AMATH 562 Assignment 10

Tyler Chen

Exercise 10.1

Let $P = (P_t)_{t\geq 0}$ be a Poisson process with intensity λ.

(a) What is the Lévy Measure ν of P.

(b) Let $\mathrm{d}X_t = \mathrm{d}P_t$. Define $u(x,t) := \mathbb{E}\left[\varphi(X_T)|X_t = x\right]$. Find $u(t,x)$ and verify it solves the Kolmogorov Backward equation.

Solution

(a) We have,

$$\nu(U) = \mathbb{E}\left[N(1,U)\right] = \mathbb{E}\left[\sum_{0\leq s\leq 1} \mathbb{1}_{\Delta P_s \in U}\right] = \mathbb{E}\left[\sum_{i=1}^{P_1} \mathbb{1}_{1\in U}\right] = \mathbb{E}\left[P_1\right]\mathbb{1}_{1\in U} = \lambda \mathbb{1}_{1\in U}$$

(b) Integrating $\mathrm{d}X_t = \mathrm{d}P_t$ from 0 to t gives, $X_t - X_0 = P_t - P_0$. Since $P_0 = 0$ we have,

$$X_t = X_0 + P_t$$

First observe,

$$\mathbb{P}(X_T = k|X_t = x) = \mathbb{P}(X_0 + P_T = k|X_0 + P_t = x) = \mathbb{P}(P_T = k - X_0|P_t = x - X_0)$$

Since P has independent increments, and since P is Markov,

$$\mathbb{P}(P_T = k - X_0|P_t = x - X_0) = \mathbb{P}(P_{T-t} = k - x) = \frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Thus,

$$u(t,x) = \mathbb{E}\left[\varphi(X_T)|X_t = x\right] = \sum_{k=x}^{\infty}\varphi(k)\mathbb{P}(X_T = k|X_t = x) = \sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!}e^{-\lambda(T-t)}$$

Reindexing with $n = k - x$,

$$u(t,x) = e^{-\lambda(T-t)}\sum_{k=x}^{\infty}\varphi(k)\frac{(\lambda(T-t))^{k-x}}{(k-x)!} = e^{-\lambda(T-t)}\sum_{n=0}^{\infty}\varphi(n+x)\frac{(\lambda(T-t))^{n}}{n!}$$

We now compute the generator $\mathcal{A}(t)$ for P. By definition,

$$\mathcal{A}(t)\varphi(x) = \lim_{s\to t^+}\frac{1}{s-t}\left[\mathcal{P}(t,s)\varphi(x) - \varphi(x)\right] = \lim_{s\to t^+}\frac{1}{s-t}\left[\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x)\right]$$

In a small interval $\mathrm{d}t$ the probability $X_{t+\mathrm{d}t} = X_t + 1$ is $\lambda\mathrm{d}t$ and probability $X_{t+\mathrm{d}t} = X_t$ is $(1-\lambda)\mathrm{d}t$. Therefore,

$$\mathcal{A}(t)\varphi(x) = \frac{1}{\mathrm{d}t}\left[\varphi(x+1)\lambda + \varphi(x)(1-\lambda) - \varphi(x)\right] = \lambda(\varphi(x+1) - \varphi(x))$$

Since the t-derivative of the $n = 0$ term is zero,

$$\begin{aligned}\sum_{n=0}^{\infty} \varphi(n+x)\partial_t \left[\frac{(\lambda(T-t))^n}{n!}\right] &= \sum_{n=1}^{\infty} \varphi(n+x)\partial_t \left[\frac{(\lambda(T-t))^n}{n!}\right] \\ &= \sum_{n=1}^{\infty} \varphi(n+x)(n)(-\lambda)\frac{(\lambda(T-t))^{n-1}}{n!} \\ &= -\lambda \sum_{n=1}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!}\end{aligned}$$

Observe, by the chain rule and assuming we can bring a derivative through a sum,

$$\begin{aligned}\partial_t u(t,x) &= \left[\partial_t e^{-\lambda(T-t)}\right] \sum_{n=0}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^n}{n!} + e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \varphi(n+x)\partial_t \left[\frac{(\lambda(T-t))^n}{n!}\right] \\ &= \lambda e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)} \sum_{n=1}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \varphi(n+x)\frac{(\lambda(T-t))^n}{n!} - \lambda e^{-\lambda(T-t)} \sum_{n=m}^{\infty} \varphi(m+1+x)\frac{(\lambda(T-t))^m}{m!} \\ &= \lambda(u(t,x) - u(t,x+1))\end{aligned}$$

