

AMATH 585 Assignment 2

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Problem 1 (Inverse matrix and Green's functions)

- (a) Write out the 5×5 matrix A from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 0$ for $h = 0.25$.
- (b) Write out the 5×5 inverse matrix A^{-1} explicitly for this problem.
- (c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the five Green's functions whose sum gives this solution.

Solution

- (a) We have,

$$A = \frac{1}{h^2} \begin{bmatrix} h^2 & 0 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 0 & h^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & & & \\ 16 & -32 & 16 & & \\ & 16 & -32 & 16 & \\ & & 16 & -32 & 16 \\ & & & 0 & 1 \end{bmatrix}$$

- (b) Since $h = 1/4$ we have,

$$x_0 = 0 \quad x_1 = 1/4 \quad x_2 = 2/4 \quad x_3 = 3/4 \quad x_4 = 1$$

In the first and last columns we have,

$$B_{i,0} = 1 - x_i \quad B_{i,4} = x_i$$

In the middle columns we have,

$$B_{i,j} = hG(x_i; x_j) = \begin{cases} h(x_j - 1)x_i, & i = 0, 1, \dots, j \\ h(x_i - 1)x_j, & i = j, j + 1, \dots, m + 1 \end{cases}$$

Thus,

$$B = \begin{bmatrix} 1 - x_0 & h(x_1 - 1)x_0 & h(x_2 - 1)x_0 & h(x_3 - 1)x_0 & x_0 \\ 1 - x_1 & h(x_1 - 1)x_1 & h(x_2 - 1)x_1 & h(x_3 - 1)x_1 & x_1 \\ 1 - x_2 & h(x_2 - 1)x_1 & h(x_2 - 1)x_2 & h(x_3 - 1)x_2 & x_2 \\ 1 - x_3 & h(x_3 - 1)x_1 & h(x_3 - 1)x_2 & h(x_3 - 1)x_3 & x_3 \\ 1 - x_4 & h(x_4 - 1)x_1 & h(x_4 - 1)x_2 & h(x_4 - 1)x_3 & x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/4 & -3/64 & -1/32 & -1/64 & 1/4 \\ 1/2 & -1/32 & -1/16 & -1/32 & 1/2 \\ 1/4 & -1/64 & -1/32 & -3/64 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We easily verify that $B = A^{-1}$ using Mathematica.

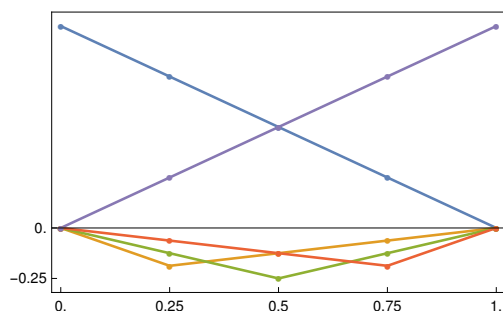
(c) We have,

$$F = \begin{bmatrix} \alpha \\ x_1 \\ x_2 \\ x_3 \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ 1/2 \\ 3/4 \\ 0 \end{bmatrix}$$

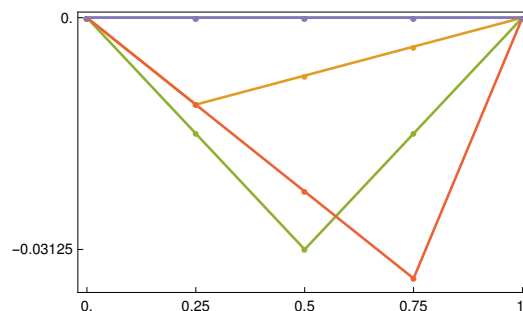
We solve $AU = F$, where $U = [U_0, \dots, U_4]^T$ yielding,

$$U = \alpha B_0 + \beta B_4 + \sum_{j=1}^3 x_j B_j = \sum_{j=1}^3 x_j B_j = \begin{bmatrix} 0 \\ -5/128 \\ -1/16 \\ -7/128 \\ 0 \end{bmatrix}$$

