Theory of Relativity
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11 The Schwarzschild Solution

11.1 The static, isotropic metric

In this section, I want to find a static, isotropic, metric. What does this mean? Starting from the metric interval,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

we want to find a set of coordinates x^μ in which the $g_{\mu\nu}$ do not depend on time t (*i.e.* the metric is static) and in which ds^2 depends *only* on rotational invariants of the spatial coordinates (*i.e.* the metric is isotropic). We can form the following rotational invariants from the coordinates:

$$\left.\begin{array}{c} d\mathbf{x}^2 \\ \mathbf{x}\cdot d\mathbf{x} \\ \mathbf{x}^2 \equiv r^2 \end{array}\right\} \text{ ($\mathbf{x}$ are the spatial coordinates)}$$

and so the general form of a static, isotropic, metric must be

$$ds^2 = A(r)dt^2 - B(r)dt\, \mathbf{x}\cdot d\mathbf{x} - C(r)(\mathbf{x}\cdot d\mathbf{x})^2 - D(r)d\mathbf{x}^2, \tag{1}$$

where A, B, C and D are arbitrary function of the radius r. The metric must have the form (1) because we seek a metric that is both quadratic in the coordinate differentials and rotationally invariant. Now transform to spherical polar coordinates

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi, \\ x^2 &= r\sin\theta\sin\phi, \\ x^3 &= r\cos\theta. \end{aligned}$$

Since

$$\begin{aligned} \mathbf{x}^2 &= r^2, \quad \mathbf{x}\cdot d\mathbf{x} = rdr, \\ d\mathbf{x}^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \end{aligned}$$

the general metric (1) looks like

$$ds^2 = A(r)dt^2 - B(r)dt\, rdr - C(r)r^2dr^2 - D(r)\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{2}$$

in spherical polar coordinates. Now define a new time coordinate

$$\begin{aligned} t' &= t + \Phi(r), \\ dt' &= dt + d\Phi(r), \end{aligned}$$

and let's look at the first two terms in (2)

$$A(r)\left[dt'-d\Phi\right]^2 - B(r)\left[dt'-d\Phi\right] r dr$$
$$= A(r)dt'^2 + A(r)\left(\frac{d\Phi}{dr}\right)^2 dr^2 - \underbrace{2A(r)dt'\frac{d\Phi}{dr}dr} - \underbrace{B(r)rdt'dr} + B(r)\frac{d\Phi}{dr}rdr.$$

So if we set

$$\frac{d\Phi}{dr} = \frac{rB(r)}{2A(r)}$$

, then the metric will become diagonal,

$$ds^2 = A(r)dt'^2 - \left[C(r) - \frac{1}{4}\frac{B^2(r)}{A(r)}\right] r^2dr^2 - D(r)\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right].$$

If we redefine the radius r

$$r'^2 = D(r)r^2$$

then we can write the metric as

$$ds^2 = \alpha(r')dt'^2 - \beta(r')dr'^2 - [r'^2d\theta^2 + r'^2\sin^2\theta d\phi^2],$$

and there is no need to retain the prime on the radial variable so we can rewrite the metric as

$$ds^2 = \alpha(r)dt^2 - \beta(r)dr^2 - [r^2d\theta^2 + r^2\sin^2\theta d\phi^2]. \tag{3}$$

The general static, isotropic, metric is thus specified by two functions functions, $\alpha(r)$ and $\beta(r)$ which we need to determine by solving the Einstein field equations.

The components of the metric are

$$\begin{array}{lllllllllll}
g_{00} & = & g_{tt} & = & \alpha(r) & \quad & g^{00} & = & g^{tt} & = & 1/\alpha(r)\\
g_{11} & = & g_{rr} & = & -\beta(r) & & g^{11} & = & g^{rr} & = & -1/\beta(r)\\
g_{22} & = & g_{\theta\theta} & = & -r^2 & & g^{22} & = & g^{\theta\theta} & = & -1/r^2\\
g_{33} & = & g_{\phi\phi} & = & -r^2\sin^2\theta & & g^{33} & = & g^{\phi\phi} & = & -\frac{1}{r^2\sin^2\theta}
\end{array}$$

The affine connection is, by definition,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right),$$

and so the components are:

$$\begin{aligned}
\Gamma^0_{00} &= 0,\\
\Gamma^i_{00} &= \frac{1}{2}g^{i\rho}\left\{-\frac{\partial g_{00}}{\partial x^\rho}\right\} \Longrightarrow \Gamma^r_{00} = -\frac{1}{2}g^{rr}\frac{\partial g_{00}}{\partial x^r} = \frac{1}{2}\frac{1}{\beta(r)}\frac{d\alpha(r)}{dr},\\
\Gamma^0_{0i} &= \frac{1}{2}g^{0\rho}\left\{\frac{\partial g_{\rho 0}}{\partial x^i} + \frac{\partial g_{\rho i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\rho}\right\}\\
&= \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^i}\right\} \Longrightarrow \Gamma^0_{0r} = \frac{1}{2}\,\frac{1}{\alpha(r)}\frac{d\alpha(r)}{dr},\\
\Gamma^0_{ij} &= 0,
\end{aligned}$$

$$
\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2}g^{i\rho}\left\{\frac{\partial g_{\rho i}}{\partial x^i}+\frac{\partial g_{\rho i}}{\partial x^i}-\frac{\partial g_{ii}}{\partial x^\rho}\right\} \\
&= \frac{1}{2}g^{ii}\left\{\frac{\partial g_{ii}}{\partial x^i}\right\} \Rightarrow \Gamma^r_{rr} = \frac{1}{2}\frac{1}{\beta(r)}\frac{d\beta(r)}{dr}, \\
\Gamma^r_{\theta\theta} &= \frac{1}{2}g^{rr}\left\{\frac{\partial g_{r\theta}}{\partial x^\theta}+\frac{\partial g_{r\theta}}{\partial x^\theta}-\frac{\partial g_{\theta\theta}}{\partial x^r}\right\} \\
&= -\frac{1}{2}\frac{1}{\beta(r)}2r \Longrightarrow \Gamma^r_{\theta\theta} = -\frac{r}{\beta(r)}, \\
\Gamma^r_{\phi\phi} &= -\frac{1}{2}g^{rr}\frac{\partial g_{\phi\phi}}{\partial r} = -\frac{r\sin^2\theta}{\beta(r)}, \\
\Gamma^\theta_{\theta r} &= \frac{1}{2}g^{\theta\theta}\left\{\frac{\partial g_{\theta\theta}}{\partial x^r}\right\} = \frac{1}{r}, \\
\Gamma^\theta_{\phi\phi} &= \frac{1}{2}g^{\theta\theta}\left\{-\frac{\partial g_{\phi\phi}}{\partial x^\theta}\right\} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{\phi r} &= \frac{1}{2}g^{\phi\phi}\left\{\frac{\partial g_{\phi\phi}}{\partial x^r}\right\} = \frac{1}{2}(-)\frac{1}{r^2\sin^2\theta}(-)2r\sin^2\theta = \frac{1}{r}, \\
\Gamma^\phi_{\phi\theta} &= \frac{1}{2}g^{\phi\phi}\left\{\frac{\partial g_{\phi\phi}}{\partial x^\theta}\right\} = \frac{1}{2}(-)\frac{1}{r^2\sin^2\theta}(-)r^2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}.
\end{aligned}
$$

