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Solving circle packing problems by global optimization: Numerical results and industrial applications

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Abstract

A (general) circle packing is an optimized arrangement of N arbitrary sized circles inside a container (e.g., a rectangle or a circle) such that no two circles overlap. In this paper, we present several circle packing problems, review their industrial applications, and some exact and heuristic strategies for their solution. We also present illustrative numerical results using 'generic' global optimization software packages. Our work highlights the relevance of global optimization in solving circle packing problems, and points towards the necessary advancements in both theory and numerical practice.

Keywords: Circle packing problems; Global optimization; Industrial applications; Numerical examples

1. Introduction

In a general setting, a circle packing is an optimized arrangement of N arbitrary sized circles inside a container (e.g., a rectangle or a circle) such that no two circles overlap. The quality of the packing is typically measured by: (1) the size of the container, (2) the weighted average pairwise distance between the centers of the circles, or (3) a linear combination of criteria (1) and (2).

Circle packings (mostly of uniform size circles) have received a considerable amount of attention in the 'pure' mathematics literature. At the same time, these problems so far have received only limited attention in the operations research literature, in spite of a broad range of existing and potential applications.

Packings also pose interesting modeling and numerical challenges. In its full generality, a circle packing is a difficult optimization problem that cannot be tackled effectively by purely analytical approaches. The 'simplest' cases of packing uniform sized circles inside a square or inside a circle are provably solved to theoretical optimality only for a few instances (up to tens of circles), in spite of the significant effort spent on variants of the problem in recent decades.

It is impossible to offer a detailed overview on the existing solution strategies and numerical results, within the framework of a single paper. Instead,

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we will present several circle packing problems, highlight their industrial applications, and then review some exact and heuristic strategies for their solution. We also provide illustrative numerical results using several readily available global optimization software packages, namely LINGO (Lindo Systems, 2004), NMinimize (Wolfram Research, 2005), and MathOptimizer Professional (Pintér and Kampas, 2006). The latter is an implementation of the Lipschitz Global Optimizer (LGO) global/local solver suite (Pintér and LGO, 2005) for the Mathematica platform (Wolfram Research, 2005). The theoretical foundations that lead to the global solvers embedded in LGO are discussed in detail by Pintér (1996).

In the following models, the center of circle i is denoted by (x_i, y_i) and the radius of circle i is denoted by $r_i, i = 1, \ldots, N$. If uniform sized circles are considered, then simply $r_i = r, i = 1, \ldots, N$. The pairwise distance between the centers of circles i and j $(i < j)$, measured using the l_2 norm (Euclidean distance), is denoted by $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, $i, j = 1, \ldots, N$. Additional notation will be introduced as needed.

2. Generic circle packing problems

First, several circle packing problems are considered with no explicit industrial application: we refer to these problems as *generic*. These problems are classified as uniform sized and arbitrary sized circle problems.

2.1. Uniform sized circle packings

There are three main variants of the uniform sized circle packing problem (Szabó et al., 2005):

1. Find the minimum radius r_0 of a circular container so it can hold N uniform sized and non-overlapping circles. Assuming that the container is centered at the origin, this problem is formulated as follows:

$$\text{UCP1}: \quad \min \quad r_0 \tag{1}$$

$$\text{s.t.} \quad \sqrt{x_i^2 + y_i^2} + r \leqslant r_0, \quad i = 1, \ldots, N, \tag{2}$$

$$2r \leqslant d_{ij}, \; i, \quad j = 1, \ldots, N; i < j. \tag{3}$$

   Note that UCP1 can be standardized by setting $r = 1$. (Alternatively, one can set $r_0 = 1$, and then maximize r.) Observe that UCP1 has N convex constraints and $N(N-1)/2$ nonconvex constraints.

2. Find the maximum radius r of N uniform sized and non-overlapping circles belonging to the unit square. This problem is formulated as follows:

$$\text{UCP2}: \quad \max \quad r \tag{4}$$

$$\text{s.t.} \quad r \leqslant x_i \leqslant 1 - r, \quad i = 1, \ldots, N, \tag{5}$$

$$r \leqslant y_i \leqslant 1 - r, \quad i = 1, \ldots, N, \tag{6}$$

$$2r \leqslant d_{ij}, \; i, \quad j = 1, \ldots, N; i < j. \tag{7}$$

   Observe that UCP2 has $4N$ linear constraints and $N(N-1)/2$ nonconvex constraints.

3. Find the positions of N points inside the unit square in such a way that their minimum pairwise distance d is maximized. This problem is formulated as follows:

$$\text{UCP3}: \quad \max \quad d \tag{8}$$

$$\text{s.t.} \quad 0 \leqslant x_i \leqslant 1, \quad i = 1, \ldots, N, \tag{9}$$

$$0 \leqslant y_i \leqslant 1, \quad i = 1, \ldots, N, \tag{10}$$

$$d \leqslant d_{ij}, \quad i, j = 1, \ldots, N; i < j. \tag{11}$$

Observe that UCP3 has $2N$ box constraints and $N(N-1)/2$ nonconvex constraints.

Note that problems UCP2 and UCP3 are closely related. In fact, the optimal solution for one problem can be obtained from the optimal solution of the other problem as follows. Denote the optimal objective function values of UCP2 and UCP3, respectively as $r^\star$ and $d^\star$. Denote the optimal coordinates of the N circles in UCP2 by (a_i, b_i), and the optimal coordinates of the N points in UCP3 by (c_i, d_i). Then, the optimal solutions to UCP2 and UCP3 in a unit square are related as $d^\star = 2r^\star/(1 - 2r^\star)$, $c_i = (a_i - r^\star)/(1 - 2r^\star)$, and $d_i = (b_i - r^\star)/(1 - 2r^\star)$.

Let us point out that there are two main categories of studies dealing with uniform sized circle packings. One of these approaches is to prove the optimality of suggested packings, either (purely) theoretically, or with the help of computers: consult Szabó et al. (2001, 2005) for detailed discussions and further references. The other approach is to develop efficient numerical solution strategies that can be readily applied, but (in general) without proven optimality of the results obtained. This paper is focused on the latter approach, noting that it can be readily applied to 'arbitrary' packings without substantial modifications. We also provide pointers to the more theoretical approach in the following brief review.

Kravitz (1967) is perhaps one of the first researchers to consider one of the main variants of the uniform sized circle packing problem: UCP1. The problem involves determining the minimum radius of a circular container so that it can pack N uniform sized, unit circles. Solutions are proved optimal for $N = 1$–10. Additional (best known numerical) solutions are shown for $N = 11$–19. The reader is referred to Melissen et al. (1997) for additional reviews of the early circle packing research, and to the topical website http://www.packomania.com/ that cites best known solutions and other relevant references.

More recently, Theodoracatos and Grimsley (1995) consider the geometric packing problem, which is a generalization of the circle packing problem. The geometric packing problem involves finding the optimal, non-overlapping configuration of objects inside a two-dimensional container such that the area of the container is best utilized. The authors implement a solution strategy based on simulated annealing. Numerical results using the proposed solution strategy involve the problem of packing uniform sized circles inside a square. The packing of arbitrary polygons is also considered. For the circle packing problem, the objective function has two terms: (1) a cost-related term that considers the area covered by the circles, and (2) a penalty-related term that considers the overlap among the circles. The authors conjecture that when the case of packing circles is considered, a constant penalty is sufficient. Packing arbitrary polygons, however, requires a variable penalty method. Theodoracatos and Grimsley (1995) also propose several rules that improve the efficiency of the simulated annealing solution strategy.

