Theory of Relativity
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14 White Holes, Wormholes and Rotating Black Holes[1]

14.1 Kruskal Coordinates

When the Schwarzschild metric and black holes were discussed in Section 12, Eddington–Finkelstein coordinates were introduced to make ingoing light rays straight in a space–time diagram. We write this here as,

$$ct' = \begin{cases} ct + 2\mu \ln(r/2\mu - 1) & r > 2\mu \\ ct + 2\mu \ln(1 - r/2\mu) & r < 2\mu \end{cases}$$

$$i.e. \qquad ct' = -r.$$

With these coordinates, the Schwarzschild metric took the form

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt'^2 - \frac{4\mu}{r} c dt' dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 d\Omega^2. \tag{1}$$

We can simplify this metric still further by introducing an *advanced time* parameter

$$c\nu = ct' + r.$$

In terms of the parameter ν, the metric looks like

$$ds^2 = (1 - 2\mu/r)\, c^2 d\nu^2 - 2c d\nu dr - r^2 d\Omega^2 \tag{2}$$

and so takes quite a simple form. Why did I choose to straighten out only the *ingoing* null rays? The answer is because I wanted to stress the “membrane-like” character of the even horizon for infalling particles. The resulting space-time diagram in Eddington-Finkelstein coordinates (Figure 5 of Section 12) shows the fundamental differences in the light cone structure inside and outside the Schwarzschild radius and, of course, the ingoing null rays are straight lines *by construction*. However, the diagram is still unattractive because the *outgoing* null rays are discontinuous.

I could have chosen a time coordinate t^* that straightened out the outgoing light rays,

$$ct^* = ct - 2\mu \ln|r/2\mu - 1|,$$

[1]This material extends beyond the syllabus and is not examinable

and defined a *retarded time* parameter

$$cw = ct^* - r,$$

for which the Schwarzschild metric looks like

$$ds^2 = (1 - 2\mu/r)\, c^2 dw^2 + 2cdwdr - r^2 d\Omega^2. \tag{3}$$

But in these coordinates, the *ingoing* null rays are discontinuous at $r = 2\mu$.

Neither coordinate system is satisfactory. In the advanced coordinates (2) the outgoing null rays are discontinuous, and in the retarded coordinates (3) the ingoing null rays are discontinuous. Let me pose a mathematical problem: can I choose coordinates that straighten both *ingoing* and *outgoing* null rays?. The answer is yes.

By definition

$$\frac{1}{2}c\,(\nu - w) = r + 2\mu \ln|r/2\mu - 1|,$$

and so I can remove the "*dr*" terms from the metrics (2) and (3) to get

$$ds^2 = (1 - 2\mu/r)\, c^2 d\nu dw - r^2 d\Omega^2 \tag{4}$$

The 2–space defined by $\theta = \text{const}, \phi = \text{const}$ i.e., $d\Omega = 0$, has the metric

$$ds^2 = (1 - 2\mu/r)\, c^2 d\nu dw \tag{5}$$

and is *conformally flat.* To see this, transform to

$$\begin{cases} ct &= 1/2c(\nu + w) \\ x &= 1/2c(\nu - w) \end{cases}$$

then equation (5) becomes,

$$\begin{aligned} ds^2 &= (1 - 2\mu/r) \quad \left(c^2 dt^2 - dx^2\right) \\ &= \Omega^2(x)\eta_{\alpha\beta}dx^\alpha dx^\beta. \end{aligned}$$

The metric is that of a Minkowski 2-space (which is spatially flat) but multiplied by what mathematicians call a *conformal* scaling factor $\Omega^2(x)$. The 2-space itself is curved, because the derivatives of the function $\Omega(x)$ enter into the calculation of the components of the curvature tensor. Although the space is curved, the metric of the 2-space is *conformally flat.*

We can find lots of different coordinate transformations that preserve the conformal nature of the 2-space defined by equation (5). Transform to the variables

$$\nu'(\nu) \quad , \quad w'(w), \tag{6}$$

then the metric (5) is

$$ds^2 = (1 - 2\mu/r)\, c^2 \frac{d\nu}{d\nu'} \, \frac{dw}{dw'} d\nu' dw'. \tag{7}$$

