60 RIGIDITY AND SCENE ANALYSIS
Walter Whiteley

INTRODUCTION

Rigidity and flexibility of frameworks (motions preserving lengths of bars) and scene analysis (liftings from plane polyhedral pictures to spatial polyhedra) are two core examples of a general class of geometric problems:

(a) Given a discrete configuration of points, lines, planes, ... in Euclidean space, and a set of geometric constraints (fixed lengths for rigidity, fixed incidences, and fixed projections of points for scene analysis), what is the set of solutions and what is its local form: discrete? k-dimensional?

(b) Given a structure satisfying the constraints, is it unique, or at least locally unique, up to trivial changes, such as congruences for rigidity, or vertical scale for liftings?

(c) How does this answer depend on the combinatorics of the structure and how does it depend on the specific geometry of the initial data or object?

The rigidity of frameworks examines points constrained by fixed distances between pairs, using vocabulary and linear techniques drawn from structural engineering: bars and joints, first-order rigidity and first-order flexes, and static rigidity and static self-stresses (Section 60.1). Scene analysis and the dual concept of parallel drawings are described in Section 60.2. Finally, reciprocal diagrams form a fundamental geometric connection between liftings of polyhedral pictures and self-stresses in frameworks (Section 60.3).

These core problems have a wide range of applications across many areas of applied geometry. The methods used and the results obtained for these problems serve as a model for what might be hoped for other sets of constraints (plane first-order results) and as a warning of the complexity that does arise (higher dimensions and broader forms of rigidity). The subject has a rich history, stretching back into at least the middle of the 19th century, in structural and mechanical engineering. Other independent rediscoveries and connections have arisen in crystallography and scene analysis. Some other geometric problems with related mathematical and algorithmic patterns are mentioned in Sections 60.1.5, 60.2.3, and 60.3. For more general geometric reconstruction problems, see Chapter 29.

60.1 RIGIDITY OF BAR FRAMEWORKS

Given a set of points in space, with certain distances to be preserved, what other configurations have the same distances? If we make small changes in the distances, will there be a small (linear scale) change in the position? What is the structure, locally and globally, of the algebraic variety of these "realizations"?

We begin with the simplest linear theory: first-order rigidity, and the equivalent dual static rigidity, which are the linearized (and therefore linear algebra) version of rigidity. Generic rigidity refers to first-order rigidity of "almost all" geometric positions of the underlying combinatorial structure. After the initial results presenting first-order rigidity (Section 60.1.1), the study divides into the combinatorics of generic rigidity, using graphs (Section 60.1.2); the geometry of special positions in first-order rigidity, using projective geometry (Section 60.1.3); more general concepts of rigidity (Section 60.1.4); and extensions to tensegrity frameworks, using geometry and minima of energy functions for rigidity (Section 60.1.5).

60.1.1 FIRST-ORDER RIGIDITY

GLOSSARY

Configuration of points in d-space: An assignment $p = (p_1, \ldots, p_v)$ of points $p_i \in \mathbb{R}^d$ to an index set V, where $v = |V|$.

Congruent configurations: Two configurations p and q in d-space, on the same set V, related by an isometry T of $\mathbb{R}^d$ (with $T(p_i) = q_i$ for all $i \in V$).

Bar framework in d-space $G(p)$ (or **framework**): A graph $G = (V; E)$ (no loops or multiple edges) and a configuration p in d-space for the vertices V (Figure 60.1.1A).

Bar: An edge $\{i, j\} \in E$ for a framework $G(p)$.

First-order flex or **infinitesimal motion:** For a bar framework $G(p)$, an assignment of velocities $p' : V \to \mathbb{R}^d$, such that for each edge $\{i, j\} \in E$: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$ (Figure 60.1.1.C,D, where the arrows represent nonzero velocities).

Trivial first-order flex: A first-order flex p' that is the derivative of a flex of congruent frameworks (Figure 60.1.1C). (There is a fixed skew-symmetric matrix S (a rotation) and a fixed vector t (a translation) such that, for all vertices $i \in V$, $p'_i = p_i S + t$.)

First-order flexible framework: A framework $G(p)$ with a nontrivial first-order flex (Figure 60.1.1D).

First-order rigid framework: A bar framework $G(p)$ for which every first-order flex is trivial (Figures 60.1.1A, 60.1.2A).

Rigidity matrix: For a framework $G(p)$ in d-space, $R_G(p)$ is the $|E| \times d|V|$ matrix for the system of equations: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$ in the unknown velocities p'_i. The first-order flex equations are expressed as

$$
R_G(p)p'^T = \begin{bmatrix} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \end{bmatrix} \times p'^T = 0^T.
$$

Self-stress: For a framework $G(p)$, a row dependence ω for the rigidity matrix: $\omega R_G(p) = 0$. Equivalently, an assignment of scalars ω_{ij} to the edges such that

at each vertex i, $\sum_{\{j|\ \{i,j\}\in E\ \}} \omega_{ij}(p_i - p_j) = 0$ (placing these "internal forces" $\omega_{ij}(p_i - p_j)$ in equilibrium at vertex i). $\omega_{ij} < 0$ is tension, $\omega_{ij} > 0$ is compression.

Independent framework: A bar framework $G(p)$ for which the rigidity matrix has independent rows. Equivalently, there is only the zero self-stress.

Isostatic framework: A framework $G(p)$ that is first-order rigid and independent.

Generically rigid graph in d-space: A graph G for which the frameworks $G(p)$ are first-order rigid on an open dense subset of configurations p in d-space (Figures 60.1.1A, 60.1.2A).

FIGURE 60.1.1
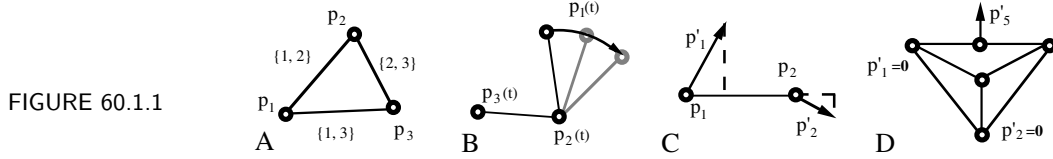
Generic d-circuit: A graph G such that with the deletion of any edge e, $G - e$ is generically rigid in d-space.

BASIC CONNECTIONS

Because the constraints $|p_i - p_j| = |q_i - q_j|$ are algebraic in the coordinates of the points (after squaring), we can work with the Jacobian matrix formed by the partial derivatives of these equations—the rigidity matrix of the framework.

The dimension of the space of trivial first-order motions of a framework in d-space is $\binom{d+1}{2}$ provided $|V| \geq d$ (the velocities generated by d translations and by $\binom{d}{2}$ rotations form a basis).

THEOREM 60.1.1 *First-order Rank*

A framework $G(p)$ with $|V| \geq d$ is first-order rigid if and only if the rigidity matrix $R_G(p)$ has rank $d|V| - \binom{d+1}{2}$.

A framework $G(p)$ with few vertices, $|V| \leq d$, is isostatic if and only if the rigidity matrix $R_G(p)$ has rank $\binom{v}{2}$ (if and only if G is the complete graph on V and the points p_i do not lie in an affine space of dimension $|V| - 2$).

First-order rigidity is linear algebra, with first-order rigid frameworks, self-stresses, and isostatic frameworks playing the roles of spanning sets, linear dependence, and bases of the row space for the rigidity matrix of the complete graph on the configuration p.

There is a dual theory of static rigidity for bar frameworks. Where first-order rigidity focuses on the kernel of the rigidity matrix (first-order flexes) and on the column space and column rank, static rigidity focuses on the cokernel of the rigidity matrix (the self-stresses) and on the row space of the rigidity matrix (the resolvable static loads). Methods from both approaches are widely used [CW82, Whi84, Whi96], although in this chapter we present the results primarily in the vocabulary of first-order rigidity.

THEOREM 60.1.2 *Isostatic Frameworks*

For a framework $G(p)$ in d-space, with $|V| \geq d$, the following are equivalent:

(a) *$G(p)$ is isostatic (first-order rigid and independent);*

(b) *$G(p)$ is first-order rigid with $|E| = d|V| - \binom{d+1}{2}$;*

(c) *$G(p)$ is independent with $|E| = d|V| - \binom{d+1}{2}$;*

(d) *$G(p)$ is first-order rigid, and removing any one bar (but no vertices) leaves a first-order flexible framework.*

First-order rigidity of a framework $G(p)$ is a robust property: a small change in the configuration p preserves this rigidity. Independence implies that the distances are robust: any small change in these distances can be realized by a nearby configuration. On the other hand, self-stresses mean that one of the distances is algebraically dependent on the others: many small changes in the distances will have no realizations, or no nearby realizations.

Figure 60.1.2 illustrates a single graph with plane configurations that produce: (A) a first-order rigid framework; (B) a first-order flexible, but rigid, framework, and (C) a flexible framework (see Section 60.1.4). The graph itself is generically 2-rigid.

