-p$).

Define $g(h_1h_2h_3) = p^k(1-p)^{3-k}$, where k is the number of h_1, h_2, h_3 which are heads.

Define $\mathbb{P} : \mathcal{F} \to [0,1]$ as $\mathbb{P}(A) = \sum_{d \in A} g(d)$.

As a quick test, let $A = \{\mathrm{HHH}, \mathrm{HHT}\}$, the set asking the question "were the first two coin tosses heads?". We have $\mathbb{P}(A) = g(\mathrm{HHH}) + g(\mathrm{HHT}) = ppp + pp(1-p) = p^2$ which corresponds to our natural understanding of the answer to this question.

Suppose $A, B \subseteq \Omega$ are disjoint. Then,

$$\mathbb{P}(A \cup B) = \sum_{d \in A \cup B} g(d) = \sum_{d \in A} g(d) + \sum_{d \in B} g(d) = \mathbb{P}(A) + \mathbb{P}(B)$$

Clearly $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$. Therefore $\mathbb{P}$ is a well defined probability measure on $(\Omega, \mathcal{F})$.

(b) $\Omega = \{b_1b_2 : b_i \in \{\mathrm{R}, \mathrm{B}\}\} = \{\mathrm{RR}, \mathrm{RB}, \mathrm{BR}, \mathrm{BB}\}$

$\mathcal{F}$ contain sets which correspond to questions we could ask about the color of the two balls. For instance, "was the first ball red?" ({RB,RR}),"was the second ball blue?" ({RB,BB}), "was the first ball red and the second ball blue?" ({RB}), etc.

Explicitly, $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω.

Define $g(b_1b_2) = \begin{cases} 1/6 & b_1 = b_2 \\ 1/3 & b_1 \neq b_2 \end{cases}$

Then define $\mathbb{P} : \mathcal{F} \to [0,1]$ as $\mathbb{P}(A) = \sum_{d \in A} g(d)$.

As a quick test, let $A = \{\mathrm{RB}, \mathrm{RR}\}$, the set asking the question "was the first ball red?". We have $\mathbb{P}(A) = g(\mathrm{RB}) + g(\mathrm{RR}) = 1/3 + 1/6 = 1/2$ which corresponds to our natural understanding of the answer to this question.

Suppose $A, B \subseteq \Omega$ are disjoint.

$$\mathbb{P}(A \cup B) = \sum_{d \in A \cup B} g(d) = \sum_{d \in A} g(d) + \sum_{d \in B} g(d) = \mathbb{P}(A) + \mathbb{P}(B)$$

Clearly $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = g(\mathrm{RR}) + g(\mathrm{RB}) + g(\mathrm{BR}) + g(\mathrm{BB}) = 1/6 + 1/3 + 1/3 + 1/6 = 1$. Therefore $\mathbb{P}$ is a well defined probability measure on $(\Omega, \mathcal{F})$.

(c) $\Omega = \{w_1w_2....w_{n-1}w_n : w_1, ..., w_{n-1} = \mathrm{T}, w_n = \mathrm{H}\}$

Again, $\mathcal{F}$ contains sets which correspond to questions we could ask about the coin tosses. For instance, "was the first toss a tails?" ({TH,TTH,TTTH,...}), "was the last toss a heads?" ({H,TH,TTH,TTTH, ...}$= \Omega$), etc.

Explicitly, $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω.

Suppose the coin is biased such that the probability of heads is $(1 - q)$ (and the probability of tails is q).

Define $g(w_1w_1...w_n) = q^{n-1}(1 - q)$, where $g() = 0$.

Define $\mathbb{P} : \mathcal{F} \to [0, 1]$ as $\mathbb{P}(A) = \sum_{d\in A} g(d)$.

As a quick test, let $A = \{\mathrm{TTTTH}, \mathrm{TTH}\}$. Then $\mathbb{P}(A) = g(\mathrm{TTTTH}) + g(\mathrm{TTH}) = q^4(1 - q) + q^2(1 - q)$ as expected.

Suppose $A, B \subseteq \Omega$ are disjoint. Then, since A, B are countable,

$$\mathbb{P}(A \cup B) = \sum_{d\in A\cup B} g(d) = \sum_{d\in A} g(d) + \sum_{d\in B} g(d) = \mathbb{P}(A) + \mathbb{P}(B)$$

By definition $\mathbb{P}(\emptyset) = 0$. Finally $\mathbb{P}(\Omega) = \sum_{n=1}^{\infty} q^{n-1}(1 - q)$ which we verify is equal to 1 using Mathematica (`Sum[p^(n-1)(1-p), {n,1,Infinity}]//FullSimplify`). Therefore $\mathbb{P}$ is a well defined probability measure on $(\Omega, \mathcal{F})$.

Exercise 1.4

Suppose X is a continuous random variable with distribution F_X. Let g be a strictly increasing continuous function. Define $Y = g(X)$.