Therefore the KBE is satisfied as

$$[\partial_t + \mathcal{A}]u(t,x) = \lambda(u(t,x) - u(t,x+1)) - \lambda(u(t,x+1) - u(t,x)) = 0, \qquad u(T,x) = \varphi(x)$$

Exercise 10.2

Solution

Exercise 10.3

Let $X = (X_t)_{t\geq 0}$ be a process defined by,

$$\begin{aligned}
\mathrm{d}X_t &= \mu_t X_t \mathrm{d}t + \sigma_t X_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{\gamma_t(z)} - 1\right) X_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
\mathrm{d}Y_t &= b_t Y_t \mathrm{d}t + a_t Y_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{g_t(z)} - 1\right) Y_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z)
\end{aligned}$$

where W is a one-dimensional Brownian motion, $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$, and $\mu, b, \sigma, a, \gamma, g$ are $\mathbb{F}$-adapted stochastic processes.

(a) Define $Z_t := X_t/Y_t$. Compute the differential $\mathrm{d}Z_t$. Your answer should not involve X_t or Y_t.

(b) Find μ_t so that Z is a martingale.

Solution

(a) Define $f(x, y) = x/y$. Then $Z_t = f(X_t, Y_t)$.
We have,

$$[(e^{\gamma_t(z)} - 1)X_t; (e^{g_t(z)} - 1)Y_t] \cdot \nabla f(X_{t^-}, Y_{t^-}) = (e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) + (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-})$$

We use Itô's formula to compute,

$$\begin{aligned}
\mathrm{d}Z_t = \mathrm{d}f(X_t, Y_t) = &\left(\mu_t X_t f_x + b_t Y_t f_y + \frac{1}{2}\left((\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_t) f_{xy} + (a_t Y_t)^2 f_{yy}\right)\right) \mathrm{d}t \\
&+ (\sigma_t X_t f_x + a_t Y_t f_y)\, \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-})\right) \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&+ \int_{\mathbb{R}} \Big(f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-}) \\
&\qquad - (e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) - (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-})\Big)\nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}$$

Now, using $f_x = 1/y, f_y = -x/y^2, f_{xy} = -1/y^2, f_{xx} = 0, f_{yy} = 2x/y^3$ we have,

$$\mu_t X_t f_x + b_t Y_t f_y = \mu_t X_t \left(\frac{1}{Y_t}\right) + b_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \mu_t Z_t - b_t Z_t$$

$$(\sigma_t X_t)^2 f_{xx} + 2(\sigma_t X_t)(a_t Y_t) f_{xy} + (a_t Y_t)^2 f_{yy} = 2(\sigma_t X_t)(a_t Y_t)\left(\frac{-1}{Y_t^2}\right) + a_t^2 Y_t^2 \left(\frac{2X_t}{Y_t^3}\right) = -2\sigma_t a_t Z_t + 2a_t^2 Z_t$$

$$\sigma_t X_t f_x + a_t Y_t f_y = \sigma_t X_t \left(\frac{1}{Y_t}\right) + a_t Y_t \left(\frac{-X_t}{Y_t^2}\right) = \sigma_t Z_t - a_t Z_t$$

$$f\left(X_{t^-} + (e^{\gamma_t(z)} - 1)X_{t^-}, Y_{t^-} + (e^{g_t(z)} - 1)Y_{t^-}\right) - f(X_{t^-}, Y_{t^-}) = \frac{e^{\gamma_t(z)}}{e^{g_t(z)}} Z_{t^-} - Z_{t^-}$$

$$\begin{aligned}
&(e^{\gamma_t(z)} - 1)X_{t^-} f_x(X_{t^-}, Y_{t^-}) + (e^{g_t(z)} - 1)Y_{t^-} f_y(X_{t^-}, Y_{t^-}) \\
&= (e^{\gamma_t(z)} - 1)X_{t^-} \left(\frac{1}{Y_{t^-}}\right) + (e^{g_t(z)} - 1)Y_{t^-} \left(\frac{-X_{t^-}}{Y_{t^-}^2}\right) \\
&= (e^{\gamma_t(z)} - 1)Z_{t^-} - (e^{g_t(z)} - 1)Z_{t^-}
\end{aligned}$$

Inserting these evaluated expressions into the original expression for $\mathrm{d}Z_t$ gives,

$$\begin{aligned}
\mathrm{d}Z_t = &\left(\mu_t - b_t - \sigma_t a_t + a_t^2\right) Z_t \mathrm{d}t + (\sigma_t - a_t)\, Z_t \mathrm{d}W_t \\
&+ \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - 1\right) Z_{t^-} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&\quad + \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) Z_{t^-} \nu(\mathrm{d}z)\mathrm{d}t
\end{aligned}$$