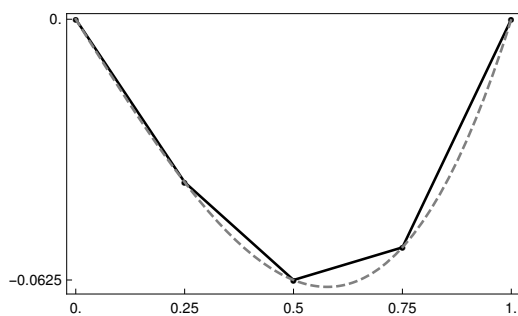
Figure ?? shows the 5 greens functions corresponding to the 5 points in F . Figure ?? shows the Green's functions after they have been scaled using the coefficients of F . Figure ?? shows the sum of the scaled Green's functions along with the actual solution $u(x) = (x^3 - x)/6$.



(a) Green's functions



(b) Green's functions with scaled axes



(c) Scaled: Green's functions

blue	$G_0(x)$
orange	$G(x, 1/4)$
green	$G(x, 1/2)$
red	$G(x, 3/4)$
purple	$G_1(x)$
black	linear combination
grey	actual solution

(d) legend

Problem 2 (Another way of analyzing the error using Green's functions)

The *composite trapezoid rule* for integration approximates the integral from a to b of a function g by dividing the interval into segments of length h and approximating the integral over each segment by the integral of the linear function that matches g at the endpoints of the segment. (For $g > 0$, this is the area of the trapezoid with height $g(x_j)$ at the left endpoint x_j and height $g(x_{j+1})$ at the right endpoint x_{j+1} .) Letting $h = (b - a)/(m + 1)$ and $x_j = a + jh$, $j = 0, 1, \dots, m, m + 1$:

$$\int_a^b g(x) dx \approx h \sum_{j=0}^m \frac{g(x_j) + g(x_{j+1})}{2} = h \left[\frac{g(x_0)}{2} + \sum_{j=1}^m g(x_j) + \frac{g(x_{m+1})}{2} \right].$$

- (a) Assuming that g is sufficiently smooth, show that the error in the composite trapezoid rule approximation to the integral is $O(h^2)$. [Hint: Show that the error on each subinterval is $O(h^3)$.]
- (b) Recall that the true solution of the boundary value problem $u''(x) = f(x)$, $u(0) = u(1) = 0$ can be written as

$$u(x) = \int_0^1 f(\bar{x})G(x; \bar{x}) d\bar{x}, \quad (1)$$

where $G(x; \bar{x})$ is the Green's function corresponding to \bar{x} . The finite difference approximation u_i to $u(x_i)$, using the centered finite difference scheme in (2.43), is

$$u_i = h \sum_{j=1}^m f(x_j)G(x_i; x_j), \quad i = 1, \dots, m. \quad (2)$$

Show that formula (??) is the trapezoid rule approximation to the integral in (??) when $x = x_i$, and conclude from this that the error in the finite difference approximation is $O(h^2)$ at each node x_i . [Recall: The Green's function $G(x; x_j)$ has a *discontinuous* derivative at $x = x_j$. Why does this not degrade the accuracy of the composite trapezoid rule?]

Solution

- (a) Since g is sufficiently smooth, it is continuous and therefore has antiderivative G on (a, b) .

Let $x = x_j$ for some $j = 0, 1, \dots, m$. We consider the error in using a single trapezoid to approximate the integral of the function on the interval $(x, x + h) = (x_j, x_{j+1})$.

Expand $G(x + h)$ about x as,

$$\begin{aligned} G(x + h) &= G(x) + G'(x)h + G''(x)\frac{h^2}{2!} + G'''(x)\frac{h^3}{3!} + \mathcal{O}(h^4) \\ &= G(x) + g(x)h + g'(x)\frac{h^2}{2!} + g''(x)\frac{h^3}{3!} + \mathcal{O}(h^4) \end{aligned}$$

By the Fundamental Theorem of Calculus we then have,

$$\int_x^{x+h} g(t)dt = G(x + h) - G(x) = g(x)h + g'(x)\frac{h^2}{2!} + g''(x)\frac{h^3}{3!} + \mathcal{O}(h^4)$$