Now transform to

$$u = \frac{1}{2} c \,(\nu' + w') \quad , \quad v = \frac{1}{2} c \,(\nu' - w') \,,$$

then the metric (7) becomes

$$ds^2 = F^2(u, v)(du^2 - dv^2), \tag{8}$$

where the function $F(u, v)$ depends on the choice of the functions ν' and w' in (6). The metric (8) is conformally flat, but we have freedom to chose the conformal scaling factor F by chosing the functions ν' and w'. Kruskal chose the following functions:

$$\nu' = \exp\left(c\nu/4\mu\right), \qquad w' = -\exp\left(-cw/4\mu\right).$$

In these *Kruskal coordinates*, the Schwarzschild metric looks like

$$ds^2 = \frac{32\mu^3}{r} \exp\left(\frac{-r}{2\mu}\right) \, \left(du^2 - dv^2\right) - r^2 d\Omega^2. \tag{9}$$

Why are Kruskal coordinates attractive? They succeed in straightening out both *ingoing* and *outgoing* null rays which we were unable to do using either advanced or retarded Eddington-Finkelstein coordinates. You can see that this is true because the 2-space with $d\Omega = 0$ is conformally flat. In our usual representation of space-time diagrams, the light cone structure will look just like that in Minkowski space (the conformal factor is just a scaling factor and does not change the light cone structure). Ingoing and outgoing null rays are defined by straight lines at $\pm 45\,\mathrm{deg}$ ($c = 1$) in these coordinates.

Figure 1 shows the space-time diagram in Kruskal coordinates.

This is quite a complicated diagram but notice that a few simple coordinate transformations have produced a space-time diagram with some fairly self-evident, but also some curious features:

- The null rays defining light cones are at $\pm 45^\circ$ to the axes, just as in a space-time diagram of Minkowski space.
- There is a past singularity (a white hole). Light rays can emanate from this past singularity!
- The diagonal lines ($r = 2\mu$, $t = \infty$), ($r = 2\mu$, $t = -\infty$) define event horizons separating two regions of space II and II$'$ from two other regions I and I$'$ which tend assymptotically to Minkowski space as $r \to \infty$.

- There are two “Minkowski” regions (I and I′), so apparently there are two universes.
- The two universes I and I′ are actually connected by a wormhole at the origin.

What has happened here? How can a few simple coordinate transformations lead to what is apparently new physics? What we have done is to mathematically extend the Schwarzschild solution. Mathematicians would call this a *maximal extension* of the Schwarzschild solution because *all* geodesics either extend to infinity or end at a past or future singularity. The extended Schwarzschild metric is a solution of Einstein's theory and hence is allowed by classical GR. Classical GR thus allows the existence of ‘white holes’. Photons or particles could, in principle, emanate from a past singularity. But as you can see from the Kruskal space-time diagram, you cannot ‘fall into’ a white hole – a white hole can only exist in your past. Can a white hole really exist? The answer is that we don't know for sure. Classical GR *must break down* at singularities. We would expect quantum effects to become important at ultra-short distances and ultra-high energies. In fact, from the three fundamental constants G, $\hbar$ and c we can form the following energy, mass, time, length and density scales: [2]

$$\text{Planck energy} \qquad E_{Pl} = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.22 \times 10^{19}\ \text{GeV},$$

$$\text{Planck mass} \qquad m_{Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5}\ \text{g},$$

$$\text{Planck time} \qquad t_{Pl} = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44}\ \text{sec},$$

$$\text{Planck length} \qquad l_{Pl} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-33}\ \text{cm},$$

$$\text{Planck density} \qquad \rho_{Pl} = \left(\frac{c^5}{\hbar G^2}\right) = 5.16 \times 10^{93}\ \text{g cm}^{-3}.$$

These *Planck scales* define the characterstic energies, lengths, times *etc when we expect quantum gravitational effects to become important*. To put it into some kind of perspective, an elementary particle with the Planck mass would weigh about the same as a small bacterium.