Summarising, we end up with nine non-vanishing components of Γ:

$$
\begin{aligned}
&(1) \qquad & \Gamma^r_{rr} &= \frac{\beta'}{2\beta}, \\
&(2) & \Gamma^r_{\phi\phi} &= \frac{-r\sin^2\theta}{\beta}, \\
&(3) & \Gamma^\theta_{r\theta} &= \frac{1}{r} \\
&(4) & \Gamma^\phi_{\phi r} &= \frac{1}{r} \\
&(5) & \Gamma^0_{0r} &= \frac{\alpha'}{2\alpha} \\
&(6) & \Gamma^r_{\theta\theta} &= -\frac{r}{\beta} \\
&(7) & \Gamma^r_{00} &= \frac{\alpha'}{2\beta} \\
&(8) & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
&(9) & \Gamma^\phi_{\phi\theta} &= \frac{\cos\theta}{\sin\theta},
\end{aligned}
$$

where the primes on α and β denote derivatives with respect to r,

$$
\beta' = \frac{d\beta}{dr}, \qquad \alpha' = \frac{d\alpha}{dr}.
$$

The Ricci tensor is, by definition,

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}.$$

It requires quite a lot of tedious (but simple) algebra to work out the components of the Ricci tensor. I will work out one of these components in full so that you can see what is involved and then state the results for the others:

$$R_{rr} = \frac{\partial \Gamma^{\lambda}_{r\lambda}}{\partial x^{r}} - \frac{\partial \Gamma^{\lambda}_{rr}}{\partial x^{\lambda}} + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta}.$$

$$\begin{aligned}
\text{Term [1]} \quad & \frac{\partial \Gamma^{\lambda}_{r\lambda}}{\partial x^{r}} = \frac{\partial \Gamma^{r}_{rr}}{\partial r} + \frac{\partial \Gamma^{\theta}_{r\theta}}{\partial r} + \frac{\partial \Gamma^{\phi}_{r\phi}}{\partial r} + \frac{\partial \Gamma^{0}_{r0}}{\partial r}, \\
\text{Term [2]} \quad & \frac{\partial \Gamma^{\lambda}_{rr}}{\partial x^{\lambda}} = \frac{\partial \Gamma^{r}_{rr}}{\partial r}, \\
\text{Term [3]} \quad & \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} = \Gamma^{0}_{r\lambda}\Gamma^{\lambda}_{r0} + \Gamma^{i}_{r\lambda}\Gamma^{\lambda}_{ri} \\
& = \left(\Gamma^{0}_{r0}\right)^2 + \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{rr} + \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta} + \Gamma^{\phi}_{r\lambda}\Gamma^{\lambda}_{r\phi} \\
& = (\Gamma^{0}_{r0})^2 + (\Gamma^{r}_{rr})^2 + (\Gamma^{\theta}_{r\theta})^2 + (\Gamma^{\phi}_{r\phi})^2, \\
\text{Term [4]} \quad & \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta} = \Gamma^{r}_{rr}\Gamma^{\lambda}_{\lambda r} = \Gamma^{r}_{rr}\left[\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r} + \Gamma^{0}_{0r}\right].
\end{aligned}$$

Hence, summing these terms

$$R_{rr} = \frac{\partial \Gamma^{0}_{0r}}{\partial r} + \frac{\partial \Gamma^{\theta}_{\theta r}}{\partial r} + \frac{\partial \Gamma^{\phi}_{\phi r}}{\partial r} + \left(\Gamma^{0}_{0r}\right)^2 + \left(\Gamma^{r}_{rr}\right)^2 + \left(\Gamma^{\theta}_{r\theta}\right)^2 + \left(\Gamma^{\phi}_{r\phi}\right)^2 - \Gamma^{r}_{rr}\left[\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r} + \Gamma^{0}_{0r}\right],$$

and so evaluating this expression using the components of the affine connection that we worked out before

$$R_{rr} = -\frac{1}{2}\frac{\alpha'^2}{\alpha^2} + \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{2}{r^2} + \frac{1}{4}\frac{\alpha'^2}{\alpha^2} + \frac{1}{4}\frac{\beta'^2}{\beta^2} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{\beta'}{2\beta}\left[\frac{\beta'}{2\beta} + \frac{1}{r} + \frac{\alpha'}{2\alpha} + \frac{1}{r}\right],$$

Hence,

$$R_{rr} = \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{\alpha'}{4\alpha}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r}\,\frac{\beta'}{\beta}, \tag{4a}$$

and, in a similar way, one can show that the rest of the components of the Ricci tensor are,

$$\begin{aligned}
R_{\theta\theta} &= -1 - \frac{r}{2\beta}\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right) + \frac{1}{\beta}, && (4b) \\
R_{\phi\phi} &= \sin^2\theta\; R_{\theta\theta}, && (4c) \\
R_{00} &= -\frac{\alpha''}{2\beta} + \frac{1}{4}\left(\frac{\alpha'}{\beta}\right)\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r}\left(\frac{\alpha'}{\beta}\right), && (4d) \\
R_{\mu\nu} &= 0, \quad \mu \neq \nu. && (4e)
\end{aligned}$$

We showed in the last lecture that Einstein's field equations in vacuum are

$$R_{\mu\nu} = 0.$$

Hence *all* of the components of the Ricci tensor (4a) to (4d) *must vanish.* First, let's eliminate α'' from (4a) and (4d),

$$\frac{R_{rr}}{\beta} + \frac{R_{00}}{\alpha} = -\frac{\alpha'}{4\alpha\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) + \frac{\alpha'}{4\alpha\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) = 0,$$

i.e.,

$$\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha} = 0,$$

which requires that

$$\alpha(r)\beta(r) = \text{constant}.$$

Recall that the functions α and β appear in the metric as

$$ds^2 = \alpha(r)dt^2 - \beta(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

and so if we want to match on to the Minkowski metric as $r \to \infty$, the functions α and β must satisfy

$$\left.\begin{array}{c} \alpha(r) \to c^2 \\ \beta(r) \to 1 \end{array}\right\} \text{ as } r \to \infty, \quad \text{therefore } \alpha(r) = c^2/\beta(r).$$