Nurmela and Östergård (1997) present a solution strategy for finding packings of up to 50 uniform sized circles in a square. The variants of the problem considered are UCP2 and UCP3. The authors transform the original constrained optimization problem into a box-constrained one by introducing the energy function $E = \sum_i \sum_{j>i} (\lambda/d_{ij}^2)^m$, where λ is a scaling factor and m is a positive integer, and propose an iterative solution strategy where the value of m is increased at each step. The final solution generated by this strategy is further improved by solving a system of nonlinear equations related to the contacts between the circles and the boundary of the square container. Nurmela and Östergård (1997) present solutions for $N = 21$–50. The authors also discuss various similarities and properties of the packings.

Lubachevsky and Graham (1997) investigate the packing of $N = h(k)$ uniform sized circles inside a circular container, where $h(k) = 3k(k+1) + 1$ corresponds to a hexagonal number of circles for $k \geqslant 1$. Thus, $h(k)$-packings are referred to as curved hexagonal packings. The authors propose a solution strategy based on the billiards simulation algorithm. The algorithm works by simulating the movement of a system of N perfectly elastic circles as they move along fixed lines colliding with each other and with the boundary of the container circle. Gravity and friction are assumed nonexistent, while the non-overlapping constraint is always maintained. The circles are gradually increased in size until no significant growth can occur, at which point a packing is formed. For $N = 19$ (i.e., $k = 2$), the packings are proven to be optimal. The authors also show that there exists at most $m(k) = \max\{(k-1)!/2, 1\}$ different curved hexagonal packings that produce the same density. For $k \leqslant 5$, these packings generate optimal results. However, for $k > 5$, higher quality packings have been found that those exhibited by curved hexagonal packings.

Graham et al. (1998) consider UCP2 and UCP3. The optimal packings for up to $N = 18$ uniform sized circles are discussed. The main motivation for the authors comes from the lack of attention paid (up to that point in time) to the cases for $N > 20$. Two solution strategies are presented. The first solution strategy is iterative in nature and similar to that of Nurmela and Östergård (1997) as it introduces an energy function. The second solution strategy is similar to that of Lubachevsky and Graham (1997) as it is based on the billiards simulation algorithm. Both solution strategies tend to 'slow down' when approaching the (unknown) optimal solution; thus, a second phase is implemented where the precision of the final solution is increased. For $N = 21$–54, best-known packings are presented. Graham et al. (1998) report that the two solution strategies are similar in terms of CPU time. However, the strategy based on the billiards simulation algorithm performs better in terms of finding the optimal solutions.

Boll et al. (2000) consider UCP2 and UCP3 and discuss a 2-phase solution strategy to find optimized packings. The first phase of the strategy produces an approximate configuration that is later used as a starting point for the second phase, which consists of using the billiards simulation algorithm. The proposed 2-phase solution strategy is able to find optimal packings faster than using the billiards simulation algorithm only. Boll et al. (2000) find several solutions that have been previously proved to be optimal, and present improved solutions for the cases $N = 32$, 37, 48, and 50.

Locatelli and Raber (2002) consider UCP2 and UCP3. The authors transform UCP3 into a (nonconvex) quadratic optimization model and propose a branch-and-bound solution strategy to exploit the special structure of the problem. Furthermore, two structural properties that all best-known solutions satisfy are presented and later used to improve the efficiency of the solution strategy. The authors note that optimal solutions to the quadratic optimization program are algebraically different but geometrically the same, and propose techniques to avoid multiple optimal solutions, thereby eliminating nodes from further consideration in the branch-and-bound solution strategy. Locatelli and Raber (2002) present solutions for $N = 10$–35, $N = 38$, and $N = 39$. The solutions are shown to be equivalent to the optimal or best-known solutions for all cases.

Mladenovic et al. (2005) propose a general reformulation descent heuristic solution strategy to solve the uniform sized circle packing problem. Each time the solution algorithm reaches a stationary point, the problem is reformulated using an alternative coordinate system. Cartesian and polar coordinates are used to reformulate the problem. Without the reformulation, the solution strategy would stop at a stationary point. The authors present solutions using both the unit square and the unit circle as containers. Mladenovic et al. (2005) show that the reformulation strategy performs better than the usage of Cartesian or polar coordinates only. Solutions using the suggested strategy are compared to solutions found with a Newton-type solution approach. It is reported that the two search methods find solutions of comparable quality; however, the reformulation descent strategy performs better in terms of CPU time.

Szabó et al. (2005) provide a survey of research done on the uniform sized circle packing problem inside a square. A brief history of the optimality proofs for various values of N is presented. The authors show that, for UCP3, the properties given by Locatelli and Raber (2002) can be used to compute lower and upper bounds for the minimum pairwise distance d. The authors also review several solution strategies that have been proposed in the literature: (1) energy function minimization, (2) billiards simulation, (3) modified billiards simulation, (4) perturbation method, (5) simulated annealing, (6) deterministic optimization based on a linear programming relaxation, and (7) interval arithmetic optimization. For $N = 1$–30 and for $N = 36$ optimal packings are known. Putative solutions, however, are known for up to $N = 200$. Szabó et al. (2005) also present an algebraic description of a given packing by determining its minimal polynomial. A list of known polynomials (not necessarily optimal) for the uniform sized circle packing problem inside a square is provided for $N = 2$–100.

Finally, Markót and Csendes (2005) propose a verified optimization solution strategy for UCP3 based on interval analysis. As a demonstration of the capabilities of the proposed strategy, the authors solve problems for $N = 28$, 29, and 30 to high precision.

2.2. Arbitrary sized circle packings

The arbitrary sized circle packing problem is a significant generalization of the uniform sized case, where each circle can have a different radius. Here, we consider two main variants of this problem, as formulated below.

1. Find the minimum radius r_0 of a circular container so it can pack N arbitrary sized and non-overlapping circles. Assuming that the container is centered at the origin, this problem is formulated as follows:

$$\text{ACP1}: \min \quad r_0 \tag{12}$$

$$\text{s.t.} \quad \sqrt{x_i^2 + y_i^2} + r_i \leqslant r_0, \quad i = 1, \ldots, N \tag{13}$$

$$r_i + r_j \leqslant d_{ij}, \quad i, j = 1, \ldots, N; i < j. \tag{14}$$

Observe that ACP1 has N convex constraints and $N(N-1)/2$ nonconvex constraints.

2. Find the minimum area of a rectangular container so it can pack N arbitrary sized and non-overlapping circles. Assuming that the width of the container, which is centered at the origin, is

denoted by A and that the height of the container is denoted by B, this problem is formulated as follows:

$$\text{ACP2}: \min \quad AB \tag{15}$$

$$\text{s.t.} \quad x_i + r_i - \frac{1}{2}A \leqslant 0, \quad i = 1,\ldots,N, \tag{16}$$

$$r_i - x_i - \frac{1}{2}A \leqslant 0, \quad i = 1,\ldots,N, \tag{17}$$

$$y_i + r_i - \frac{1}{2}B \leqslant 0, \quad i = 1,\ldots,N, \tag{18}$$

$$r_i - y_i - \frac{1}{2}B \leqslant 0, \quad i = 1,\ldots,N, \tag{19}$$

$$r_i + r_j \leqslant d_{ij}, \quad i,j = 1,\ldots,N, i < j, \tag{20}$$

$$A^{\text{low}} \leqslant A \leqslant A^{\text{up}}, \tag{21}$$

$$B^{\text{low}} \leqslant B \leqslant B^{\text{up}}, \tag{22}$$

where A^{low} and A^{up} are the width requirements (with $A^{\text{low}} \leqslant A^{\text{up}}$) and B^{low} and B^{up} are the height requirements (with $B^{\text{low}} \leqslant B^{\text{up}}$) of the rectangular container. If the actual width $\overline{A}$ or the height $\overline{B}$ of the container are known in advance, such dimensions can be enforced by setting $A^{\text{low}} = A^{\text{up}} = \overline{A}$ or $B^{\text{low}} = B^{\text{up}} = \overline{B}$. Observe that ACP2 has $4N$ linear constraints, $N(N-1)/2$ nonconvex constraints, and two pairs of additional simple bounds.

