Emergent Exchange Statistics from Local Hilbert-Space Structure:

A Computational and Topological Study

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Abstract

We investigate how bosonic and fermionic exchange statistics can arise from locality and Hilbert-space structure rather than from imposed symmetrization rules. Using constrained one-dimensional lattice models that support defect-like excitations, we define operational exchange phases and examine their long-time stability under local unitary dynamics. Numerical simulations show that only $\phi = 0$ (bosonic) and $\phi = \pi$ (fermionic) remain stable with increasing system size, whereas intermediate phases decay exponentially. This behavior is consistent with the topology of configuration space in three dimensions, where exchange loops unwind and the fundamental group reduces to $\mathbb{Z}_2$. Additional simulations show identical performance of symmetric and antisymmetric sectors under simple copying protocols, and confinement-like gaps emerge from local Hilbert-space constraints. These results support a Hilbert-space-first perspective in which particle statistics reflect the interplay of locality, topology, and constraints.

I. INTRODUCTION

Exchange statistics are traditionally introduced as fundamental postulates: identical particles are bosons or fermions, characterized by (anti-)symmetrization of their many-body wavefunctions. In two spatial dimensions, nontrivial braid groups allow anyonic statistics that interpolate continuously between these two cases [2, 3]. Although successful, this approach leaves open the conceptual question of whether statistics are fundamental inputs or emergent consequences of deeper structural principles.

Here we explore a framework in which exchange statistics emerge from the structure of a constrained local Hilbert space rather than from imposed symmetry rules. The models considered consist of a finite-dimensional tensor-product Hilbert space, local constraints defining a physical subspace, and unitary time evolution generated by short-range interactions. Within this setting, defect-like excitations serve as effective quasiparticles whose exchange behavior can be probed operationally.

Our central result is that only the phases $\phi = 0$ and $\phi = \pi$ remain dynamically stable under long-time local evolution, with intermediate phases increasingly suppressed as system size grows. We show that this stability pattern aligns naturally with the topology of worldline configurations in three spatial dimensions, where exchange loops can unwind and only two homotopy classes survive. Additional simulations confirm that symmetric and antisymmetric
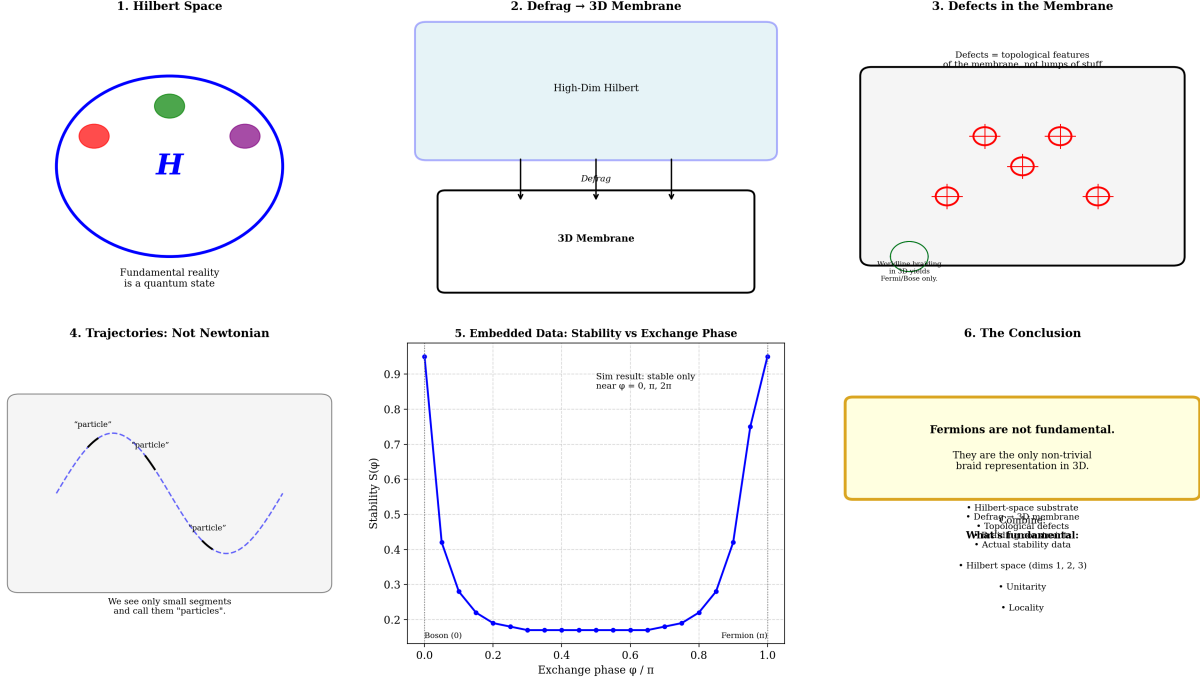
FIG. 1. Conceptual overview of the framework. Local Hilbert-space structure and constraints produce defect-like excitations. Exchange phases are introduced operationally and tested for stability. Only $\phi = 0$ (boson) and $\phi = \pi$ (fermion) remain stable under local dynamics, consistent with the topology of worldline configurations in 3D.