Nobody really expects the centres of black holes to harbour true singularities. What we expect is that close to the classical singularity, quantum gravitational effects will occur that will prevent the divergences of classical GR. We do not yet have a complete theory of quantum gravity, though many people hope that M-theory (formerly known as superstring theory) may one day provide such a theory. Theorists have, however, developed semi-classical

[2]Try this. Use dimensional arguments to create quantities with the required dimensions from the constants G, $\hbar$, c.

type theories which might (or might not) contain some of the features of a complete theory of quantum gravity. Such calculations suggest that white holes would be unstable and could not exist for more than about a Planck time. It is interesting that within a few pages of deriving the field equations of Einstein's relativity, we have pushed the theory to the edge of known physics.

Let us now look at the 'wormhole' at the origin of Figure 1. It is not obvious from this figure that regions I and I′ are connected at this "point". To understand this, you must realize that the coordinates θ and ϕ have been suppressed in this figure. Each point in figure 1 actually represents a two-sphere. Look at the metric (7) at $t' = 0$, $\theta = \pi/2$. It takes the form

$$ds^2 = -F^2 v^2 - r^2 d\phi^2,$$

and if we sketch this metric you can now see the 'bridge' (sometimes called the *Einstein-Rosen bridge*) between the two universes (see Figure 2).

Figure 3 shows the time evolution of the bridge. Can wormholes exist in Nature? Can they connect different universe, or different parts of the same universe? Again, nobody knows for sure. Many theorists would argue that we need to understand quantum gravity to understand wormholes. Wormholes are probably unstable, but 'virtual' wormholes are a feature of some formulations of quantum gravity.

14.2 Rotating Black Holes

14.2.1 The Kerr metric

The Schwarzschild solution describes a spherically symmetric black hole, characterized only by its mass M. What happens if a black hole rotates? The Schwarzschild solution cannot apply in this case, because the rotation axis of the black hole defines a special direction, so destroying the isotropy of the solution.

In 1963 Kerr discovered a remarkable exact solution of the vacuum field equations that describes rotating black holes. It is interesting, firstly, because it is an *exact* solution and so can be used to study the strong gravitational regime and, secondly, because we will see that there are some important differences between rotating and non-rotating black holes leading to new physical effects.

I will not go through the mathematics of the Kerr metric in any detail. One can't really derive it from first principles, though one can motivate the particular functional form that Kerr investigated. You can, if you like, verify that the metric satisfies the vacuum field equations. In Schwarzschild–like coordinates (called Boyer–Lindquist coordinates) the Kerr metric looks like

$$ds^2 = \frac{\Delta}{\rho^2}\left(cdt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - acdt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \tag{13}$$

where Δ and ρ are

$$\left.\begin{array}{rcl} \Delta & = & r^2 - 2\mu r + a^2, \\ \rho^2 & = & r^2 + a^2\cos^2\theta. \end{array}\right\}$$

Comparing the Kerr metric to the Schwarzschild metric you can see that Kerr solution contains an additional parameter a and that in the limit $a \to 0$,

$$\left.\begin{array}{rcl} \Delta & \longrightarrow & r^2\left(1-\frac{2\mu}{r}\right), \\ \rho^2 & \longrightarrow & r^2, \end{array}\right\}$$

and the Kerr metric tends to the standard Schwarzschild form,

$$ds^2 \longrightarrow \left(1-2\mu/r\right)c^2dt^2 - \left(1-2\mu/r\right)^{-1}dr^2 - r^2 d\Omega^2.$$

The components of the Kerr metric are independent of the coordinates t and ϕ, so the solution is:

- stationary,
- axially symmetric.