Stoyan and Yaskov (1998) consider the general optimization placement problem in which a specified placement quality criterion reaches the optimum, whereas the objects are subject to restrictions on their relative (mutual) position and on the position of the object within the placement region. This general verbal model formulation includes the arbitrary sized circle packing problem. The authors also consider packing rectangles and other objects. The circle packing problem that Stoyan and Yaskov (1998) consider can be stated as follows: find a non-overlapping placement of N arbitrary sized circles in a container rectangle of fixed width so that the height of the container rectangle is minimal. A combination of a branch-and-bound algorithm and a reduced gradient algorithm is used as the solution procedure. First an initial solution is obtained using branch-and-bound and then improved upon by the reduced gradient method, which moves from one extreme point in the feasible region to another until it reaches a locally optimal solution. The authors provide an example of packing $N = 100$ arbitrary sized circles inside a container rectangle.

Wang et al. (2002) consider the problem ACP1 of packing arbitrary sized circles into a circular container. The authors suggest, but do not investigate, two possible applications: (1) placing optical fibers into a tube such that the radius of the tube is minimized, and (2) transporting pipes of various sizes inside a container. The proposed solution strategy is a combination of two individual approaches proposed previously. The combined strategy, which will report a solution in a sufficiently long time if a circle packing problem instance is solvable, is shown to be more efficient than the individual approaches.

Huang et al. (2003) consider the problem of packing arbitrary sized circles inside a circular container with fixed radius. The authors propose a solution strategy that starts by placing two circles inside the container. Each circle that can be placed is then examined and a rule is applied to determine which one is to be packed next. The authors state that solutions were obtained but little information is actually reported in the paper. Moreover, Huang et al. (2003) only offer a solution strategy for providing feasible packings, and do not provide a strategy for the problem of minimizing the radius of the container circle.

Stoyan and Yaskov (2004) consider a problem similar to that of Stoyan and Yaskov (1998). The authors identify several structural properties specific to the problem. Using these properties, the authors propose a solution strategy that moves from one local minimum to another by increasing the dimension of the problem before choosing the movement direction to a new local minimum. The authors note that this method will not work for the packing of uniform sized circles.

Huang et al. (forthcoming) consider an arbitrary sized circle packing similar to that of Stoyan and Yaskov (1998, 2004). The authors propose two greedy solution strategies to solve the problem. The first strategy is similar to that of Huang et al. (2003). The second strategy uses a ‘look-ahead’ search. This strategy examines the relation between all circles and not just the ones already packed. The solutions found using the two strategies are compared to those found by Stoyan and Yaskov (2004).

Zhang and Deng (2005) consider ACP1. The authors propose a hybrid solution strategy that combines the features of simulated annealing and tabu search. The proposed strategy is somewhat related to the approach found in Wang et al.

(2002). Given a packing, in order to generate neighbor solutions, one circle is selected and randomly placed in a different position. Since there is an infinite number of possibilities for the new random placement, the authors use a strategy that simulates the physical movement of the circles to reduce the range of the neighborhood. Solutions are given for nine benchmark problems: five instances with uniform sized circles and four instances with arbitrary sized circles.

Pintér and Kampas (2005) consider – in addition to UCP1 and UCP2 – the model ACP1. The authors provide a brief review of the LGO global/local solver suite and its Mathematica related implementation. Illustrative numerical results are presented on several circle packing instances in the unit square and the unit circle (reproducing several best known results from www.packomania.com), followed by examples with arbitrary sized circles in an optimized circumscribed circle.

3. Circle packing applications in industry

In this section, we present a collection of industrial circle packing problems. In recent years, several interesting application areas have been investigated in the literature, including circular cutting problems, communication networks, facility location and dashboard layout. Let us note here that Dyckhoff (1990), Haessler and Sweeney (1991), and Dowsland and Dowsland (1992) also discuss geometric packing and cutting stock applications. However, these authors deal mainly with the packing of rectangular objects as opposed to circles.

3.1. Circular cutting

Hifi et al. (2004) consider the unconstrained and constrained circular cutting problems. The problem involves cutting out from a rectangular plate R as many circular pieces as possible of N different types of values v_i and radii r_i. The vector $\{w_1, \ldots, w_N\}$ corresponds to a feasible cutting pattern if it is possible to produce w_i pieces of type i in plate R without overlap. The unconstrained circular cutting problem is formulated as follows: $\max\sum_{i=1}^{N} w_i v_i$ s.t. $\{w_1, \ldots, w_N\}$ is a feasible cutting pattern. When the value v_i represents the surface area of circle type i, the problem becomes that of maximizing the surface of the rectangular plate that is covered by the $\sum_{i=1}^{N} w_i$ pieces. The constrained circular cutting problem places a restriction on the maximum number of pieces to be cut such that $\{w_1, \ldots, w_N\} \leqslant \{b_1, \ldots, b_N\}$, where $b_i, i = 1, \ldots, N$ is the maximum number of times piece i can be cut in the pattern. The authors point out that in its full generality the circular cutting problem is NP-hard (since it is a generalization of the cutting stock problem, which is a generalization of the knapsack problem), and propose a simulated annealing solution strategy to solve it. The solution strategy is initialized with a (most likely infeasible) solution composed by placing as many circular pieces into the rectangular plate so that they cover the entire surface. The pieces are then rearranged or removed, to produce a feasible solution with no overlaps. In order to generate neighboring solutions, several transformations on the positions of the circular pieces are proposed. The solution strategy often produces infeasible final solutions. To overcome this limitation, Hifi et al. (2004) propose a post-processing procedure to ensure a feasible final solution. Two series of numerical results are presented. The first series involves randomly generated instances solved by the proposed solution strategy and three other circular cutting solution strategies implemented by the authors. The second series involves benchmarking the proposed strategy on instances for which optimal or best-known solutions exist using data from Stoyan and Yaskov (1998).

Hifi and M'Hallah (2004) consider the constrained circular cutting problem and propose two heuristic solution strategies to solve it: (1) a constructive procedure-based heuristic and (2) a genetic algorithm-based heuristic. Numerical results using data from Stoyan and Yaskov (1998) are presented in order to evaluate the efficiency of the proposed strategies. Hifi and M'Hallah (2004) report that the genetic algorithm-based heuristics is more efficient than the constructive procedure-based heuristic. Moreover, the authors report that both proposed heuristic solution strategies are more efficient than the simulated annealing solution strategy proposed by Hifi et al. (2004).

Cui (2005) proposes optimal patterns for cutting circular blanks used to built electric motors. The circular blanks are cut from silicon steel and thus can be expensive. To increase the efficiency of material usage, optimal cutting patterns are needed. The author focuses on the T-shape pattern that is formed by placing rows of strips on a plate. Some rows are arranged horizontally for a portion of the sheet and the remaining rows are arranged vertically so that the resulting pattern forms a T-like shape. Circular blanks of the same size are usually cut in the same row. The author mentions that these patterns are slightly more difficult to make but give higher material usage rates. The problem is formulated as a constrained circular cutting problem with the purpose of minimizing the amount of material wasted. Cui (2005) proposes a heuristic solution strategy to find optimal cutting patterns along with numerical results for 500 randomly generated problem instances.