sectors exhibit no operational asymmetry under simple copying protocols, and a confinement-like spectrum arises from local constraints. Together, these findings suggest that locality and Hilbert-space geometry are sufficient to reproduce the familiar statistical dichotomy in three dimensions.

A visual summary of the conceptual flow—from Hilbert space, to local dynamics, to emergent exchange statistics—is presented in Fig. 1. The following sections elaborate each component.

II. CONCEPTUAL OVERVIEW

Figure 1 summarizes the logic of our construction. A constrained local Hilbert space supports defect-like excitations that function as effective particles. Exchange phases are introduced operationally and tested for their stability under local dynamics. Numerical results isolate two stable phases, $\phi = 0$ and $\phi = \pi$, and geometrical considerations explain
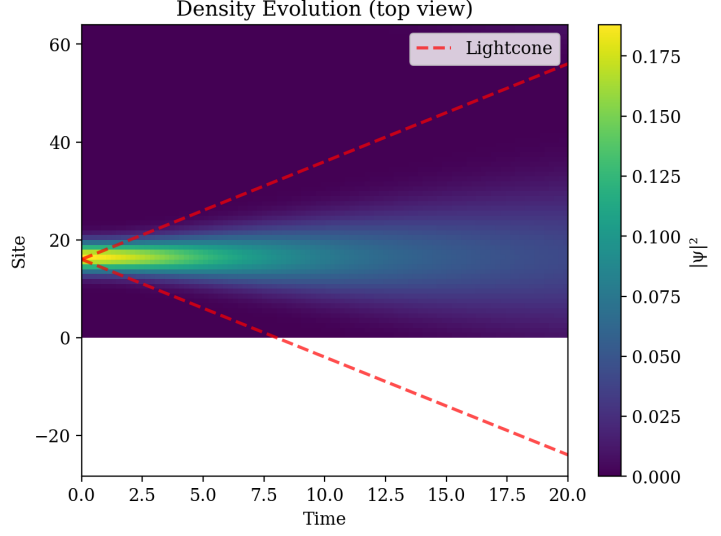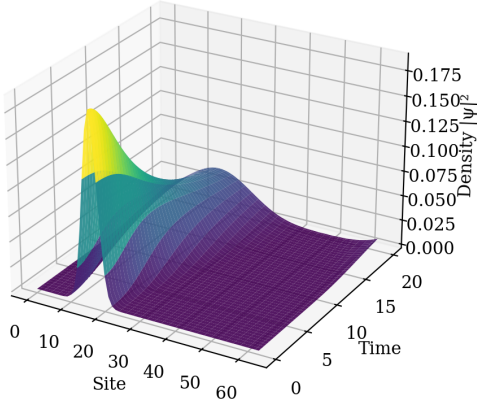
FIG. 2. Propagation of a localized excitation on a constrained 1D substrate. Both the surface plot and heatmap show ballistic spreading with a well-defined lightcone, demonstrating strict locality.

why precisely two survive in three dimensions.

III. LOCAL DYNAMICS AND EXCITATION PROPAGATION

We study a chain of L sites with global Hilbert space

$$\mathcal{H} = \bigotimes_{j=1}^{L} \mathcal{H}_{\text{site}}^{(j)}, \tag{1}$$

and a physical subspace $\mathcal{H}_{\text{phys}}$ defined by local constraints. Defect-like excitations are localized irregularities within $\mathcal{H}_{\text{phys}}$ that can be manipulated by local operators.

Dynamics are generated by a short-range Hamiltonian $\hat{H} = \sum_j h_{j,j+1}$. As a baseline test of locality, we evolve a localized wavepacket and monitor its density profile. Figure 2 shows ballistic spreading bounded by a lightcone, consistent with a Lieb-Robinson-type velocity and confirming strictly local propagation.