FIGURE 60.1.2

A   B   C
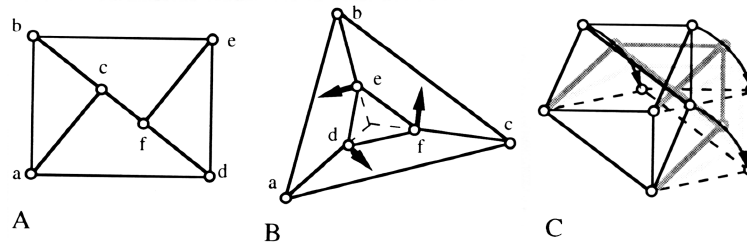
THEOREM 60.1.3 *Generic Rigidity Theorem*

For a graph G and a fixed dimension d the following are equivalent:

(a) *G is generically rigid in d-space;*

(b) *for each configuration $p \in \mathbb{R}^{dv}$ using algebraically independent numbers over the rationals as coordinates, the framework $G(p)$ is first-order rigid;*

(c) *$G(p)$ is first-order rigid for some configuration $p \in \mathbb{R}^{dv}$.*

60.1.2 COMBINATORICS FOR GENERIC RIGIDITY

The major goal in generic rigidity is a combinatorial characterization of graphs that are generically rigid in d-space. The companion problem is to find efficient combinatorial algorithms to test graphs for generic rigidity. For the plane (and the line), this is solved. Beyond the plane the results are essentially incomplete, but some significant partial results are available.

GLOSSARY

Generically d-independent: A graph G for which some (equivalently, almost all) configurations p produce independent frameworks in d-space.

Generically d-isostatic graph: A graph G for which some (equivalently, almost all) configurations p produce isostatic frameworks in d-space.

Generic d-circuit: A graph G that is dependent for all configurations p in d-space but for all edges $\{i, j\} \in E$, $G - \{i, j\}$ is generically independent in d-space.

Complete bipartite graph: A graph $K_{m,n} = (A \cup B, A \times B)$, where A and B are disjoint sets of cardinality $|A| = m$ and $|B| = n$.

Triangulated d-pseudomanifold: A finite set of d-simplices (complete graphs on $d + 1$ points) with the property that each d subset (facet) occurs in exactly two simplices, any two simplices are connected by a path of simplices and shared facets, and any two simplices sharing a vertex are connected through other simplices at this vertex. (For example, the triangles, edges, and vertices of a closed triangulated 2-surface without boundary, such as a sphere or torus, form a 2-pseudomanifold.) Cf. Section 18.3.

Henneberg d-construction for a graph G: A sequence $(V_d, E_d), ..., (V_n, E_n)$ of graphs, such that:

(i) For each index $d < j \leq n$, (V_j, E_j) is obtained from (V_{j-1}, E_{j-1}) by

vertex addition: attaching a new vertex by d edges (Figure 60.1.4A for $d = 2$), or

edge splitting: replacing an edge from (V_{j-1}, E_{j-1}) with a new vertex joined to its ends and to $d - 1$ other vertices (Figure 60.1.4B for $d = 2$); and

(ii) (V_d, E_d) is the complete graph on d vertices, and $(V_n, E_n) = G$ (Figure 60.1.6A).

Proper 3Tree2 partition: A partition of the edges of a graph into three trees, such that each vertex is attached to exactly two of these trees and no nontrivial subtrees of distinct trees T_i have the same support (i.e., the same vertices) (Figure 60.1.6B).

Proper 2Tree partition: A partition of the edges of a graph into two spanning trees, such that no nontrivial subtrees of distinct trees T_i have the same support (i.e., the same vertices) (Figure 60.1.6C).

d-connected graph: A graph G such that removing any $d - 1$ vertices (and all incident edges) leaves a connected graph. (Equivalently, a graph such that any two vertices can be connected by at least d paths that are vertex-disjoint except for their endpoints.)
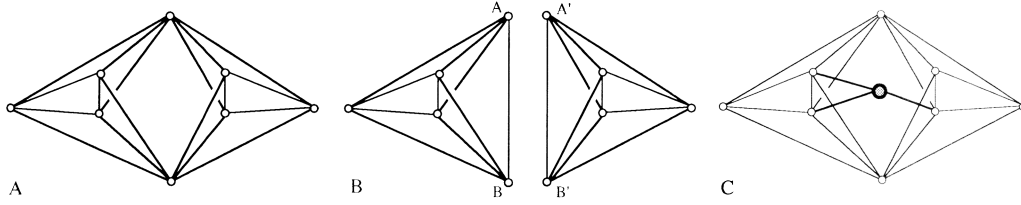
BASIC PROPERTIES IN ALL DIMENSIONS

THEOREM 60.1.4 *Necessary Counts and Connectivity Theorem*

If a graph G is generically d-isostatic, then, if $V \geq d$, $|E| \leq d|V| - \binom{d+1}{2}$ and for every subgraph on $|V'| \geq d$ vertices with edges E' in $V' \times V'$, $|E'| \leq d|V'| - \binom{d+1}{2}$.

If $G = (V, E)$ is a generically d-isostatic graph with $|V| > d$, then (V, E) is a d-connected graph.

FIGURE 60.1.3

For dimensions 1 and 2, the first count alone is sufficient for generic rigidity (see below). For dimensions $d > 2$, these two conditions are not enough to characterize the generically d-isostatic graphs. Figure 60.1.3A shows a generically flexible counterexample for the sufficiency of the counts in dimension 3. This example is generated by a "circuit exchange" on two over-counted graphs (Figure 60.1.3B). Figure 60.1.3C adds 3-connectivity, but preserves the flexibility and the counts.

THEOREM 60.1.5 *Bipartite Graphs*

A complete bipartite graph $K_{m,n}$, with $m > 1$, is generically rigid in dimension d if and only if $m + n \geq \binom{d+2}{2}$ and $m, n > d$.

INDUCTIVE CONSTRUCTIONS FOR ISOSTATIC GRAPHS

Inductive constructions for graphs that preserve generic rigidity are used both to prove theorems for general classes of frameworks and to analyze particular graphs.

THEOREM 60.1.6 *Vertex Addition Theorem*

Let $G = (V, E)$ be a graph with a vertex i of valence d; let $H = (U, F)$ denote the subgraph obtained by deleting i and the edges incident with it. Then G is generically d-isostatic if and only if H is generically d-isostatic (Figure 60.1.4A for $d = 2$).

THEOREM 60.1.7 *Edge Split Theorem*

Let $G = (V, E)$ be a graph with a vertex i of valence $d+1$, let S be the set of vertices adjacent to i, and let $H = (U, F)$ be the subgraph obtained by deleting i and its $d+1$ incident edges. Then G is generically d-isostatic if and only if there is a pair j, k of vertices of V such that the edge $\{j, k\}$ is not in F and the graph $H' = (U, F \cup \{j, k\})$ is generically d-isostatic (Figure 60.1.4B for $d = 2$).

FIGURE 60.1.4
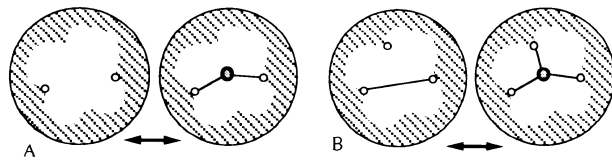
THEOREM 60.1.8 *Construction Theorem*

If a graph G is obtained by a Henneberg d-construction, then G is generically d-isostatic (Figure 60.1.6A for $d = 2$).

THEOREM 60.1.9 *Gluing Theorem*

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are generically d-rigid graphs sharing at least d vertices, then $G = (V_1 \cup V_2, E_1 \cup E_2)$ is generically d-rigid.

THEOREM 60.1.10 *Vertex Splitting Theorem*

If the graph G' is a vertex split of a generically d-isostatic graph G on d edges (Figure 60.1.5A for $d = 3$) or a vertex split on $d - 1$ edges (Figure 60.1.5B for $d = 3$), then G' is generically d-isostatic.

FIGURE 60.1.5

A Split on *d* edges B Split on *d*-1 edges
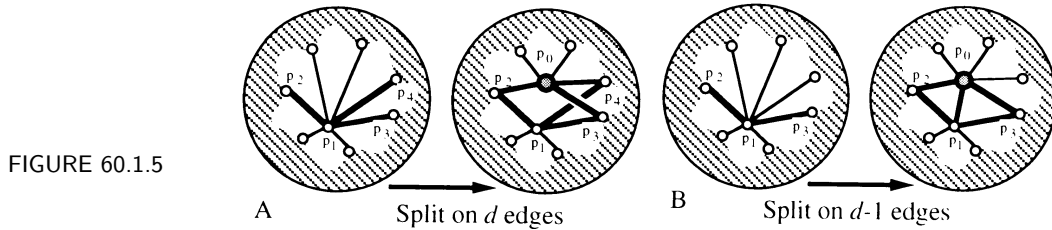
PLANE ISOSTATIC GRAPHS

Many plane results are expressed in terms of trees in the graph, building on a simpler correspondence between rigidity on the line and the connectivity of the graph.