3.2. Container loading

Fraser and George (1994) consider a container loading problem developed for the paper and pulp industry. In this industry, products are delivered as either flat sheets or reels. Reels of variable diameter are assumed to stand on end when being transported; thus, the problem of packing reels in a two-dimensional plane is equivalent to the circle packing problem ACP2. The authors remark that depending on the size of the container and the different diameters of the reels, the optimal shipping pattern will vary. A heuristic solution strategy is implemented to determine the shipping pattern for transport containers. Given an instance of the problem, the solution strategy, which is based on several pre-specified patterns and placement rules, generates several different feasible shipping patterns. The shipping pattern with the best fit is selected as the final solution. It must be noted that the authors do not actually search for an optimal shipping pattern, but the solution strategy appears to generate good results for its intended purpose. Fraser and George (1994) further consider the stacking of reels. The authors formulate this problem as a bin-packing problem given the height constraint of the container and the various widths of the reels. This problem, however, does not have a simple (two-dimensional) circle packing equivalent.

George et al. (1995) consider the problem of packing pipes of different diameters into a shipping container. The problem, also referred to as the cylindrical bin-packing problem, is composed of three parts: (1) how to nest pipes inside one another, (2) how to best pack a given number and type of pipes into a container, and (3) how to allocate pipes to various shipping containers. The authors focus on (2) and show that in two-dimensions the problem can be reduced to a circle packing problem. George et al. (1995) formulate the problem as the following non-linear mixed integer programming model.

$$\text{ICP1}: \max \quad \sum_{i=1}^{N} \delta_i W_i \tag{23}$$

$$\text{s.t.} \quad \delta_i r_i \leqslant x_i \leqslant \delta_i (A - r_i), \quad i = 1, \ldots, N, \tag{24}$$

$$\delta_i r_i \leqslant y_i \leqslant \delta_i (B - r_i), \quad i = 1, \ldots, N, \tag{25}$$

$$\delta_i \delta_j (r_i + r_j) \leqslant d_{ij}, \quad i < j = 1, \ldots, N, \tag{26}$$

$$\delta_i \in \{0, 1\}, \quad i < j = 1, \ldots, N, \tag{27}$$

where W_i is the weight of pipe i and A and B are, respectively, the horizontal and vertical dimensions of the rectangular shipping container. The binary decision variable, δ_i, takes the value of 1 if pipe i is placed in the shipping container and 0 if it is not. The authors define the weight W_i as the density of pipe i and calculate it by: $W_i = (1/AB)\pi r_i^2$. Observe the ICP1 has $4N$ linear constraints and $N(N-1)/2$ nonconvex constraints, for a fixed setting of all binary variables. All the constraints include the binary decision variable δ_i, which adds to the model complexity significantly. George et al. (1995) propose two heuristic solution strategies. The solution strategies differ in terms of the solution-building rules that they use. Among other considerations, these rules include: (1) order and use pipes from largest to smallest, (2) pack larger pipes in the corners, and (3) pack equal pipes together. Numerical results are finally presented to evaluate the heuristic solution strategies.

George et al. (1995, 1996) consider the multiple container loading problem. In industry, the loads are usually packed in a greedy fashion with each container being packed successively using a single container loading solution strategy. However, as noted by the authors, this sequential strategy may lead to the first few containers being packed very efficiently, but the last ones being packed poorly. The formulation of the multiple container loading problem is similar to that of the cutting stock problem, where the packing of loads in a single container is equivalent to a circle packing problem. The objective is to find the minimum number of containers to pack at least the specified load assuming that a set of feasible packing patterns is available. Thus, the proposed integer programming model uses a pattern selection approach. The following alternative solution strategies are discussed by the authors: (1) sequential packing using a single container strategy (i.e., the usual greedy fashion), (2) pre-allocate loads to certain containers, and (3) simultaneous packing using an integer programming model in a multiple container strategy. The numerical results apply the alternative solution strategies on randomly generated instances. As expected, packing loads simultaneously following a multiple container strategy is the superior alternative.

3.3. Cylinder packing

Dowsland (1991) considers the problem of packing cylindrical units into a rectangular container with respect to the *palletisation efficiency* of the resultant container. When the circular cross-sections of the cylinders are considered, the problem is equivalent to the circle packing problem ACP2. In the problem considered by the author, cylinders of equal diameter are packed into boxes that are then loaded onto pallets. The problem involves optimizing the palletisation of these cylinder-filled boxes and the minimization of the area of the container. When packing cylinders, it is suggested that shippers often insist on regular arrangements that are easy to produce. Dowsland (1991) notes that the size and shape of the surrounding container depends not only on the number of rows and the number of cylinders per row but also on the angle between the centers of cylinders in adjacent rows. Regular arrangements require the angle between the centers to be constant. The author analyzes the effect of the angle and proposes a solution strategy to determine its optimal value. Clearly, due to the restriction on possible arrangements, only certain types of regular arrangements are considered. The regular arrangements are represented by an analytical expression that relate the angle between rows, the number of rows in the arrangement, the number of cylinders per row, and the type of arrangement. Numerical results on randomly generated instances are presented and the values of the angle that are the most often used in practice are identified.

Correia et al. (2000, 2001) consider the cylinder packing problem and propose simulated annealing solution strategies to solve it. Cylinders of identical radii are to be placed onto a rectangular pallet in the densest possible way. The authors mention that this problem is also referred to as the pallet loading problem (although in the generalized pallet loading problem, identical rectangles or other shapes can instead be packed). As mentioned before, the cylinder packing problem considered is equivalent to the circle packing problem. Numerical results using simulated annealing are presented. When the number of circles to pack is small, the strategies are quite efficient. The authors propose an extension of one of the strategies to solve the problem with arbitrary sized circles. Although only slight adjustments to the strategy are needed to handle the problem with arbitrary sized circles, Correia et al. (2000) do not present numerical results.

Birgin et al. (2005) consider the container loading problem in which equal cylinders need to be packed into a rectangular container. This is equivalent to the cylinder packing problem found in literature, and hence equivalent to the circle packing problem. The authors note that this problem is of economic importance since packing more items into a container leads to a reduction in costs: therefore they aim for the densest packing of cylinders. A nonlinear optimization model is formulated to solve the problem. The decision to be made is as follows: given N circles of radius r and a rectangular container with horizontal and vertical dimensions A and B, respectively, is it possible to locate all the circles inside the container or not? The formulation is given by

$$\text{ICP2}: \quad \min \quad \sum_{i=1}^{N}\sum_{j>i}^{N} \max(0, 4r^2 - d_{ij}^2)^2 \tag{28}$$

$$\text{s.t.} \quad r \leqslant x_i \leqslant A - r, \quad i = 1, \ldots, N, \tag{29}$$

$$r \leqslant y_i \leqslant B - r, \quad i = 1, \ldots, N. \tag{30}$$

If the objective function is equal to zero then the answer to the decision problem is *shape yes*, the N circles can be packed inside the container. Otherwise, the answer is *shape no*. Numerical results are presented and compared to results from previously published examples. Two extensions are proposed: (1) packing circles into circles and (2) packing arbitrary sized circles inside a rectangular container. The first extension uses the same objective function with a modified constraint set. The second extension requires a different objective function and used the appropriate constraint set depending on whether the container is rectangular or circular. Birgin et al. (2005) note that their proposed strategy is no longer applicable to these extensions. Thus, no numerical results are presented.