The metric is also invariant under the *joint* transformation

$$t \to -t \;\;,\;\; \phi \to -\phi,$$

suggesting that the metric has something to do with spin. In fact, the light–cone structure (in particular, the frame–dragging effect described later) suggests that we make the following identifications:

$$\begin{array}{ccl} M(\text{with } \mu = GM/c^2) & \longrightarrow & \text{geometric mass of the black hole,} \\ \text{Mac} & \longrightarrow & \text{angular momentum of the black hole.} \end{array}$$

14.2.2 Structure of a Kerr black hole

We can show that according to the Kerr solution, the Riemann curvature scalar R diverges when

$$\rho^2 = r^2 + a^2\cos^2\theta = 0,$$

$$i.e., \;\text{ when }\; r = 0, \;\; \cos\theta = 0.$$

The coordinates $r,\ \theta,\ \phi$ are not the normal spherical polars, but are related to Cartesian–type coordinates by the transformations

$$\begin{aligned} x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi, \\ y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi, \\ z &= r\cos\theta, \end{aligned}$$

and so the singularity is at

$$x^2 + y^2 = a^2, \qquad z = 0,$$

i.e. the singularity lies on a *ring* in the equatorial plane.

The g_{00} component of the Kerr metric is

$$g_{00} = \frac{\Delta}{\rho^2} - \frac{a^2\sin^2\theta}{\rho^2} = \frac{r^2 - 2\mu r + a^2\cos^2\theta}{\rho^2}.$$

Surfaces of *infinite redshift* are given by settin $g_{00} = 0$, *i.e.*

$$r_{S\pm} = \mu \pm \left(\mu^2 - a^2\cos^2\theta\right)^{1/2}.$$

Evidently, there are two such surfaces, S^+ and S^-. The surface S^- lies within S^+, and S^+ defines an ellipsoid with a minor axis of 2μ and a major axis of $\mu + (\mu^2 - a^2)^{1/2}$ (see Figure 4).

To find the event horizons, locate the surface where $r =$ constant becomes null. This is given by setting $g^{11} = 0$ [3]

$$g^{11} = -\frac{\Delta}{\rho^2} = -\frac{(r^2 - 2\mu r + a^2)}{(r^2 + a^2\cos^2\theta)} = 0.$$

So there are *two* event horizons located at

$$r = r_\pm = \mu \pm \left(\mu^2 - a^2\right)^{1/2}.$$

The structure of the Kerr black hole is quite complicated and is shown below.

We can define 3 distinct regions in the Kerr solution bounded by the event horizons:

- Region [I] $\quad r_+ < r < \infty$
- Region [II] $\quad r_- < r < r_+$
- Region [III] $\quad 0 < r < r_-$.

Figure 5 shows the three regions plotted in a space-time diagram along the equator of the black hole using advanced Eddington -Finkelstein coordinates in which ingoing radial null rays are straight lines.

[3] We haven't proved this, but the proof is quite simple, see standard textbooks

As in the Schwarzschild solution, the event horizon at r^+ marks a surface of 'no return'. Once you have crossed the event horizon, your future is directed towards region III which contains the singularity – you can never return back to region I. Unlike the Schwarzschild solution, the singularity in the Kerr solution is *timelike* (the singularity in the Schwazchild solution is *spacelike*. In theory, this means that it is possible to *avoid* the singularity by moving along a timelike path, in other words, if we were in a spaceship (and ignoring the intense tidal forces which would make this experiment impractical) we could manouvre along a path to avoid the singularity. However, you should not take the internal structure of the Kerr solution too seriously. Region III also contains closed timelike curves, which are very bad news because they violate causality. Most theorists would hope that quantum gravity comes to the rescue and prevents causality violation. At present we do not really know what happens within region III.

The volume between the outer infinite redshift surface S^+ and the event horizon r^+ is called the *ergosphere* (from the Greek word *ergo* meaning work). Penrose showed that it is possible to extract the rotational energy of the black hole within the ergosphere. Imagine that a particle is fired into the ergosphere where it decays into two particles, one of which falls through the event horizon while the other escapes back out to infinity. Under certain circumstances, the outgoing particle can have more energy than the original particle. This extra energy comes from the rotational energy of the black hole. Models have been developed to explain powerful radio galaxies that utilise magnetohydrodynamic extraction of the rotational energy of a Kerr black hole.

