AMATH 562 Assignment 9

Tyler Chen

Exercise 9.1

Solution

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Exercise 9.2

Let X be a solution to the following SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \delta\sqrt{X_t}\mathrm{d}W_t$$

Define

$$u(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T X_s \mathrm{d}s\right)\middle| X_t = x\right]$$

Derive a PDE for the function u. To solve the PDE for u, try a solution of the form

$$u(t,x) = \exp(-xA(t) - B(t)),$$

where A and B are deterministic functions of t. Show that A and B must satisfy a pair of coupled ODEs (with appropriate terminal conditions at time T). Bonus question: solve the ODEs (it may be helpful to note that one of the ODEs is a Riccati equation).

Solution (direct)

First observe that $u(t,X_t)$ is not a martingale as,

$$\begin{aligned}
\mathbb{E}[u(t,X_t)|\mathcal{F}_s] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_t^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_s\right] \\
&\neq \mathbb{E}\left[\exp\left(-\int_s^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_s\right] \\
&= u(s,X_s)
\end{aligned}$$

Define,

$$M_t = \exp\left(-\int_0^t X_z \mathrm{d}z\right)u(t,X_t) = \mathbb{E}\left[\exp\left(-\int_0^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_t\right]$$

where we have used the fact that $\exp(-\int_0^t X_z \mathrm{d}z) \in \mathcal{F}_t$.

Clearly,

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_0^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\exp\left(-\int_0^T X_z \mathrm{d}z\right)\middle|\mathcal{F}_s\right] = M_s$$

Therefore M_t is a martingale.

Note that,

$$\begin{aligned}
\mathrm{d}u(t,X_t) &= \partial_t u(t,X_t)\mathrm{d}t + \partial_x u(t,X_t)\mathrm{d}X_t + \frac{1}{2}\partial_x^2 u(t,X_t)\mathrm{d}[X,X]_t \\
&= \left(\partial_t + \mu(t,X_t)\partial_x + \frac{1}{2}\sigma^2(t,X_t)\partial_x^2\right)u(t,X_t)\mathrm{d}t + \sigma(t,X_t)\partial_x u(t,X_t)\mathrm{d}W_t
\end{aligned}$$

Write $v(t,X_t) = \exp(-\int_0^t X_z \mathrm{d}z)$. This does not depend on X_t so,

$$\begin{aligned}\mathrm{d}v(t,X_t) &= \partial_t v(t,X_t)\mathrm{d}t + \partial_x v(t,X_t)\mathrm{d}X_t + \frac{1}{2}\partial_x^2 v(t,X_t)\mathrm{d}[X,X]_t \\ &= -X_t v(t,X_t)\mathrm{d}t\end{aligned}$$

We supress the arguments to u, v, μ, σ and compute,

$$\begin{aligned}\mathrm{d}M_t &= u(t,X_t)\mathrm{d}v(t,X_t) + v(t,X_t)\mathrm{d}u(t,X_t) + \mathrm{d}[u,v]_t \\ &= u(-X_t v)\mathrm{d}t + v\left(\partial_t + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2\right)u\mathrm{d}t + (\cdots)\,\mathrm{d}W_t \\ &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_x^2 + \mu\partial_x - X_t\right)u\mathrm{d}t + (\cdots)\,\mathrm{d}W_t\end{aligned}$$

Since M_t is a martingale, the dt term must be zero. Moreover, v is always positive. Therefore,

$$\left[\left(\partial_t + \frac{1}{2}\sigma^2(t,x)\partial_x^2 + \mu(t,x)\partial_x - x\right)u(t,x)\right]_{t=t,x=X_t} = 0$$

The boundary condition is obtained by,

$$u(T,X_T) = \mathbb{E}\left[\exp\left(-\int_T^T X_z\mathrm{d}z\right)\middle|\mathcal{F}_T\right] = 1$$

The rest of the solution is included below.