THEOREM 60.1.11 *Line Rigidity*

For graph G and configuration p on the line with $p_i \neq p_j$ for all $\{i, j\} \in E$, the following are equivalent:

 (a) *$G(p)$ is minimal among rigid frameworks on the line with these vertices;*

 (b) *$G(p)$ is isostatic on the line;*

 (c) *G is a spanning tree on the vertices;*

 (d) *$|E| = |V| - 1$ and for every nonempty subset E' with vertices V', $|E'| \leq |V'| - 1$.*

THEOREM 60.1.12 *Plane Isostatic Graphs Theorem*

For a graph G with $|V| \geq 2$, the following are equivalent:

 (a) *G is generically isostatic in the plane;*

 (b) *$|E| = 2|V| - 3$, and for every subgraph (V', E') with $|V'| \geq 2$ vertices, $|E'| \leq 2|V'| - 3$ (Laman's theorem);*

 (c) *there is a Henneberg 2-construction for G (Henneberg's theorem);*

 (d) *E has a proper 3Tree2 partition (Crapo's theorem);*

 (e) *for each $\{i, j\} \in E$, the multigraph obtained by doubling the edge $\{i, j\}$ is the union of two spanning trees (Recski's theorem).*
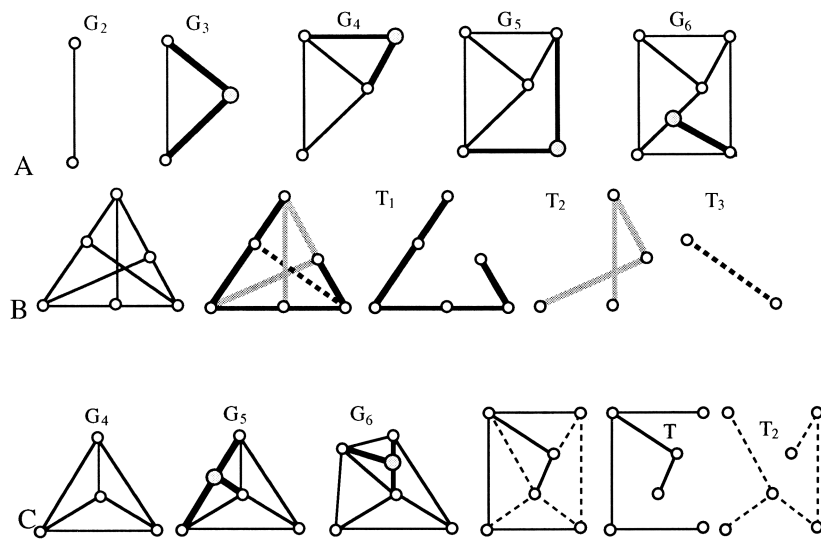
FIGURE 60.1.6

Figure 60.1.6A shows the Henneberg plane construction for the isostatic graph of Figure 60.1.2. Figure 60.1.6B shows a proper 3Tree2 partition of the isostatic complete bipartite graph $K_{3,3}$. With an added edge, joining T_2 to T_3, this partition creates several of the pairs of spanning trees predicted by Recski's theorem.

THEOREM 60.1.13 *Plane 2-Circuits Theorem*

For a graph G with $|V| \geq 2$, the following are equivalent:

(a) *G is a generic 2-circuit;*

(b) *$|E| = 2|V| - 2$, and for every proper subset E' on vertices V', $|E'| \leq 2|V'| - 3$;*

(c) *there is a construction for G from K_4, using only edge splitting and gluing; (Berg and Jordan's theorem);*

(d) *E has a proper 2Tree partition.*

Figure 60.1.6C shows the construction for a 2-circuit, and an associated 2Tree partition. For 2-circuits with planar graphs, the planar dual is also a 2-circuit. The inductive techniques given above, and others, form dual pairs of constructions for these planar 2-circuits [BCW02].

THEOREM 60.1.14 *Sufficient Connectivity*

If a graph G is 6-connected, then G is generically rigid in the plane.

There are 5-connected graphs that are not generically rigid in the plane.

ALGORITHMS FOR GENERIC 2-RIGIDITY

Each of the combinatorial characterizations has an associated algorithm for verifying whether a graph is generically 2-isostatic:

(i) Counts: This can be checked by an $O(|V|^2)$ algorithm based on bipartite matchings or network flows on an associated graph [Sug86].

(ii) 2-construction: Existence of a 2-construction can be checked by an $O(2^{|V|})$ algorithm, but a proposed 2-construction can be verified in $O(|V|)$ time.

(iii) 3Tree2 covering: Existence can be checked by an $O(|V|^2)$ matroid partition algorithm [Cra].

(iv) Double tree partition: All required double-tree partitions can be found by a matroidal algorithm of order $O(|V|^3)$.

GENERICALLY RIGID GRAPHS IN HIGHER DIMENSIONS

Most of the results are covered by the initial summary for all dimensions d. Special results apply to the graphs of triangulated polytopes, as well as more general surfaces.

THEOREM 60.1.15   *Triangulated Pseudomanifolds Theorem*

For $d \geq 2$, the graph of a triangulated d-pseudomanifold is generically $(d+1)$-rigid.

In particular, the graph of any closed triangulated 2-surface without boundary is generically rigid in 3-space (Fogelsanger's theorem), and the graph of any triangulated sphere is generically 3-isostatic (Gluck's theorem). Beyond the triangulated spheres in 3-space, most of these graphs are not isostatic, but are dependent.

OPEN PROBLEMS

There is no combinatorial characterization of generically 3-isostatic graphs. There are several related conjectures, due to Dress, Graver, and Tay and Whiteley, that may be correct but are unproven. We offer one of these.
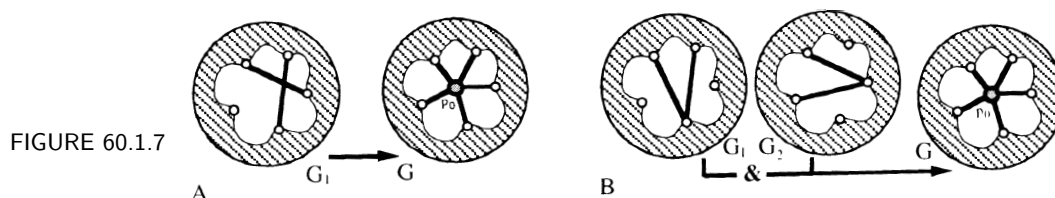

FIGURE 60.1.7

CONJECTURE 60.1.16   *3-D Replacement Conjecture*

The X-replacement in Figure 60.1.7A takes a graph G_1 that is generically rigid in 3-space to a graph G that is generically rigid in 3-space.

   The double V-replacement in Figure 60.1.7B takes two graphs G_1, G_2 that are generically rigid in 3-space to a graph G that is generically rigid in 3-space.

Every 3-isostatic graph is generated by an "extended Henneberg 3-construction," which adds these two moves to the simpler edge splitting and vertex addition. What is unproven is that *only* 3-isostatic graphs are generated in this way.

The plane analogue of X-replacement is true for plane generic rigidity (without adding the fifth bar) [BCW02], and the 4-space analogue is false for some graphs (with two extra bars added in this analogue). If these conjectured steps prove correct in 3-space, then we would have inductive techniques to generate the graphs of all isostatic frameworks in 3-space, but the algorithm would be exponential.

For 4-space, there is no conjecture that has held up against the known counterexamples based on generically 4-flexible complete bipartite graphs such as $K_{7,7}$.

CONJECTURE 60.1.17 *Sufficient Connectivity Conjecture*

If a graph G is 12-connected, then G is generically rigid in 3-space.

A graph can be checked for generic 3-rigidity by a "brute force" $O(2^{2^{|V|}})$ algorithm. Assign the points independent variables as coordinates, form the rigidity matrix, then check the rank by symbolic computation. On the other hand, if numerical coordinates are chosen for the points "at random," then the rank of this numerical matrix $(O(|E|^3))$ will be the generic value, with probability 1. This problem has a randomized polynomial-time algorithm, but there is no known deterministic algorithm that runs in polynomial, or even exponential, time.

60.1.3 GEOMETRY OF FIRST-ORDER RIGIDITY

GLOSSARY

Special position of a graph G in d-space: Any configuration $p \in \mathbb{R}^{dv}$ such that the rigidity matrix $R_G(p)$, or any submatrix, has rank smaller than the maximum rank (the rank at a configuration with algebraically independent coordinates).

Projective transform of a d-configuration p: A d-configuration q on the same vertices, such that there is an invertible matrix T of size $d+1 \times d+1$ making $T(p_i, 1) = \lambda_i(q_i, 1)$ (where $(p_i, 1)$ is the vector p_i extended with an additional 1 — the affine coordinates of p_i).

Affine spanning set for d-space: A configuration p of points such that every point $q_0 \in \mathbb{R}^d$ can be expressed as an affine combination of the p_i: $q_0 = \sum_i \lambda_i p_i$, with $\sum_i \lambda_i = 1$. (Equivalently, the affine coordinates $(p_i, 1)$ span the vector space $\mathbb{R}^{d+1}$.)

Cone graph: The graph $G * u$ obtained from $G = (V, E)$ by adding a new vertex u and the $|V|$ edges (u, i) for all vertices $i \in V$.

Cone projection from p_0:   For a $(d+1)$-configuration p on V, a configuration $q = \Pi_0(p)$ in d-space (placed as a hyperplane in $(d+1)$-space) on the vertices $V \backslash 0$, such that $p_i \neq p_0$ is on the line $q_i p_0$ for all $i \neq 0$.

BASIC RESULTS

THEOREM 60.1.18 *First-order Flex Test*

If the points of a configuration p on the vertices V affinely span d-space, then a

first-order motion p' is nontrivial if and only if there is some pair h, k (not a bar) such that: $(p_h - p_k) \cdot (p'_h - p'_k) \neq 0$.