The identification of the parameter a with the rotation of a black hole is perhaps clearest if we look at the light cone structure pole on. This is shown in Figure 6:

Timelike paths must of course lie within the light cone. As we approach the infinite redshift surface S^+, these timelike curves are dragged to the edge of the light cone. This effect is called frame dragging. The light cones tip within the ergosphere and as you can see, at the event horizon r^+, they tip so far that the future is directed towards region II.

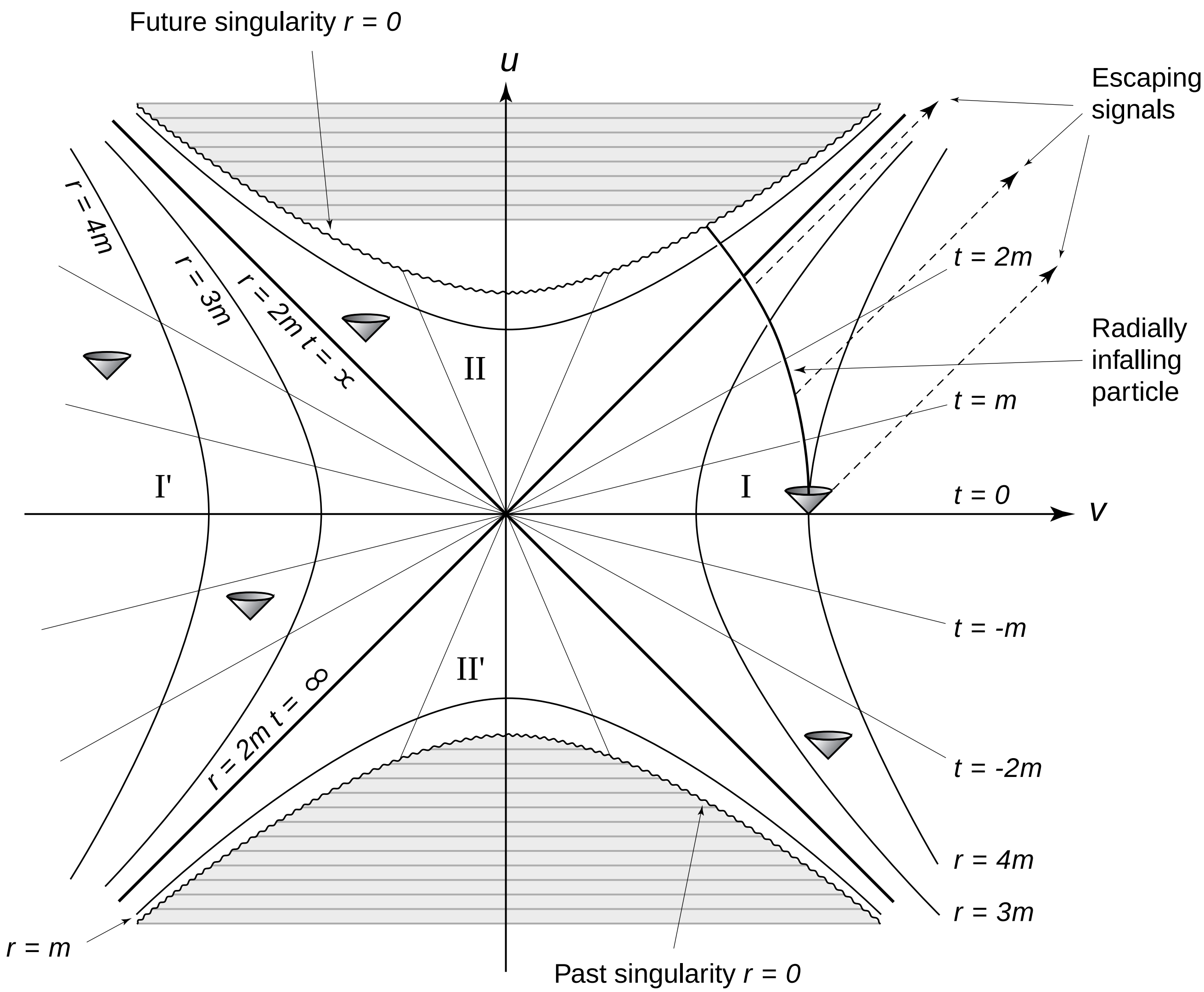

Figure 1: Space time diagram of the Schwarzschild solution in Kruskal coordinates.