Now look at the other components:

$$\begin{aligned} R_{\theta\theta} &= -1 - \frac{r}{2}\frac{\alpha}{c^2}\left(-2\frac{\alpha'}{\alpha}\right) + \frac{\alpha}{c^2} \\ &= -1 + r\frac{\alpha'}{c^2} + \frac{\alpha}{c^2} = 0, \qquad \text{so} \quad \frac{d\left(r\alpha(r)/c^2\right)}{dr} = 1, \qquad i.e. \;\; r\frac{\alpha(r)}{c^2} = r + \text{constant}. \end{aligned} \tag{5}$$

$$R_{rr} = \frac{1}{2}\frac{\alpha''}{\alpha} + \frac{1}{r}\frac{\alpha'}{\alpha} = \frac{(r\alpha'' + 2\alpha')}{2r\alpha} = \frac{1}{2r\alpha}\frac{dR_{\theta\theta}}{dr},$$

and so setting $R_{\theta\theta} = 0$ over all space guarantees that $R_{rr} = 0$. Now, in the weak field limit we know that

$$\alpha(r) \to (1 + \underbrace{2\phi}_{\nearrow} /c^2)c^2.$$

Newtonian potential

So, a long way from a mass M,

$$\alpha(r) \to \left(1 - \frac{2GM}{rc^2}\right)c^2$$

hence the constant in (5) must be $\underline{-2GM/c^2}$ and so

$$\alpha(r) = \left(1 - 2\frac{GM}{rc^2}\right) c^2$$

Our metric is therefore

$$ds^2 = \left[1 - \frac{2GM}{rc^2}\right] c^2 dt^2 - \left[1 - \frac{2GM}{rc^2}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{6}$$

This is called the *Schwarzschild* metric after its discoverer. This metric is extremely important in General Relativity. It is an *exact* solution of the field equations and represents the metric around a spherical compact object of mass M. By construction, the metric reproduces the Newtonian potential $\phi = -GM/r$ in the weak field limit. However, since the metric (6) is an exact solution of the field equations, it is also valid in the *strong* gravitational field limit $GM/r \gg c^2$. We can therefore use the metric (6) to describe the orbits of planets around the Sun (a weak field application of GR) or to describe the orbits of particles around a compact object such as a black hole (a strong field application of GR). We will investigate both of these problems in the next few Sections.

11.2 Equations of motion for the Schwarzschild metric

In the previous section, I derived the Schwarzschild metric

$$ds^2 = \overbrace{\left[1 - \frac{2GM}{rc^2}\right] c^2}^{\alpha(r)} dt^2 - \overbrace{\left[1 - \frac{2GM}{rc^2}\right]^{-1}}^{\beta(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{7}$$

In this lecture, I will derive the equations of motion for a freely falling particle in the metric (7). The geodesic equations of motion are

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0, \tag{8}$$

where p is an affine parameter that varies along the path. We derived the components of the affine connection for the Schwarzschild metric in the previous lecture, so it is straightforward to substitute these into (8) and derive the equations of motion for a freely falling particle. The equations of motion are:

$$\frac{d^2x^0}{dp^2} + \Gamma^0_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = \frac{d^2x^0}{dp^2} + 2\Gamma^0_{0r}\frac{dx^0}{dp}\frac{dx^r}{dp},$$

$$\text{So,} \qquad \boxed{\frac{d^2t}{dp^2} + \frac{\alpha'}{\alpha}\frac{dt}{dp}\frac{dr}{dp} = 0} \tag{9a}$$

$$\frac{d^2r}{dp^2} + \Gamma^r_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = \frac{d^2r}{dp^2} + \Gamma^r_{rr}\left(\frac{dr}{dp}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{dp}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dp}\right)^2 + \Gamma^r_{00}\left(\frac{dt}{dp}\right)^2 = 0,$$

$$\text{So,} \qquad \boxed{\frac{d^2r}{dp^2} + \frac{1}{2}\frac{\beta'}{\beta}\left(\frac{dr}{dp}\right)^2 - \frac{r}{\beta}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{\beta}\left(\frac{d\phi}{dp}\right)^2 + \frac{1}{2}\frac{\alpha'}{\beta}\left(\frac{dt}{dp}\right)^2 = 0} \tag{9b}$$

$$\frac{d^2\theta}{dp^2} + \Gamma^\theta_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = \frac{d^2\theta}{dp^2} + 2\Gamma^\theta_{r\theta}\left(\frac{dr}{dp}\right)\left(\frac{d\theta}{dp}\right) + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{dp}\right)^2 = 0,$$

$$\text{So,} \qquad \boxed{\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2 = 0.} \tag{9c}$$

$$\frac{d^2\phi}{dp^2} + \Gamma^\phi_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = \frac{d^2\phi}{dp^2} + 2\Gamma^\phi_{\phi r}\frac{d\phi}{dp}\frac{dr}{dp} + 2\Gamma^\phi_{\phi\theta}\frac{d\phi}{dp}\frac{d\theta}{dp} = 0.$$

$$\text{So,} \qquad \boxed{\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} + \frac{2\cos\theta}{\sin\theta}\frac{d\phi}{dp}\frac{d\theta}{dp} = 0.} \tag{9d}$$

Since the field is isotropic, we can take the orbit to lie in the equatorial plane without loss of generality, *i.e.*, $\theta = \pi/2$. Equation (9c) becomes

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} = 0.$$

and so is satisfied if $\theta =$ constant. The equations of motion simplify to

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{\beta'}{\beta}\left(\frac{dr}{dp}\right)^2 - \frac{r}{\beta}\left(\frac{d\phi}{dp}\right)^2 + \frac{1}{2}\;\frac{\alpha'}{\beta}\left(\frac{dt}{dp}\right)^2 = 0, \tag{10a}$$

$$\frac{d^2t}{dp^2} + \frac{\alpha'}{\alpha}\frac{dt}{dp}\frac{dr}{dp} = 0, \tag{10b}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} = 0. \tag{10c}$$

Look at equation (4c):

$$\text{write} \;\; \dot{\phi} = \frac{d\phi}{dp},$$

then (10c) becomes

$$\frac{1}{\dot{\phi}}\frac{d\dot{\phi}}{dp} + \frac{2}{r}\frac{dr}{dp} = 0,$$

$$i.e. \;\; \frac{d\left[\ln\dot{\phi} + \ln r^2\right]}{dp} = 0.$$

Now look at (10b):

$$\text{write} \;\; \dot{t} = dt/dp,$$

then (10b) becomes

$$\frac{1}{\dot{t}}\frac{d\dot{t}}{dp} + \frac{1}{\alpha}\frac{d\alpha}{dp} = 0,$$

$$i.e. \;\; \frac{d\left[\ln\dot{t} + \ln\alpha\right]}{dp} = 0.$$

So from these equations,

$$r^2\dot{\phi} = r^2\frac{d\phi}{dp} = \text{constant},$$

$$\alpha\frac{dt}{dp} = \text{constant},$$

but, we can choose the normalisation of p so that

$$\alpha\frac{dt}{dp} = c^2, \qquad i.e. \qquad dt = \frac{dp}{\left[1 - 2\underbrace{GM/(rc^2)}\right]} \tag{11a}$$

$$\uparrow$$

if this is small $p \approx t$

We can write the other constant of integration as

$$r^2\frac{d\phi}{dp} = J, \tag{11b}$$

where J is the GR equivalent of specific angular momentum.

