AMATH 562 Assignment 7

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Exercise 7.1

Let W be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration for W. Show that $W(t)^2 - t$ is a martingale with respect to the filtration $\mathbb{F}$.

Solution

Suppose $X \sim \mathcal{N}(0, \sigma^2)$. Then,

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 0^2 = \mathbb{E}[X^2]$$

Let $0 \leq s \leq t$. By the definition of a filtration, $(W(t) - W(s))$ is independent of $\mathcal{F}_s$. Moreover, by the definition of Brownian Motion we have $W(t) - W(s) \sim \mathcal{N}(0, t-s)$. Thus,

$$\mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] = \mathbb{E}\left[(W(t) - W(s))^2\right] = (t - s)$$

Since $W(s) \in \mathcal{F}_s$, by "taking out what is known" we have,

$$\begin{aligned}
\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] &= W(s)\mathbb{E}\left[W(t) \middle| \mathcal{F}_s\right] = W(s)W(s) = W(s)^2 \\
\mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] &= W(s)\mathbb{E}\left[W(s) \middle| \mathcal{F}_2\right] = W(s)W(s) = W(s)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[W(t)^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[(W(t) - W(s) + W(s))^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W(s)^2 - t\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2 \middle| \mathcal{F}_s\right] + 2\mathbb{E}\left[W(t)W(s) \middle| \mathcal{F}_s\right] - \mathbb{E}\left[W(s)^2 \middle| \mathcal{F}_2\right] - \mathbb{E}\left[t\right] \\
&= (t - s) + 2W(s)^2 - W(s)^2 - t \\
&= W(s)^2 - s
\end{aligned}$$

This proves $W(t) - t$ is a martingale with respect to the filtration $\mathbb{F}$. $\square$

Exercise 7.2

Compute the characteristic function of $W(N(t))$ where N is a Poisson process with intensity λ and the Brownian motion W is independent of the Poisson process N.

Solution

The characteristic function is defined as,

$$\phi(s) = \mathbb{E}e^{isW(N(t))}$$

We condition on $N(t)$ using iterated conditioning,

$$\mathbb{E}\left[e^{isW(N(t))}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right]$$

The characteristic function of $Z \sim \mathcal{N}(\mu, \sigma^2)$ is $\phi_Z(s) = \exp(i\mu s - \sigma^2 s^2/2)$. At time t, $W(t)$ is normally distributed with mean zero and variance t. Thus,

$$\mathbb{E}\left[\mathbb{E}\left[e^{isW(N(t))}\middle|N(t)\right]\right] = \mathbb{E}\left[e^{-N(t)s^2/2}\right]$$

Since $N(t)$ is a Poisson process with parameter λ, then $N(t) = k$ with probability $(\lambda t)^k e^{-\lambda t}/k!$. Thus,

$$\mathbb{E}\left[e^{-N(t)s^2/2}\right] \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} e^{-ks^2/2} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k$$

Simplifying yields,

$$e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{-s^2/2}\right)^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda t e^{-s^2/2}\right)^k = e^{-\lambda t} \exp\left(\lambda t e^{-s^2/2}\right) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

That is, the characteristic function $\phi(s)$ of $W(N(t))$ is,

$$\phi(s) = \exp\left(\lambda t \left(e^{-s^2/2} - 1\right)\right)$$

Exercise 7.3

The n-th variation of a function f, over the interval $[0,T]$ is defined as,

$$V_T(n,f) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^n, \quad \Pi = \{0 = t_0, t_1, \ldots, t_m = T\}, \quad \|\Pi\| = \max_j (t_{j+1} - t_j)$$

Show that $V_T(1,W) = \infty$ and $V_T(3,W) = 0$, where W is a Brownian motion.

Solution

We first prove that if $f_n \to 0$ and $|g_n| \leq M$ for some $|M| < \infty$ then $(f_n g_n) \to 0$.

Indeed, fix $\varepsilon > 0$. Then, by convergence of f_n there is some $N \in \mathbb{N}$ such that $|f_n| < \varepsilon/M$ for all $n \geq N$. Then,

$$|f_n g_n| = |f_n||g_n| \leq |f_n| M < (\varepsilon/M)M = \varepsilon$$

This proves $f_n g_n \to 0$. $\square$

Write,

$$V_T(k+1,W) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^{k+1} = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)|$$

Let, $M_\Pi = \max_j |W(t_{j+1}) - W(t_j)|$ for a given partition Π. Then,

$$\begin{aligned} \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k |W(t_{j+1}) - W(t_j)| &\leq \lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k M_\Pi \\ &= \lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^k \end{aligned}$$

Provided, $|V_T(k,T)| = V_T(k,T)$ is not infinite,

$$\lim_{\|\Pi\|\to 0} M_\Pi \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right) \left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2\right)$$

Since $W(t)$ is continuous, $|W(t_{j+1}) - W(t_j)| \to 0$ as $\|\Pi\| \to 0$ since $t_{j+1} - t_j \to 0$. In particular, this means that $M_\Pi \to 0$ as $\|\Pi\| \to 0$.