Now expand $g(x + h)$ about x as,

$$g(x + h) = g(x) + g'(x)h + g''(x)\frac{h^2}{2!} + \mathcal{O}(h^3)$$

The trapezoid approximation on the interval $(x, x+h)$ is then,

$$\begin{aligned}\frac{h}{2}[g(x) + g(x+h)] &= \frac{h}{2} \left[2g(x) + g'(x)h + g''(x)\frac{h^2}{2!} + \mathcal{O}(h^3) \right] \\ &= g(x)h + g'(x)\frac{h^2}{2} + g''(x)\frac{h^3}{2 \cdot 3!} + \mathcal{O}(h^4)\end{aligned}$$

Thus, the error for a single interval of width h is,

$$\left(\int_x^{x+h} g(t)dt \right) - \frac{h}{2}[g(x) + g(x+h)] = g''(x)\frac{h^3}{3!} - g''(x)\frac{h^3}{2 \cdot 3!} + \mathcal{O}(h^4) = g''(x)\frac{h^3}{12} + \mathcal{O}(h^4)$$

This proves that on an interval of width h , the trapezoid rule has an error order h^3 .

To approximate the integral on the interval (a, b) we add up the results of $m+1 = (b-a)/h$ trapezoid approximations for intervals of width h . The total error is then at most,

$$(m+1) \left(\max_{x \in (a,b)} g''(x) \frac{h^3}{12} + \mathcal{O}(h^4) \right) = \mathcal{O}(h^2) \quad \square$$

- (b) Suppose f is “sufficiently smooth”. Then $g(\bar{x}) := f(\bar{x})G(x; \bar{x})$ is piecewise “sufficiently smooth”, with a possible discontinuity in some derivative of $g(\bar{x})$ at $\bar{x} = x$ (since $G(x; \bar{x})$ is analytic except at $\bar{x} = x$).

Our above analysis only requires that the function be smooth on the interval $(x, x+h)$ for the error to be $\mathcal{O}(h^3)$.

Thus, if $x = x_i$, the discontinuity in $g(\bar{x})$ will be at one of the partition points of $(0, 1)$, so the analysis above is still applicable.

Using the definition of the trapezoid rule, where $x_j = jh$ and $h = 1/(m+1)$, we have

$$u(x) = \int_0^1 f(\bar{x})G(x; \bar{x})d\bar{x} \approx h \left[\frac{f(x_0)G(x; x_0)}{2} + \sum_{j=1}^m f(x_j)G(x; x_j) + \frac{f(x_{m+1})G(x; x_{m+1})}{2} \right]$$

Note that $G(x, x_0) = 0(1-0) = 0$ and $G(x, x_{m+1}) = 1(1-1) = 0$. Thus, at $x = x_i$,

$$u(x_i) \approx h \sum_{j=1}^m f(x_j)G(x_i; x_j)$$

In particular, since the trapezoid approximation is equal to $u(x)$ up to order h^2 for any x ,

$$u(x_i) - h \sum_{j=1}^m f(x_j)G(x_i; x_j) = \mathcal{O}(h^2) \quad \square$$

Problem 3 (Green's function with Neumann boundary conditions)

- (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e., find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

- (b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 5×5 matrices A and A^{-1} for the case $h = 0.25$.
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Solution

- (a) Integrating once yields a solution which is piecewise constant with a single discontinuity at $x = \bar{x}$. Moreover, since the integral over an interval containing the delta function is equal to one, the jump at this discontinuity must be one. We are given the condition $u'(0) = 0$ so,

$$G'(x; \bar{x}) = \begin{cases} 0 & 0 \leq x \leq \bar{x} \\ 1 & \bar{x} \leq x \leq 1 \end{cases}$$

Integrating again yields a piecewise linear function with a discontinuity in the derivative at $x = \bar{x}$. That is,

$$G(x; \bar{x}) = \begin{cases} c & 0 \leq x \leq \bar{x} \\ x + d & \bar{x} \leq x \leq 1 \end{cases}$$

Since the function is continuous we have, $c = \bar{x} + d$. Moreover, we require $u(1) = 0$ which forces $d = -1$. Thus,

$$G(x; \bar{x}) = \begin{cases} \bar{x} - 1 & 0 \leq x \leq \bar{x} \\ x - 1 & \bar{x} \leq x \leq 1 \end{cases}$$

By inspection it is clear that,

$$G_0(x) = x - 1 \qquad G_1(x) = 1$$

- (b) We have,

$$\frac{1}{h^2} \begin{bmatrix} -h & h & & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

We use the Greens functions derived above to write the inverse.