3.4. Facility dispersion and communication networks

Erkut (1990) introduces the p-dispersion problem in which the optimized location of a set of points that represent facilities is sought. Drezner and Erkut (1995) consider the continuous p-dispersion problem and its relationship to the circle packing problem. The objective of the continuous p-dispersion problem (refer to UCP3) is to select N points from a feasible region such that the minimum distance between the selected points is maximized; thus, the facilities are dispersed as much as possible. The authors suggest locating undesirable facilities or fast-food franchises in a fixed-area as possible applications. As mentioned throughout, the uniform sized circle packing problem (refer to UCP2) is concerned with packing N equal circles in a convex set (circle or square) such that the radius of the circles is maximized. Drezner and Erkut (1995) show that the optimal solution to the continuous p-dispersion problem and to the corresponding circle packing problem are equivalent. The local nonlinear solver AMPL/MINOS is used as the solution strategy. Two computational strategies are used to reduce the chances of being trapped in a local minimum. One strategy involves the reformulation of the last constraint into its square-root form. The second involves starting the problem several times with random starting solutions. Numerical results are presented for several instances.

Martin (2004) considers the robot communication problem. Essentially, in a mob of robots, each has circular communication coverage with a range of R meters but when they are within r meters of another robot they cannot communicate with that robot. The problem attempts to answer the question of how many robots can be used such that this constraint is met. The author suggests that the problem is equivalent to the circle packing problem, where one wants to determine the smallest circle in which N unit circles can be placed without overlap. Unfortunately, no information regarding numerical results is known.

Adickes et al. (2002) consider the optimization of indoor wireless communications network layouts. A major problem identified is where to place the transceivers in the layout so that they meet coverage requirements. Engineers usually do the placement manually, but they first need to check if a location is suitable. The authors mention that the manual work involved is both time consuming and costly; and it may also lead to areas that remain uncovered when not enough transceivers are allocated. The authors present a solution strategy for determining the optimal number of transceivers and their placement. Transceiver coverage is modeled as a geometric circle covering problem that attempts to place the minimum number of circles so that the specified area is covered. This problem slightly differs from the circle packing problem in that there are no boundary or overlapping constraints. However, it may show a situation where the circle packing problem can be applied if the non-overlapping constraint is relaxed. In determining the placement of the transceivers three different objectives are used: (1) maximize facility coverage, (2) maximize data transmission rates as requested, and (3) maximize signal strength. Adickes et al. (2002) implement a genetic algorithm based strategy to handle this problem. It is interesting to note that once transceivers are installed their coverage region may no longer be circular. The actual coverage areas form a polygon shape due to obstructions in the area. Interferences are accounted for in the methodology of the solution strategy.

3.5. Facility and dashboard layout

Drezner (1980) considers the facility layout (or floor planning) problem. This problem often involves locating a given number of facilities such that they do not overlap and the cost of interactions between facilities is minimized. When facilities are modeled as circular shapes, the problem can be modeled as a circle packing problem. Furthermore, the cost of interactions is simply the weighted distance between facilities, or circles. The facility layout problem is formulated as follows:

$$\text{ICP3}: \quad \min \quad \sum_{i=1}^{N} \sum_{j>i}^{N} c_{ij} d_{ij} \tag{31}$$

$$\text{s.t.} \quad r_i + r_j \leqslant d_{ij}, \quad i < j = 1, \ldots, N, \tag{32}$$

where c_{ij} is the cost per unit distance between facilities i and j. The author proposes the DISCON solution strategy to solve the facility layout problem. The solution strategy uses a Langrangian differential gradient method and is composed of two phases: (1) DISpersion phase and (2) CONcentration phase. The dispersion phase is needed to find a suitable initial solution for the second phase. The dispersion phase starts by placing all circles at the origin where the overlapping constraint is violated. This forces the circles to be dispersed outwards. The solution generated provides a starting point for the second phase. The concentration phase moves the facilities closer together so that a dense arrangement is achieved. Numerical results for several equal and arbitrary sized circle instances are presented. Results are compared to those of the CRAFT (Buffa et al., 1964) solution strategy. Drezner (1980) suggests that CRAFT can be used additionally, to improve the results obtained by the DISCON solution approach.

Anjos and Vannelli (2002) consider the facility layout problem. The problem involves finding the optimal positions for a given set of modules within a facility such that the distances between the pairs of modules are minimized. As before, when the modules are circular, the problem can be modeled as a circle packing problem. The authors propose a heuristic Attractor–Repeller (AR) solution procedure to handle this problem. The objective of AR is to 'attract' the modules so that they are as close together as possible, but also to 'repel' them so that the desired separation is attained (i.e., the modules do not overlap). The concept of target distances is important as it defines the desired separation of the modules. To enforce the target distances, a penalty function is included in the objective function. Two other features found in the AR method are: (1) modules can be distinguished as either being fixed or variable (i.e., movable) and (2) bounds of the facility dimensions can be specified. The authors note that the resulting optimization problem only has linear constraints. However, the problem is still nonconvex because of the penalty function used. Anjos and Vannelli (2002) present numerical results using AR and MINOS. Solutions are often infeasible in the sense that modules might overlap. Thus, a scaling factor is used to accentuate the repeller effect.

Castillo and Sim (2004) also consider the facility layout problem and propose a spring-embedding (SE) solution strategy to solve it. In contrast to DISCON and AR, the resulting optimization problem in SE is convex. SE is based on a dynamic system in which N particles of a given area are mutually connected by springs. Assume that each of the springs connecting pairs of the N particles has different strengths. From an initial arrangement of the particle locations, the system oscillates until it stabilizes at a minimum-energy arrangement resulting in a dynamically balanced system. When the N particles represent the N facilities or modules and the spring strengths represent the cost of interactions between facilities or modules, then SE can be applied to solve the facility layout problem. SE is formulated as follows:

$$\text{ICP4}: \min \quad \sum_{i=1}^{N}\sum_{j>i}^{N}\frac{1}{2}c_{ij}d_{ij}^2 + \sum_{i=1}^{N}\sum_{j>i}^{N}\max\{0, K_{ij}(r_i+r_j-d_{ij})\} \tag{33}$$

$$\text{s.t.} \quad x_i + r_i - \frac{1}{2}A \leqslant 0, \quad i=1,\ldots,N, \tag{34}$$

$$r_i - x_i - \frac{1}{2}A \leqslant 0, \quad i=1,\ldots,N, \tag{35}$$

$$y_i + r_i - \frac{1}{2}B \leqslant 0, \quad i=1,\ldots,N, \tag{36}$$

$$r_i - y_i - \frac{1}{2}B \leqslant 0, \quad i=1,\ldots,N, \tag{37}$$

$$A^{\text{low}} \leqslant A \leqslant A^{\text{up}}, \tag{38}$$

$$B^{\text{low}} \leqslant B \leqslant B^{\text{up}}, \tag{39}$$

where $K_{ij} > 0, \forall 1 \leqslant i < j \leqslant N$, A^{low} and A^{up} are the floor area width requirements with $A^{\text{low}} \leqslant A^{\text{up}}$, and B^{low} and B^{up} are the floor area height requirements with $B^{\text{low}} \leqslant B^{\text{up}}$. Observe that ICP4 has $4N$ linear constraints and additional simple bounds. The $N(N-1)/2$ nonconvex constraints are removed from the constraint set by extending the objective function with an additional convex term. The authors present numerical results using an augmented Langrangian multiplier method for several equal and arbitrary sized circle instances. Results are compared with those of DISCON and AR. Castillo and Sim (2004) note that DISCON is more efficient than SE in terms of CPU time; however, facility layout problems are planned for the long-term and thus longer run times are often acceptable. Moreover, SE solutions, in contrast to those of AR, are guaranteed to be feasible.