THEOREM 60.1.19   *Projective Invariance*

If a framework $G(p)$ is first-order rigid (isostatic, independent) and $q = T(p)$ is a projective transform of p, then $G(q)$ is first-order rigid (isostatic, independent, respectively).

The following result provides an alternate proof of projective invariance as well as a corresponding generic result for cones.

THEOREM 60.1.20   *Coning Theorem*

A framework $G(\Pi_0 p)$ is first-order rigid (isostatic, independent) in d-space if and only if the cone $(G * u)(p)$ is first-order rigid (isostatic, independent, respectively) in $(d+1)$-space.

The special positions of a graph in d-space are rare, since they form a proper algebraic variety (essentially generated by minors of the rigidity matrix with variables for the coordinates of points). For a generically isostatic graph, this set of special positions can be described by the zeros of a single polynomial [WW83].

THEOREM 60.1.21   *Pure Condition*

For any graph G that is generically isostatic in d-space, there is a homogeneous polynomial $C_G(x_{1,1}, \ldots, x_{1,d}, \ldots, x_{|V|,1}, \ldots, x_{|V|,d})$ such that $G(p)$ is first-order flexible if and only if $C_G(p_1, \ldots, p_{|V|}) = 0$. C_G is of degree $(\mathrm{val}_i + 1 - d)$ in the variables $(x_{i,1}, \ldots, x_{i,d})$ for each vertex i of valence val_i in the graph.

FIGURE 60.1.8
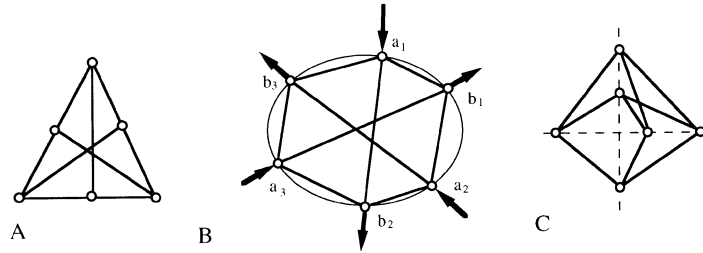
Since Grassmann algebra (Chapter 59) is the appropriate language for these projective properties, these pure conditions C_G are polynomials in the Grassmann algebra. Section 59.4 contains several examples of these polynomial conditions.

THEOREM 60.1.22   *Quadratics for Bipartite Graphs*

For a complete bipartite graph $K_{m,n}$ and $d > 1$, the framework $K_{m,n}(p)$, with $p(A)$ and $p(B)$ each affinely spanning d-space, is first-order flexible if and only if all the points $p(A \cup B)$ lie on a quadric surface of d-space (Figure 60.1.8).

The following classical result describes an important open set of configurations that are not special for triangulated spheres.

### THEOREM 60.1.23   *Extended Cauchy Theorem*

If $G(p)$ consists of the vertices and edges of a convex simplicial d-polytope, then

G(p) is first-order rigid in d-space.

 If G(p) consists of the vertices and edges of a strictly convex polyhedron in 3-space, then G(p) is independent.

We recall that Steinitz's theorem guarantees that every 3-connected planar graph has a realization as the edges of a strictly convex polyhedron in 3-space, which gives Gluck's theorem. There are numerous example of nonconvex simplicial polytopes that are not first-order rigid. Connelly [Con78] gives a nonconvex (but not self-intersecting) triangulated sphere (with nine vertices) that is flexible (see the definition below). For many graphs, such as a triangulated torus (Theorem 60.1.15), we do not have even one specific configuration that gives a first-order rigid framework, only the guarantee that "almost all" configurations will work.

FIGURE 60.1.9
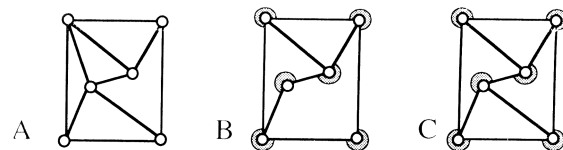
Recent papers [Str03, HOR+02] suggest that pseudotriangulations play a role for planar graphs in plane rigidity analogous to the role of convex polyedra for planar graphs in 3-space. ***Pseudotriangulations*** were defined in Chapter 5, as plane-embedded graphs with a convex polygonal boundary, all interior regions being polygons with exactly three interior angles that are $< \pi$ (Figure 60.1.9A,C). A plane-embedded graph is ***pointed*** if at each vertex there is an angle that is embedded as $> \pi$ (Figure 60.1.9B,C). The following are some of these recent results.

THEOREM 60.1.24 *Counts on Pseudotriangulations*

For a general position configuration p, the following properties are equivalent:

 (a) G(p) is a pointed pseudotriangulation;

 (b) G(p) is a pseudotriangulation with $|E| = 2|V| - 3$;

 (c) G(p) is a noncrossing pointed graph with $|E| = 2|V| - 3$;

 (d) G(p) is a noncrossing pointed graph and is maximal with this property, with the given vertices.

THEOREM 60.1.25 *Rigidity of Pseudotriangulations*

A pseudotriangulation G(p), realized as a bar framework, is first-order rigid. A pointed noncrossing graph G(p) is an independent bar framework.

 A planar graph G is generically 2-isostatic if and only if it has a realization as a pointed pseudotriangulation.

There are further significant consequences of the underlying projective geometry of first-order rigidity [CW82]. The concepts of first-order rigidity and first-order flexibility, as well as the dual statics, can be expressed in any of the Cayley-Klein metrics that are extracted from the shared underlying projective space. This family includes the spherical metric, the hyperbolic metric, and others. It is possible to

express first-order rigidity in entirely projective terms that are essentially independent of the metric. In this way, the points "at infinity" in the Euclidean space can be fully integrated into first-order rigidity. However, in some metrics such as the hyperbolic metric, there is a singular set (the sphere at infinity, also known as the *absolute*) on which rigidity equations have distinct properties. This transfer goes back to Pogorelov and has been reworked in [SW02].

THEOREM 60.1.26   *Transfer of Metrics*

For a given graph G and a fixed point p in projective space of dimension d, the framework G(p) is first-order rigid in Euclidean space if and only if G(p) is first-order rigid in any alternate Cayley-Klein metric, with p not containing points on the absolute.

The most extreme projective transformation is a polarity, in which points and hyperplanes (e.g., planes in 3-space) switch roles. For Euclidean 3-space, there are translations of first-order rigidity results to these dual "sheet" structures [Whi87]. For other metrics, the duality in three dimensions changes distance constraints on pairs of points into angle constraints on pairs of planes [SW02].

OTHER RELATED STRUCTURES

A number of related structures have also been investigated for first-order rigidity. One, which appears in engineering, robotics, and chemistry, is the "body-and-hinge framework." Rigid bodies, indexed by V, are connected in pairs along hinges (lines in 3-space), indexed by edges of a graph. The bodies each move, preserving the contacts at the hinges. Such hinged frameworks could be modeled as bar-and-joint frameworks, with each hinge replaced by a pair of joints and each body replaced by a first-order rigid framework on the joints of its hinges (and other joints if desired); cf. Sections 48.1 and 59.4. Unlike the unsolved problems for generic rigidity of frameworks in 3-space, the generic behavior of body-and-hinge structures has been completely solved. We state two sample results and a related conjecture.

THEOREM 60.1.27   *Tay's Theorem*

For a graph G the following are equivalent:

 (a) *for some hinge assignment of lines $h_{i,j}$ in 3-space to the edges $\{i, j\}$ of G, the body-and-hinge framework $G(h)$ is first-order rigid;*

 (b) *for almost all hinge assignments h, the body-and-hinge framework $G(h)$ is first-order rigid;*

 (c) *if each edge of the graph is replaced by five copies, the resulting multigraph contains six edge-disjoint spanning trees.*

Tay's theorem extends directly to all dimensions d (finding $\binom{d+1}{2}$ edge-disjoint spanning trees inside $\binom{d+1}{2} - 1$ copies of the graph).

THEOREM 60.1.28   *Spherical Flexes and Stresses*

Given an abstract spherical structure (see Section 60.3) $S = (V, F; \underline{E})$, and an assignment of distinct points $p_i \in \mathbb{R}^3$ to the vertices, the following two conditions are equivalent:

    (a) *the bar framework $G(p)$ on $G = (V, E)$ has a nontrivial self-stress;*

    (b) *the body-and-hinge framework on the dual graph $G^* = (F, E^*)$ with hinge lines $p_i p_j$ for each edge $\{i, j\}$ of G is first-order flexible.*

A second "model" treats the atoms of a molecule as the bodies, and the lines of the bond lines as hinges. Such structures are geometrically singular since the lines of all bonds of an atom are concurrent in the center of the atom. This model, and the equivelant bar frameworks, are central to applications of rigidity to protein structures with thousands of atoms [Whi99].

CONJECTURE 60.1.29 *Molecular Conjecture*

If a graph G is realized as the atoms (points) and bonds (lines) of a molecular structure, then the molecular structure is generically rigid if, and only if, when each edge of the graph G is replaced by five copies, the resulting multigraph contains six edge-disjoint spanning trees.