In the first and last columns we have,

$$B_{i,0} = x_i - 1 \qquad B_{i,m+1} = 1$$

In the middle columns we have,

$$B_{i,j} = hG(x_i; x_j) = \begin{cases} h(x_j - 1) & i = 0, 1, \dots, j \\ h(x_i - 1) & i = j, j + 1, \dots, m + 1 \end{cases}$$

Thus, when $h = 1/4$,

$$A = \begin{bmatrix} -4 & 4 & & & \\ 16 & -32 & 16 & & \\ & 16 & -32 & 16 & \\ & & 16 & -32 & 16 \\ & & & & 1 \end{bmatrix}$$

Again we have,

$$x_0 = 0 \qquad x_1 = 1/4 \qquad x_2 = 2/4 \qquad x_3 = 3/4 \qquad x_4 = 1$$

Thus,

$$B = \begin{bmatrix} x_0 - 1 & h(x_1 - 1) & h(x_2 - 1) & h(x_3 - 1) & 1 \\ x_1 - 1 & h(x_1 - 1) & h(x_2 - 1) & h(x_3 - 1) & 1 \\ x_2 - 1 & h(x_2 - 1) & h(x_2 - 1) & h(x_3 - 1) & 1 \\ x_3 - 1 & h(x_3 - 1) & h(x_3 - 1) & h(x_3 - 1) & 1 \\ x_4 - 1 & h(x_4 - 1) & h(x_4 - 1) & h(x_4 - 1) & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3/16 & -1/8 & -1/16 & 1 \\ -3/4 & -3/16 & -1/8 & -1/16 & 1 \\ -1/2 & -1/8 & -1/8 & -1/16 & 1 \\ -1/4 & -1/16 & -1/16 & -1/16 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We easily verify that $B = A^{-1}$ using Mathematica.

Problem 4 (Solvability condition for Neumann problem)

Determine the null space of the matrix A^T , where A is given in equation (2.58), and verify that the condition (2.62) must hold for the linear system to have solutions.

Solution

We have,

$$A^T = \begin{bmatrix} -h & 1 & & & & & \\ & h & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & 1 & -2 & h \\ & & & & & & 1 & -h \end{bmatrix}$$

Supposed $x \in \ker A^T$. Then, multiplying by h^2 for simplicity,

$$0 = h^2 A^T x = \begin{bmatrix} -h & 1 & & & & & \\ & h & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & 1 & -2 & h \\ & & & & & & 1 & -h \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \\ x_{m+1} \end{bmatrix} = \begin{bmatrix} -hx_0 + x_1 \\ hx_0 - 2x_1 + x_2 \\ x_1 - 2x_2 + x_3 \\ x_2 - 2x_3 + x_4 \\ \vdots \\ x_{m-1} - 2x_m + hx_{m+1} \\ x_m - hx_{m+1} \end{bmatrix}$$

Thus,

$$\begin{aligned} x_1 &= hx_0 \\ x_2 &= 2x_1 - hx_0 = hx_0 \\ x_3 &= 2x_2 - x_1 = hx_0 \\ x_4 &= 2x_3 - x_2 = hx_0 \\ &\vdots \\ x_m &= 2x_{m-1} - x_{m-2} = hx_0 \\ x_{m+1} &= (2x_m - x_{m-1})/h = x_0 \\ x_{m+1} &= x_m/h = x_0 \end{aligned}$$

The null space of A^T is then, $\text{span}\{[1, h, h, \dots, h, 1]\}$

Recall that the range of A is perpendicular to the null space of A^T . Clearly $AU = F$ has a solution if and only if $F \in \text{range } A = (\text{null } A^T)^\perp$.

That is, the system $AU = F$ has solutions if and only if F is orthogonal to the null space of A^T .

Since $\text{null } A^T$ is spanned by $y = [1, h, h, \dots, h, 1]$ every vector has the form cy for some $c \in \mathbb{R}$. It is then sufficient to show that $y^T F = 0$ as this will imply $(cy)^T F = 0$ for any vector $cy \in \text{null } A^T$.