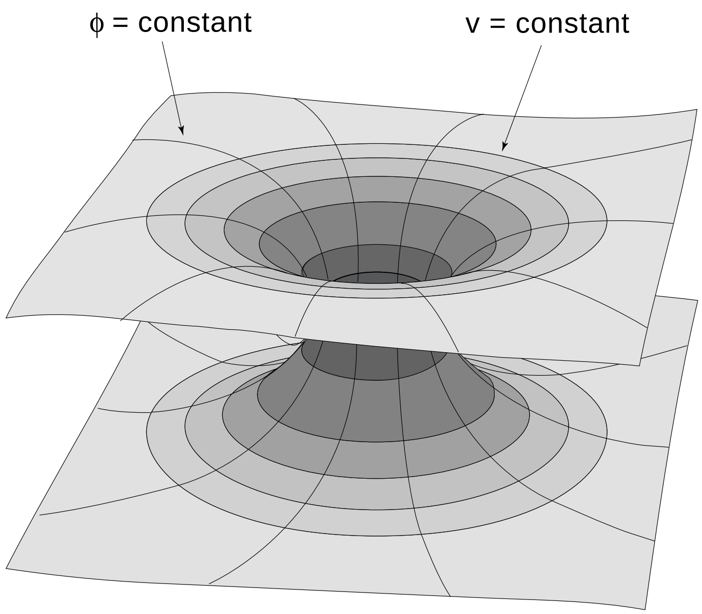

Figure 2: Structure of the wormhole of Figure 1.

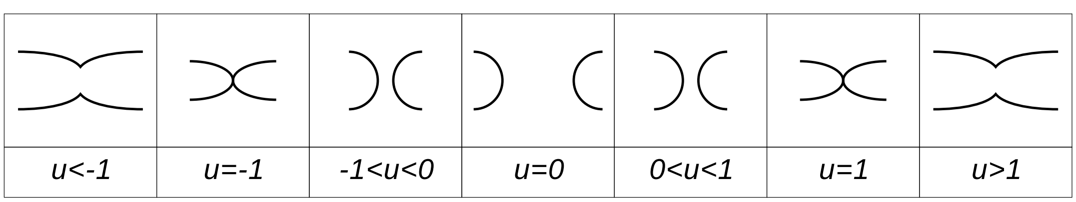

Figure 3: Time evolution of the Einstein-Rosen bridge.