Riskin et al. (2003) consider the placement of a given number of circular dials inside a fixed-area dashboard. This problem is known as the dashboard planning problem. The design of flight panels found in airplanes and control panels at nuclear plants are suggested as possible applications. The objective of the problem is that of minimizing the mental workload that the operation of the dashboard requires. The problem can easily be seen as yet another application of the circle packing problem. The authors propose a logarithmic barrier ($\log \mathrm{B}$) solution strategy. Including the barrier forces the dials from overlapping. A scaling factor is used to control the effect of the barrier so that large undesirable distances do not separate dials. Riskin et al. (2003) embed $\log \mathrm{B}$ in an iterative procedure that progressively reduces the scaling factor until a desirable solution is obtained. Numerical examples are presented using a single instance.

4. Classification and summary

So far, we have introduced and classified the circle packing problems that have appeared in the literature as generic and industrial problems. In the case of the generic problems, we have also classified them based on the size of the circles: uniform and arbitrary. Two additional classifications are possible. One of the possible classifications is based on the interpretation of the objective function and the other is based on the solution strategy used to solve the respective circle packing problems.

Table 1 summarizes the generic circle packing problems. In general, these problems have a linear objective function that either minimizes the radius of the circular container or maximizes the common radius of the circles to be packed. Aside from constraints that are derived from simple geometrical considerations, all problems have $N(N-1)/2$ non-overlapping, nonconvex constraints. Both exact and heuristic solution strategies have been proposed in the literature.

Table 2 summarizes the circle packing applications. Here, it is interesting to note that some applications involve the 'simplest' case of packing uniform sized circles. The circle packing applications can also be classified based on the interpretation of the objective function. The objective function of the applications that consider relationships between the circles to be packed has a clear economic interpretation; e.g., for the facility layout problem the economic interpretation is that of minimizing the total material handling cost. Other applications consider circles that are independent of each other and thus concentrate on the size of the container. It can be argued that the minimization of the size of the container, however, can be related economically to the amount of material used. As with the generic circle packing problems, aside from constraints that are derived from simple geometrical considerations, most problems have $N(N-1)/2$ non-overlapping, nonconvex constraints. There are three interesting exceptions: (1) the container loading problem considered by George et al. (1995), where all the constraints include a binary decision variable; (2) the container loading problem considered by Birgin et al. (2005), where all the constraints are linear; and (3) the facility layout problem considered by Castillo and Sim (2004), where all the constraints are also linear. As with the generic circle packing problems, both exact and heuristic solution strategies have been proposed in the literature.

As mentioned above, in general circle packing models lead to difficult optimization problems that cannot be tackled effectively by analytical approaches that would lead to provably globally optimal results. Moreover, the above circle packing problem formulations imply that the complexity of the packing models increases rapidly as N increases. Thus, numerical – and often heuristic – solution strategies appear to be preferred when dealing with larger problem instances ($N \geqslant 30$).

Table 1
Summary of generic circle packing problems

Reference	Size of circles		Solution strategy	
	Uniform	Arbitrary	Exact	Heuristic
Boll et al. (2000)	✓			✓
Graham et al. (1998)	✓		✓	✓
Huang et al. (2003)		✓		✓
Huang et al. (forthcoming)		✓		✓
Locatelli and Raber (2002)	✓		✓	
Lubachevsky and Graham (1997)	✓			✓
Markót and Csendes (2005)	✓		✓	
Mladenovic et al. (2005)	✓		✓	
Nurmela and Österga·rd (1997)	✓		✓	
Pintér and Kampas (2005)		✓	✓	
Stoyan and Yaskov (2004)		✓	✓	
Stoyan and Yaskov (1998)		✓	✓	
Szabó et al. (2005)	✓		✓	✓
Theodoracatos and Grimsley (1995)	✓			✓
Wang et al. (2002)		✓		✓
Zhang and Deng (2005)		✓		✓

Table 2
Summary of circle packing applications

Reference	Industrial application	Size of circles		Objective function		Solution strategy	
		Uniform	Arbitrary	Container	Economic	Exact	Heuristic
Adickes et al. (2002)	Comm. networks	✓			✓		✓
Anjos and Vannelli (2002)	Facility layout		✓		✓	✓	
Birgin et al. (2005)	Cylinder packing	✓		✓			✓
Castillo and Sim (2004)	Facility layout		✓		✓	✓	
Correia et al. (2000)	Cylinder packing	✓		✓			✓
Correia et al. (2001)	Cylinder packing	✓		✓			✓
Cui (2005)	Circular cutting		✓		✓		✓
Dowsland (1991)	Cylinder packing	✓		✓			✓
Drezner (1980)	Facility layout		✓		✓		✓
Drezner and Erkut (1995)	Facility dispersion	✓		✓		✓	
Fraser and George (1994)	Container loading		✓	✓			✓
George et al. (1995)	Container loading		✓		✓		✓
George (1996)	Container loading		✓		✓		✓
Hifi et al. (2004)	Circular cutting		✓		✓		✓
Hifi and M'Hallah (2004)	Circular cutting		✓		✓		✓
Martin (2004)	Comm. networks	✓		✓		N/A	N/A
Riskin et al. (2003)	Dashboard layout		✓		✓		✓

From an optimization standpoint, packing arbitrary sized circles is a much more complex problem than packing uniform sized circles since often there exist many local minima (maxima) close in value to the global minimum (maximum). Many of these local minima (maxima) can be regarded as arising from interchanging two circles in the global solution and then suitably 'readjusting' the other circles. Evidently, similar interchanges have no effect on the solution when uniform sized circles are packed.

5. Illustrative numerical results

In this section, numerical results are presented for various circle packing problem instances that have appeared in the literature. Unfortunately, most of the data that is based on industrial applications is not publicly available. Since results for packing uniform sized circles are abundant in the literature, we concentrate on instances related to the packing of arbitrary sized circles; that is, ACP1, ACP2, and ICP3/ICP4.

We acknowledge that there is clearly a trade-off between solution quality, flexibility, and speed depending on the solution strategy used. Specialized solution strategies tend to be more efficient than generic solution strategies. However, we will not exploit any prior structural considerations and initial arrangements that could help the optimization process, since we wish to illustrate the 'off-the-shelf' capabilities of generic global optimization solution strategies and software. In practice, the potential trade-offs should be considered depending on the actual industrial needs and resource limitations. To the best of our knowledge, the instances introduced in this section have not been investigated by other researchers in global optimization (when this paper was written).

First, we evaluate the performance of three generic global optimization software packages on a set of 14 instances of ACP1 with different values of N. For these instances, the size of the circles to be packed is defined by the function $r_i = i^{-1/2}$, $i = 1, \dots, N$. We note that we are not aware of the existence of optimal or best known solutions for these particular instances.

The generic global solvers are LINGO (Lindo Systems, 2004), NMinimize (Wolfram Research, 2005), and MathOptimizer Professional (Pintér and Kampas, 2006). The solvers require setting explicit bounds for all decision variables: such explicit bounds were set as needed. Default settings were used for all solvers with the exception of the maximum number of iterations for LINGO, which was set to match the default maximum number of iterations for MathOptimizer Professional. MathOptimizer Professional was used in conjunction with the Salford FORTRAN FTN 95 compiler. Our computing platform is a Windows XP laptop with a Centrino 2.00 GHz processor and 2 GB of RAM. CPU times are given in seconds.

