# **AMATH 584** Assignment 3

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### Exercise 6.1

If $P$ is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraiclly, and give a geometric interpretation.

### Solution

Suppose $P$ is an orthogonal projector. Then $P^2 = P = P^*$. Thus,

$$(I-2P)(I-2P)^* = (I-2P)(I^*-2P^*) = (I-2P)(I-2P) = I^2 - 2P - 2P + 4P^2 = I - 4P + 4P = I$$

This proves $I - 2P$ is unitary.

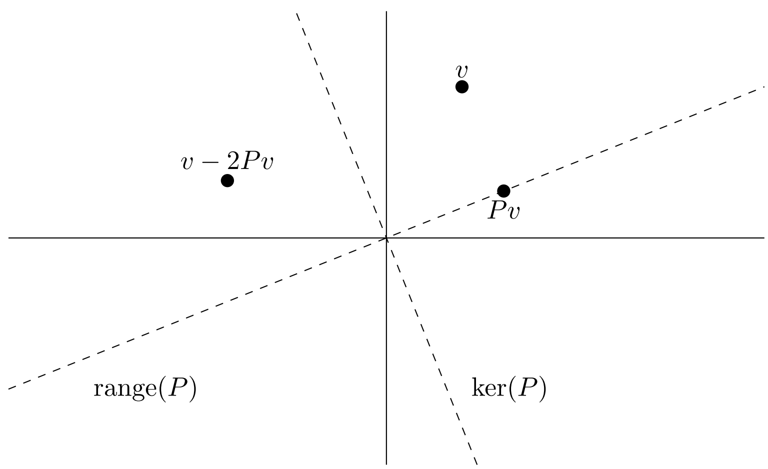


Figure 1: Image of $I - 2P$ acts on $v$

Using Figure 1 it is clear that $I-2P$ reflects points about orthogonal compliment of range($P$). Reflecting across $(\text{range}(P))^{\perp} = \ker(P)$ twice will do nothing. Since $(I-2P)^2 = (I-2P)(I-2P)^* = I$, this coincides with the algebraic proof above.

**Exercise 6.4**

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer the following questions by hand calculation.

(a) What is the orthogonal projector $P$ onto range($A$), and what is the image under $P$ of the vector $(1, 2, 3)^*$?

(b) Same question for $B$

**Solution**

(a) First observe,

$$(A^*A)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$P_A = A(A^*A)^{-1}A^* = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

So,

$$P_A(1, 2, 3)^* = (2, 2, 2)^*$$

(b) First observe,

$$(B^*B)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Thus,

$$P_B = B(B^*B)^{-1}B^* = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

So,

$$P_B(1, 2, 3)^* = (2, 0, 2)^*$$

### Exercise 7.1

Consider again the matrices $A$ and $B$ of Exercise 6.4.

(a) Using any method you like, determine (on paper) a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

(b) Again using any method you like, determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

### Solution

The book gives the following algorithm for calculating a reduced QR decomposition.

```
1  for j = 1 to n
2      v_j = a_j
3      for i = 1 to j − 1
4          r_ij = q_i^* a_j
5          v_j = v_j − r_ij q_i
6      r_jj = ||v_j||_2
7      q_j = v_j / r_jj
```

(a) We have $a_1 = (1,0,1)^*, a_2 = (0,1,0)^*$. We use the algorithm listed above:

(1) with $j = 1$:

(2) $v_1 = a_1$

(6) $r_{11} = \|v_1\|_2 = \sqrt{2}$.