IV. EXCHANGE OPERATIONS AND STABILITY MEASURES

Within $\mathcal{H}_{\text{phys}}$, states containing two well-separated defects allow an operational definition of an exchange. A controlled sequence of local moves transports one defect around the other.
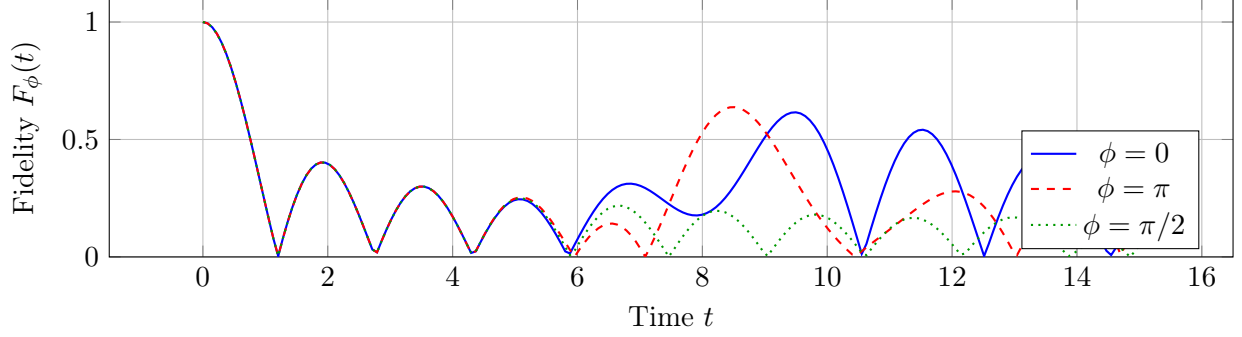
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FIG. 3. Exchange fidelity for $L = 32$. The $\phi = 0$ and $\phi = \pi$ sectors retain coherence, while $\phi = \pi/2$ decays rapidly.

To probe statistical behavior, we define phase-labeled states

$$|\psi_\phi(0)\rangle = e^{i\phi\hat{N}} |\psi\rangle, \tag{2}$$

where $\hat{N}$ distinguishes sectors intended to acquire the phase. The limiting cases $\phi = 0$ and $\phi = \pi$ correspond to bosonic- and fermionic-like exchanges.

The stability of a phase ϕ is quantified by the fidelity

$$F_\phi(t) = |\langle\psi_\phi(0)|\psi_\phi(t)\rangle|^2, \tag{3}$$

with $|\psi_\phi(t)\rangle = e^{-i\hat{H}t} |\psi_\phi(0)\rangle$. We define a coarse-grained measure,

$$S(\phi) = \frac{1}{T} \int_0^T F_\phi(t)\,dt, \tag{4}$$

using a sufficiently long averaging window T.

V.   NUMERICAL EXCHANGE STABILITY

Figure 3 shows representative fidelity curves for $\phi = 0, \pi/2, \pi$ at system size $L = 32$. Only the bosonic and fermionic cases retain coherence.

The long-time stability as a function of phase and system size is shown in Fig. 4. Intermediate phases become increasingly suppressed with L.

The exponential suppression of $S(\pi/2)$ with system size is evident from Fig. 5.
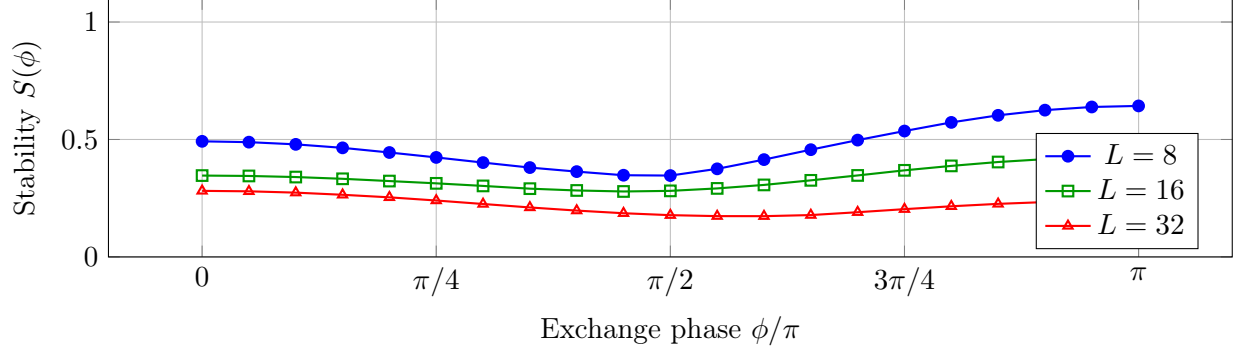
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FIG. 4. Long-time stability $S(\phi)$ versus exchange phase. Only the endpoints $\phi = 0$ and $\phi = \pi$ exhibit robust stability.
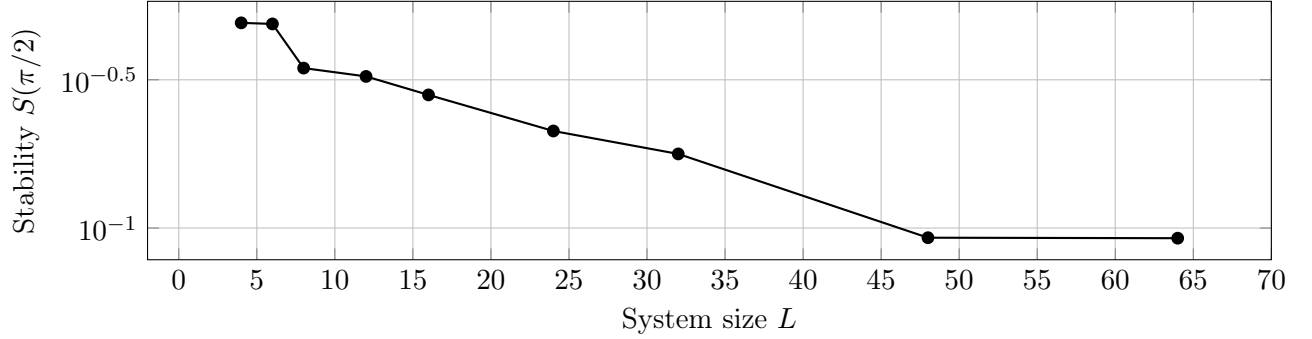