Table 3 summarizes the results for these 14 instances. In order to have a sense of the different circle sizes, the mean and the standard

Table 3
Numerical results for ACP1 ($r_i = i^{-1/2},\ i = 1,\ldots,N$)

Problem number	Number of circles, N	Mean of radii	Std. dev. of radii	LINGO		NMinimize		MathOptimizer Pro.	
				Objective	CPUs	Objective	CPUs	Objective	CPUs
1	5	0.6463	0.2206	1.7516[a]	3	1.7734	2	1.7516[a]	3
2	6	0.6067	0.2199	1.8236	5	1.8473	2	1.8101[a]	1
3	7	0.5740	0.2186	1.8476	8	1.8921	3	1.8387[a]	2
4	8	0.5464	0.2169	1.9095	11	1.9377	4	1.8796[a]	3
5	9	0.5228	0.2149	1.9201[a]	15	1.9758	9	1.9221	4
6	10	0.5021	0.2129	1.9553	16	1.9647	14	1.9382[a]	6
7	12	0.4676	0.2088	2.0038	31	2.0617	32	1.9902[a]	11
8	14	0.4397	0.2047	2.0317	43	2.0663	65	2.0316[a]	18
9	16	0.4165	0.2009	2.0562[a]	78	2.1166	160	2.0661	28
10	18	0.3968	0.1972	2.1142	90	2.1213	302	2.0700[a]	50
11	20	0.3798	0.1938	2.1382	144	2.1392	522	2.1255[a]	59
12	25	0.3456	0.1860	2.1821	248	2.2521	1913	2.1669[a]	792
13	30	0.3195	0.1793	2.2112[a]	906	2.2185	5,273	2.2149	263
14	35	0.2987	0.1735	2.2645	1367	2.3428	12,311	2.2597[a]	464

[a] Best known solution.

deviation of the radii of the circles to be packed is provided. Note that the total area needed to pack the circles is slowly divergent as N increases. Indeed, the total area needed is $\pi\sum_{i=1}^{N} 1/i$; thus, the optimized radius of the container also increases as N increases. Table 3 indicates that, in most cases, MathOptimizer Professional ran faster and gave better results than LINGO and NMinimize for these instances.

Several of the generic and applied studies reviewed in Sections 2 and 3 have emphasized the analysis of prior structural considerations and initial arrangements. The expectation is that solution strategies that exploit such issues may deliver even better results. In an attempt to improve the MathOptimizer Professional results presented in Table 3, we implemented an *a posteriori* strategy that, given a (supposedly) near-optimal initial arrangement, swaps all pairs of adjacent sized circles, taking the best result until there is no additional benefit. Table 4 summarizes the results on the set of 14 instances of ACP1 with different values of N, where 10 near optimal initial arrangement were used for each instance. Note that, in this case, the CPU time considers both the generation of the initial arrangement and the time it takes to evaluate swapping all

Table 4
Improved numerical results for ACP1 ($r_i = i^{-1/2},\ i = 1,\ldots,N$)

Problem number	Number of circles, N	MathOptimizer Pro.		Improved MathOptimizer Pro. by swapping			
		Objective	CPUs	Best objective	Mean objective	Std. dev. objective	Mean CPUs
1	5	1.7516	3	1.7516	1.7516	0.0000	3
2	6	1.8101	1	1.8101	1.8118	0.0027	6
3	7	1.8387	2	1.8387	1.8452	0.0084	10
4	8	1.8796	3	1.8613	1.8761	0.0076	16
5	9	1.9221	4	1.8900	1.9099	0.0125	19
6	10	1.9382	6	1.9244	1.9369	0.0112	48
7	12	1.9902	11	1.9696	1.9947	0.0183	35
8	14	2.0316	18	2.0173	2.0340	0.0141	30
9	16	2.0661	28	2.0464	2.0740	0.0118	140
10	18	2.0700	50	2.0664	2.0972	0.0216	243
11	20	2.1255	59	2.1050	2.1263	0.0147	166
12	25	2.1669	792	2.1642	2.1813	0.0111	265
13	30	2.2149	263	2.2008	2.2238	0.0211	825
14	35	2.2597	464	2.2259	2.2631	0.0170	1312

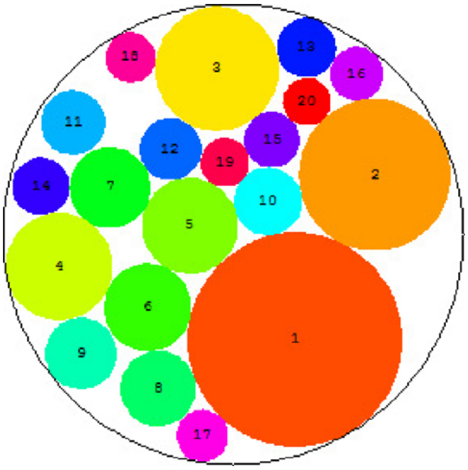

Fig. 1. Initial near-optimal arrangement for problem 11.

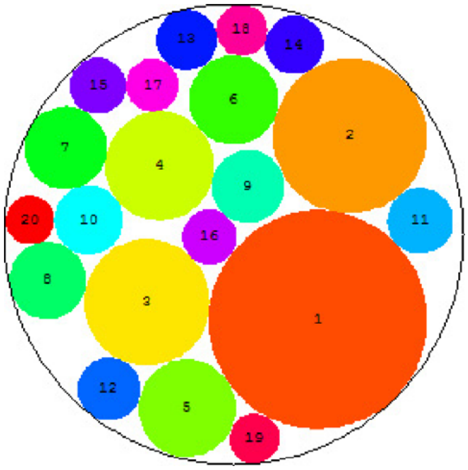

Fig. 2. Final arrangement after the *a posteriori* swapping strategy for problem 11.

pairs of adjacent circles. Despite its simplicity, the outlined *a posteriori* strategy improves upon all the original results. For illustrative purposes, Figs. 1 and 2 show the arrangements for the initial near-optimal solution and the final solution after the *a posteriori* swapping strategy, respectively, for problem 11 ($N = 20$).

At this point, let us mention Zimmermann's recent circle packing contest, where numerous, highly specialized solution strategies were 'competing' against each other on a set of instances related to the packing of arbitrary sized circles in a circle (ACP1), similarly to the problems solved above. We note that the results presented in this paper (with the exception of those presented in Table 4) were obtained before the contest. The Zimmermann contest site is available at http://www.recmath.org/contest/CirclePacking/index.php.