(7) $q_1 = v_1/r_{11} = (1/\sqrt{2}, 0, 1/\sqrt{2})^*$

(1) with $j = 2$

(2) $v_2 = a_2$

(3) with $i = 1$

(4) $r_{21} = q_1^* a_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})(0,1,0) = 0$

(5) $v_2 = v_2 - 0q_1 = (0,1,0)$

(6) $r_{22} = \|v_2\|_2 = 1$

(7) $q_2 = v_2/r_{22} = (0,1,0)$

This gives reduced QR factorization,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

We extend this to a full QR factorization by finding a vector orthogonal to $q_1, q_2$. First,

$$\begin{aligned} 0 &= (1/\sqrt{2}, 0, 1/\sqrt{2})(a,b,c)^* = (a+c)/\sqrt{2} \\ 0 &= (0,1,0)(a,b,c)^* = b \\ 1 &= \sqrt{a^2+b^2+c^2} \end{aligned}$$

Thus $q_3 = (a, b, c) = (1/\sqrt{2}, 0, -1/\sqrt{2})$ so

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) We have $b_1 = (1, 0, 1)^*, b_2 = (2, 1, 0)^*$ We use the algorithm listed above:

- (1) with $j = 1$:
  - (2) $v_1 = b_1$
  - (6) $r_{11} = \|v_1\|_2 = \sqrt{2}$.
  - (7) $q_1 = v_1/r_{11} = (1/\sqrt{2}, 0, 1/\sqrt{2})^*$
- (1) with $i = 2$
  - (2) $v_2 = b_2$
  - (3) with $i = 1$
    - (4) $r_{12} = q_1^* a_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})(2, 1, 0) = 2/\sqrt{2}$
    - (5) $v_2 = v_2 - r_{12} q_1 = (1, 1, -1)$
  - (6) $r_{22} = \|v_2\|_2 = \sqrt{3}$
  - (7) $q_2 = v_2/r_{22} = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$

This gives reduced QR factorization,

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{2}{\sqrt{2}} \\ 0 & \sqrt{3} \end{bmatrix}$$

We extend this to a full QR factorization by finding a vector orthogonal to $q_1, q_2$. First,

$$\begin{aligned} 0 &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)(a, b, c)^* = \frac{a + c}{\sqrt{2}} \\ 0 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \begin{bmatrix} a \\ b \\ c \end{bmatrix} (a, b, c)^* = \frac{a + b - c}{\sqrt{3}} \\ 1 &= \sqrt{a^2 + b^2 + c^2} \end{aligned}$$

Thus $q_3 = (a, b, c) = (-1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$ so,
We extend this to a full QR factorization,

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

**Exercise 7.5**

Let $A$ be a $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that $A$ has rank $n$ if and only if all the diagonal entries of $\hat{R}$ are nonzero.

(b) Suppose $\hat{R}$ has $k$ nonzero diagonal entries for some $k$ with $0 \leq k \leq n$. What does this imply about the rank of $A$? Exactly $k$? At least $k$? At most $k$? Give a precise answer, and prove it.

**Solution**

We first prove the following: *If $F$ is rank $m$ then $FA$ and $A$ have the same rank.*

Indeed, let $F$ be a rank $m$ matrix compatible with $A$. By the rank-nullity theorem $\dim(\ker(F)) + \operatorname{rank}(F) = \dim \operatorname{dom}(F)$ so $\ker(F) = \{0\}$. That is, $Fu = 0 \Leftrightarrow u = 0$.

Then,

$$w \in \ker(A) \Leftrightarrow Aw = 0 \Leftrightarrow FAw = 0 \Leftrightarrow w \in \ker(FA)$$

Thus $\ker(A) = \ker(FA)$, so by the rank-nullity theorem, $A$ and $FA$ have the same rank.

With this is mind, let $A$ be a $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization. Then $\hat{Q}$ is full rank and $\hat{R}$ is upper triangular.

(a) By the above result, the fact that the determinant of a triangular matrix is the product of the diagonal, and by the invertible matrix theorem, the following are equivalent:

- $\hat{R}$ has no nonzero entries
- $\hat{R}$ has nonzero determinant
- $\hat{R}$ has rank $n$
- $A$ has rank $n$

This proves $A$ has rank $n$ if and only if all the diagonal entries of $\hat{R}$ are nonzero. □

(b) Suppose $\hat{R}$ has $k$ nonzero diagonal entries. Consider the $k$ columns corresponding to the nonzero diagonal entries labeled $c_1, c_2, ..., c_k$. Observe $c_j$ has a nonzero component with higher index than any $c_i$ with $i < j$. Therefore $c_j$ is not in the span of $c_1, ..., c_{j-1}$. By induction it is clear that $c_1, ..., c_k$ are linearly independent.