Solution

With $\gamma(u,x) = x$, $\phi(x) = 1$, $g(u,x) = 0$ this is a subcase of an example in the notes. We then know $u(t,x)$ solves,

$$(\partial_t + \mathcal{A})u + g = 0, \qquad u(T,\cdot) = \phi, \qquad \mathcal{A} = \frac{1}{2}\sigma^2\partial_x^2 + \mu\partial_x - \gamma = 0$$

Assume u has the form,

$$u(t,x) = \exp\left(-xA(t) - B(t)\right)$$

First compute,

$$\partial_t u = (-xA' - B')u \qquad \partial_x u = -Au \qquad \partial_x^2 u = A^2 u$$

This gives,

$$\begin{aligned}0 &= \left[\partial_t + \frac{1}{2}\delta^2 x\partial_x^2 + \kappa(\theta - x)\partial_x - x\right]u \\ &= \left[-xA' - B' + \frac{1}{2}\delta^2 xA^2 + \kappa(\theta - x)(-A) - x\right]u \\ &= \left[\left(-A' + \frac{1}{2}\delta^2 A^2 + \kappa A - 1\right)x + (-B' - \kappa\theta A)\right]u\end{aligned}$$

Observe $u(t,x) > 0$ for all t, x. Therefore we require the bracketed term above to be zero for all x, t. Setting the coefficients of the x terms and constant terms to zero gives a coupled pair of ODEs,

$$\begin{cases}-A'(t) + \frac{1}{2}\delta^2 A^2(t) + \kappa A(t) - 1 = 0 \\ -B'(t) - \kappa\theta A(t) = 0\end{cases}$$

We have,

$$1 = \varphi(x) = u(T,x) = \exp\left(-xA(T) - B(T)\right)$$

This gives terminal condition,

$$A(T) = 0 \qquad\qquad B(T) = 0$$

We solve this in Mathematica without boundary conditions using,

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DSolve[{-D[A[t],t]+1/2 \[Delta]^2 A[t]^2+\[Kappa] A[t] - 1 ==0 , -D[B[t],t]-\[
    Kappa] \[Theta] A[t]==0},{A,B},t]
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This gives solution,

$$A(t) = \frac{\sqrt{-2\delta^2-\kappa^2}\tan\left(\frac{1}{2}\left(2c_1\sqrt{-2\delta^2-\kappa^2}+t\sqrt{-2\delta^2-\kappa^2}\right)\right)-\kappa}{\delta^2}$$
$$B(t) = \frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2-\kappa^2}+\frac{1}{2}t\sqrt{-2\delta^2-\kappa^2}\right)\right)+\kappa t\right)}{\delta^2}+c_2$$

where, using the boundary conditions we find,

$$c_1 = \frac{1}{2\sqrt{-2\delta^2-\kappa^2}}\left[2\arctan\left(\frac{\kappa}{\sqrt{-2\delta^2-\kappa}}\right)-T\sqrt{-2\delta^2-\kappa^2}\right]$$
$$c_2 = -\frac{\theta\kappa\left(2\log\left(\cos\left(c_1\sqrt{-2\delta^2-\kappa^2}+\frac{1}{2}T\sqrt{-2\delta^2-\kappa^2}\right)\right)+\kappa T\right)}{\delta^2}$$

Exercise 9.3

For $i = 1, 2, \ldots, d$ let $X^{(i)}$ satisfy,

$$\mathrm{d}X_t^{(i)} = -\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t^{(i)}$$

where $(W_t^{(i)})_{i=1}^d$ are independent Brownian motions. Define

$$R_t := \sum_{i=1}^d \left(X_t^{(i)}\right)^2, \qquad B_t := \sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}$$

Show that B is a Brownian motion. Derive an SDE for R that involves only $\mathrm{d}t$ and $\mathrm{d}B_t$ terms (i.e., no $\mathrm{d}W_t^{(i)}$ terms should appear).

Solution

We use the Lévy characterization of Brownian motion. In particular, we must show B is a martingale, B has continuous sample paths, and $B_0 = 0$ with $[B, B]_t = t$ for all $t \geq 0$.

