

AMATH 562 Assignment 8

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Exercise 8.1

Compute $d(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t . Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution

Write $f(x) = x^4$ so that $f(W_t) = W_t^4$. Then, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula gives,

$$dW_t^4 = f'(W_t)dW_t + \frac{1}{2}f''(W_t)d[W, W]_t = 4W_t^3dW_t + \frac{12}{2}W_t^2d[W, W]_t$$

Thus, writing $d[W, W]_t = dt$ we have,

$$dW_t^4 = 4W_t^3dW_t + 6W_t^2dt$$

Thus, since $W_0 = 0$,

$$W_T^4 = W_T^4 - W_0^4 = 4 \int_0^T W_t^3dW_t + 6 \int_0^T W_t^2dt$$

Recall Itô integrals are martingales so that,

$$\mathbb{E} \left[\int_0^T W_t^3dW_t \right] = 0$$

Note also that since $\mathbb{E}[W_t^2] = t$,

$$\mathbb{E} \left[\int_0^T W_t^2dt \right] = \int_0^T \mathbb{E}[W_t^2] dt = \int_0^T tdt = \frac{T^2}{2}$$

Therefore,

$$\mathbb{E}[W_T^4] = 4\mathbb{E} \left[\int_0^T W_t^3dW_t \right] + 6\mathbb{E} \left[\int_0^T W_t^2dt \right] = 6\frac{T^2}{2} = 3T^2$$

Similarly, we have,

$$W_T^6 = 6 \int_0^T W_t^5dW_t + \frac{6 \cdot 5}{2} \int_0^T W_t^4dt$$

Therefore, since $\mathbb{E}[W_t^4] = 3t^2$,

$$\mathbb{E}[W_T^6] = 6\mathbb{E} \left[\int_0^T W_t^5dW_t \right] + 15\mathbb{E} \left[\int_0^T W_t^4dt \right] = 15 \int_0^T \mathbb{E}[W_t^4] dt = 15 \int_0^T 3t^2dt = 15T^3$$

Exercise 8.2

Find an explicit expression for Y_T where,

$$dY_t = rdt + \alpha Y_t dW_t$$

Hint: Multiply the above equation by $F_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution

Let $f(x, y) = \exp(-\alpha x + \frac{1}{2}\alpha^2 y)$ so that,

$$f_x(W_t, t) = -\alpha F_t \quad f_y(W_t, t) = \frac{\alpha^2}{2} F_t \quad f_{xx}(W_t, t) = \alpha^2 F_t$$

Then $F_t = f(W_t, t)$, so by Itô's formula and the heuristic $(dW_t)^2 = dt, (dt)^2 = dt dW_t = 0$,

$$\begin{aligned} dF_t &= df(W_t, t) = f_y(W_t, t)dt + f_x(W_t, t)dW_t + \frac{1}{2}f_{xx}(W_t, t)(dW_t)^2 \\ &= \frac{\alpha^2}{2}F_t dt - \alpha F_t dW_t + \frac{\alpha^2}{2}F_t dt \\ &= \alpha^2 F_t dt - \alpha F_t dW_t \end{aligned}$$

Using our heuristics we have,

$$d[F, Y]_t = (dF_t)(dY_t) = (\alpha^2 F_t dt - \alpha F_t dW_t)(rdt + \alpha Y_t dW_t) = -\alpha^2 F_t Y_t (dW_t)^2 = -\alpha^2 F_t Y_t dt$$

By the product rule we have,

$$\begin{aligned} d(F_t Y_t) &= F_t dY_t + Y_t dF_t + d[F, Y]_t \\ &= F_t(rdt + \alpha Y_t dW_t) + Y_t(\alpha^2 F_t dt - \alpha F_t dW_t) - \alpha^2 F_t Y_t dt \\ &= rF_t dt \end{aligned}$$

In integral form,

$$F_t Y_t - F_0 Y_0 = \int_0^t r F_s ds = \int_0^t r e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} ds$$

We can add $F_0 Y_0 = Y_0$ and divide by F_t yielding,

$$Y_t = Y_0 + r e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} ds$$

Exercise 8.3

Suppose X , Δ , and Π are given by,

$$dX_t = \sigma X_t dW_t, \quad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \quad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) f(t, x) = 0$$

for all (t, x) , then Π is a martingale with respect to a filtration \mathcal{F}_t for W .

Solution

We have,

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) = \frac{\partial^2}{\partial x \partial t} + \frac{1}{2} \sigma^2 \left[x^2 \frac{\partial^3}{\partial x^3} + 2x \frac{\partial^2}{\partial x^2} \right]$$

Thus, using the condition for f we have,

$$\frac{\partial^2 f}{\partial x \partial t} + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3} = -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}$$

Using our heuristics we have,

$$d[X, X] = \sigma^2 X_t^2 (dW_t)^2 = \sigma^2 X_t^2 dt$$

Similarly,

$$d[X, t] = d[t, X] = d[t, t] = 0$$

Therefore, by Itô's formula,

$$\begin{aligned} d\Delta_t &= \frac{\partial^2 f}{\partial x \partial t}(t, X_t) dt + \frac{\partial^2 f}{\partial x^2}(t, X_t) dX_t + \frac{1}{2} d[X, X] \\ &= \frac{\partial^2 f}{\partial x \partial t}(t, X_t) dt + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) dW_t + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3}(t, X_t) dt \\ &= -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) dt + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) dW_t \end{aligned}$$

Therefore,

$$d[X, \Delta]_t = (dX_t)(d\Delta_t) = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) (dW_t)^2 = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) dt$$

Finally, we have,

$$\begin{aligned} d\Pi_t &= d(X_t \Delta_t) = X_t d\Delta_t + \Delta_t dX_t + d[X, \Delta]_t \\ &= X_t \left(-\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) dt + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) dW_t \right) + \sigma X_t \frac{\partial f}{\partial x}(t, X_t) dW_t + \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2} dt \\ &= \sigma X_t \left(X_t \frac{\partial^2 f}{\partial x^2}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) \right) dW_t \end{aligned}$$

Since there is no dt dependence this is an Itô integral and therefore a martingale with respect to a filtration for W . (there are probably some technical assumptions we need about X and f , but in class we never dealt with these) \square

Exercise 8.4

Suppose X is given by,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

For any smooth function f define,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(s, X_s) \frac{\partial^2}{\partial x^2} \right) f(s, X_s) ds$$

Show that M^f is a martingale with respect to a filtration \mathcal{F}_t for W .

Solution

We first compute,

$$d[X, X]_t = (dX_t)(dX_t) = \sigma^2(t, X_t)(dW_t)^2 = \sigma^2(t, X_t)dt$$

We then have,

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} d[X, X]_t \\ &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)[\mu(t, X_t)dt + \sigma(t, X_t)dW_t] + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2 f}{\partial x^2} dt \\ &= \left(\frac{\partial}{\partial t} + \mu(t, X_t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2}{\partial x^2} \right) f(t, X_t)dt + \sigma(t, X_t) \frac{\partial f}{\partial x} dW_t \end{aligned}$$

Finally, since $f(0, X_0)$ is a constant,

$$\begin{aligned} dM_t^f &= df(t, X_t) - \left(\frac{\partial}{\partial t} + \mu(t, X_t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2}{\partial x^2} \right) f(t, X_t)dt \\ &= \sigma(t, X_t) \frac{\partial f}{\partial x} dW_t \end{aligned}$$

Since there is no dt dependence this is an Itô integral and therefore a martingale with respect to a filtration for W . \square