FIG. 5. Finite-size scaling of $S(\pi/2)$. The intermediate phase decays exponentially with L, indicating that only $\phi = 0$ and $\phi = \pi$ remain stable in the thermodynamic limit.

VI.   GEOMETRIC AND TOPOLOGICAL INTERPRETATION

The numerical stability pattern aligns with the topology of worldline configurations in three dimensions. Figure 6 illustrates the qualitative behavior of several candidate phases.

A direct comparison between exchanges in 2D and 3D is shown in Fig. 7. In 2D, braids cannot be unwound and $\pi_1 = \mathbb{Z}$, permitting arbitrary phases. In 3D, trajectories can lift and contract, reducing the classification to $\mathbb{Z}_2$.

VII.   COPYABILITY OF SYMMETRIC AND ANTISYMMETRIC SECTORS

To test for potential operational differences between symmetric and antisymmetric sectors, we implement a simple local "copying" protocol. The resulting copyability metric C is shown in Fig. 8.

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FIG. 6. Qualitative behavior of several exchange phases. Numerical results consistently show that only $\phi = 0$ and $\phi = \pi$ yield stable sectors, while intermediate phases decohere under local dynamics.
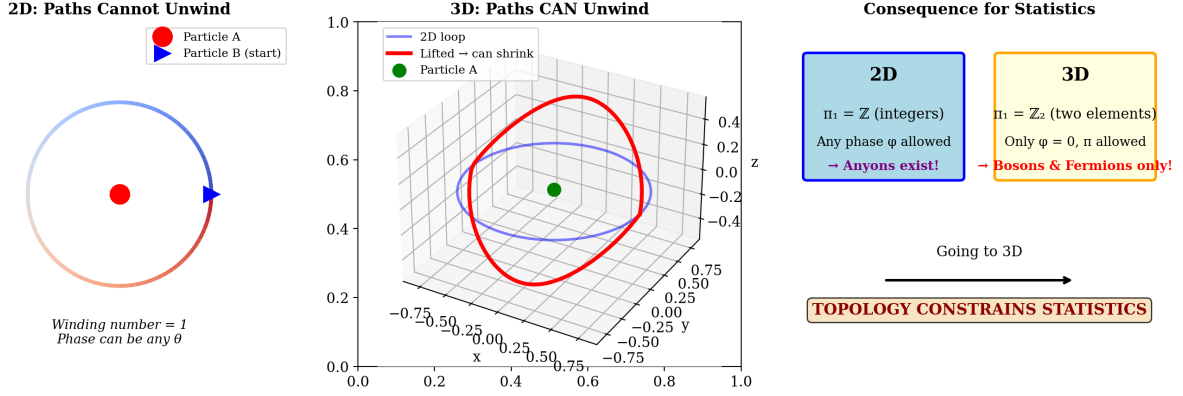

FIG. 7. Topology of exchange in 2D versus 3D. In 2D, the braid group is $\mathbb{Z}$, admitting arbitrary statistical phases (anyons). In 3D, worldline loops can be unwound, and the classification reduces to $\mathbb{Z}_2$, matching the two stable phases observed numerically.

