Theory of Relativity
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3 Relativistic Kinematics and Doppler effect

3.1 The four-velocity

The proper time interval between two events is defined by

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{(dx^2 + dy^2 + dz^2)}{c^2}.$$

For an ideal clock at rest

$$d\tau^2 = dt^2,$$

but for an ideal clock moving at speed u with respect to an observer,

$$\begin{aligned} d\tau^2 &= dt^2 - u^2/c^2 dt^2, \\ &= dt^2 \left(1 - u^2/c^2\right), \end{aligned}$$

so

$$dt = d\tau/(1 - u^2/c^2)^{1/2}, \qquad \text{(time dilation).}$$

The proper time interval $d\tau$ is invariant with respect to Lorentz transformations and so can be used to form derivatives of tensors that also transform like tensors. Rank 1 Tensors in Minkowski space are called **four-vectors**. Consider a particle with world line $x^\mu(\tau)$, then

$$\frac{dx^\mu}{d\tau} \qquad \text{is a four} - \text{vector :} \qquad \text{the four} - \text{velocity } \mathbf{U},$$

$$\frac{d^2x^\mu}{d\tau^2} \qquad \text{is also a four} - \text{vector :} \quad \text{the four} - \text{acceleration } \mathbf{A}.$$

In contrast, the derivative dx^μ/dt does not transform as a four-vector. In fact, the four-velocity of the particle is related to the derivative dx^μ/dt by

$$\mathbf{U} = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma c,\ \gamma \mathbf{u}),$$

where the three-velocity $\mathbf{u}$ is

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}.$$

3.2 Relativistic velocity addition

Recall the Lorentz transformation of the coordinates between two inertial frames,

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The four-velocity $\mathbf{U}$ is a four-vector, and so according to the transformation rules for tensors will transform just like the components of the coordinates.

The transformation formulae for the four-velocity will tell us how velocities transform between inertial frames *i.e.* they will give general formulae for *velocity addition* in Special Relativity. Inserting the components of the four-velocity $(\gamma c, \gamma \mathbf{u})$ into the Lorentz transformation equations for the coordinates gives

$$\begin{aligned} \gamma(u') &= \gamma(v)\left[\gamma(u) - v\gamma(u)u_1/c^2\right], \\ \gamma(u')u_1' &= \gamma(v)\left[\gamma(u)u_1 - v\gamma(u)\right], \\ \gamma(u')u_2' &= \gamma(u)u_2, \\ \gamma(u')u_3' &= \gamma(u)u_3. \end{aligned}$$

Pay careful attention to the velocities u, u' and v appearing in these formulae: v is the relative speed of the two frames S and S' directed along the $x - axis$, u is the magnitude of the three-velocity in frame S and u' is the magnitude of the three-velocity in frame S'. From the first of these equations

$$\frac{\gamma(u)}{\gamma(u')} = \frac{1}{\gamma(v)} \frac{1}{(1 - u_1 v/c^2)}.$$

And so the velocity addition law in Special Relativity is:

$$\begin{aligned} u_1' &= \frac{(u_1 - v)}{(1 - u_1 v/c^2)}, \\ u_2' &= \frac{u_2}{\gamma(v)\,(1 - u_1 v/c^2)}, \\ u_3' &= \frac{u_3}{\gamma(v)\,(1 - u_1 v/c^2)}. \end{aligned}$$

Note that the u_2' and u_3' components are affected by the transformation even though the relative motion of the two frames is in the x (component 1) direction. This is a result of time dilation.

Notice that in this example, the tensorial transformation rules allowed us to derive the velocity addition law in an almost trivial way by inspection of the Lorentz transformation formulae for the coordinates.

The Fizeau experiment:

In 1851 Fizeau measured the velocity of light in a flowing liquid and found it to be related to the refractive index of the liquid. If the velocity of light in the liquid at rest is u' and the liquid moves with velocity v the velocity of light relative to the outside was found to be

$$u = u' + kv, \qquad\qquad k = 1 - 1/n^2,$$

where $n = c/u'$ is the refractive index of the fluid. This result was at odds with the expectation for the luciferous aether believed to exist at the time. Within SR it can be simply explained with the velocity addition law,

$$u = \frac{u' + v}{1 + u'v/c^2} \approx (u' + v)(1 - \frac{u'v}{c^2}) \approx u' + v(1 - \frac{u'^2}{c^2}) = u' + kv,$$

where terms of order v^2/c^2 have been neglected.