We are left with one equation of motion for the radial coordinate:

$$\frac{d^2r}{dp^2}+\frac{1}{2}\frac{\beta'}{\beta}\left(\frac{dr}{dp}\right)^2-\frac{J^2}{\beta r^3}+\frac{\alpha' c^4}{2\beta\alpha^2}=0.$$

Multiply this equation by $2\beta(dr/dp)$

$$2\beta\frac{dr}{dp}\frac{d^2r}{dp^2}+\beta'\left(\frac{dr}{dp}\right)^3-\frac{2J^2}{r^3}\left(\frac{dr}{dp}\right)+\frac{\alpha'}{\alpha^2}\,\frac{dr}{dp}c^4=0,$$

$$i.e.\quad \frac{d}{dp}\left(\beta\left(\frac{dr}{dp}\right)^2+\frac{J^2}{r^2}-\frac{c^4}{\alpha}\right)=0.$$

Hence

$$\beta\left(\frac{dr}{dp}\right)^2+\frac{J^2}{r^2}-\frac{c^4}{\alpha}=-\mathcal{E}=\text{constant},\qquad (11c)$$

where $\mathcal{E}$ is an integration constant which is related to the GR "analogue" of specific energy. From equation (11c) and using (11a) and (11b) we must have

$$\beta dr^2+\overbrace{r^2d\phi^2}^{J=r^2d\phi/dp}-\alpha dt^2=-\mathcal{E}\;dp^2,$$

and so comparing with the Schwarzschild metric (7), we must have

$$c^2d\tau^2=\mathcal{E}dp^2$$

and so,

$$\mathcal{E}>0\quad\text{for massive particles,}\quad(ds^2>0),$$

$$\mathcal{E}=0\quad\text{for photons,}\qquad\qquad(ds^2=0).$$

Notice that in a weak gravitational field the equations of motion become,

$$\begin{aligned} r^2\frac{d\phi}{dt} &= J,\\ \left(\frac{dr}{dt}\right)^2+\frac{J^2}{r^2}-\frac{c^2}{(1+2\phi/c^2)} &= -\mathcal{E}, \end{aligned}$$

i.e.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2+\frac{1}{2}\,\frac{J^2}{r^2}+\phi=\left(\frac{c^2-\mathcal{E}}{2}\right),$$

which is identical to the Newtonian equation with the specific energy replaced by $\left(\frac{c^2-\mathcal{E}}{2}\right)$.

11.3 Deflection of light by the Sun

Let's start with the equations of motion that we derived in the previous section:

$$\begin{aligned} \beta \left(\frac{dr}{dp} \right)^2 + \frac{J^2}{r^2} - \frac{c^4}{\alpha} &= -\mathcal{E}, \\ r^2 \frac{d\phi}{dp} &= J. \end{aligned}$$

Eliminate p from the first of these equations,

$$\frac{\beta}{r^4} J^2 \left(\frac{dr}{d\phi} \right)^2 + \frac{J^2}{r^2} - \frac{c^4}{\alpha} = -\mathcal{E},$$

and integrating,

$$\phi = \pm \int \frac{\beta^{1/2} dr}{r^2 \left[\frac{c^4}{J^2 \alpha} - \frac{1}{r^2} - \frac{\mathcal{E}}{J^2} \right]^{1/2}}. \tag{12a}$$

For a photon ($\mathcal{E} = 0$) this simplifies to

$$\phi = \pm \int \frac{\beta^{1/2} dr}{r^2 \left[\frac{c^4}{J^2 \alpha} - \frac{1}{r^2} \right]^{1/2}}. \tag{12b}$$

I will now use equation (12b) to analyse the deflection of light by the Sun. The relevant angles and distances are defined in Figure 1. The length b is the *impact parameter* – this is the distance of closest approach for a straight line orbit, *i.e.* if the photon travelled in a straight line, undeflected by the gravity of the Sun. In fact, the photon will be deflected by the gravity of the Sun and so it will move along a curved path as shown in the diagram. The distance r_0 is the *distance of closest approach* and this differs from the impact parameter as you can see from the diagram.

At large distances, the photon will move along a straight line with a speed c. I will use relativistic units ($c = 1$), in which case the straight line orbit can be written

$$b = r \sin \left(\phi - \phi_\infty \right) \approx r \left(\phi - \phi_\infty \right). \tag{13}$$

Differentiating this equation with respect to time t,

$$\dot{r} \left(\phi - \phi_\infty \right) + r \dot{\phi} = 0, \qquad \dot{r} = -c,$$

and so from equation (13)

$$r^2 \dot{\phi} = bc.$$

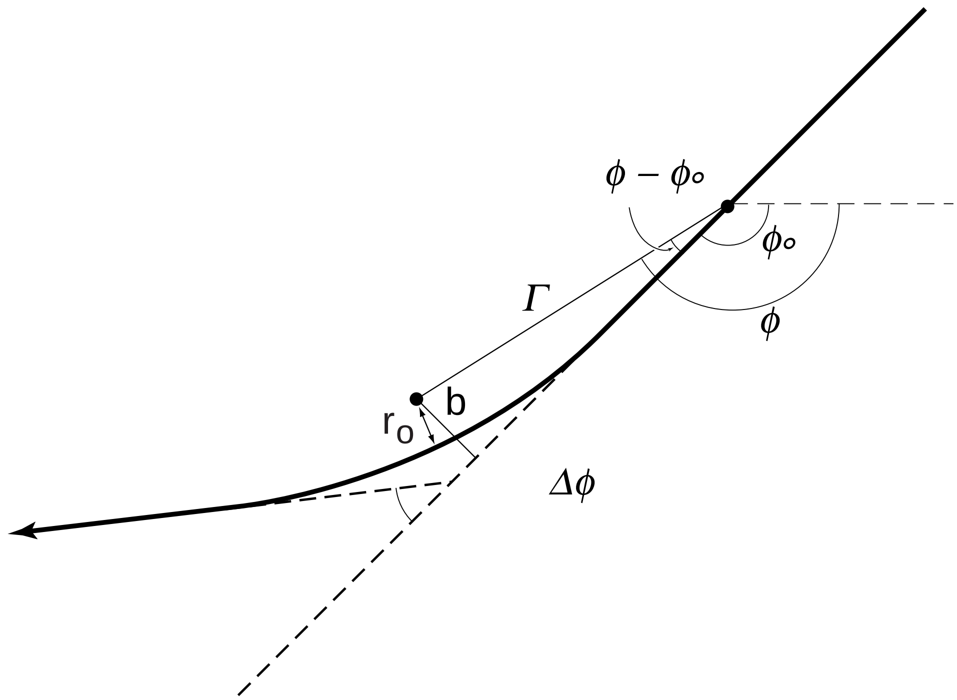

Figure 1: Angles and coordinates in the analysis of deflection of light by a point mass.