(a) What is F_Y, the distribution of Y?

(b) What is f_Y, the density of Y?

Solution

Suppose X is a continuous random variable with distribution F_X. Let $g : D \to \mathbb{R}$ be a strictly increasing continuous function.

(a) Since g is strictly increasing and continuous, then it has an inverse g^{-1} on its range $R := g(D)$. Since g is strictly increasing, by the intermediate value theorem, we know R is an interval.

Let $a = \inf R$ and $b = \sup R$. Then, for $y \leq a$ we have $F_Y(y) = 0$ and for $y \geq b$ we have $F_Y(y) = 1$

For $y \in R$, assuming g^{-1} is differentiable, we have,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Thus,

$$F_Y(y) = \begin{cases} 1 & y \geq b \\ F_X(g^{-1}(y)) & a < y < b \\ 0 & y \leq a \end{cases}$$

Since $\{a, b\}$ is a set of measure zero it doesn't really matter how we define F_Y on these points.

(b) Recall $F_Y(y) = \int_{-\infty}^{y} f_Y(u)du$ for a continuous random variable Y. Thus for $a < y < b$, observing that $dF_X(y)/dy = f_x(y)$ by the above result, and assuming g^{-1} is differentiable,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y)) = f_x(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

For $y < a$ and $a > b$ we have $F_Y(y)$ constant so $f_Y(y)$ is zero.

Thus,

$$f_Y(y) = \begin{cases} 0 & y > b \\ f_X(g^{-1}(y))\dfrac{d}{dx}g^{-1}(y) & a < y < b \\ 0 & y < a \end{cases}$$

Since $\{a, b\}$ is a set of measure zero it doesn't really matter how we define F_Y on these points.

Exercise 1.5

Suppose X is a continuous random variable with distribution F_X. Find F_Y where Y is given by:

(a) X^2

(b) $\sqrt{|X|}$

(c) $\sin(X)$

(d) $F_X(X)$

Solution

Since X is continuous we can write $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(u)du$ for some density $f_X : \mathbb{R} \to [0, \infty)$.

In general, $\mathbb{P}(a < X < b) = \int_a^b f_X(u)du = F_X(b) - F_X(a)$ by the FTC.

(a) Let $Y = X^2$. We then have,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y)$$

If $y \leq 0$ then $F_Y(y) = \mathbb{P}(X^2 \leq y \leq 0) = 0$ since $X^2 = 0$ on a set of measure 0).
If $y > 0$ then,

$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(|X| \leq \sqrt{y}) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Finally,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \end{cases}$$

(b) Let $Y = \sqrt{|X|}$. We then have,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sqrt{|X|} \leq y)$$

If $y \leq 0$ then $F_Y(y) = \mathbb{P}(\sqrt{|X|} \leq y \leq 0) = 0$ (since $\sqrt{(|X|)} = 0$ on a set of measure 0).
If $y > 0$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sqrt{|X|} \leq y) = \mathbb{P}(|X| \leq y^2) = \mathbb{P}(-y^2 \leq X \leq y^2) = F_X(y^2) - F_X(-y^2)$$

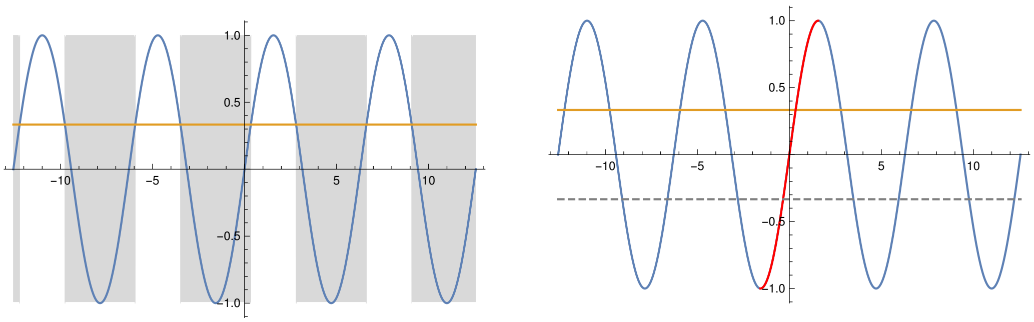
Finally,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(y^2) - F_X(-y^2) & y > 0 \end{cases}$$

(c) Let $Y = \sin(X)$. We then have,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sin(X) \leq y)$$

If $y \leq= -1$ then $F_Y(y) = \mathbb{P}(\sin(X) \leq y \leq 0) = 0$ since $\sin(X) = 1$ on a set of measure 0.



(a) intervals for which $\sin(X) < y$

(b) portion of $\sin(X)$ which is inverted to get $\arcsin(X)$ displayed

Figure 1: Exercise 1.5(c)

Similarly, if $y \geq 1$ then $F_Y(y) = \mathbb{P}(\sin(X) \leq 1 \leq y) = 1$.