(b) We need the $\mathrm{d}t$ term to be zero. Therefore pick,

$$\mu_t = b_t + \sigma_t a_t - a_t^2 - \int_{\mathbb{R}} \left(\frac{e^{\gamma_t(z)}}{e^{g_t(z)}} - e^{\gamma_t(z)} + e^{g_t(z)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}t$$

Exercise 10.4

Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy Process and define $X = (X_t)_{t\geq 0}$ by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \mathrm{d}\eta_t$$

(a) Find X_t explicitly as a function of η.

(b) Assume $\eta_t = \sigma W_t + \int_{\mathbb{R}} z\tilde{N}(t, \mathrm{d}z)$. Compute $m(t) := \mathbb{E}X_t$ and $c(t,s) := \mathbb{E}(X_t - m(t))(X_s - m(s))$.

Solution

(a) Let $Y_t = X_t - \theta$ and $Z_t = e^{\kappa t}Y_t = f(t, Y_t)$, where $f(t,y) = e^{\kappa t}y$.
Then,

$$\mathrm{d}Y_t = \mathrm{d}X_t = -\kappa Y_t \mathrm{d}t + \mathrm{d}\eta_t$$

Recall the product rule (which applies to Lévy Itô processes),

$$\mathrm{d}(U_t V_t) = U_{t^-}\mathrm{d}V_t + V_{t^-}\mathrm{d}U_t + \mathrm{d}[U,V]_t$$

Therefore,

$$\mathrm{d}Z_t = \mathrm{d}(e^{\kappa t}Y_t) = e^{\kappa t^-}\mathrm{d}Y_t + Y_{t^-}\mathrm{d}e^{\kappa t} + \mathrm{d}[e^{\kappa t}, Y]_t$$

Using our heuristics we have $\mathrm{d}(e^{\kappa t})\mathrm{d}Y_t = 0$. Therefore, since t^- and t can be "treated the same" on $\mathrm{d}t$ terms which are continuous,

$$\mathrm{d}Z_t = e^{\kappa t^-}\mathrm{d}Y_t + \kappa e^{\kappa t}Y_{t^-} = e^{\kappa t^-}\mathrm{d}\eta_t$$

Integrating we have,

$$Z_t = Z_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s$$

Therefore, since $Y_t = e^{-\kappa t}Z_t$, $Z_0 = Y_0$ so,

$$Y_t = e^{-\kappa t}\left(Y_0 + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right)$$

Finally, since $X_t = \theta + Y_t$, $Y_0 = X_0 - \theta$ so,

$$X_t = \theta + e^{-\kappa t}\left(X_0 - \theta + \int_0^t e^{\kappa s}\mathrm{d}\eta_s\right) = \theta + e^{-\kappa t}(X_0 - \theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s$$

(b) We have,

$$\mathrm{d}\eta_t = \sigma \mathrm{d}W_t + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)$$

Observe, that since integrals with respect to $\mathrm{d}W_t$ and $\int_{\mathbb{R}} \tilde{N}(\mathrm{d}t, \mathrm{d}z)$ are martingales so,

$$\mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \mathbb{E}\left[\int_0^t e^{\kappa(s-t)}\sigma \mathrm{d}W_t + \int_0^t e^{\kappa(s-t)}\int_{\mathbb{R}} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)\right] = 0$$

Therefore,

$$m(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[\theta + e^{-\kappa t}(X_0 - \theta) + \int_0^t e^{\kappa(s-t)}\mathrm{d}\eta_s\right] = \theta + e^{-\kappa t}(X_0 - \theta)$$

Clearly,

$$X_t - m(t) = \int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u$$

Without loss of generality assume $t \geq s$. Then, using the independent increments property to write the expectation of a product as the product of expectations,

$$\begin{aligned}
\mathbb{E}\left[\left(X_t - m(t)\right)\left(X_s - m(s)\right)\right] &= \mathbb{E}\left[\left(\int_0^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^s e^{\kappa(u-t)}\mathrm{d}\eta_u + \int_s^t e^{\kappa(u-t)}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa(v-s)}\mathrm{d}\eta_v\right)\right] \\
&= \mathbb{E}\left[e^{-\kappa(t+s)}\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2 + e^{-\kappa(t+s)}\left(\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right)\left(\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right)\right] \\
&= e^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] + e^{-\kappa(t+s)}\mathbb{E}\left[\int_s^t e^{\kappa u}\mathrm{d}\eta_u\right]\mathbb{E}\left[\int_0^s e^{\kappa v}\mathrm{d}\eta_v\right]
\end{aligned}$$