Write,

$$\mathrm{d}B_t = \mathrm{d}\left[\sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}\right] = \sum_{i=1}^d \frac{1}{\sqrt{R_t}} X_t^{(i)} \mathrm{d}W_t^{(i)}$$

As B_t is an Itô integral it is a martingale with respect to a filtration $\mathbb{F} = (\mathcal{F}_\sqcup)_{t\geq 0}$ for $W_t^{(i)}$.

Similarly, B_t has continuous sample paths as $W_t^{(i)}$ have continuous sample paths.

By our definition of B_t we have $B_0 = 0$. Now,

$$\begin{aligned}
(\mathrm{d}B_t)(\mathrm{d}B_t) &= \frac{1}{R_t} \sum_{i=1}^d \sum_{j=1}^d X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)} \\
&= \frac{1}{R_t}\left(\sum_{j=1}^d \left(X_t^{(i)} \mathrm{d}W_t^{(i)}\right)^2 + 2\sum_{i=1}^d \sum_{j=1}^i X_t^{(i)} X_t^{(j)} \mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)}\right)
\end{aligned}$$

Using the heuristic, $\mathrm{d}W_t^{(i)}\mathrm{d}W_t^{(j)} = \delta_{ij}\mathrm{d}t$ and the definition of R_t we have,

$$\mathrm{d}[B, B]_t = \frac{1}{R_t} \sum_{i=1}^d \left(X_t^{(i)}\right)^2 \mathrm{d}t = \mathrm{d}t$$

Therefore, $[B, B]_t = t$.

This proves B is a Brownian motion. $\square$

Compute, using Itô's formula,

$$\mathrm{d}R_t = \mathrm{d}\left[\sum_{i=1}^d \left(X_t^{(i)}\right)^2\right] = \sum_{i=1}^d 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \frac{1}{2}2\mathrm{d}[X^{(i)}, X^{(i)}]_t = \sum_{i=1}^d 2X_t^{(i)}\mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)}, X^{(i)}]_t$$

Using our heuristics we have,

$$\mathrm{d}[X^{(i)}, X^{(i)}]_t = \left(\mathrm{d}X_t^{(i)}\right)\left(\mathrm{d}X_t^{(i)}\right) = \left(-\frac{b}{2}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t^{(i)}\right)^2 = \frac{\sigma^2}{4}\mathrm{d}t$$

Now,

$$\begin{aligned}
\sum_{i=1}^{d} 2X_t^{(i)} \mathrm{d}X_t^{(i)} + \mathrm{d}[X^{(i)}, X^{(i)}]_t &= \sum_{i=1}^{d} 2X_t^{(i)} \left(-\frac{b}{2} X_t^{(i)} \mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t^{(i)} \right) + \frac{\sigma^2}{4}\mathrm{d}t \\
&= \sum_{i=1}^{d} \left(\frac{\sigma^2}{4} - b\left(X_t^{(i)}\right)^2 \right) \mathrm{d}t + \sigma\sqrt{R_t}\frac{1}{\sqrt{R_t}} X_t^{(i)} \mathrm{d}W_t^{(i)}
\end{aligned}$$

Therefore, simplifying slightly we have,

$$\mathrm{d}R_t = (d\sigma^2/4 - bR_t)\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}B_t$$

Exercise 9.4

Solution

Exercise 9.5

Consider a diffusion $X = (X_t)_{t\geq 0}$ that lives on a finite interval (l, r), $0 < l < r < \infty$ and satisfies the SDE

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

One can easily check that the endpoints l and r are regular (you do not have to prove it here). Assume both endpoints are killing. Find the transition density $\Gamma(t, x; T, y)$ of X.

Solution

We have, $\Gamma(\cdot, \cdot; T, y)$ satisfies,

$$(\partial_t + \mathcal{A}(t))\Gamma(\cdot, t; T, y) = 0 \qquad\qquad \Gamma(T, \cdot; T, y) = \delta_y$$

where the infinitesimal generator $\mathcal{A}$ is,

$$\mathcal{A} = \mu x \partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2$$

We seek a spectral representation for $\mathcal{A}$. That is, a basis $\{\Psi_n\}_{n\geq 0}$ for such that $\mathcal{A}\Psi_n = \lambda_n \Psi_n$.