But recall that the specific angular momentum is defined by $r^2\dot{\phi} = J$, so for a photon the two integration constants of the orbit, $\mathcal{E}$ and J are,

$$\left.\begin{aligned} J &= bc, \\ \mathcal{E} &= 0. \end{aligned}\right\}$$

At the point of closest approach, r_0,

$$\left(\frac{dr}{d\phi}\right)_{r_0} = 0$$

so, from the equations of motion,

$$\frac{J^2}{r_0^2} = \frac{c^4}{\alpha(r_0)},$$

and so, the deflection angle is given by

$$\phi(r) = \phi_\infty + \int_r^\infty \frac{\beta^{1/2} dr}{r^2 \left[\frac{\alpha(r_0)}{\alpha(r)}\frac{1}{r_0^2} - \frac{1}{r^2}\right]^{1/2}}, \tag{14}$$

where

$$\begin{aligned}\alpha(r) &= c^2\left(1-\frac{2GM}{rc^2}\right),\\ \beta(r) &= \left(1-\frac{2GM}{rc^2}\right)^{-1}.\end{aligned}$$

The deflection from a straight line is

$$\triangle\phi = 2\left[\phi(r_0)-\phi_\infty\right]-\pi,$$

so all that remains is to evaluate the integral in equation (14). As it stands, equation (14) is a non-trivial integral, but we can reduce it to elementary integrals by expanding α and β in powers of $\frac{GM}{rc^2}$,

$$\int_r^\infty \frac{dr}{r}\frac{1}{\left(r^2/r_0^2-1\right)^{1/2}}\left[1+\frac{GM}{rc^2}+\frac{GM}{c^2r_0}\frac{r}{(r+r_0)}+\ldots\ldots\right],$$

Put $x=r/r_0$, $\gamma=\frac{GM}{r_0c^2}$, this integral becomes

$$\int_1^\infty \frac{dx}{x}\frac{1}{\left(x^2-1\right)^{1/2}}\left[1+\frac{\gamma}{x}+\gamma\frac{x}{(1+x)}+\mathcal{O}(\gamma^2)\ldots\right].$$

The first term integrates to $\pi/2$, and each of the next two terms integates to γ. [1] Hence the deflection angle is

$$\triangle\phi = 4\gamma = \frac{4GM}{r_0c^2},$$

and so for light grazing the Sun, General Relativity predicts a deflection of

$$\triangle\phi = 1.75''.$$

The 1919 eclipse expedition led by Eddington gave two sets of results

$$\begin{aligned}\triangle\phi &= 1.98\pm0.16'',\\ \triangle\phi &= 1.61\pm0.4'',\end{aligned}$$

consistent with the theory. This provided the first experimental verification of Einstein's theory and turned Einstein into a scientific superstar. (The media had a great story. Remember this was just after the end of the first world war, and so the headlines read something like 'Newton's theory of gravity overthrown by German physicist, verified by British scientists'). Some historians have argued that Eddington 'fiddled' the results to agree with the theory. If Eddington did indeed massage the results, then he gambled correctly. Modern radio experiments using VLBI (very long baseline interferometry) have been done to measure the

[1] The integral of the third term is $\gamma\frac{(x-1)^{1/2}}{(x+1)^{1/2}}$.

gravitational deflection of the positions of radio quasars as they are eclipsed by the Sun. Such experiments can be done to an accuracy of better than $\sim 10^{-4}$ arcseconds. The following diagram is from the book by Clifford Will (*Theory and Experiment in Gravitational Physics*, CUP, 1981). It summarizes the results of VLBI measurements of the deflection angle $\Delta\phi$ from experiments conducted over the period $1969 - 1975$. The results are in excellent agreement with the predictions of General Relativity.

11.4 Precession of Planetary Orbits

In the preceeding section we derived the equation of motion

$$\frac{\beta}{r^4} J^2 \left(\frac{dr}{d\phi}\right)^2 + \frac{J^2}{r^2} - \frac{c^4}{\alpha} = -\mathcal{E}. \tag{15}$$

Now, make the substitution $u = 1/r$ and write $\mu = \frac{GM}{c^2}$ in (15). This gives

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = -\frac{\mathcal{E}}{J^2}(1 - 2\mu u) + 2\mu u^3 + \frac{c^2}{J^2}. \tag{16}$$

Now recall that

$$r^2 \frac{d\phi}{d\tau} = \frac{Jc}{\sqrt{\mathcal{E}}} = h = \text{constant},$$

and so differentiating (16) with respect to ϕ we get the simple equation of motion

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu c^2}{h^2} + 3\mu u^2. \tag{17}$$

This equation gives the exact General Relativistic equation of motion for a particle around a point mass M. How does this equation compare with the Newtonian equation of motion? Assume, without loss of generality, that the orbit is in the plane

$$\theta = \text{constant} = \pi/2.$$

The Lagrangian of a point mass m is

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) + \frac{GMm}{r},$$

and so from Lagrange's equations we can readily derive the the Newtonian equations of motion:

$$r^2 \dot{\phi} = h, \tag{18a}$$

$$\ddot{r} = \frac{h}{r^3} - \frac{GM}{r^2}. \tag{18b}$$

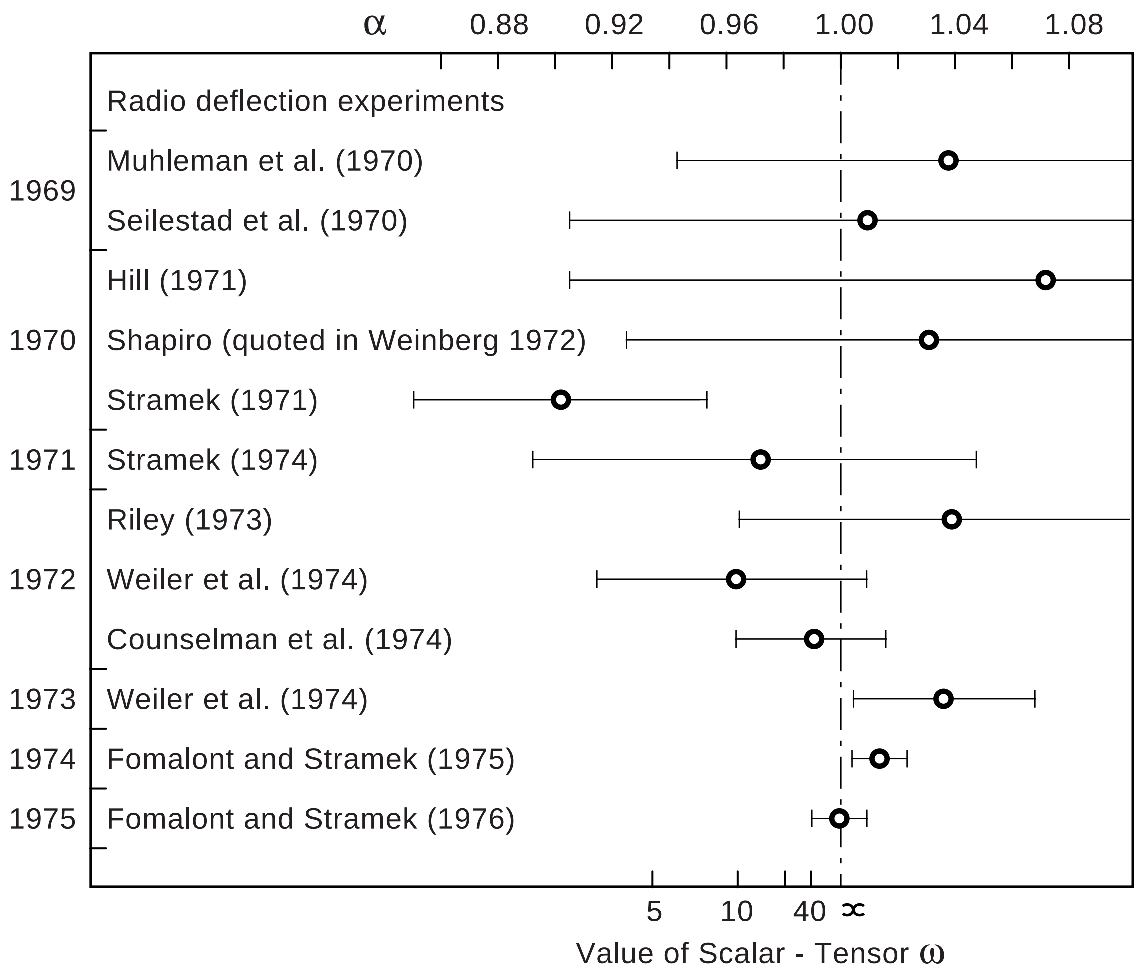

Figure 2: Results of radio-wave deflection measurements of the positions of quasars in the period 1969-1975. I have written the deflection angle as $\Delta\phi = \alpha(4GM/r_0c^2)$ and plotted the error bars on the parameter α. If GR is correct, we expect $\alpha = 1$.

You will recognise the integration constant h in (18a) as the specific angular momentum of the particle. Now substitute $u = 1/r$ and eliminate the time variable in equation (18b). The Newtonian equation of motion is

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu c^2}{h^2}. \tag{19}$$

Compare this with the General Relativistic equation of motion (17). It is *identical* except for the extra term $3\mu u^2$ on the right hand side. In fact we can write the GR equation as

$$\begin{aligned} \frac{d^2u}{d\phi^2} + u &= \frac{\mu c^2}{h^2} + 3\mu u^2 \\ &= \frac{\mu c^2}{h^2}\left[1 + \frac{3h^2}{r^2c^2}\right] \\ &= \frac{\mu c^2}{h^2}\left[1 + \frac{3r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2\right]. \end{aligned}$$

Now, if the gravitational field is *weak* $\tau \approx t$ and we can replace $d\phi/d\tau$ by $d\phi/dt$. But $rd\phi/dt$ is just the *transverse* velocity, v_t. For example, if we consider the orbit of the Earth around the Sun, the transverse velocity is $v_t = 30\text{km/s}$. The Newtonian and General Relativistic equations of motion of the Earth around the Sun therefore differ by a tiny amount

$$\frac{\mu c^2}{h^2}\left[1 + \frac{3r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2\right] = \frac{\mu c^2}{h^2}\left[1 + \frac{3v_t^2}{c^2}\right] \approx \frac{\mu c^2}{h^2}\left[1 + \mathcal{O}\left(3 \times 10^{-8}\right)\right].$$

It should, of course, be no surprise that the deviation from the Newtonian equation of motion is so small. Newtonian gravity is a very good approximation for the orbits of planets – after all, that is how the inverse square law of Newtonian gravity was first deduced! Evidently, General Relativity implies small deviations from Newtonian orbits in the weak gravitational regime. But what about close to a massive object when the gravitational field is strong, $\mu u \approx 1$? In this regime we can derive a suprising result very simply.

11.4.1 Circular Orbits

In Newtonian mechanics you are used to thinking about circular orbits as a limiting case for particles with high angular momentum, *i.e.* when the velocity becomes purely tangential. What about circular orbits in General Relativity? For a circular orbit,

$$u = \text{constant},$$

and the equations of motion become,

$$\begin{aligned} u &= \frac{\mu c^2}{h^2} + 3\mu u^2, \\ \frac{d\phi}{d\tau} &= hu^2. \end{aligned}$$