3.3 Conservation of Four-momentum

Let us suppose that the particle has a certain property, its rest-mass m_0, which we use to define a **four-momentum**, $\mathbf{P}$:

$$\begin{aligned} \mathbf{P} &= m_0\mathbf{U} = m_0(\gamma c,\ \gamma \mathbf{u}), \\ &= (mc,\ \mathbf{p}). \end{aligned}$$

The components of the four-momentum have the following interpretation:

$$\begin{aligned} m &= \gamma(u)m_0 && \leftarrow \text{relativistic mass}, \\ \mathbf{p} &= m\mathbf{u} && \leftarrow \text{relativistic three} - \text{momentum}. \end{aligned}$$

In analogy with conservation of momentum in Newtonian mechanics, we can introduce a new principle into Special Relativity, namely that *four-momentum* is conserved in interactions.

Suppose that some particles go into a little box, where they experience various interactions. Particles (not necessarily the same ones) come out of the box leaving nothing behind. Conservation of four-momentum requires

$$\sum_i \mathbf{P}_i^{\text{in}} = \sum_i \mathbf{P}_i^{\text{out}},$$

where the sums extend over all particles. *Each component of the four-momentum must satisfy this equation separately.* Thus the relativistic masses before and after must balance, as must the relativistic three-momenta:

$$\sum_i m_i^{\rm in} = \sum_i m_i^{\rm out}, \tag{1}$$

$$\sum_i \mathbf{p}_i^{\rm in} = \sum_i \mathbf{p}_i^{\rm out}. \tag{2}$$

Now let us look at the first of these equations. It expresses the conservation of the quantity

$$m = m_0(1 - u^2/c^2)^{-1/2}.$$

For speeds much smaller than the speed of light, $u \ll c$, we can expand the term in brackets

$$m \approx m_0 + \tfrac{1}{2} m_0 u^2/c^2 + \ldots\ldots$$

classical kinetic energy

Evidently, our conservation law is equivalent to the conservation of *energy* in the non-relativistic limit. This suggests that we identify mc^2 with the *energy* of a particle leading to the most famous equation in physics:

$$\boxed{E = mc^2}$$

Furthermore, this identification tells us that a particle has a *rest energy*

$$E_0 = m_0 c^2. \tag{3}$$

Equation (1) expresses the conservation of energy in special relativity. The conclusion that particles have a rest energy (equation 3) is profound and follows from equation (1) since rest energy can be converted into kinetic energy and vice-versa (depending on what happens in the interaction). This conclusion follows from the requirement that the laws of nature have a tensorial form. (We generalized the principle of conservation of momentum to apply to the four-momentum $\mathbf{P}$).

