AMATH 562 Assignment 8

Tyler Chen

Exercise 8.1

Compute $\mathrm{d}(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution

Write $f(x) = x^4$ so that $f(W_t) = W_t^4$. Then, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, Itô's formula gives,

$$\mathrm{d}W_t^4 = f'(W_t)\mathrm{d}W_t + \frac{1}{2}f''(W_t)\mathrm{d}[W,W]_t = 4W_t^3\mathrm{d}W_t + \frac{12}{2}W_t^2\mathrm{d}[W,W]_t$$

Thus, writing $\mathrm{d}[W,W]_t = \mathrm{d}t$ we have,

$$\mathrm{d}W_t^4 = 4W_t^3\mathrm{d}W_t + 6W_t^2\mathrm{d}t$$

Thus, since $W_0 = 0$,

$$W_T^4 = W_T^4 - W_0^4 = 4\int_0^T W_t^3\mathrm{d}W_t + 6\int_0^T W_t^2\mathrm{d}t$$

Recall Itô integrals are martingales so that,

$$\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] = 0$$

Note also that since $\mathbb{E}\left[W_t^2\right] = t$,

$$\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = \int_0^T \mathbb{E}\left[W_t^2\right]\mathrm{d}t = \int_0^T t\mathrm{d}t = \frac{T^2}{2}$$

Therefore,

$$\mathbb{E}\left[W_T^4\right] = 4\mathbb{E}\left[\int_0^T W_t^3\mathrm{d}W_t\right] + 6\mathbb{E}\left[\int_0^T W_t^2\mathrm{d}t\right] = 6\frac{T^2}{2} = 3T^2$$

Similarly, we have,

$$W_T^6 = 6\int_0^T W_t^5\mathrm{d}W_t + \frac{6\cdot 5}{2}\int_0^T W_t^4\mathrm{d}t$$

Therefore, since $\mathbb{E}\left[W_t^4\right] = 3t^2$,

$$\mathbb{E}\left[W_T^6\right] = 6\mathbb{E}\left[\int_0^T W_t^5\mathrm{d}W_t\right] + 15\mathbb{E}\left[\int_0^T W_t^4\mathrm{d}t\right] = 15\int_0^T \mathbb{E}\left[W_t^4\right]\mathrm{d}t = 15\int_0^T 3t^2\mathrm{d}t = 15T^3$$

Exercise 8.2

Find an explicit expression for Y_T where,

$$\mathrm{d}Y_t = r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t$$

Hint: Multiply the above equation by $F_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution

Let $f(x, y) = \exp(-\alpha x + \frac{1}{2}\lambda^2 y)$ so that,

$$f_x(W_t, t) = -\alpha F_t \qquad f_y(W_t, t) = \frac{\alpha^2}{2} F_t \qquad f_{xx}(W_t, t) = \alpha^2 F_t$$

Then $F_t = f(W_t, t)$, so by Itô's formula and the heuristic $(\mathrm{d}W_t)^2 = \mathrm{d}t, (\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W_t = 0$,

$$\begin{aligned}
\mathrm{d}F_t = \mathrm{d}f(W_t, t) &= f_y(W_t, t)\mathrm{d}t + f_x(W_t, t)\mathrm{d}W_t + \frac{1}{2} f_{xx}(W_t, t)(\mathrm{d}W_t)^2 \\
&= \frac{\alpha^2}{2} F_t \mathrm{d}t - \alpha F_t \mathrm{d}W_t + \frac{\alpha^2}{2} F_t \mathrm{d}t \\
&= \alpha^2 F_t \mathrm{d}t - \alpha F_t \mathrm{d}W_t
\end{aligned}$$

Using our heuristics we have,

$$\mathrm{d}[F, Y]_t = (\mathrm{d}F_t)(\mathrm{d}Y_t) = \left(\alpha^2 F_t \mathrm{d}t - \alpha F_t \mathrm{d}W_t\right)(r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t) = -\alpha^2 F_t Y_t (\mathrm{d}W_t)^2 = -\alpha^2 F_t Y_t \mathrm{d}t$$

