AMATH 514 Assignment 6

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Problem 8.8

Let A be a totally unimodular matrix. Show that the columns of A can be split into two classes such that the sum of the columns in one class, minus the sum of the columns in the other class, gives a vector with entries 0, +1, and −1 only.

Let e be the vector of all ones. Let $b = \lfloor \frac{1}{2}(Ae+1) \rfloor$ and $b' = \lfloor \frac{1}{2}(1-Ae) \rfloor$. Define a polytope,

$$P = \left\{ x : \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ b' \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then P is bounded as $x_i \in [0,1]$ for all i. Moreover, for all integers, $2k, 2k+1$,

$$\begin{aligned} \lfloor ((2k)+1)/2 \rfloor &= \lfloor k+1/2 \rfloor = k \geq (2k)/2, & \lfloor ((2k+1)+1)/2 \rfloor &= \lfloor k+1 \rfloor = k+1 \geq (2k+1)/2, \\ \lfloor (1-(2k))/2 \rfloor &= \lfloor 1/2-k \rfloor = -k \geq -(2k)/2, & \lfloor (1-(2k+1))/2 \rfloor &= \lfloor -k \rfloor = -k \geq -(2k+1)/2 \end{aligned}$$

Therefore, since Ae is an integer, $\frac{1}{2}e \in P$.

Since P is nonempty and bounded P has a vertex v. The matrix $[A; -A; I; -I]$ is totally unimodular since A is totally unimodular meaning v is integer. In particular, this means $v_i \in \{0,1\}$ for all i and, since $v \in P$,

$$\begin{aligned} Av \leq b = \left\lfloor \frac{1}{2}(Ae+1) \right\rfloor \leq \frac{1}{2}(Ae+1) \qquad &\Longrightarrow \qquad Ae - 2Av \geq -1 \\ -Av \leq b' = \left\lfloor \frac{1}{2}(1-Ae) \right\rfloor \leq \frac{1}{2}(1-Ae) \qquad &\Longrightarrow \qquad Ae - 2Av \leq 1 \end{aligned}$$

Now define $z = 1 - 2v$. Clearly z is integer with entries in $\{-1, 1\}$. Therefore Az is integer as A and z are each integer. Moreover, since $Az = A(e-2v) = Ae - 2Av$, by above we have, $-1 \leq Az \leq 1$. Together these mean Az has entries in $\{-1, 0, 1\}$.

Finally take one class as the rows corresponding to 1 entries in z and the other class corresponding to -1 entries in z. Then the result is proved. $\square$

I got a hint online to use these floor functions, but derived the proof without more.

Problem 8.9

Let A be a totally unimodular matrix and let b be an integer vector. Let x be an integer vector satisfying $x \geq 0; Ax \leq 2b$. Show that there exists integer vectors $x' \geq 0$ and $x'' \geq 0$ such that $Ax' \leq b$, $Ax'' \leq b$ and $x = x' + x''$.

Define,

$$P = \left\{ z : \begin{bmatrix} A \\ I \\ -I \\ -A \end{bmatrix} z \leq \begin{bmatrix} b \\ x \\ 0 \\ b - Ax \end{bmatrix} \right\}$$

Clearly P is bounded. We have $A(x/2) = (Ax)/2 \leq 2b/2 = b$. Then $A(x - x/2) \leq b$ so $-A(x/2) \leq Ax - b$. Clearly $0 \leq x/2 \leq x$. Therefore $x/2 \in P$.

Since P is nonempty and bounded P has a vertex x'. The matrix $[A; I; -I; -A]$ is totally unimodular since A is totally unimodular meaning x' is integer.

Define $x'' = x - x'$. Since $x' \in P$ and x is integer we have x'' integer with $0 \leq x'' \leq x$. Moreover, since $-Ax' \leq b - Ax$ we have $Ax - Ax' \leq b$ so that $Ax'' = A(x - x') \leq b$. $\square$

Problem 4.15

Let $D = (V, A)$ be a directed graph, and let $f : A \to \mathbb{R}_+$. Let $\mathcal{C}$ be the collection of directed circuits in D. For each directed circuit C in D let χ^C be the incidence vector of C. That is, $\chi^C : A \to \{0, 1\}$, with $\chi^C(a) = 1$ if C transverses a and $\chi^C(a) = 0$ otherwise.

Show that f is a non-negative circulation if and only if there exists a function $\lambda : \mathcal{C} \to \mathbb{R}_+$ such that,

$$f = \sum_{C \in \mathcal{C}} \lambda(C)\chi^C$$

That is, the non-negative circulations form the code generated by $\{\chi^C : C \in \mathcal{C}\}$.

Fix $\lambda : A \to \mathbb{R}_+$ and let $f = \sum_{C\in\mathcal{C}} \lambda(C)\chi^C$. Consider the flux into and out of a vertex $v \in V$. We have,

$$\begin{aligned}\sum_{a\in\delta^{\text{out}}(v)} f(a) - \sum_{a\in\delta^{\text{in}}(a)} f(a) &= \sum_{a\in\delta^{\text{out}}(v)} \sum_{C\in\mathcal{C}} \lambda(C)\chi^C(a) - \sum_{a\in\delta^{\text{in}}(a)} \sum_{C\in\mathcal{C}} \lambda(c)\chi^C(a) \\ &= \sum_{C\in\mathcal{C}} \left[\sum_{a\in\delta^{\text{out}}(v)} \lambda(C)\chi^C(a) - \sum_{a\in\delta^{\text{in}}(a)} \lambda(C)\chi^C(a) = \right]\end{aligned}$$

Fix $C \in \mathcal{C}$. If C does not pass through v then $\chi^C(a) = 0$ for all $a \in \delta^{\text{in}}(v) \cup \delta^{\text{out}}(v)$. If C does pass through v, then $\chi^C(a) = 1$ for exactly one $a \in \delta^{\text{in}}(v)$ and exactly one $a \in \delta^{\text{out}}(v)$. Moreover, since $\lambda(C)$ is constant (if C is fixed), then the term $\lambda(C)\chi^C(a)$ appears in both sums.

Therefore the difference of the two sums is zero. This proves f is a circulation.

We provide an algorithm to find $\lambda : A \to \mathbb{R}_+$ such that $f = \sum_{C\in\mathcal{C}} \lambda(C)\chi^C$ for a non-negative circulation f.

At the k-th step, start with a circulation $f^{[k-1]}$. If the circulation on each edge of every directed circuit in D is zero then terminate.

Otherwise, at step k find a directed circuit C_k with $f(a) \neq 0$ for all $a \in C_k$. Define,

$$\lambda(C_k) = \min_{a\in C_k} f(a)$$

Now, define a new circulation $f^{[k]} : A \to \mathbb{R}_+$ by,

$$f^{[k]}(a) = \begin{cases} f^{[k-1]}(a) - \lambda(C_k) & a \in C_k \\ f^{[k-1]}(a) & \text{otherwise} \end{cases}$$

Then clearly $f^{[k]}$ is a circulation. Moreover, $f^{[k]}$ has at least one fewer non-zero edge than $f^{[k-1]}$ since $f^{[k-1]}(a) = \lambda(C_k)$ for some $a \in C_k$. Since $|A| < \infty$ this means the algorithm will terminate (in less than $|A|$ steps).

Then, starting with $f^{[0]} = f$ the algorithm will terminate and give us $\lambda(C_k)$ such that,

$$f = \sum_k \lambda(C_k)\chi^{C_k}(a)$$