This conjecture is embedded in the FIRST algorithm for protein flexibility [JRKT01]. In polar form, the conjecture states that if each body is realized with all hinges of each body coplanar (plate structures), the generic rigidity is still measured by the existence of six spanning trees.

60.1.4 RIGID AND FLEXIBLE FRAMEWORKS

GLOSSARY

Bar equivalence: Two frameworks $G(p)$ and $G(q)$ such that all bars have the same length in both configurations: $|p_i - p_j| = |q_i - q_j|$ for all bars $\{i, j\} \in E$.

Analytic flex: An analytic function $p(t) : [0, 1) \to \mathbb{R}^{vd}$ such that $G(p(0))$ is bar-equivalent to $G(p(t))$ for all t (i.e., all bars have constant length).

Flexible framework: A bar framework $G(p)$ in $\mathbb{R}^d$ with an analytic flex $p(t)$ such that $p(0) = p$ but p is not congruent to $p(t)$ for all $0 < t$ (Figure 60.1.1B).

Rigid framework: A bar framework $G(p)$ in d-space that is not flexible (Figure 60.1.1A,D).

BASIC CONNECTIONS

Because the constraints $|p_i - p_j| = |q_i - q_j|$ are algebraic in the coordinates of the points (after squaring), many alternate definitions of a "rigid framework" are equivalent. These connections depend on results such as the curve selection theorem of algebraic geometry or the inverse function theorem.

THEOREM 60.1.30 *Alternate Rigidity Definitions*

For a bar framework $G(p)$ the following conditions are equivalent:

    (a) *the framework is rigid;*

(b) *for every continuous path, or **continuous flex** of $G(p)$, $p(t) \in \mathbb{R}^{vd}$, $0 \le t < 1$ and $p(0) = p$, such that $G(p(t))$ is bar-equivalent to $G(p)$ for all t, $p(t)$ is congruent to p for all t;*

(c) *there is an $\epsilon > 0$ such that if $G(p)$ and $G(q)$ are bar-equivalent and $|p-q| < \epsilon$, then p is congruent to q.*

Essentially, the first derivative of a nontrivial analytic flex is a nontrivial first-order flex: $D_t\big((p_i(t) - p_j(t))^2 = c_{ij}\big)\big|_{t=0} \Rightarrow 2(p_i - p_j) \cdot (p'_i - p'_j) = 0$. (If this first derivative is trivial, then the earliest nontrivial derivative is a first-order motion.) This result is related to general forms of the inverse function theorem.

THEOREM 60.1.31   *First-order Rigid to Rigid*

If a bar framework $G(p)$ is first-order rigid, then $G(p)$ is rigid.

Some first-order flexes are not the derivatives of analytic flexes (Figures 60.1.1D and 60.1.2B). However, a nontrivial first-order flex for a framework does guarantee a pair of nearby noncongruent, bar-equivalent frameworks.

THEOREM 60.1.32   *Averaging Theorem*

If the points of a configuration p affinely span d-space, then the assignment p' is a nontrivial first-order flex of $G(p)$ if and only if the frameworks $G(p + p')$ and $G(p - p')$ are bar-equivalent and not congruent.

Rigidity and first-order rigidity are equivalent in some situations.

THEOREM 60.1.33   *Rigid to First-order Rigid*

If bar framework $G(p)$ is independent, then $G(p)$ is first-order rigid if and only if $G(p)$ is rigid.

The recent solution of the Carpenter's Rule problem on straightening plane-embedded polygonal paths and convexifying plane-embedded polygons [CDR03, Str03] uses independence of appropriate bar frameworks, and resulting paths. The independence is proven using Maxwell's theorem (see Section 60.3). See Chapter 9 for more connections. The following is one form of this connection [RSS03].

THEOREM 60.1.34   *Expansive Motions*

If one edge of the boundary polygon of a pointed pseudotriangulation $G(p)$ is removed, and its two vertices are spread apart in a motion, then the resulting path (unique up to congruences) is expansive—all pairs of joints are either moving apart or remaining at a constant distance.

Whereas first-order rigidity is projectively invariant, rigidity itself is not projectively invariant—or even affinely invariant. It is a purely Euclidean property.

THEOREM 60.1.35   *Generic Rigidity Theorem II*

For a graph G and a fixed dimension d the following are equivalent:

(a) *G is generically rigid in d-space;*

(b) *for all $q \in U \subset \mathbb{R}^{dv}$, U some nonempty open set, $G(q)$ is rigid;*

(c) *for all $q \in W \subset \mathbb{R}^{dv}$, W some open dense set, $G(q)$ is first-order rigid.*

60.1.5 TENSEGRITY FRAMEWORKS

In a tensegrity framework, we replace some (or all) of the equalities for bars with inequalities for the distances—corresponding to *cables* (the distance can shrink but not expand) and *struts* (the distance can expand but not shrink). The study of these inequalities introduces techniques from linear programming.

GLOSSARY

Signed graph: A graph with a partition of the edges into three classes, written $G_\pm = (V; E_-, E_0, E_+)$.

Tensegrity framework $G_\pm(p)$ **in** $\mathbb{R}^d$: A signed graph $G_\pm = (V; E_-, E_0, E_+)$ and a configuration p on V.

Cables, bars, struts: For a tensegrity framework $G_\pm(p)$, the members of E_-, of E_0, and of E_+, respectively. In figures, cables are indicated by dashed lines, struts by double thin lines, and bars by single thick lines (see Figure 60.1.10).

FIGURE 60.1.10
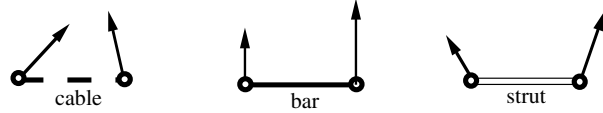
$G_\pm(p)$ **dominates** $G_\pm(q)$: For each edge, the appropriate condition holds:

$$|p_i - p_j| \geq |q_i - q_j| \quad \text{when} \quad \{i, j\} \in E_-$$
$$|p_i - p_j| = |q_i - q_j| \quad \text{when} \quad \{i, j\} \in E_0$$
$$|p_i - p_j| \leq |q_i - q_j| \quad \text{when} \quad \{i, j\} \in E_+.$$

Rigid tensegrity framework $G_\pm(p)$: For every analytic path $p(t)$ in $\mathbb{R}^{vd}$, $0 \leq t < 1$, if $p(0) = p$ and $G(p)$ dominates $G(p(t))$ for all t, then p is congruent to $p(t)$ for all t.

First-order flex of a tensegrity framework $G_\pm$: An assignment $p' : V \to \mathbb{R}^d$ of velocities to the vertices such that, for each edge $\{i, j\} \in E$ (Figure 60.1.10),

$$(p_j - p_i) \cdot (p_j' - p_i') \leq 0 \quad \text{for cables} \quad \{i, j\} \in E_-$$
$$(p_j - p_i) \cdot (p_j' - p_i') = 0 \quad \text{for bars} \quad \{i, j\} \in E_0$$
$$(p_j - p_i) \cdot (p_j' - p_i') \geq 0 \quad \text{for struts} \quad \{i, j\} \in E_+.$$

Trivial first-order flex: A first-order flex p' of a tensegrity framework $G_\pm(p)$ such that $p_i' = Sp_i + t$ for all vertices i, with a fixed skew-symmetric matrix S and vector t.

First-order rigid: A tensegrity framework $G_\pm(p)$ is first-order rigid if every first-order flex is trivial, and **first-order flexible** otherwise.

**Proper self-stress** on a tensegrity framework $G_\pm(p)$: An assignment ω of scalars to the edges of G such that:

(a) $\omega_{ij} \geq 0$ for cables $\{i, j\} \in E_-$;

(b) $\omega_{ij} \le 0$ for struts $\{i,j\} \in E_+$; and

(c)   for each vertex i, $\sum_{\{j \ | \ \{i,j\} \in E\}} \omega_{ij}(p_j - p_i) = 0$.

Strict self-stress: A proper self-stress ω with the inequalities in (a) and (b) strict.

Underlying bar framework: For a tensegrity framework $G_\pm(p)$, the bar framework $G(p)$ on the unsigned graph $G = (V, E)$, where $E = E_- \cup E_0 \cup E_-$ (Figure 60.1.11A,B).

FIGURE 60.1.11
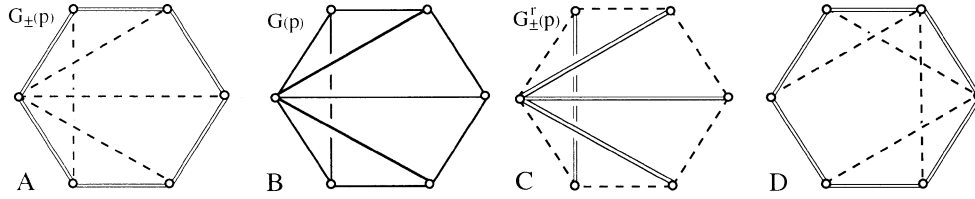
BASIC RESULTS

The equivalent definitions of "rigidity" and the basic connections between rigidity and first-order rigidity all transfer directly to tensegrity frameworks [RW81].

THEOREM 60.1.36   *First-order Stress Test*

A tensegrity framework $G_\pm(p)$ is first-order rigid if and only if the underlying bar framework $G(p)$ is first-order rigid and there is a strict self-stress on $G_\pm(p)$ (Figure 60.1.11A,B).