The identification $E = mc^2$ allows us to write the four-momentum as

$$\mathbf{P} = (E/c,\ \mathbf{p})$$

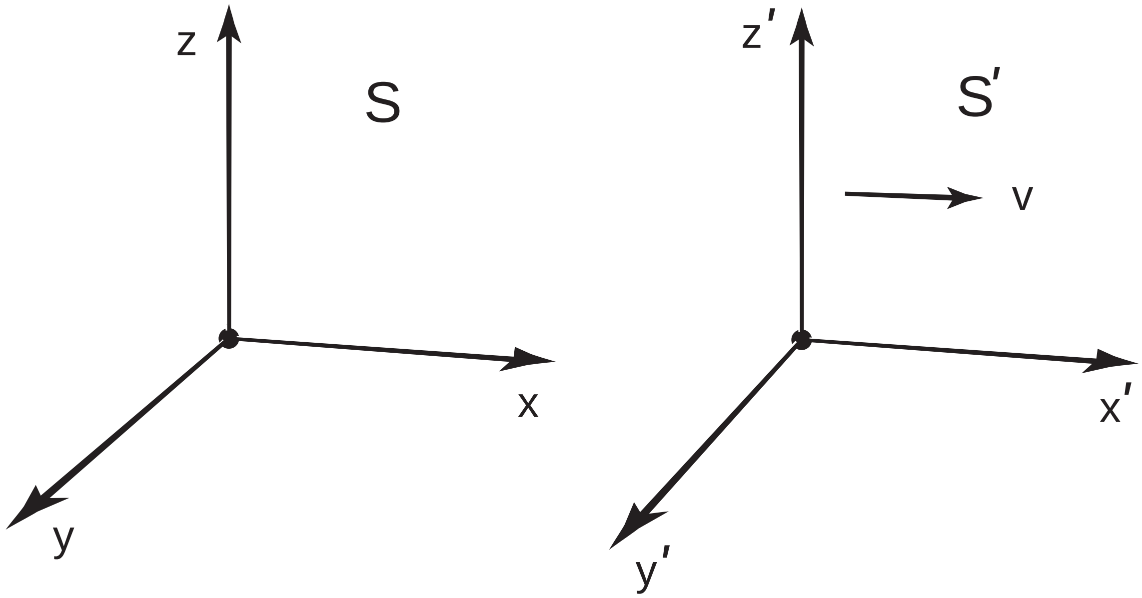

Figure 1:

and note that $\mathbf{P}^2$ is invariant

$$P^2 = E^2/c^2 - p^2 = m^2c^2 - p^2,$$

or equivalently

$$E^2 = p^2c^2 + m_0^2c^4. \tag{4}$$

(Think about how to get equation (4) from the previous equation without having to substitute for the relativistic mass and three-momentum.)

3.4 Photons and Planck's Hypothesis

Equation (4) applies to particles with a rest mass m_0. But what if $m_0 = 0$? Equation (4) tells us that the energy of the particle is

$$E = pc.$$

Suppose, as before, that we have two inertial frames S and S', with S' moving at speed v relative to S as in Figure 1.

Let the energy of a zero mass particle travelling in the negative x'-direction in S' be E_0, then, transforming the components of the four-momentum to S (remembering that they transform just like the coordinates)

$$E = \gamma(E_0 + vp'_x)$$

But, according to equation (4), for a zero mass particle $p'_x = -E_0/c$ (the minus sign comes about because the particle is travelling in the negative x direction), so

$$E = \gamma E_0(1 - v/c) = E_0 \frac{(1 - v/c)^{1/2}}{(1 + v/c)^{1/2}}. \tag{5}$$

Now, if we identify the zero mass particle with light of frequency ν_0 in S', then we know that its frequency in S will differ because of the relativistic radial Doppler effect,

$$\frac{\nu_0}{\nu} = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}. \tag{6}$$

Comparing equations (5) and (6) we find the startling result that

$$\frac{E}{E_0} = \frac{\nu}{\nu_0}.$$

In other words, the *energy* of the particle is proportional to its *frequency*,

$$E = h\nu,$$

with a constant of proportionality h that we now know as Planck's constant.

As I am sure many of you know, Einstein published a quantum theory of the photo-electric effect in 1905, the same year in which he published his fundamental paper on Special Relativity. In explaining the photo-electric effect, Einstein introduced the notion of the photon, a particle with zero mass and energy given by Planck's formula $E = h\nu$. In fact, Einstein was awarded the Nobel prize for his explanation of the photo-electric effect, not for developing the theory of relativity.

In various books, Einstein's introduction of the photon and Special Relativity are portrayed as completely disconnected theories. Yet, the arguments given in this section show that Special Relativity *requires* Planck's hypothesis. Einstein *must* have known about this, and that will have given him added confidence in his concept of the photon.

3.5 Two examples of kinematic calculations

Students often have trouble doing kinematic calculations in Special Relativity. In fact most of these calculations are not as difficult as they may appear. The key principle is to choose inertial frames in which the problem is simplified and in which invariant quantities take obvious values. This principle saves lots of time and algebra. The following two examples will illustrate this.