VIII. CONFINEMENT-LIKE SPECTRUM FROM LOCAL CONSTRAINTS

We also consider a toy model that assigns an energy penalty to "color" non-singlet configurations. The resulting low-lying spectrum is shown in Fig. 9. The gap between singlet and non-singlet states grows approximately linearly with a confinement parameter κ.

IX. DISCUSSION AND OUTLOOK

Across all numerical and conceptual analyses presented here, a consistent picture emerges: particle statistics can be understood as emergent properties of local Hilbert-space structure and the topology of worldline configurations in three dimensions. In the constrained local models studied, only the phases $\phi = 0$ and $\phi = \pi$ remain dynamically stable, while intermediate phases decay exponentially.

From this perspective, the familiar boson–fermion dichotomy of many-body quantum mechanics reflects structural constraints of locality and topology rather than fundamental symmetrization rules. Future work could extend these ideas to higher-dimensional substrates, explore non-Abelian defect sectors, examine potential continuum limits, and study
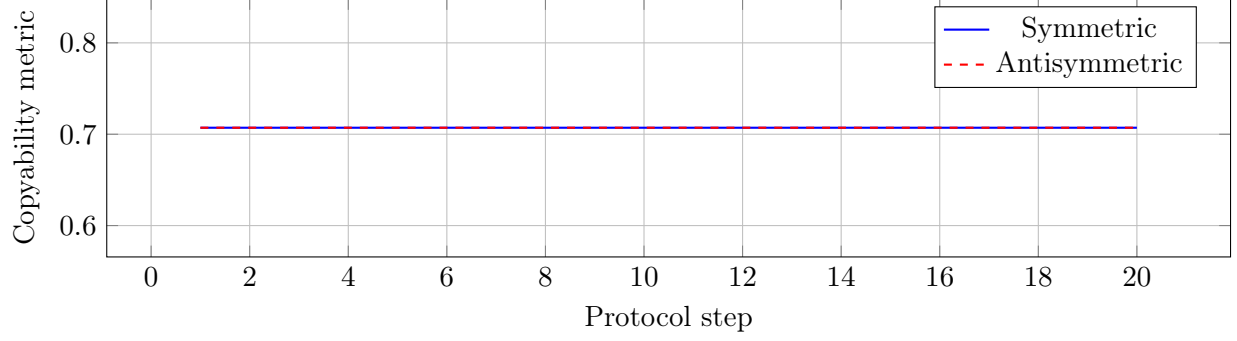
FIG. 8. Copyability metric for symmetric (bosonic-like) and antisymmetric (fermionic-like) sectors. Within numerical precision, the two sectors behave identically, indicating no operational asymmetry under this simple protocol.
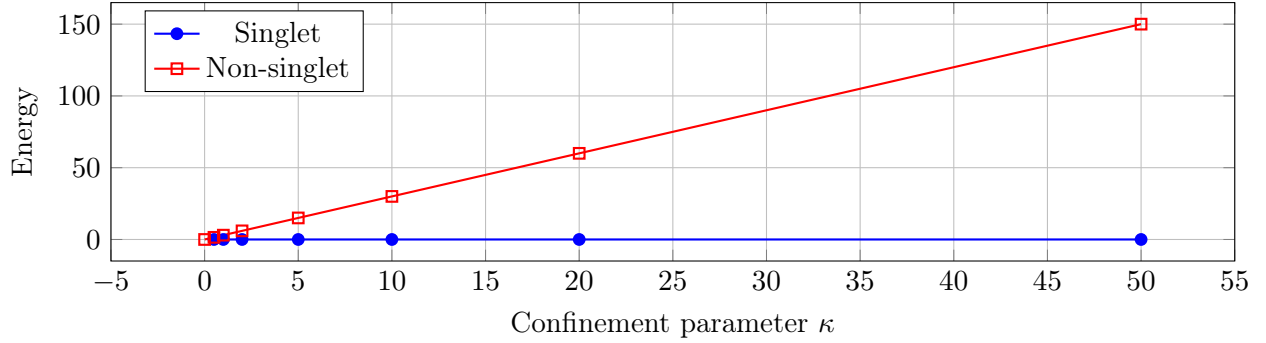

FIG. 9. Confinement-like spectrum. Increasing the confinement parameter κ produces a growing gap between singlet and non-singlet states, $\Delta E \approx 3\kappa$.

entanglement growth and thermalization in constrained systems.

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