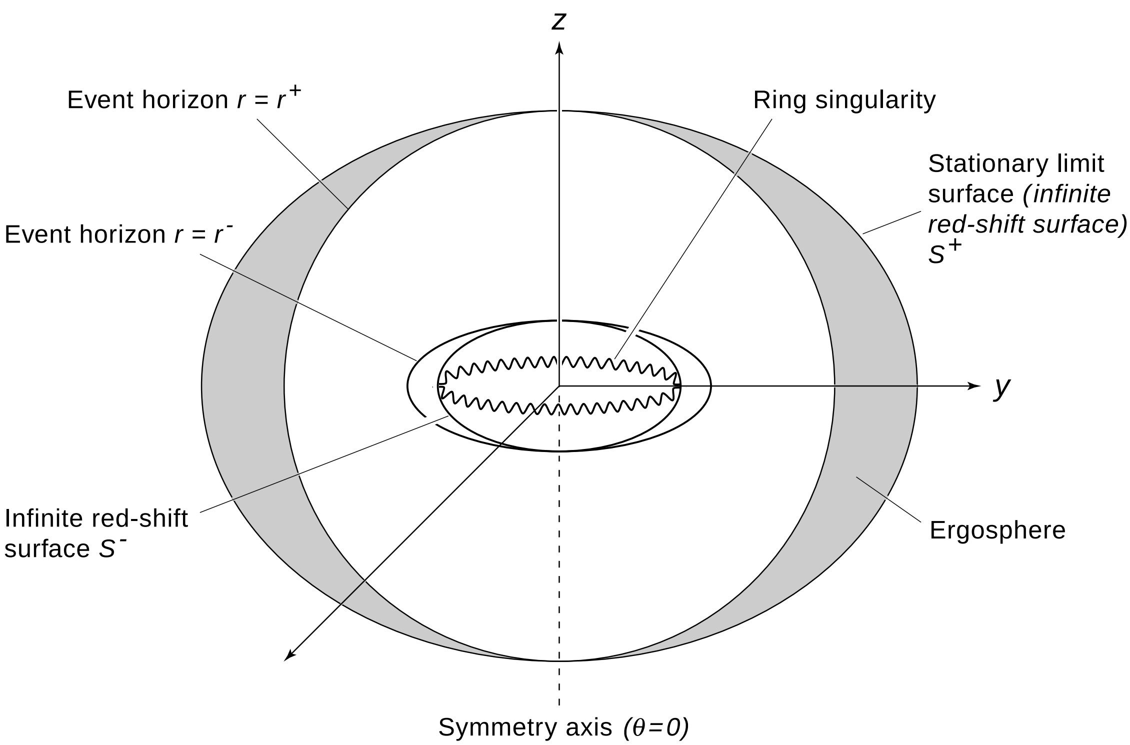

Figure 4: The structure of a Kerr black hole.