3.5.1 Example I: Threshold Energies

$$\begin{array}{ccc} & \mathbf{P}_1 & \mathbf{P}_2 \\ \bullet & - - - - - - - - \longrightarrow & \bullet \\ & & \uparrow \end{array}$$

target particle at rest

Figure 2

Consider the collision of two particles to produce a third. Particle 1 (with four-momentum $\mathbf{P}_1$) is moving towards particle 2 (four-momentum $\mathbf{P}_2$) which is at rest as shown in the Figure above. Let the post-collision momentum be $\mathbf{P}$. Conservation of four-momentum requires that

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}.$$

Now square this equation,

$$\mathbf{P}_1^2 + \mathbf{P}_2^2 + 2\mathbf{P}_1 \cdot \mathbf{P}_2 = \mathbf{P}^2.$$

Now we can write down what $\mathbf{P}_1^2$ is,

$$\mathbf{P}_1^2 = m^2c^2 - p_1^2 = m_{0_1}^2 c^2,$$

where m_{0_1} is the rest mass of particle 1. You see, it doesn't matter that the particle is moving, $\mathbf{P}_1^2$ is an invariant and so we can evaluate it in the particles rest frame. Likewise,

$$\mathbf{P}_2^2 = m_{0_2}^2 c^2.$$

The dot product $\mathbf{P}_1 \cdot \mathbf{P}_2$ can be written

$$\begin{aligned} \mathbf{P}_1 \cdot \mathbf{P}_2 &= c^2 m_{0_1} m_2 = c^2 m_1 m_{0_2} \\ &= c^2 \gamma(v) m_{0_1} m_{0_2} \end{aligned}$$

where v is the relative speed between the two particles. So, we can write the post-collision momentum as

$$\begin{aligned} \mathbf{P}^2 &= \mathbf{P}_1^2 + \mathbf{P}_2^2 + 2\mathbf{P}_1 \cdot \mathbf{P}_2 \\ &= m_{0_1}^2 + m_{0_2}^2 + 2m_{0_1} m_{0_2} \gamma(v). \end{aligned}$$

Let us now pose a specific problem. Suppose, for example, that we have two protons that interact to produce a pion:

$$p + p \rightarrow p + p + \pi^0$$

what is the threshold energy for the production of a pion? In this example, the relevant rest masses are

$$\begin{aligned} m_{0_1} &= m_p &= M, \\ m_{0_2} &= m_p &= M, \\ m_{0_3} &= m_\pi &= m. \end{aligned}$$

As derived above, conservation of four-momentum requires

$$M^2 + M^2 + 2M^2\gamma(v) = \bar{m}_{\rm cm}^2,$$

where $\bar{m}_{\rm cm}^2$ is the sum of the post-collision relativistic masses in the post-collision center of momentum frame (*i.e.* the frame in which the post-collision four-momentum is $\mathbf{P} = (\bar{m}_{\rm cm}c, 0)$. Now at the threshold for the production of a pion

$$\bar{m}_{\rm cm} = (M + M + m),$$

(the three post-collision particles are at rest with respect to each other). Hence we can calculate the threshold value of γ for the production of a pion,

$$M^2 + M^2 + 2M^2\gamma(v) = 4M^2 + 4Mm + m^2,$$

i.e.

$$\gamma(v) \geq 1 + \frac{2m}{M} + \frac{m^2}{2M^2}.$$

If we use a particle accelerator to accelerate the proton towards a stationary target, we therefore need to supply an energy of

$$\begin{aligned} E_{\rm Accelerator} &= \gamma Mc^2 - Mc^2 \\ &= (\gamma - 1)Mc^2 \end{aligned}$$

with γ set to the threshold value if we are to create a particle of mass m. If we want to create a very massive particle $m \gg M$, we need to supply an energy

$$E_{\rm Accelerator} \geq \left(\frac{m}{2M}\right) mc^2,$$

i.e. much greater than the rest mass energy of the particle that we want to create. This is why particle physicists have developed colliders which accelerate oppositely charged particles (*e.g.* e^+ and e^-) towards each other.

3.5.2 Example II: Elastic collisions

In our second example, we will consider an elastic collision between two particles of equal mass, one of which is at rest (see Figure 3).