Then $\hat{R}$ has at least $k$ linearly independent columns. That is, the rank of $\hat{R}$ is at least $k$.

Equality is not always attained. For instance, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is rank 1. However the QR factorization is $A = IA$, which has no nonzero diagonal entires on $\hat{R} = A$.

Therefore, since $\hat{Q}$ is full rank, the rank of $A$ is at least $k$. □

### Exercise 8.1

Let $A$ be an $m \times n$ matrix. Determine the exact number of floating point additions, subtractions, multiplications, and divisions involved in computing the factorization $A = \hat{Q}\hat{R}$ by Algorithm 8.1.

### Solution

Let $A$ be an $m \times n$ matrix. Algorithm 8.1 is displayed below, along with line numbering.

```
1  for  i = 1 to n
2        v_i = a_i
3  for i = 1 to n
4        r_ii = ||v_i||
5        q_i = v_i / r_ii
6        for j = i + 1 to n
7              r_ij = q_i^* v_j
8              v_j = v_j - r_ij q_i
```

First observe $a_i, v_i, q_i$ are all vectors in $\mathbb{C}^m$.

The first for loop simply reassigns $v_i$ to $a_i$. This does not require any floating point operations, however it does require memory allocation.

In line 4 we assign $r_i i$ to $\|v_i\|$. Calculating the norm of $v_i$ takes $m$ products, $m-1$ sums, and then one square root. Thus, this link takes $m + (m-1) + 1 = 2m$ flops.

In line 5 we assign $q_i$ to $v_i/r_{ii}$. We have calculated $r_{ii}$ in the previous line, so this requires $m$ divisions.

In line 7 we assign $r_{ij}$ to $q_i^* v_j$.This inner product takes $m$ multiplications and $m-1$ additions. Thus, this line takes $m + (m-1) = 2m - 1$ flops.

In line 8 we assign $v_j = v_j - r_{ij}q_i$. We have already calculated $r_{ij}$ and $q_i$ so this takes $m$ multiplications.We then have $m$ subtractions.

For a fixed $i$, lines 7 and 8 occur at each $j = i+1, i+2, ..., n$.

Lines 4 through 8 occur for $i = 1, 2, ..., n$.

The total number of flops is then give by,

$$\begin{aligned}
\#\text{ of flops} &= \sum_{i=1}^{n}\left[m + (m-1) + 1 + m + \sum_{j=i+1}^{n}[m + (m-1) + m + m]\right] \\
&= \sum_{i=1}^{n}\left[3m + \sum_{j=i+1}^{n}[4m-1]\right] \\
&= \left(3m\sum_{i=1}^{n}1\right) + \left((4m-1)\sum_{i=1}^{n}\sum_{j=i+1}^{n}1\right) \\
&= 3mn + (4m-1)(n(n-1)/2)
\end{aligned}$$

Alternatively, highlighting the specific floating point operations,

$$
\begin{aligned}
\#\text{ of flops} &= (\#\text{ of addition} + \#\text{ of subtraction} + \#\text{ of multiplication} + \#\text{ of division})\\
&= \sum_{i=1}^{n}\left[m + (m-1) + 1 + m + \sum_{j=i+1}^{n}[m + (m-1) + m + m]\right]\\
&= (m-1)n + (m-1)(n(n-1)/2) + mn(n-1)/2 + mn + 2m(n(n-1)/2) + mn + n\\
&= (m-1)(n(n+1)/2) + mn(n-1)/2 + mn^2 + mn + n
\end{aligned}
$$

$$
\begin{aligned}
\#\text{ of addition} &= (m-1)(n(n+1))/2\\
\#\text{ of subtraction} &= mn(n-1)/2\\
\#\text{ of multiplication} &= mn^2\\
\#\text{ of division} &= mn\\
\#\text{ of others} &= n
\end{aligned}
$$