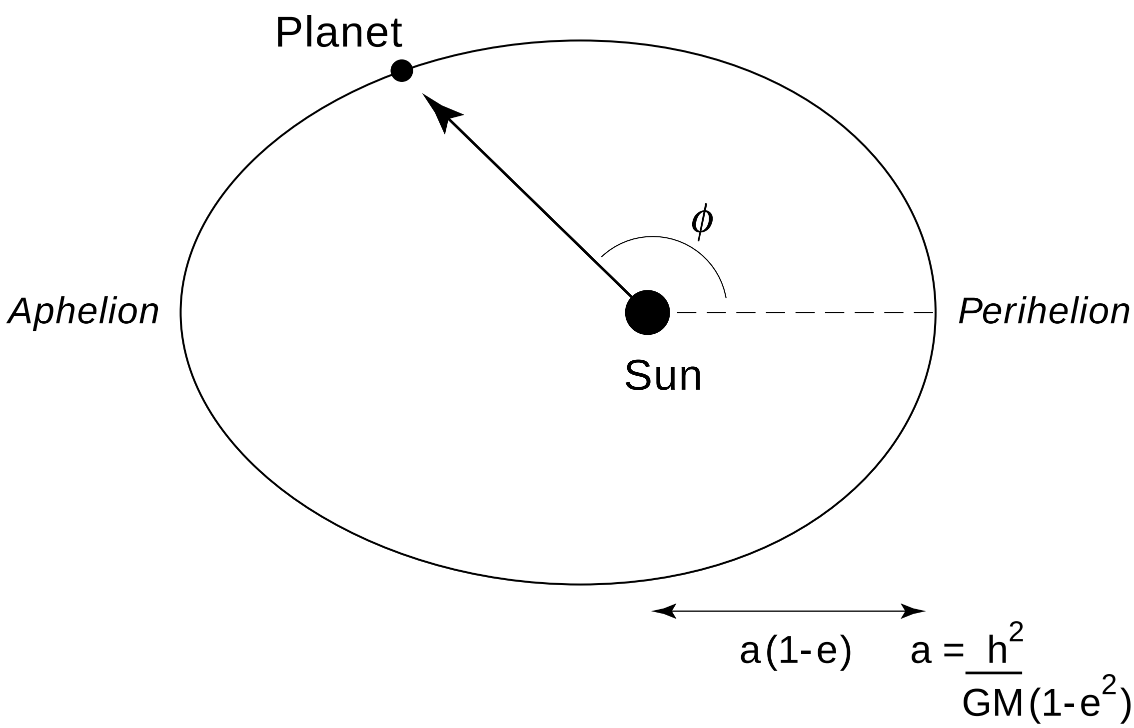

Figure 3: The elliptical orbit of a planet around the Sun.

From the first of these equations,

$$h^2\left(u - 3\mu u^2\right) = \mu c^2,$$

and so from the second,

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\mu u^4 c^2}{(u - 3\mu u^2)} = \frac{\mu c^2}{r^2\,(r - 3GM/c^2)}.$$

What is the significane of the last equation? Evidently the last equation *cannot be satisfied* for circular orbits with $r < 3GM/c^2$! Such orbits cannot be geodesics (because they do not satisfy the geodesic equations) and so cannot be followed by freely falling particles. According to General Relativity, there are no stable circular orbits close to a massive body, no matter how large the angular momentum of a particle. This is very different from Newtonian theory.

11.4.2 Non-circular orbits

For a general non-circular orbit, Newton's equation

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu c^2}{h^2},$$

has a solution

$$u = \frac{\mu c^2}{h^2}(1 + e\cos\phi). \tag{20}$$

The last equation describes an ellipse, so for example, we can draw the orbit of a planet around the Sun as in the Figure 3.

The parameter e measures the ellipticity of the orbit. We can write the distance of closest approach (*perihelion*) as $r_1 = a(1-e)$ and the distance of furthest approach (*aphelion*) as $r_2 = a(1+e)$. The equation of motion then requires that

$$a = \frac{h^2}{GM(1-e^2)}.$$

If the gravitational field is weak, as it is for planetary orbits, then we expect Newtonian gravity to provide an excellent approximation to the motion in General Relativity. We can therefore treat the Newtonian solution (6) as the zeroth order solution to the GR equations of motion. Let's write the GR solution as

$$u = \frac{\mu c^2}{h^2}(1 + e\cos\phi) + \Delta u,$$

where Δu is a perturbation. To first order in the perturbation Δu, the GR equation (17) gives

$$\frac{d^2\Delta u}{d\phi^2} + \Delta u = \frac{3\mu^3 c^4}{h^4}\left(1 + e^2\cos^2\phi + 2e\cos\phi\right),$$

where $\Delta u = \Delta u_1 + \Delta u_2 + \Delta u_3$ with:

$$\begin{aligned}
\Delta u_1 &= 3\mu^3 c^4/h^4, && \longleftarrow \text{tiny shift in orbit radius} \\
\Delta u_2 &= \tfrac{3}{2}\,\tfrac{\mu^3 c^4 e^2}{h^4} - \tfrac{1}{2}\,\tfrac{\mu^3 c^4 e^2}{h^4}\cos 2\phi, && \longleftarrow \text{tiny shift in orbit radius and oscillation} \\
\Delta u_3 &= \tfrac{3\mu^3 c^4}{h^4} e\phi\sin\phi, && \longleftarrow \text{cumulative change in shape of orbit}
\end{aligned}$$

The first two terms Δu_1 and Δu_2 are immeasurably small, and are of no use in testing the theory. The third term Δu_3 is also small, but it is *cumulative*. It adds up over a long time interval to produce a measurable change in the shape of the orbit. To see this, add Δu_3 to the Newtonian solution:

$$u = \frac{\mu c^2}{h^2}\left[1 + e\left(\cos\phi + \underbrace{\frac{3\mu^2 c^2\phi}{h^2}}_{\substack{\downarrow \\ \ll\ 1\ \text{approximate as} \\ \sin\alpha,\quad \alpha \simeq \frac{3\mu^2 c^2 \phi}{h^2}}}\sin\phi\right)\right],$$

and using the relation

$$\cos\left(\beta - \alpha\right) = \cos\beta\cos\alpha + \sin\beta\sin\alpha,$$

we can write the solution as

$$\begin{aligned} u &\approx \frac{\mu c^2}{h^2}\left[1 + e\cos(\phi - \alpha)\right] \\ &\approx \frac{\mu c^2}{h^2}\left[1 + e\cos\left(\phi\left(1 - 3\mu^2 c^2/h^2\right)\right)\right], \end{aligned}$$

so the ellipse "closes" when

$$\Delta\phi = \frac{2\pi}{(1 - 3\mu^2 c^2/h^2)} > 2\pi,$$

i.e. we have to rotate the angle ϕ by *more* than 2π for the ellipse to "close". The result is that the ellipse cannot maintain a fixed shape, in other words the ellipse *precesses*:

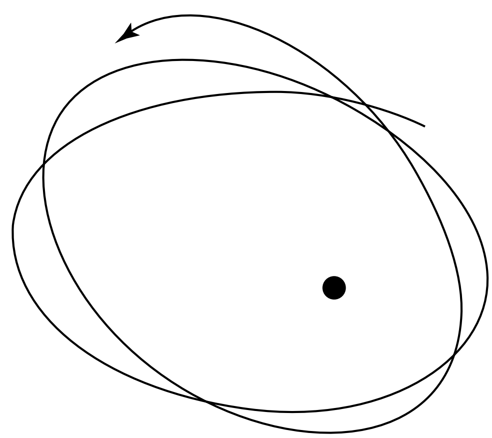

Figure 4: Precession of an elliptical orbit (greatly exaggerated).