Since the endpoints are killing we also require,

$$\Psi_n(l) = 0, \qquad\qquad \Psi_n(r) = 0$$

We make a change of variables. Let $z = \log(x)$. Then,

$$\partial_x = \frac{1}{x}\partial_z, \qquad\qquad \partial_x^2 = -\frac{1}{x^2}\partial_z + \frac{1}{x}\partial_z^2$$

Then, in terms of z we have generator,

$$\mathcal{A}_z = \left(\mu - \frac{\sigma^2}{2}\right)\partial_z + \frac{1}{2}\sigma^2 \partial_z^2$$

This equation is very similar to a damped harmonic oscillator. We therefore guess that the eigenfunctions have the form,

$$\psi_n(z) = \exp(\gamma_n z)\left[A \sin\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right) + B\cos\left(\frac{n\pi(z - \log(l)}{\log(r) - \log(l)}\right)\right]$$

In order to satisfy the boundary conditions listed above we need $B = 0$. The constant A will be determined by the normalization of ψ_n, so we will leave it off until the end.

For convenience, write,

$$\psi = \psi_n, \qquad \gamma = \gamma_n, \qquad k = \frac{n\pi}{\log(l/r)}, \qquad \cos(z') = \cos(k(z - \log l))$$

We then have,

$$\begin{aligned}
\partial_z \psi(z) &= \gamma\psi + \exp(\gamma z) k \cos(z') \\
\partial_z^2 \psi(z) &= \gamma^2 \psi + \gamma \exp(\gamma z) k \cos(z')) + \gamma \exp(\gamma z) k \cos(z') - k^2 \psi \\
&= \gamma^2 \psi + 2\gamma \exp(\gamma z) k \cos(z') - k^2 \psi
\end{aligned}$$

We seek γ such that $\mathcal{A}_z\psi = \lambda\psi$ for some constant λ. That is, in our expression of $\mathcal{A}_z\psi$ we require the terms not containing a ψ be zero. Thus,

$$\begin{aligned} 0 &= \left(\mu - \frac{\sigma^2}{2}\right)\exp(\gamma z)k\cos(z') + \left(\frac{\sigma^2}{2}\right)2\gamma\exp(\gamma z)k\cos(z') \\ &= \left[\left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma\right]\exp(\gamma z)\cos(z') \end{aligned}$$

Suppose $k \neq 0$ (i.e. that the solution is non-trivial). Since $\exp(\gamma z)$ and $\cos(z') \neq 0$ we have,

$$0 = \left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\gamma$$

Solving for γ we have,

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2}$$

The eigenvalues are,

$$\lambda_n = \left(\mu - \frac{\sigma^2}{2}\right)\gamma + \left(\frac{\sigma^2}{2}\right)\left(\gamma^2 - k^2\right) = -\frac{\sigma^2}{2}[k^2 + \gamma^2]$$

Transforming back to x we have, $\hat{\Psi}_n(x) = \psi_n(\log(x))$ satisfies,

$$\mathcal{A}\hat{\Psi}_n(x) = \lambda_n\hat{\Psi}_n(x), \qquad\qquad \mathcal{A} = \mu x\partial_x + \frac{1}{2}\sigma^2x^2\partial_x^2$$

Define,

$$m(y) = \frac{2}{\sigma^2y^2}\exp\left(\int \mathrm{d}y\frac{2\mu y}{\sigma^2y^2}\right) = \frac{2}{\sigma^2y^2}\exp\left(\frac{2\mu}{\sigma^2}\log(y)\right) = \frac{2}{\sigma^2}y^{2\mu/\sigma^2-2} = \frac{2}{\sigma^2}y^{-2\gamma-1}$$

It is clear that the $\hat{\Psi}_n$ are orthogonal (properties of sines). We compute,

$$\langle\hat{\Psi}_n(x), \hat{\Psi}_n(x)\rangle_m = \int_l^r \Psi_n(x)^2m(x)\mathrm{d}x = \log(r/l)/\sigma^2$$