This connection to self-stresses means that any first-order rigid tensegrity framework with at least one cable or strut has $|E| > d|V| - \binom{d+1}{2}$ edges.

THEOREM 60.1.37   *Reversal Theorem*

A tensegrity framework $G_\pm(p)$ is first-order rigid if and only if the reversed framework $G_\pm^r(p)$ is first-order rigid, where the graph $G_\pm^r$ interchanges cables and struts (Figure 60.1.11A,C).

There is no single "generic" behavior for a signed graph $G_\pm$. If some configuration produces a first-order rigid framework for a graph $G_\pm$, then the set of all such configurations is open but not dense. The algebraic variety of "special positions" of the underlying unsigned graph divides the configuration space into open subsets, in some of which all configurations are rigid, and in others, none are. The required sign pattern for a self-stress can change as you cross such a boundary [WW83].

The first-order rigidity of a tensegrity framework is projectively invariant, with the proviso that a cable (strut) $\{i,j\}$ is switched to a strut (cable) whenever $\lambda_i \lambda_j < 0$ for the projective transformation.

THEOREM 60.1.38   *Stress Existence*

If a tensegrity framework $G_\pm(p)$ with at least one cable or strut is rigid, then there is a nonzero proper self-stress.

A number of results relate minima of quadratic energy functions to the rigidity of tensegrity frameworks. These energy results are not invariant under projective transformations, but such rigidity is preserved under "small" affine transformations. This is one result, drawn from results on second-order rigidity [CW96].

THEOREM 60.1.39   *Rigidity Stress Test*

A tensegrity framework $G_\pm(p)$ is rigid if, for each nontrivial first-order motion p' of $G_\pm(p)$, there is a proper self-stress $\omega^{p'}$ making $\sum_{ij} \omega_{ij}^{p'}(p_i' - p_j') \cdot (p_i' - p_j') > 0$.

A special result for modified frameworks—with some vertices fixed or pinned—further illustrates the role of tensegrity frameworks. A ***spiderweb*** is a partitioned graph $G_- = (V_0, V_1, E_-)$, with pinned vertices V_0, with $E_- \cdot_1 V_1 \times [V_0 \cup V_1]$ and a configuration p for $V_0 \cup V_1$. A ***spiderweb self-stress*** for $G_-(p)$ is an assignment ω of nonnegative scalars to E_- such that for each unpinned vertex $i \in V_1$, $\sum_{\{j|\{i,j\}\in E_-\}} \omega_{ij}(p_j - p_i) = 0$. A ***spiderweb flex*** for $G_-(p)$ is a flex $p(t)$ of the induced tensegrity framework on the spiderweb, with all pinned vertices fixed $(p_k(t) = p_k)$ (Figure 60.1.12).
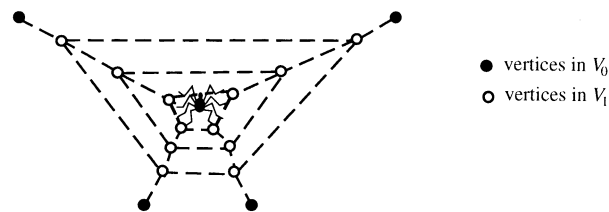

FIGURE 60.1.12

● vertices in V_0
○ vertices in V_1

THEOREM 60.1.40   *Spiderweb Rigidity*

Any spiderweb $G_-(p)$ in d-space with a spiderweb self-stress, positive on all cables, is rigid in d-space.

All critical points of functions of squared edge lengths correspond to proper self-stresses of a tensegrity framework, with members E_- for positive coefficients and E_+ for negative coefficients in the energy function. As a corollary, graph drawing programs (Chapter 52) that use minima (or critical points) of such energy functions will generate polyhedral pictures for planar graphs.

In the spiderweb energies, there is a global minimum of energy. This means that the configuration is globally rigid—no other realizations have the same edge lengths. In general, global rigidity has a distinct theory with some specific overlaps to the theory presented here.

Related to sphere packings (Chapter 61) are "reversed spiderwebs": tensegrity frameworks with vertices at the centers of the spheres (fixed joints for external pressures or constraints) and struts when two spheres contact. Such strut frameworks are rigid (corresponding to locally maximal density of the packing) if and only if they are first-order rigid (again with vertices in V_0 fixed) [Con88].

60.2 SCENE ANALYSIS

The problem of reconstructing spatial objects (polyhedra or polyhedral surfaces) from a single plane picture is basic to several applications. This section summarizes the combinatorial results for "generic pictures" (Section 60.2.1). Section 60.2.2 presents a polar "parallel configurations" interpretation of the same abstract mathematics and Section 60.2.3 presents connections to other fields of discrete geometry.

60.2.1 COMBINATORICS OF PLANE POLYHEDRAL PICTURES

### GLOSSARY

Polyhedral incidence structure S: An abstract set of **vertices** V, an abstract set of **faces** F and a set of **incidences** $I \subset V \times F$.

d-scene for an incidence structure $S = (V, F; I)$: A pair of location maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \ldots, z_i, w_i)$ and $P : F \to \mathbb{R}^d$, $P^j = (A^j, \ldots, C^j, D^j)$, such that, for each $(i, j) \in I$: $A^j x_i + \ldots + C^j z_i + w_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, 0, \ldots, 0, 1)$.)

(d−1)-picture of an incidence structure S: A location map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, \ldots, z_i)$ (Figure 60.2.1A).

Lifting of a $(d-1)$-picture $S(r)$: A d-scene $S(p, P)$ with vertical projection $\Pi(p) = r$ (Figure 60.2.1B). (I.e., if $p_i = (x_i, \ldots, z_i, w_i)$, then $r_i = (x_i, \ldots, z_i) = \Pi(p_i)$).

FIGURE 60.2.1

$$|I| = 14$$
$$|V| = 6$$
$$|F| = 4$$
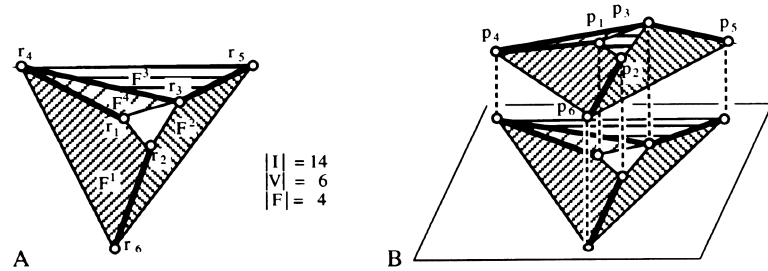
Lifting matrix for a picture $S(r)$: The $|I| \times (|V| + d|F|)$ coefficient matrix $M_S(r)$ of the system of equations for liftings of a picture $S(r)$: for each $(i, j) \in I$, $A^j x_i + \ldots + C^j z_i + w_i + D^j = 0$, where the variables are ordered:

$$\ldots, w_i, \ldots \; ; \; \ldots, A^j, \ldots, C^j, D^j, \ldots.$$

Sharp picture: A $(d-1)$-picture $S(r)$ that has a lifting $S(p, P)$ with a distinct hyperplane for each face (Figure 60.2.1A,B).

BASIC RESULTS

Since the incidence equations are linear, there is no distinction between "continuous liftings" and "first-order liftings." Since the rank of the lifting matrix is determined

by a polynomial process on the entries, "generic properties" of pictures have several characterizations.

THEOREM 60.2.1   *Generic Pictures*

For a structure S and a dimension d, the following are equivalent:

(a) *the structure is generically sharp in d-space (an open dense subset of configurations r produce sharp d-pictures);*

(b) *S(r) is sharp for a configuration r with algebraically independent coordinates.*

The generic properties of a structure are robust: all small changes in such a sharp picture are also sharp pictures and small changes in the points of a sharp picture require only small changes in the sharp lifting. Even special positions of such structures will always have nontrivial liftings, although these may not be sharp. However, up to numerical round-off, all pictures "are generic." Other structures that are not generically sharp (Figure 60.2.2A) may have sharp pictures in special positions (Figure 60.2.2B), but a small change in the position of even one point can destroy this sharpness.

FIGURE 60.2.2

$$|I|=12$$
$$=6+3(3)\ -3$$
$$=|V|+3|F|-3$$

$$|I|=14$$
$$=6+3(4)\ -4$$
$$=|V|+3|F|-4$$
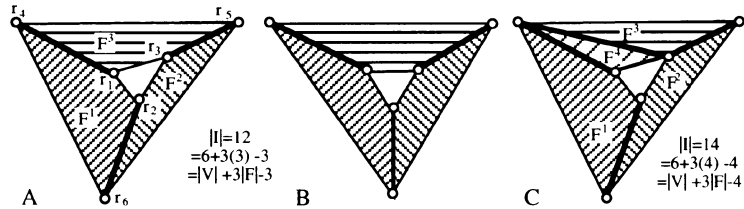
The incidence equations allow certain "trivial" changes to a lifted scene that will preserve the picture—generated by adding a single plane H^0 to all existing planes: $P_*^j = H^0 + P^j$; and by changes in vertical scale in the scene: $w_i^* = \lambda w_i$. This space of **lifting equivalences** has dimension $d+1$, provided the points of the scene do not lie in a single hyperplane.