By the product rule we have,

$$\begin{aligned}
\mathrm{d}(F_t Y_t) &= F_t \mathrm{d}Y_t + Y_t \mathrm{d}F_t + \mathrm{d}[F, Y]_t \\
&= F_t(r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t) + Y_t(\alpha^2 F_t \mathrm{d}t - \alpha F_t \mathrm{d}W_t) - \alpha^2 F_t Y_t \mathrm{d}t \\
&= r F_t \mathrm{d}t
\end{aligned}$$

In integral form,

$$F_t Y_t - F_0 Y_0 = \int_0^t r F_s \mathrm{d}s = \int_0^t r e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} \mathrm{d}s$$

We can add $F_0 Y_0 = Y_0$ and divide by F_t yielding,

$$Y_t = Y_0 + r e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} \mathrm{d}s$$

Exercise 8.3

Suppose X, Δ, and Π are given by,

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, \qquad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \qquad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) f(t,x) = 0$$

for all (t,x), then Π is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We have,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) = \frac{\partial^2}{\partial x \partial t} + \frac{1}{2}\sigma^2 \left[x^2 \frac{\partial^3}{\partial x^3} + 2x \frac{\partial^2}{\partial x^2} \right]$$

Thus, using the condition for f we have,

$$\frac{\partial^2 f}{\partial x \partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3} = -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}$$

Using our heuristics we have,

$$\mathrm{d}[X,X] = \sigma^2 X_t^2 (\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \mathrm{d}t$$

Similarly,

$$\mathrm{d}[X,t] = \mathrm{d}[t,X] = \mathrm{d}[t,t] = 0$$

Therefore, by Itô's formula,

$$\begin{aligned}
\mathrm{d}\Delta_t &= \frac{\partial^2 f}{\partial x \partial t}(t,X_t)\mathrm{d}t + \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}X_t + \frac{1}{2}\mathrm{d}[X,X] \\
&= \frac{\partial^2 f}{\partial x \partial t}(t,X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}W_t + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^3 f}{\partial x^3}(t,X_t)\mathrm{d}t \\
&= -\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}W_t
\end{aligned}$$

Therefore,

$$\mathrm{d}[X,\Delta]_t = (\mathrm{d}X_t)(\mathrm{d}\Delta_t) = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)(\mathrm{d}W_t)^2 = \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}t$$

Finally, we have,

$$
\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}(X_t\Delta_t) = X_t \mathrm{d}\Delta_t + \Delta_t \mathrm{d}X_t + \mathrm{d}[X,\Delta]_t \\
&= X_t\left(-\sigma^2 X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}t + \sigma X_t \frac{\partial^2 f}{\partial x^2}(t,X_t)\mathrm{d}W_t\right) + \sigma X_t \frac{\partial f}{\partial x}(t,X_t)\mathrm{d}W_t + \sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \sigma X_t \left(X_t \frac{\partial^2 f}{\partial x^2}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\right)\mathrm{d}W_t
\end{aligned}
$$

Since there is no dt dependence this is an Itô integral and therefore a martingale with respect to a filtration for W. (there are probably some technical assumptions we need about X and f, but in class we never dealt with these) $\square$

Exercise 8.4

Suppose X is given by,

$$\mathrm{d}X_t = \mu(t, X_t)dt + \sigma(t, X_t)\mathrm{d}W_t$$

For any smooth function f define,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2} \right) f(s, X_s)ds$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution

We first compute,

$$\mathrm{d}[X,X]_t = (\mathrm{d}X_t)(\mathrm{d}X_t) = \sigma^2(t, X_t)(\mathrm{d}W_t)^2 = \sigma^2(t, X_t)\mathrm{d}t$$

We then have,

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}[X,X]_t \\
&= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)[\mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t] + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\mathrm{d}t \\
&= \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t + \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Finally, since $f(0, X_0)$ is a constant,

$$\begin{aligned}
\mathrm{d}M_t^f &= \mathrm{d}f(t, X_t) - \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t \\
&= \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}W_t
\end{aligned}$$

Since there is no $\mathrm{d}t$ dependence this an Itô integral and therefore a martingale with respect to a filtration for W. $\square$