Then, F is orthogonal to y (and $\text{null } A^T$) if and only if,

$$0 = y^T F = 1 \left(\sigma_0 + \frac{h}{2} f(x_0) \right) + h(f(x_1)) + h(f(x_2)) + \dots + h(f(x_m)) + 1 \left(-\sigma_1 + \frac{h}{2} f(x_{m+1}) \right)$$

Equivalently, moving σ_0 and $-\sigma_1$ to the other side, and using sum notation to condense the middle terms,

$$\frac{h}{2}f(x_0) + h \sum_{i=1}^m f(x_i) + \frac{h}{2}f(x_{m+1}) = \sigma_1 - \sigma_0$$

This is exactly condition 2.62.

□

Problem 5 (Symmetric tridiagonal matrices)

- (a) Consider the **Second approach** described on p. 31 for dealing with a Neumann boundary condition. If we use this technique to approximate the solution to the boundary value problem $u''(x) = f(x)$, $0 \leq x \leq 1$, $u'(0) = \sigma$, $u(1) = \beta$, then the resulting linear system $A\mathbf{u} = \mathbf{f}$ has the following form:

$$\frac{1}{h^2} \begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix} = \begin{bmatrix} \sigma + (h/2)f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}$$

Show that the above matrix is similar to a symmetric tridiagonal matrix via a *diagonal* similarity transformation; that is, there is a diagonal matrix D such that DAD^{-1} is symmetric.

- (b) Consider the **Third approach** described on pp. 31-32 for dealing with a Neumann boundary condition. [Note: If you have an older edition of the text, there is a typo in the matrix (2.57) on p. 32. There should be a row above what is written there that has entries $\frac{3}{2}h$, $-2h$, and $\frac{1}{2}h$ in columns 1 through 3 and 0's elsewhere. I believe this was corrected in newer editions.] Show that if we use that first equation (given at the bottom of p. 31) to eliminate u_0 and we also eliminate u_{m+1} from the equations by setting it equal to β and modifying the right-hand side vector accordingly, then we obtain an m by m linear system $A\mathbf{u} = \mathbf{f}$, where A is similar to a symmetric tridiagonal matrix via a diagonal similarity transformation.
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Solution

- (a) The only non-symmetric entries are the (0,1) and (1,0) entries. We then expect D to be mostly the identity except the first entry. Pick, $D = \text{diag}([1/\sqrt{h}, 1, 1, \dots, 1])$. Then, $D^{-1} = \text{diag}([\sqrt{h}, 1, 1, \dots, 1])$ so clearly,

$$DAD^{-1} = \frac{1}{h^2} \begin{bmatrix} -h & \sqrt{h} & & & \\ \sqrt{h} & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

- (b) Suppose $U_{m+1} = \beta$, along with the boundary equation,

$$\frac{1}{h} \left(\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2 \right) = \sigma \quad (*)$$

We then have the system,

$$\frac{1}{h^2} \begin{bmatrix} \frac{3h}{2} & -2h & \frac{h}{2} & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

We eliminate U_0 by subtracting $2/3h$ times row zero from row one, and we eliminate U_{m+1} by subtracting $1/h^2$ times row $m+1$ from row m . Thus,

$$\frac{1}{h^2} \begin{bmatrix} \frac{3h}{2} & -2h & \frac{h}{2} & & & & \\ 0 & -\frac{2}{3} & \frac{2}{3} & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 0 \\ & & & & & 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_{m-1} \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) - \frac{2}{3h}\sigma \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \frac{1}{h^2}\beta \\ \beta \end{bmatrix}$$

Thus, given equation (*) and $U_{m+1} = \beta$ we have the $m \times m$ system,

$$\frac{1}{h^2} \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{m-1} \\ U_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \frac{2}{3h}\sigma \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \frac{1}{h^2}\beta \end{bmatrix}$$

Pick, $D = \text{diag}([\sqrt{3/2}, 1, 1, \dots, 1])$. Then, $D^{-1} = \text{diag}([\sqrt{2/3}, 1, 1, \dots, 1])$ so clearly,

$$DAD^{-1} = \frac{1}{h^2} \begin{bmatrix} -\frac{2}{3} & \sqrt{\frac{2}{3}} & & & & \\ \sqrt{\frac{2}{3}} & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & \end{bmatrix}$$