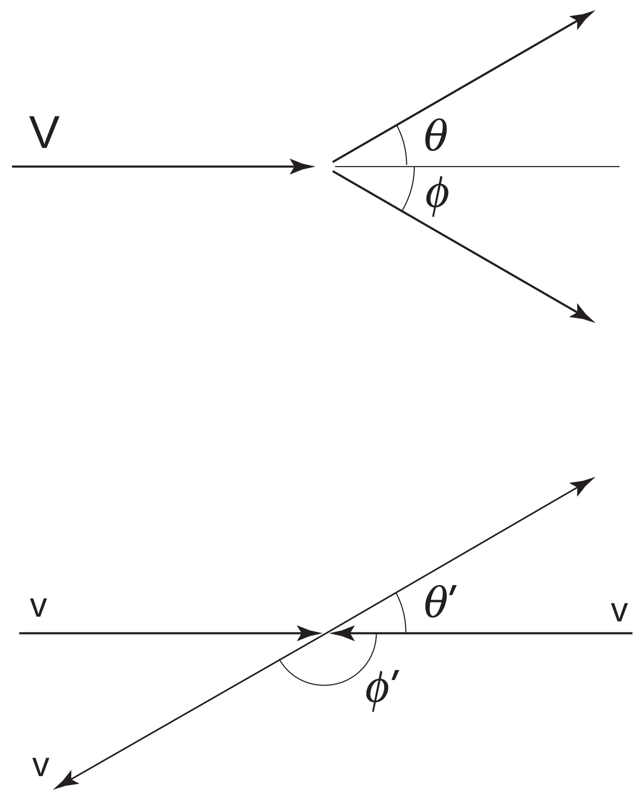

Figure 3: The upper figure shows how the collision appears in the laboratory frame. The lower figure shows the collision in the centre-of-momentum frame.

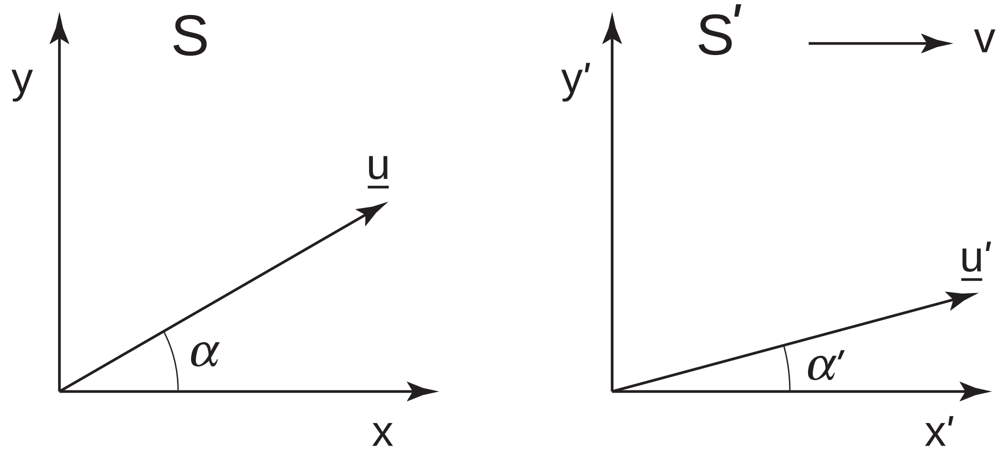

Figure 4:

Our goal is to relate the two angles θ and ϕ of the scattered particles in the laboratory frame to the speed V of the incoming particle.

Now the problem looks much more symmetrical in the centre-of-momentum (*com*) frame. In the *com* frame, the particles (each moving at speed v) approach each other and scatter at some angle θ'. Their post collision speed is v, because the collision is assumed to be elastic. To relate the situation in the laboratory frame to the simpler situation in the *com* frame, we must apply a Lorentz transformation between the two frames.

Let us first look at the following simple problem. A particle is moving with a velocity $\mathbf{u}$ in the $x-y$ plane in the inertial frame S. What is its velocity in frame S' moving at speed v relative to S in the x-direction (see Figure 4). We have already solved this problem. The components of $\mathbf{u}'$ in S' are given by the velocity addition formulae:

$$\begin{aligned} u'_x &= \frac{u_x - v}{(1 - u_x v/c^2)}, \\ u'_y &= \frac{u_y}{\gamma(v)\,(1 - u_x v/c^2)}. \end{aligned}$$

The angle α' of the velocity $\mathbf{u}'$ with respect to the x' axis in frame S', is therefore given by

$$\tan\alpha' = \frac{u'_y}{u'_x} = \frac{1}{\gamma(v)} \quad \frac{\sin\alpha}{(\cos\alpha - v/u)\,.} \tag{1}$$

This is sometimes called the *particle aberration* formula.