We now note that, Lévy processes without a $\mathrm{d}t$ term are martingales so that,

$$\mathbb{E}\left[\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right] = \mathbb{E}\left[\int_0^s e^{\kappa u}\left(\sigma \mathrm{d}W_u + \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}u, \mathrm{d}z)\right)\right] = 0$$

Define,

$$Z_s = \int_0^s e^{\kappa u}\mathrm{d}\eta_u$$

Then,

$$\mathrm{d}Z_s = e^{\kappa s}\mathrm{d}\eta_s = \sigma e^{\kappa s}\mathrm{d}W_s + \int_{\mathbb{R}} e^{\kappa s} z\tilde{N}(\mathrm{d}s, \mathrm{d}z)$$

Using Itô's isometry we have,

$$\mathbb{E}\left[\left(\int_0^s e^{\kappa u}\mathrm{d}\eta_u\right)^2\right] = \mathbb{E}\left[\int_0^s \left(\sigma^2 e^{2\kappa u} + \int_{\mathbb{R}} e^{2\kappa u} z^2\nu(\mathrm{d}z)\right)\mathrm{d}u\right] = \mathbb{E}\left[\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)\frac{e^{2\kappa s} - 1}{2\kappa}\right]$$

Therefore,

$$c(t,s) = e^{-\kappa(t+s)}\frac{e^{2\kappa s} - 1}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right) = \frac{e^{\kappa(s-t)} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

We can remove our assumption that $t \geq s$ and write,

$$c(t,s) = \frac{e^{-\kappa|t-s|} - e^{-\kappa(t+s)}}{2\kappa}\left(\sigma^2 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)\right)$$

Exercise 10.5

Let X be the following one-dimensional jump-diffusion

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t + \int_{\mathbb{R}} \gamma(t, X_{t^-}, z)\tilde{N}(t, \mathrm{d}z),$$

where W is a one-dimensional Brownian motion and $\tilde{N}$ is a one-dimensional compensated Poisson random measure on $\mathbb{R}$. Derive using the Lévy-Itô formula the infinitesimal generator $\mathcal{A}(t)$ of the X process,

$$\mathcal{A}(t)\varphi(x) := \lim_{s \to t^+} \frac{\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x)}{s - t}$$

Solution

Since $\mathbb{E}[\varphi(X_t)|X_t = x] = \varphi(x)$,

$$\mathbb{E}\left[\varphi(X_s)|X_t = x\right] - \varphi(x) = \mathbb{E}\left[\varphi(X_t) + \int_t^s \mathrm{d}\varphi(X_u)\right] - \varphi(x) = \mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right]$$

From the Lévy-Itô formula we have,

$$\begin{aligned}
\mathrm{d}\varphi(X_u) = &\left(\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\right)\mathrm{d}u + \sigma(u, X_u)\varphi'(X_u)\mathrm{d}W_u \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-})\Big)\tilde{N}(\mathrm{d}u, \mathrm{d}z) \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\mathrm{d}u
\end{aligned}$$

We note that as integrals with respect to W and $\tilde{N}$ are martingales that,

$$\begin{aligned}
\mathbb{E}\left[\int_t^s \mathrm{d}\varphi(X_u)\right] = \mathbb{E}\Bigg[\int_t^s \Bigg(&\mu(u, X_u)\varphi'(X_u) + \frac{1}{2}\sigma(u, X_u)^2\varphi''(X_u)\mathrm{d}u \\
&+ \int_{\mathbb{R}} \Big(\varphi(X_{u^-} + \gamma(u, X_{u^-}, z)) - \varphi(X_{u^-}) - \gamma(u, X_{u^-}, z)\varphi'(X_{u^-})\Big)\nu(\mathrm{d}z)\Bigg)\mathrm{d}u\Bigg]
\end{aligned}$$

Thus, taking the limit as $s \to t^+$,

$$\mathcal{A}(t)\varphi(x) = \left(\mu(t, X_t)\partial_x + \frac{1}{2}\sigma(t, X_t)\partial_x^2 + \int_{\mathbb{R}} \nu(\mathrm{d}z)\left(\theta_{\gamma(t, X_t, z)} - 1 - \gamma(t, X_t, z)\partial_x\right)\right)\varphi(x)$$