In one revolution, the ellipse will rotate about the focus by an amount

$$\Delta = \frac{2\pi}{\left(1 - 3\frac{G^2M^2}{c^2h^2}\right)} - 2\pi \approx 6\pi\frac{G^2M^2}{c^2h^2} \approx \frac{6\pi GM}{a\left(1 - e^2\right)c^2}, \tag{21}$$

where I have replaced h by the parameters that define the shape of the Newtonian orbit (*c.f.* Figure 3):

$$\begin{array}{ll} \text{Perihelion} & r_1 = a(1 - e), \quad \phi = 0, \\ \text{Aphelion} & r_2 = a(1 + e), \quad \phi = \pi, \end{array}$$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2GM}{h^2} = \frac{2}{a\left(1 - e^2\right)}.$$

Now let's apply equation (21) to the orbit of Mercury, which has the followng parameters

$$\left.\begin{array}{lcl} \text{Period} & = & 88 \text{ days} \\ a & = & 5.8\times 10^{10}\text{ m} \\ e & = & 0.2 \\ M_\odot & = & 2\times 10^{30}\text{ kg} \end{array}\right\} \text{ we find } \Delta = 43'' \text{ per century.}$$

In fact, the measured precession is

$$5599.7 \pm 0.4'' \text{ per century.}$$

The residual, after taking the precession of the equinoxes and perturbations by other planets into account, is in remarkable agreement with General Relativity. Here are the residuals for a number of planets (Icarus is a minor planet with a perihelion that lies within the orbit of Mercury):

	Observed Residual (arcsec per century)	Predicted Residual
Mercury	43.1 ± 0.5	43.03
Venus	8 ± 5	8.6
Earth	5 ± 1	3.8
Icarus	10 ± 1	10.3

In each case, the results are in excellent agreement with the predictions of GR. Einstein included this calculation in his 1915 paper on General Relativity. He had solved one of the major problems of celestial mechanics in the very first application of his complicated theory to an empirically testable problem. As you can imagine, this gave him tremendous confidence in his new theory.

11.5 Radar Echos

We have shown previously that photon paths obey the equation

$$\frac{1}{(1-2\mu/r)^3}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{(1-2\mu/r)} = 0, \tag{22}$$

where ($\mu \equiv GM/c^2$). (Use equations (11a) and (11c)). Now consider a photon path from Earth to a planet (say Venus) as in the diagram below:

Evidently, the photon path will be deflected by the gravitational field of the Sun. (I am, of course, assuming that the planets are in a configuration like that shown in Figure 5 where the photon has to pass close to the Sun to get to Venus). Let r_0 be the distance of closest approach of the photon to the Sun, then

$$\left(\frac{dr}{dt}\right)_{r_0} = 0,$$

and so from (22), we can write the integration constant J as

$$J^2 = \frac{c^2 r_0^2}{(1 - 2\mu/r_0)}.$$

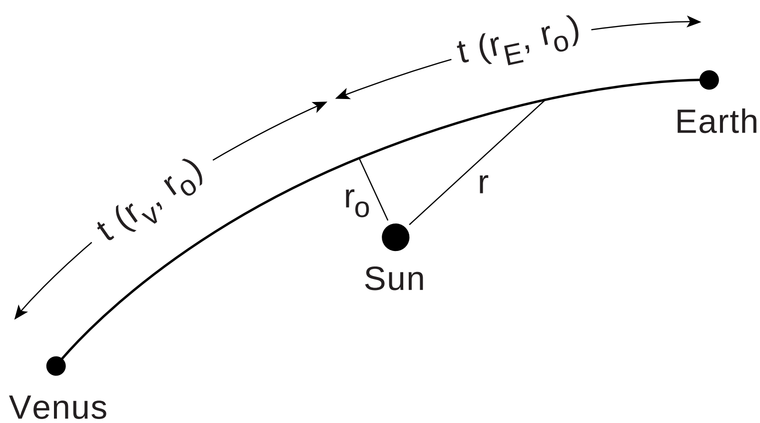

Figure 5: Photon path from Earth to Venus deflected by the Sun

We can therefore write the equation of motion (22) in the form

$$\frac{1}{(1 - 2\mu/r)^3}\left(\frac{dr}{dt}\right)^2 + \frac{r_0^2}{r^2}\frac{c^2}{(1 - 2\mu/r_0)} - \frac{c^2}{(1 - 2\mu/r)} = 0,$$

which we can integrate to give

$$ct(r, r_0) = \int_{r_0}^{r} \frac{dr}{(1 - 2\mu/r)}\left[1 - \frac{r_0^2}{r^2}\frac{(1 - 2\mu/r)}{(1 - 2\mu/r_0)}\right]^{-1/2}.$$

As in our derivation of the deflection of light by the Sun, expand this in powers of the small quantities μ/r and μ/r_0,

$$ct\,(r, r_0) = \int_{r_0}^{r} \left(1 - r_0^2/r^2\right)^{-1/2}\left[1 + \frac{2\mu}{r} + \frac{\mu r_0}{(r + r_0)}\frac{1}{r} + \ldots\ldots\right],$$

which integrates to give

$$ct(r, r_0) = \sqrt{r^2 - r_0^2} + 2\mu \ln \left[\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right] + \mu \left(\frac{r - r_0}{r + r_0} \right)^{1/2}.$$

The first term is just what we would have expected if light travelled in a straight line. The second and third term give us the extra time taken for the photon to travel along the *curved* path to the point r. So, you can see from Figure 5, that if we bounce a radar beam to Venus and back, then the excess time delay over a straight line path is

$$c\Delta t = 2 \left[ct(r_E, r_0) + ct(r_V,\ r_0) - \sqrt{r_E^2 - r_0^2} - \sqrt{r_V^2 - r_0^2} \right],$$

(where the factor of two is included because the photon has to get there and back). Since $r_E \gg r_0$, and $r_V \gg r_0$

$$ct(r_E, r_0) - \sqrt{r_E^2 - r_0^2} \approx 2\mu \ln(2r_E/r_0) + \mu,$$

(and likewise for t_V and r_V), and so the excess time delay is

$$c\Delta t \approx 4\mu \left[\ln \left(\frac{4 r_E r_V}{r_0^2} \right) + \frac{1}{2} \right].$$

For Venus, when it is opposite to the Earth from the Sun

$$\Delta t \approx 220\ \mu s.$$

The idea of the experiment is as follows. Fire an intense radar beam towards Venus when it is almost opposite to us from Earth and measure the time delay of the radar echo with a sensitive radio telescope. The excess time delay gives us a test of the principle of equivalence. This sounds straightforward, but the time delay is very small and depends on the values of $r_E,\ r_V,\ r_0$. How can one determine these parameters to the required precision? The answer, is fit the entire time interval $t(r_E, r_0) + t(r_V, r_0)$ over a long period of time with the orbital parameters r_E, r_V, μ etc., as free parameters. There are a number of technical problems that limit the accuracy of this method. In practice, the radar beam is reflected off different points on the surface of Venus (mountain peaks, valleys, *etc*) and this introduces a dispersion in the time delay of several hundred μs. This problem can be solved by bouncing the radar beams from a mirror – as has since been done using the Viking Landers on Mars. Another, more complicated problem is correcting for refraction by the Solar corona – this can be important for photon paths which graze the surface of the Sun.