AMATH 586 Numerical SDE Solvers

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I INTRODUCTION

A Stochastic Differential Equation (SDE) is an equation of the form,

$$\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W_t \tag{1}$$

where W_t denotes a standard Brownian Motion [1].

A solution to (1) is an stochastic process which satisfies (1). In particular, a solution can be written in integral form as,

$$X_T - X_0 = \int_0^T \mu(t,X_t)\mathrm{d}t + \int_0^T \sigma(t,X_t)\mathrm{d}W_t \tag{2}$$

II BROWNIAN MOTION / ITÔ PROCESSES

There are many characterizations of Brownian Motion. Perhaps the most standard is the following definition.

Definition. *A Brownian Motion is a stochastic process $W=(W_t)_{t\geq 0}$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ satisfying,*

1. $W_0 = 0$
2. $(W_d - W_c) \perp\!\!\!\perp (W_b - W_a)$ *for* $0 \leq a \leq b \leq c \leq d$*)*
3. $(W_t - W_s) \sim \mathcal{N}(0, t-s)$ *for* $0 \leq s \leq t$
4. *the map* $t \to W_t$ *is continuous almost surely*

An Itô drift-diffusion process is a process of the form,

$$X_T = X_t + \int_t^T \mu(s,X_s)\mathrm{d}s + \int_t^T \sigma(s,X_s)\mathrm{d}W_s$$

III STOCHASTIC CALCULUS

We first introduce Riemann–Stieltjes integrals.

Definition. *For real valued functions f and g the Riemann–Stieltjes integral is defined as,*

$$\int_a^b f(x)\mathrm{d}g(x) := \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a,b]$, $\|\Pi\|$ is the length of the largest subinterval, and c_i is any point in $[x_i, x_{i+1}]$.

We note that if $g(x) = x$ the Riemann–Stieltjes integral is the standard Riemann integral and that if g is continuously differentiable,

$$f(g(T)) - f(g(t)) = \int_t^T \mathrm{d}f(g(s)) = \int_t^T f'(g(s))g'(s)\mathrm{d}s$$

Brownian motion and many processes involving Brownian motion are not differentiable. Itô's Lemma gives us a way to compute the analogous result for a class of stochastic processes called Itô (drift-diffussion) processes. For our purposes we can think of Itô processes as processes with an integral with respect to t and an integral with respect to W_t.

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x^2 + \cdots$$

We can replace $\mathrm{d}x$ with $\mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$ and simplify using the heuristics,

$$\mathrm{d}t\mathrm{d}t = 0, \qquad \mathrm{d}t\mathrm{d}W_t = 0, \qquad \mathrm{d}W_t\mathrm{d}W_t = \mathrm{d}t \tag{3}$$

Thus,

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}(\mu\mathrm{d}t + \sigma\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu^2\mathrm{d}t^2 + \mu\sigma\mathrm{d}t\mathrm{d}W_t + \sigma^2\mathrm{d}W_t) \\ &= \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma\frac{\partial f}{\partial x}\mathrm{d}W_t\end{aligned}$$

Itô's Lemma can be generalized to functions and processes of higher dimension.

Lemma (Itô)**.** *For $f : \mathbb{R}^n \to \mathbb{R}^n$ sufficiently differentiable and Itô process $X_t = [X_t^1, X_t^2, \ldots X_t^n]^T$,*

$$\mathrm{d}f(X_t) = \sum_{i=1}^n \left[\frac{\partial}{\partial x_i}f(X_t)\right]\mathrm{d}X_t^i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}f(X_t)\right]\mathrm{d}[X^i, X^j]_t \tag{4}$$

Similar to before, we compute $\mathrm{d}[X^i, X^j]_t$ by expanding $(\mathrm{d}X^i)(\mathrm{d}X^j)$ and using the heuristics in (3).

Consider the special case when $n = 2$, $X_t^1 = t$, and $X_t^2 = X_t$. Using our heuristics in (3) we have $\mathrm{d}[X^1, X^2]_t = (\mathrm{d}t)(\mathrm{d}X_t) = 0$ and $\mathrm{d}[X^1, X^1] = (\mathrm{d}t)(\mathrm{d}t) = 0$. Therefore, by (4),

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t}f(t, X_t) + \frac{\partial}{\partial x}f(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(t, X_t)\mathrm{d}[X, X]_t \tag{5}$$

$$= \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)f(t, X_t)\mathrm{d}t + \frac{\partial}{\partial x}f(t, X_t)\mathrm{d}X_t \tag{6}$$

IV EXAMPLE PROCESSES

In this section we provide some results about a few important stochastic processes. Proofs of the results presented here are readily available on the internet.

For constants θ, μ, σ, an Ornstein–Uhlenbeck (OU) process satisfies,

$$\mathrm{d}X_t = \theta(\mu - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \qquad \theta > 0 \tag{7}$$

We note that if X_t is away from μ it will tend towards this value in expectation. Figure **??** shows Euler–Maruyama method applied to (7). As expected, the trajectories all end up "centered" about μ. More precisely, (7) has solution,

$$X_t = X_0 \exp(-\theta t) + \mu(1 - \exp(-\theta t)) + \sigma \int_0^t \exp(-\theta(t-s))\mathrm{d}W_s$$

The mean of X_t is,

$$\mathbb{E}[X_t] = (X_0 - \mu)\exp(-\theta t) + \mu$$

Likewise, the variance of X_t is,

$$\mathbb{E}\left[(X_t - \mathbb{E}[X_t])^2\right] = \frac{\sigma^2}{2\theta}(1 - \exp(-2\theta t))$$

For constants μ and σ a Geometric Brownian Motion satisfies,

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \qquad t \in [0, T] \tag{8}$$

The solution to (8) is,

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + W_t\right)$$

REFERENCES

[1] Matthew Lorig, *Introduction to probability and stochastic processes*, 2006.