THEOREM 60.2.2   *Picture Theorem*

A generic picture of an incidence structure $S = (V, F; I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if $|I| = |V|+d|F|-(d+1)$ and, for all subsets I' of incidences on at least two faces, $|I'| \le |V'|+d|F'|-(d+1)$ (Figure 60.2.1A,C).

A generic picture of an incidence structure $S = (V, F; I)$ has independent rows in the lifting matrix if and only if for all nonempty subsets I' of incidences, $|I'| \le |V'| + d|F'| - d$ (Figure 60.2.2A).

ALGORITHMS

Any part of a structure with $|I'| = |V'| + d|F'| - d$ independent incidences will be forced to be coplanar over a picture with algebraically independent coordinates for the points. If the structure is not generically sharp, then an effective, robust lifting algorithm consists of selecting a subset of vertices for which the incidences

are sharp, then "correcting" the position of the other vertices based on calculations in the resulting scene. This requires effective algorithms for selecting such a set of incidences. Sugihara and Imai have implemented $O(|I|^2)$ algorithms for finding generically sharp (independent) structures using modified bipartite matching on the incidence structure [Sug86].

60.2.2 PARALLEL DRAWINGS

The mathematical structure defined for polyhedral pictures has another, dual interpretation: the polar of a "point constrained by one projection" is a "hyperplane constrained by an assigned normal." Two configurations sharing the prescribed normals are "parallel drawings" of one another. These geometric patterns, used by engineering draftsmen in the nineteenth century, have reappeared in a number of branches of discrete geometry. This dual interpretation also establishes a basic connection between the geometry and combinatorics of scene analysis and the geometry and combinatorics of first-order rigidity of frameworks.

GLOSSARY

Parallel d-scenes for an incidence structure: Two d-scenes $S(p, P)$, $S(q, Q)$ such that for each face j, $P^j || Q^j$ (that is, the first $d-1$ coordinates are equal) (Figure 60.2.3). (For convenience, not necessity, we stick with the "nonvertical" scenes of the previous section.)

Nontrivially parallel d-scene for a d-scene $S(p, P)$:   A parallel d-scene $S(q, Q)$, such that the configuration q is not a translation or dilatation of the configuration p (Figure 60.2.3B for $d = 2$).
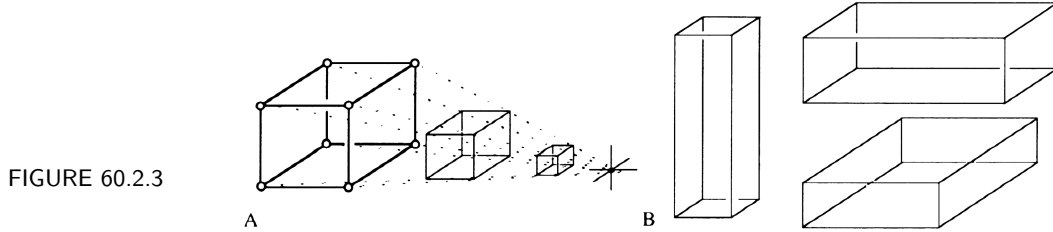

FIGURE 60.2.3

Directions for the faces: An assignment of d-vectors $D^j = (A^j, \ldots, C^j)$ to $j \in F$.

d-scene realizing directions D: A d-scene $S(p, P)$ such that for each face $j \in F$, the first $d-1$ coordinates of P^j and D^j coincide.

Parallel drawing matrix for directions D in d-space: The $|I| \times (|V| + d|F|)$ matrix $M_S(D)$ for the system of equations for each incidence $(i, j) \in I$: $A^j x_i + B^j y_i + \ldots + C^j z_i + w_i + D^j = 0$, where the variables are ordered:

$$\ldots, D^j, \ldots \ ; \ \ldots, x_i, y_i, \ldots, z_i, w_i, \ldots.$$

BASIC RESULTS

All results for polyhedral pictures dualize to parallel drawings. Again, for parallel drawings there is no distinction between continuous changes and first-order changes. The trivially parallel drawings, generated by d translations and one dilatation towards a point, form a vector space of dimension $d + 1$, provided there are at least two distinct points (Figure 60.2.3A). (A trivially parallel drawing may even have all points coincident, though the faces will still have assigned directions (Figure 60.2.3A).)

THEOREM 60.2.3   *Parallel Drawing Theorem*

For generic selections of the directions D in d-space for the faces, a structure $S = (V, F; I)$ has a realization $S(p, P)$ with all points p distinct if and only if, for every nonempty set I' of incidences involving at least two points $V(I')$ and faces $F(I')$, $|I'| \leq d|V(I')| + |F(I')| - (d+1)$ (Figure 60.2.3A).

In particular, a configuration p, P with distinct points realizing generic directions for the incidence structure is unique, up to translation and dilatation, if and only if $|I| = d|V| + |F| - (d+1)$ and $|I'| \leq d|V'| + |F'| - (d+1)$.

Of course other nontrivially parallel drawings will also occur if the rank is smaller than $d|V'| + |F'| - (d+1)$ (Figure 60.2.3 B, with a generic rank 1 less than required for $d = 2$, and a geometric rank, as drawn, 2 less than required).

Figure 60.2.3 may also be interpreted as the parallel drawings of a "cube in 3-space." For spherical polyhedra, there is an isomorphism between the nontrivially parallel drawings in 3-space (the parallel drawings modulo the trivial drawings) and the nontrivially parallel drawings in a plane projection [CW94]. Only the dimension (4 vs. 3) of the trivially parallel drawings will change with the projection.

60.2.3 CONNECTIONS TO OTHER FIELDS

FIRST-ORDER RIGIDITY

For any plane framework, if we turn the vectors of a first-order motion 90° (say clockwise), they become the vectors joining p to a parallel drawing q of the framework (Figure 60.2.4A,B). The converse is also true.

THEOREM 60.2.4

A plane framework $G(p)$ has a nontrivial first-order flex if and only if the configuration $G(p)$ has a nontrivially parallel drawing $G(q)$ (Figure 60.2.4C,D).

Because of this connection, combinatorial and geometric results for plane first-order rigidity and for plane parallel drawings have numerous deep connections. For example, Laman's theorem (Theorem 60.1.12b) is a corollary of the parallel drawing theorem, for $d = 2$. In higher dimensions, the connection is one-way: a nontrivially parallel drawing of a "framework" (the "direction of an edge" is represented by $d-1$ facets through the two points) induces one (or more) nontrivial first-order motions of the corresponding bar framework. The theory of parallel drawing in higher
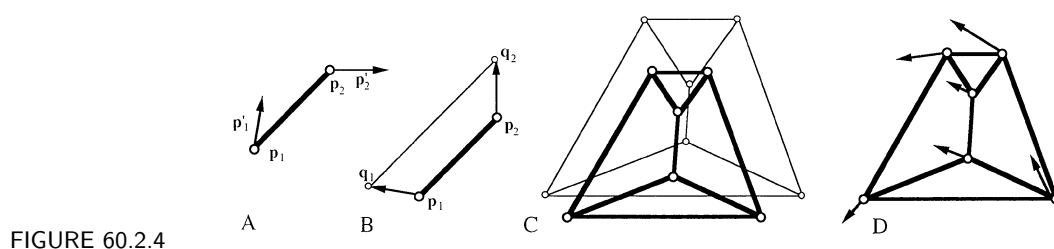
FIGURE 60.2.4

dimensions is more complete and has simpler algorithms than the theory of first-order rigidity in higher dimensions, generalizing almost all results for plane first-order rigidity and plane parallel drawings, including combinatorial characterizations using counts, tree partitions, and inductive constructions of maximimal independent sets.

MINKOWSKI DECOMPOSABILITY

By a theorem of Shephard, a polytope is decomposable as the Minkowski sum of two simpler polyhedra if and only if the faces and vertices of the polytope (or the edges and vertices of the polytope) have a nontrivially parallel drawing. Many characterizations of Minkowski indecomposable polytopes can be deduced directly from results for parallel d-scenes (or equivalently, for polyhedral pictures of the polar polytope).

ANGLES IN CAD

In plane computer-aided design, many different patterns of constraints (lengths, angles, incidences of points and lines, etc.) are used to design or describe configurations of points and lines, up to congruence or local congruence. With distances between points, the geometry becomes that of first-order rigidity. If angles and incidences are added, even the problems of "generic rigidity" of constraints are unsolved (and perhaps not solvable in polynomial time). However, special designs, mixing lengths, distances of points to lines, and trees of angles have been solved, using direct extensions of the techniques and results for plane frameworks and plane parallel drawings [SW99].

There is another connection between angles of intersections and rigidity. A recent manuscript [SW02] describes a correspondence between the first-order theory of circles of variable radius and intersection angles as constraints and distance constraints between points in Euclidean (and hyperbolic) 3-space, as well as spheres and angles in 3-space and points and distances in 4-space. As a result, the full complexity of distance constraints in 4-space is embedded inside general dimensioning in 3-space CAD. In general, geometric systems of constraints do not yield simple combinatorial counting algorithms of the type found for plane first-order rigidity.

60.3 RECIPROCAL DIAGRAMS

The reciprocal diagram is a single geometric construction that has appeared, independently over a 140-year span, in areas such as "graphical statics" (drafting techniques for resolving forces), scene analysis, and computational geometry.