Thus,

$$0 \geq V_T(k+1,W) = \left(\lim_{\|\Pi\|\to 0} M_\Pi\right) \left(\lim_{\|\Pi\|\to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^k\right) \leq 0 \cdot N = 0$$

Recall $V_T(2,W) = T < \infty$. Then, by above, $V_T(3,W) = 0$. $\square$

Suppose, for the sake of contradiction that $V_T(1,W) \neq \infty$. Clearly $V_T(1,W) \geq 0$, so $V_T(1,W)$ is bounded above and below by finite constants. Then, by above, $V_T(2,W) = 0$, a contradiction (for $T > 0$). This proves $V_T(1,W) = \infty$. $\square$

Exercise 7.4

Define

$$X_t = \mu t + W_t \qquad\qquad \tau_m := \inf\{t \geq 0 : X_t = m\}$$

Show that Z is a martingale where,

$$Z_t = \exp(\sigma X_t - (\sigma\mu + \sigma^2/2)t)$$

Assume $\mu > 0$ and $m \geq 0$. Assume further that $\tau_m < \infty$ with probability one and the stopped process $Z_{t\wedge\tau_m}$ is a martingale. Find the Laplace transform $\mathbb{E}e^{-\alpha\tau_m}$.

Solution

Let $0 \leq s \leq t$. Rewrite,

$$\mathbb{E}\left[Z_t \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(\mu t + W_t) - (\sigma\mu+\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right]$$

Now, pulling out what is known,

$$\mathbb{E}\left[e^{\sigma W_t - (\sigma^2/2)t} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma(W_t - W_s) + \sigma W_s - (\sigma^2/2)t)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right]$$

By the property of independent increments,

$$e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right] = e^{\sigma W_s - (\sigma^2/2)t}\mathbb{E}\left[e^{\sigma(W_t - W_s)}\right] = e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2}$$

Finally,

$$e^{\sigma W_s - (\sigma^2/2)t}e^{\sigma^2(t-s)/2} = e^{\sigma W_s - (\sigma^2/2)s} = e^{\sigma(\mu s + W_s) - (\sigma\mu+\sigma^2/2)s} = e^{\sigma X_2 - (\sigma\mu+\sigma^2/2)s}$$

This proves Z_t is a martingale. □

Define $s = \min\{t, \tau_m\}$. Fix $m \geq 0$ and define,

$$Z^{(m)} = \left(Z_t^{(m)}\right)_{t\geq 0}, \qquad\qquad Z_t^{(m)} = Z_s$$

Then, using the fact that Z_t is a martingale we have,

$$1 = Z_0^{(m)} = \mathbb{E}\left[Z_t^{(m)}\right] = \mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right]$$

If $\tau_m = \infty$ then $X_t < m$ for all t. Thus, since $\sigma \geq 0, \mu > 0$,

$$e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t} \leq e^{\sigma m - (\sigma\mu+\sigma^2/2)t} < \infty$$

Therefore, since $\mathbb{P}(\tau_m < \infty) = 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right) + \mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma X_s - (\sigma\mu+\sigma^2/2)s}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma X_t - (\sigma\mu+\sigma^2/2)t}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma X_{\tau_m} - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= 0 + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m - (\sigma\mu+\sigma^2/2)\tau_m}\right)\right]
\end{aligned}$$

Similarly, since $\sigma \geq 0, \mu > 0$, $e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m)} < \infty$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\tau_m=\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right) + \mathbb{1}_{\{\tau_m<\infty\}}\left(e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right)\right] \\
&= \mathbb{E}\left[e^{\sigma m-(\sigma\mu+\sigma^2/2)\tau_m}\right]
\end{aligned}$$

Then, setting $\alpha = (\sigma\mu + \sigma^2/2)$,

$$e^{-\sigma m} = \mathbb{E}\left[e^{-(\sigma\mu+\sigma^2/2)\tau_m}\right] = \mathbb{E}\left[e^{-\alpha\tau_m}\right]$$

We solve the equation, $\alpha = (\sigma\mu + \sigma^2/2)$ for σ using the quadratic equation, yielding,

$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$$

However, $\sigma, \alpha \geq 0$ so we must take $\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$. Thus,

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{\left(\mu-\sqrt{\mu^2+2\alpha}\right)m}$$