We now turn our attention to circle packing problem instances for which global or best known solutions exist. Table 5 summarizes the results for 8 instances taken from Zhang and Deng (2005) (ACP1) and Hifi and M'Hallah (2004) (circular cutting problem, ACP2). The objective function values reported for the instances taken from Hifi and M'Hallah (2004) refer to the density of the packing as given by $\sum_{i=1}^{N}\pi r_i^2/AB$, where A is the fixed width and B is the height of the container to be optimized. As before, MathOptimizer Professional mostly ran faster and gave better results than LINGO and NMinimize. For the sake of conciseness, the LINGO results are not presented. Table 5 also indicates that MathOptimizer Professional compares rather well in terms of CPU time with the heuristic results. Note that we have been able to find a better solution for problem number 17. On average, the

Table 5
Numerical results for ACP1 and ACP2

Problem number	Number of circles, N	Mean of radii	Std. dev. of radii	Heuristic		NMinimize		MathOptimizer Pro.		
				Objective	CPUs	Objective	CPUs	Objective	CPUs	% gap
15[a]	12	48.92	32.53	215.47	0	248.63	52	222.09	8	3.07
16[a]	15	8.00	4.47	39.37	12	41.27	126	39.84	14	1.19
17[a]	17	9.12	6.18	50.00	0	53.55	250	49.37	24	−1.26
18[a]	17	45.12	32.75	241.43	20	300.00	212	248.72	24	3.02
19[b]	20	1.22	0.42	79.41	35	74.93	613	77.47	33	2.44
20[b]	25	1.08	0.49	81.65	85	78.03	2,115	79.40	89	2.76
21[a]	30	1.15	0.38	80.72	100	75.91	5,513	77.58	145	3.89
22[a]	35	1.34	0.45	80.39	74	76.13	13,901	78.76	233	2.03

[a] Zhang and Deng (2005). ACP1 solved with a combined simulated annealing and tabu search based heuristic.
[b] Hifi and M'Hallah (2004). Circular cutting problem, ACP2 solved with a genetic algorithm based heuristic.

Table 6
Numerical results for ICP3/ICP4

Problem number	Number of circles, N	Mean of radii	Std. dev. of radii	Heuristic[a]		NMinimize		MathOptimizer Pro.		
				Objective	CPUs	Objective	CPUs	Objective	CPUs	% gap
23	6	0.50	0.00	38.02	1	40.20	5	38.00	1	−0.05
24	8	0.50	0.00	84.99	2	87.86	24	88.32	1	3.92
25	12	0.50	0.00	238.73	6	252.23	139	245.91	9	3.01
26	15	0.50	0.00	454.74	19	454.83	37	465.75	17	2.42
27	20	0.50	0.00	1007.88	54	–	–[b]	1054.54	49	4.63
28	30	0.50	0.00	2,397.65	246	–	–[b]	2,551.50	209	6.42
29	6	1.21	0.30	89.83	1	89.81	4	89.81	1	−0.02
30	8	1.15	0.37	195.44	2	216.83	17	195.40	2	− 0.02
31	12	1.16	0.36	530.04	11	561.18	133	552.89	7	4.31
32	15	1.18	0.33	1046.47	22	1108.77	476	1082.38	17	3.43
33	20	1.12	0.36	2,604.33	69	–	–[b]	2,412.36	50	−7.37
34	30	1.06	0.37	4,651.95	261	–	–[b]	5,159.43	757	10.91
35	6	1.21	0.30	103.23	3	112.33	27	144.66	1	40.12
36	8	1.15	0.37	235.21	7	236.77	123	233.04	2	−0.92
37	12	1.16	0.36	548.80	21	628.31	576	675.14	9	23.02
38	15	1.18	0.33	1119.55	42	–	–[b]	1226.61	23	9.56
39	20	1.12	0.36	2,685.89	113	–	–[b]	2,561.47	77	−4.63
40	30	1.06	0.37	5,032.55	376	–	–[b]	5,768.49	430	14.62

[a] Castillo and Sim (2004). Facility layout, ICP3/ICP4 solved with an augmented Lagrangian multiplier based heuristic.
[b] CPU time limit of 1000 seconds exceeded.

'gap' between the result obtained by MathOptimizer Professional and the best known solution is 2.14%.

Finally, we consider 18 instances from Castillo and Sim (2004). Table 6 summarizes the results for these facility layout instances (ICP3/ICP4) using a CPU time limit of 1000 seconds. In general, MathOptimizer Professional ran faster and gave better results than LINGO and NMinimize. Again, for the sake of conciseness, the LINGO results are not presented. The table indicates that MathOptimizer Professional again compares rather well in terms of CPU time with the heuristic results. Note that we have been able to find significantly better solutions for problem numbers 33 and 39. We conjecture that the small differences between some of the solutions are due to the precision of the various calculations. For illustrative purposes, Figs. 3 and 4 show the final arrangements for problem 39. On average, the 'gap' between MathOptimizer Professional's solution value and the best known solution is 6.30%.

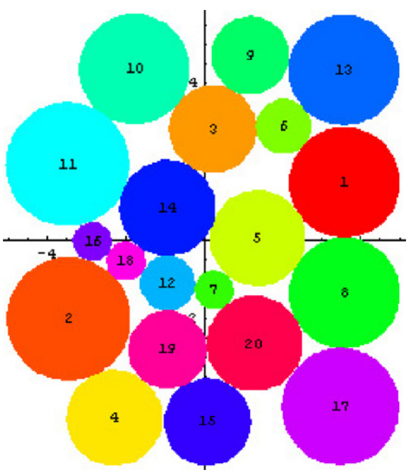

Fig. 3. Heuristic arrangement for problem 39.

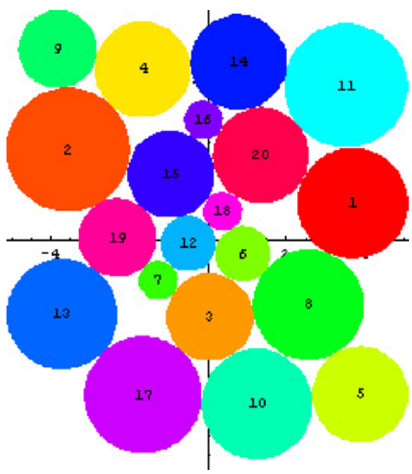

Fig. 4. MathOptimizer Professional arrangement for problem 39.

We conclude this section by noting that no attempt was made to optimize the performance of MathOptimizer Professional at this stage of our research. Optimal or near-optimal arrangements were reached in at most 15 minutes, for an arguably broad range of packing problems considered. Let us also point out that even several hours or days of run-time could be acceptable since, in practice, the modeled decisions are often meant to be part of a long-term strategy.

6. Summary and conclusions

Circle packing problems have received a considerable amount of attention in the mathematics literature, but only modest attention in the operational research literature. This is surprising, considering the numerous areas of existing and potential applications: several of these have been reviewed in this paper. In general, the reviewed industrial applications cannot be tackled effectively by purely analytical approaches. Our review of industrial circle packing problems reveals that a significant effort has been spent on the development of problem-specific heuristic solution strategies. These specialized approaches tend to be more efficient than generic solution strategies since they attempt to exploit the specific structural characteristics of a particular problem. However, such specialized strategies, as a rule, cannot be easily extended or modified to solve other packing problems. Moreover, some of the customized strategies require modeling and solution techniques that tend to complicate the actual mathematical formulation of the problem studied, and may not guarantee the feasibility of the solutions obtained.

We argue that the solution of industrial applications (especially at the design stage, or in need of quick and high-quality, perhaps alternative, feasible solutions) requires flexible, ready-to-use software used within a robust modeling environment. State-of-the-art modeling systems (such as e.g. Mathematica) and linked global solvers (such as MathOptimizer Professional, or similar other software implementations) offer suitable platforms. As our numerical results show, there is a trade-off between solution quality, flexibility, and speed: these trade-offs should be considered depending on the actual industrial needs and resource limitations.

Our illustrative numerical results highlight the relevance of generic global optimization solution strategies, and – at the same time – the necessary advancements in both theory and numerical practice. The solution of industrial (realistic) packing problems most typically also has a significant experimental component that will benefit from the analysis of prior structural considerations and initial arrangements. The development of such strategies (tailored to specific problems) is in progress, and will be reported in our forthcoming work.

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