GLOSSARY

Abstract spherical polyhedron $S = (V, F; \underline{E})$: For a 2-connected planar graph $G_S = (V, E_S)$, drawn without self-intersection on a sphere (or in the plane), we record the vertices as V and the regions as faces F, and rewrite the directed edges $\underline{E}$ as ordered 4-tuples $\underline{e} = \langle h, i; j, k \rangle$, where the edge from vertex h to vertex i has face j on the right and face k on the left. (The reversed edge $-\underline{e} = \langle i, h; k, j \rangle$ runs from i to h, with k on the right.)

FIGURE 60.3.1

A B C D Framework $G_S(\mathbf{r})$ E Reciprocal $G^S(\mathbf{u})$
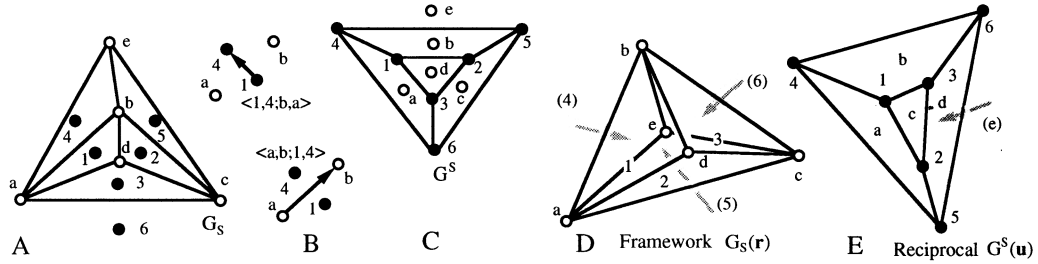
Dual abstract spherical polyhedron: The abstract spherical polyhedron S^* formed by switching the roles of V and F, and switching the pairs of indices in each ordered edge $\underline{e} = \langle h, i; j, k \rangle$ into $\underline{e}^* = \langle j, k; i, h \rangle$. (Also the abstract spherical polyhedron formed by the dual planar graph $G^S = (F, E^S)$ of the original planar graph (Figure 60.3.1A,C).)

Proper spatial spherical polyhedron: An assignment of points $p_i = (x_i, y_i, z_i)$ to the vertices and planes $P^j = (A^j, B^j, D^j)$ to the faces of an abstract spherical polyhedron $(V, F; \underline{E})$, such that if vertex i and face j share an edge, then the point lies on the plane: $A^j x_i + B^j y_i + z_i + D^j = 0$; and at each edge the two vertices are distinct points and the two faces have distinct planes.

Projection of a proper spatial polyhedron $S(p, P)$: The plane framework $G_S(r)$, where r is the vertical projection of the points p (i.e., $r_i = \Pi p_i = (x_i, y_i)$) (Figure 60.3.2).

Gradient diagram of a proper spatial polyhedron $S(p, P)$: The plane framework $G^S(s)$, where $s_j = (A^j, B^j)$ is (minus) the gradient of the plane P^j (Figure 60.3.2).

Reciprocal diagrams: For an abstract spherical polyhedron S, two frameworks $G_S(r)$ and $G^S(s)$ on the graph and the dual graph of the polyhedron, such that for each directed edge $\langle h, i; j, k \rangle \in \underline{E}$, $(r_h - r_i) \cdot (s_j - s_k) = 0$ (Figure 60.3.1D,E).

BASIC RESULTS

Reciprocal diagrams have deep connections to both of our previous topics:

(a) Given a spatial scene on a spherical structure, with no faces vertical, the vertical projection and the gradient diagram are reciprocal diagrams. (This follows because the difference of the gradients at an edge is a vector perpendicular to the vertical plane through the edge.)

(b) Given a pair of reciprocal diagrams on $S = (V, F; E)$, then for each edge $\underline{e} = \langle h, i; j, k \rangle$ the scalars ω_{ij} defined by $\omega(r_h - r_i) = (s_j - s_k)^{\perp}$ (where $^{\perp}$ means rotate by 90° clockwise) form a self-stress on the framework $G_S(r)$. (This follows because the closed polygon of a face in $G^S(s)$ is, after $^{\perp}$, the vector sum for the "vertex equilibrium" in the self-stress condition.)

These facts can be extended to other oriented polyhedra and their projections. The real surprise is that, for spherical polyhedra, the converses hold and all these concepts are equivalent (an observation dating back to Clerk Maxwell and the drafting techniques of graphical statics).

FIGURE 60.3.2
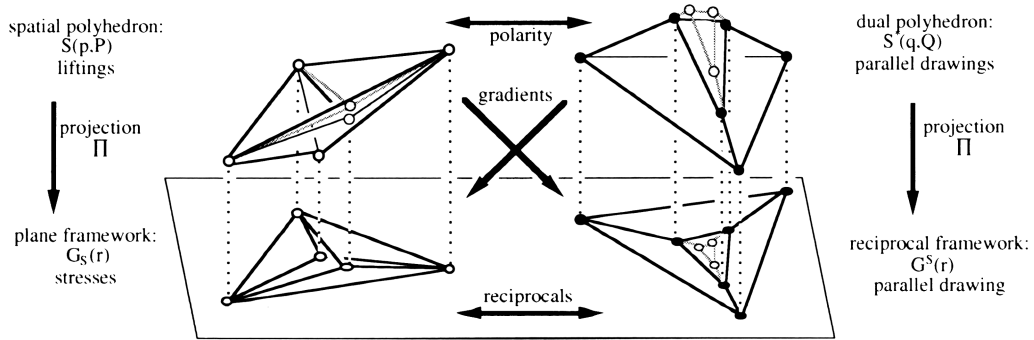
THEOREM 60.3.1 *Maxwell's Theorem*

For an abstract spherical polyhedron $(V, F; \underline{E})$, the following are equivalent:

(a) *The framework $G_S(r)$, with the vertices of each edge distinct, has a self-stress nonzero on all edges;*

(b) *$G_S(r)$ has a reciprocal framework $G^S(s)$ with the vertices of each edge distinct;*

(c) *$G_S(r)$ is the vertical projection of a proper spatial polyhedron $S(p, P)$;*

(d) *$G_S(r)$ is the gradient diagram of a proper spatial polyhedron $S^*(q, Q)$.*

There are other refinements of this theorem, that connect the space of self-stresses of $G_S(r)$ with the space of parallel drawings (and first-order flexes) of $G^S(s)$, the space of polyhedra $S(p, P)$ with the same projection, and the space of

parallel drawings of $S^*(q, Q)$ [CW94] (Figure 60.3.2). A second refinement connects the local convexity of the edge of the polyhedron with the sign of the self-stress.

THEOREM 60.3.2 *Convex Self-stress*

The vertical projection of a strictly convex polyhedron, with no faces vertical, produces a plane framework with a self-stress that is < 0 on the boundary edges (the edges bounding the infinite region of the plane) and > 0 on all edges interior to this boundary polygon.

A plane Delaunay triangulation also has a basic "reciprocal" relationship to the plane Voronoi diagram: the edges joining vertices at the centers of the regions are perpendicular to edges of the polygon of the Voronoi regions surrounding the vertex. This pair of reciprocals is directly related to the projection of a spatial convex polyhedral cap, as are generalized Voronoi diagrams. See Section 23.1.

This pattern of "reciprocal constructions" and the connection to liftings to polytopes in the next dimension generalizes to higher dimensions [CW94]. For example, for Voronoi diagrams and Delaunay simplicial complexes, the edges of one are perpendicular to facets of the other, in all dimensions. Moreover, for appropriate sphere-like homology, the existence of a reciprocal corresponds to the existence of nontrivial liftings [CW94, ERR01, Ryb99]. Such geometric structures are also related to k-rigidity and to combinatorial proofs of the g-theorem in polyhedral combinatorics [TW00]. Finally, [BGH02] makes a related connection between parallel drawings and group actions on complex manifolds.

60.4 SOURCES AND RELATED MATERIALS

SURVEYS AND BASIC SOURCES

All results not given an explicit reference can be traced through these surveys:

[CW96]: A presentation of basic results for concepts of rigidity between first-order rigidity and rigidity for tensegrity frameworks.

[CW]: A thorough introduction to a number of topics on the rigidity of frameworks, in manuscript form only.

[GSS93]: A monograph devoted to combinatorial results for the graphs of generically rigid frameworks, with an extensive bibliography on many aspects of rigidity.

[Ros00]: A recent thesis that explores in depth both topics of this chapter and their connections.

[Sug86]: A monograph on the reconstruction of spatial polyhedral objects from plane pictures.

[Whi93]: A survey of results relating first-order rigidity to matroid theory and related matroids for scene analysis, and to multivariate splines.

[Whi96]: An expository article presenting matroidal aspects of first-order rigidity, scene analysis, and multivariate splines.

RELATED CHAPTERS

Chapter 9: Geometry and topology of polygonal linkages
Chapter 18: Face numbers of polytopes and complexes
Chapter 23: Voronoi diagrams and Delaunay triangulations
Chapter 29: Geometric reconstruction problems
Chapter 48: Robotics
Chapter 52: Graph drawing
Chapter 59: Geometric applications of the Grassmann-Cayley algebra
Chapter 61: Sphere packing and coding theory

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