We then satisfy $\langle\Psi_k, \Psi_l\rangle_m = \delta_{kl}$ by defining,

$$\Psi_n(x) = \frac{\hat{\Psi}_n(x)}{\sqrt{\langle\Psi_n(x), \Psi_n(x)\rangle_m}}$$

Explicitly,

$$\Psi_n(x) = \frac{\sigma}{\sqrt{\log(r/l)}}x^\gamma\sin(k(z - \log l)) = \frac{\sigma}{\sqrt{\log(r/l)}}x^{1/2-\mu/\sigma^2}\sin\left(n\pi\frac{\log(x/l)}{\log(r/l)}\right)$$

Finally,

$$\Gamma(t, x; T, y) = m(y)\sum_n \exp((T - t)\lambda_n)\Psi_n(x)\Psi_n(y)$$

Explicitly,

$$\Gamma(t, x; T, y) = \frac{2}{\log(r/l)}\left(\frac{x}{y}\right)^{1/2-\mu/\sigma^2}y^{-1}\sum_n \exp((T - t)\lambda_n)\sin\left(n\pi\frac{\log(x/l)}{\log(r/l)}\right)\sin\left(n\pi\frac{\log(y/l)}{\log(r/l)}\right)$$

Since the Ψ_n are normalized then Γ is normalized.

We verify in Mathematica that Γ satisfies both the KFE and KBE.

Exercise 9.6

Consider a two-dimensional diffusion processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ that satisfy the SDEs

$$\mathrm{d}X_t = \mathrm{d}W_t^1 \qquad\qquad \mathrm{d}Y_t = \mathrm{d}W_t^2$$

where W_t^1 and W_t^2 are two independent Brownian motions. Define a function u as follows

$$u(x,y) = \mathbb{E}\left[\phi(X_\tau)|X_t = x, Y_t = y\right], \qquad\qquad \tau = \inf\{s \geq t : Y_s = a\}$$

1. State a PDE and boundary conditions satisfied by the function u.
2. Let us define the Fourier transform and and inverse Fourier transform, respectively, as follows

$$\text{Fourier Transform:} \qquad \hat{f}(\omega) := \int e^{-i\omega x} f(x)\mathrm{d}x$$

$$\text{Inverse Transform:} \qquad f(x) := \frac{1}{2\pi}\int e^{i\omega x}\hat{f}(\omega)\mathrm{d}\omega$$

   Use Fourier transforms and a conditioning argument to derive an expression for $u(x,y)$ as an inverse Fourier transform. Use this result to derive an explicit form for $\mathbb{P}(X_\tau \in \mathrm{d}z|X_t = x, Y_t = y)$(i.e., an expression involving no integrals).
3. Show the expression you derived in part 2 for $u(x,y)$ satisfies the PDE and BCs you stated in part 1.

Solution

1. Since there are no dt terms in either Brownian motion, and since the coefficient in both of the dW_t term is 1 we have, generator,

$$\mathcal{A} = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2$$

   The PDE satisfied by u is,

$$\mathcal{A}u = \left(\frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2\right)u = 0 \qquad \iff \qquad \left(\partial_x^2 + \partial_y^2\right)u = 0$$

   If $y = a$ then $\tau = t$ so $X_\tau = x$. We therefore have boundary condition,

$$u(x,a) = \phi(x)$$

2. Given starting position (x,y) at time t, and time τ, from the notes we know X_τ is normally distributed with mean x and variance $\tau - t$ by the independent increments property of Brownian motion. We know the characteristic function of a normally distributed random variable with distribution $\mathcal{N}(\mu,\sigma^2)$ is $e^{i\omega x - \sigma^2\omega^2/2}$. Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}\middle|\tau, X_t = x, Y_t = y\right] = e^{i\omega x - (\tau - t)\omega^2/2}$$

   Thus, using iterated conditioning,

$$\begin{aligned}
\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] &= \mathbb{E}\left[\mathbb{E}[e^{i\omega X_\tau}|\tau, X_t = x, Y_t = y]|X_t = x, Y_t = y\right] \\
&= \mathbb{E}\left[e^{i\omega x - (\tau - t)\omega^2/2}|X_t = x, Y_t = y\right] \\
&= e^{i\omega x}\mathbb{E}\left[e^{-(\tau - t)\omega^2/2}|X_t = x, Y_t = y\right]
\end{aligned}$$