Figure 1a shows the intervals for which $\sin(X) < y$ is grey, for some y drawn in orange. Figure 1b shows which part of sin the inverse is defined on in red. We see that we can find the intervals for which $\sin(X) < y$ by finding the intersection of $\sin(X)$ and y as well as the intersection of $\sin(X)$ and $-y$ and then appropriately shifting these endpoints.

If $-1 < y < 1$ then, $\sin(X) \leq y$ if and only if for some integer k,

$$\arcsin(y) + 2k\pi < X < \pi + \arcsin(-y) + 2k\pi = \pi - \arcsin(y) + 2k\pi$$

Thus, since $\mathbb{Z}$ is countable,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(\arcsin(y) + 2k\pi < X < \pi - \arcsin(y) + 2k\pi, \text{for any } k \in \mathbb{Z}) \\ &= \mathbb{P}\left(x \in \bigvee_{k\in\mathbb{Z}} (\arcsin(y) + 2k\pi < X < \pi - \arcsin(y) + 2k\pi) \right) \\ &= \sum_{k\in\mathbb{Z}} \mathbb{P}(\arcsin(y) + 2k\pi < X < \pi - \arcsin(y) + 2k\pi) \\ &= \sum_{k\in\mathbb{Z}} [F_X(\arcsin(y) + 2k\pi) - F_X(\pi - \arcsin(y) + 2k\pi)] \end{aligned}$$

Therefore,

$$F_Y(y) = \begin{cases} 0 & y \leq -1 \\ \sum_{k\in\mathbb{Z}} [F_X(\arcsin(y) + 2k\pi) - F_X(\pi - \arcsin(y) + 2k\pi)] & -1 < x < 1 \\ 1 & y \geq 1 \end{cases}$$

(d) Let $Y = F_X(X)$. We then have,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(F_X(X) \leq y)$$

Recall F_X is a (not necessarily strictly) increasing function from $\mathbb{R}$ to $[0,1]$. We deal with this in the following way: Define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \sup\{a : F_X(a) \leq y\}$. If F_X is strictly increasing then $g(y) = y$ for all y as desired. Now define $\hat{F}_X : \mathbb{R} \to [0,1]$ as $\hat{F}_X(y) = F_x(g(y))$. This function is strictly increasing and injective so therefore invertible. Denote the inverse by F_X^{-1}.

Recall F_X goes from 0 to 1. If $y < 0$ then $F_Y(y) = \mathbb{P}(F_X(X) \leq y < 0) = 0$. If $y > 1$ then $F_Y(y) = \mathbb{P}(F_X(X) \leq 1 < y) = 1$.

For $0 < y < a$, we have,

$$F_Y(y) = \mathbb{P}(F_X(X) \leq y) = \mathbb{P}(X \leq \hat{F}_X^{-1}(y)) = F_X(\hat{F}_X^{-1}(y)) = y$$

Thus,

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 < y < 1 \\ 1 & y > 1 \end{cases}$$

Again since the points $\{0, 1\}$ is a set of measure zero it doesn't really matter how we define F_Y on these points.

Exercise 1.6

Suppose X is a continuous random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let f be the density of X under $\mathbb{P}$ and assume $f > 0$. Let g be the density function of a random variable. Define $Z := g(X)/f(X)$.

(a) Show that $Z \equiv \mathrm{d}\tilde{\mathbb{P}}/\mathrm{d}\mathbb{P}$ defines a Radon-Nikodym derivative.

(b) What is the density of X under $\tilde{\mathbb{P}}$?

Solution

Suppose X is a continuous random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let f be the density of X under $\mathbb{P}$ and assume $f > 0$. Let g be the density function of a random variable. Define $Z := g(X)/f(X)$.

(a) Since g is a density function, $g \geq 0$. Thus, $Z = g(X)/f(X) \geq 0$. Moreover, since g is a density, $\int_\Omega g(x)dx = 1$. Thus,

$$\mathbb{E}Z = \int_\Omega \frac{g(x)}{f(x)} f(x)dx = \int_\Omega g(x)dx = 1$$

Then define $\tilde{\mathbb{P}}(A) = \mathbb{E}Z\mathbb{1}_A$.

Then Z is a Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with resepct to $\mathbb{P}$.

(b) We first compute the distribution of X under $\tilde{\mathbb{P}}$ assuming $\Omega = \mathbb{R}$.

$$F_X(y) = \tilde{\mathbb{P}}(X \leq y) = \mathbb{E}Z\mathbb{1}_{\{X\leq y\}} = \int_\Omega \frac{g(x)}{f(x)}\mathbb{1}_{\{X\leq y\}} f(x)dx = \int_{-\infty}^{y} g(x)dx$$

Thus,

$$f_X(y) = \frac{d}{dy}\int_{-\infty}^{y} g(x)dx = g(y)$$