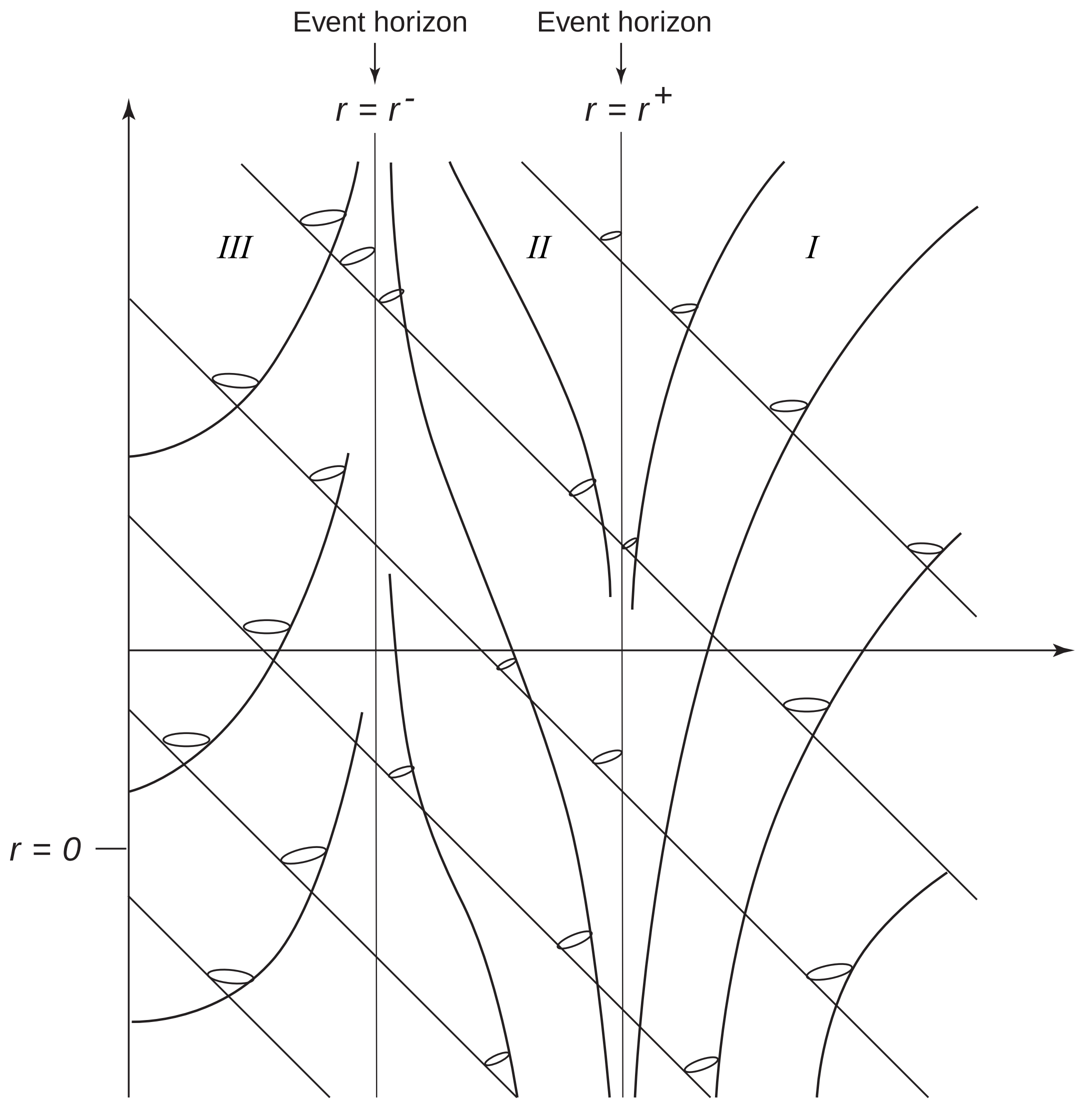

Figure 5: Space-time diagram of the Kerr solution in advanced Eddington-Finkelstein coordinates.

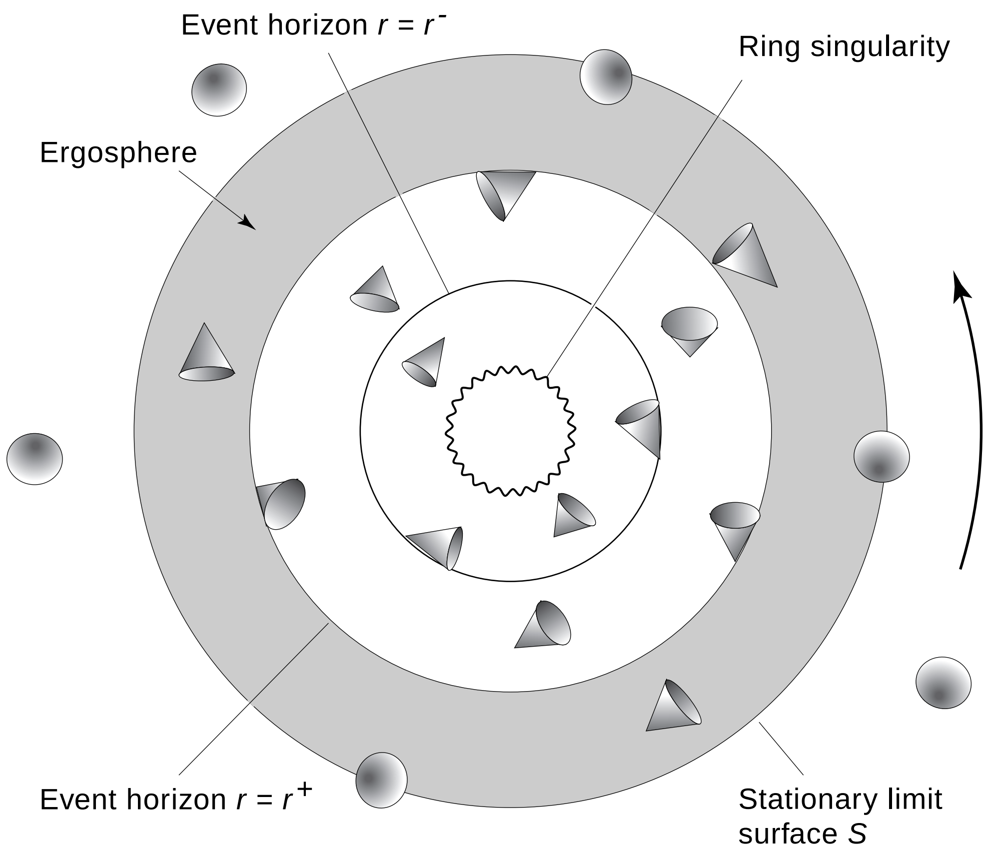

Figure 6: Frame dragging in the Kerr solution.