We have previously shown that the first hitting time of a Brownian motion τ_m satisfies,

$$\mathbb{E}\left[e^{-\lambda\tau_m}\right] = e^{-|m|\sqrt{2\lambda}}$$

where $\tau_m = \inf\{t \geq 0 : W_t = m\}$ and $W_0 = 0$.
Since we start at position y at time t (rather that position 0 and time 0 as above), we know that,

$$\mathbb{E}\left[e^{-(\omega^2/2)(\tau-t)}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Therefore,

$$\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right] = e^{-|a-y||\omega|}$$

Write,

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x}\hat{\phi}(\omega)\mathrm{d}\omega$$

Then,

$$u(x,y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y] = \mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega\middle|X_t = x, Y_t = y\right]$$

Now, bringing the expectation through the integral, and applying the above result,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega X_\tau}\hat{\phi}(\omega)\mathrm{d}\omega\middle|X_t = x, Y_t = y\right] &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)\mathbb{E}\left[e^{i\omega X_\tau}|X_t = x, Y_t = y\right]\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega\end{aligned}$$

First recall, $\mathbb{E}[\phi(X)] = \int\phi(x)f_X(x)\mathrm{d}x$ and $\mathbb{P}(X \in \mathrm{d}z) = f_X(z)\mathrm{d}z$. Then, taking $\phi(x) = \mathbb{1}_{\{x\in\mathrm{d}z\}}$ means $\mathbb{E}[\phi(X)] = f_X(z)\mathrm{d}z = \mathbb{P}(X \in \mathrm{d}z)$. Therefore,

$$u(x,y) = \mathbb{E}[\mathbb{1}_{\{X_\tau\in\mathrm{d}z\}}|X_t = x, Y_t = y] = \mathbb{P}(X_\tau \in \mathrm{d}z|X_t = x, Y_t = y)$$

In this case,

$$\hat{\phi}(\omega) = \int_{\mathbb{R}} e^{-i\omega x}\mathbb{1}_{\{x\in\mathrm{d}z\}}\mathrm{d}x = e^{-i\omega z}\mathrm{d}z$$

Thus, computing this integral by splitting it at 0,

$$\begin{aligned}u(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega z}\mathrm{d}z e^{-|a-y||\omega|}e^{i\omega x}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\left[\frac{2|a-y|}{(a-y)^2+(x-z)^2}\right]\mathrm{d}z \\ &= \frac{1}{\pi}\left[\frac{|y-a|}{(y-a)^2+(x-z)^2}\right]\mathrm{d}z\end{aligned}$$

3. First observe,

$$u(x,a) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{-|a-a|}|\omega|e^{i\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(\omega)e^{i\omega x}\mathrm{d}\omega = \phi(x)$$

Define,

$$c = \begin{cases} 1 & y \geq a \\ -1 & y < a \end{cases}$$

Now observe,

$$\partial_x^2 u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega) e^{-c(y-a)|\omega|} \partial_x^2 e^{i\omega x} \mathrm{d}\omega = \frac{(i^2\omega^2)}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega) e^{-c(y-a)|\omega|} e^{i\omega x} \mathrm{d}\omega$$

Then,

$$\partial_y^2 u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega) \partial_y^2 e^{-c(y-a)|\omega|} e^{i\omega x} \mathrm{d}\omega = \frac{c^2\omega^2}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega) e^{-c(y-a)|\omega|} e^{i\omega x} \mathrm{d}\omega$$

Thus, since $i^2 = -1$ and $c^2 = 1$,

$$(\partial_x^2 + \partial_y^2) u(x,y) = 0$$

Note there is probably some issue with the partial derivative with respect to y at $y = a$, since $|y - a|$ is not differentiable at this point.

Therefore $u(x,y) = \mathbb{E}[\phi(X_\tau)|X_t = x, Y_t = y]$ satisfies the PDE from 1.