How is this calculation related to our problem of elastic collisions? Identify S' with the laboratory frame and S with the *com* frame. We now want the inverse of the particle aberation formula to relate the angles θ and θ' (replace v by $-v$ in equation (1)),

$$\tan\theta = \frac{\sin\theta'}{\gamma(v)\left[\cos\theta' + 1\right]}$$

Likewise for the angle ϕ',

$$\tan\phi = \frac{\sin\phi'}{\gamma(v)\left[\cos\phi' + 1\right]}.$$

But since in the *com* frame $\phi' = \pi - \theta'$ (see Figure 3), the second of these equations becomes

$$\tan\phi = \frac{\sin\theta'}{\gamma(v)\left[-\cos\theta' + 1\right]}.$$

Combining the equations for $\tan\theta$ and $\tan\phi$ gives,

$$\begin{aligned}
\tan\theta\tan\phi &= \frac{\sin^2\theta'}{\gamma^2(v)\left[1 - \cos^2\theta'\right]} \qquad = \frac{1}{\gamma^2(v)} \\
&= \frac{2}{\gamma(V) + 1}.
\end{aligned}$$

The last line comes from the γ relations in the velocity addition law. Thus, we have solved our problem and derived a relation between the scattering angles θ and ϕ that *must* apply if the collision is elastic (and so can be used to check whether the collision is indeed elastic).

3.6 Relativistic Doppler effect and aberration of light

3.6.1 Equation of motion and four momentum of massless particles

For photons and other particles moving at the speed of light $d\tau = 0$. The proper time can thus not be used to parameterise their path. Assume a photon is moving into the positive x-direction of an inertial frame, $x = ct$. This could be written parametrically with parameter σ using four vectors as

$$x^\mu = U^\mu \sigma,$$

where $U^\mu = (c, c, 0, 0)$ and $U^\mu U_\mu = g_{\mu\nu} U^\mu U^\nu = 0$. With this choice of parameter the equation of motion of a photon can be written as

$$\frac{d\mathbf{U}}{d\sigma} = 0.$$

In analogy to massive particles let us now define a four-momentum which is proportional to the four-velocity. With a suitable chosen constant α,

$$\mathbf{P} = \alpha U,$$

such that in an arbitrary inertial frame S

$$\mathbf{P} = (E/c, \mathbf{p}),$$

where E is the energy of the photon as measured in S and $\mathbf{p}$ is its three-momentum.

3.6.2 Relativistic Doppler effect and relativistic aberration of light

Suppose that an observer is at rest in in some Cartesian initial frame S. Suppose further that a source of radiation is moving along the positive x-direction with velocity v and that at some event Q the observer receives a photon of frequency ν in a direction that makes an angle θ with the positive x-direction. The four momentum can then be written

$$\mathbf{P} = \frac{h\nu}{c}(1, \cos\theta, \sin\theta, 0).$$

The photon observed at event Q must have been emitted at some other event R. The four-momentum is conserved and is thus the same at Q and R. Let us denote the Cartesian inertial frame in which the radiation source is at rest by S'. The components of the photon's four momentum in the two reference frame are related by a Lorentz transformation as,

$$P'^{\mu} = \Lambda^{\mu}_{\nu} P^{\nu}.$$

The zeroth component gives the formula for the relativistic Doppler effect,

$$\frac{\nu'}{\nu} = \frac{1}{\gamma(1 - \beta\cos\theta)},$$

where $\beta = v/c$. This formulae contains all the familiar Doppler effect results as special cases. If $\theta = 0$ the source must be approaching the observer along the negative x-direction. If $\theta = \pm\pi/2$ we obtain the transverse Doppler effect. Similarly from the x- and y-components of the Lorentz transformation of the four-momentum one obtains the formula for the relativistic aberration of light,

$$\tan\theta' = \frac{\tan\theta}{\gamma[1 - (v/c)\sec\theta]}.$$

The aberration formula can be used to describe the apparent changes of the appearance of spatially extended relativistically moving sources of light.

3.7 The three-force and four-force

In non-relativistic mechanics, Newton's second law is

$$\mathbf{f} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}.$$

This suggests that in Special Relativity we write

$$\mathbf{F} = m_0 \frac{d\mathbf{U}}{d\tau} = \frac{d\mathbf{P}}{d\tau}$$

defining a *four-force* $\mathbf{F}$. Now since the four-momentum is

$$\mathbf{P} = m_0(\gamma c, \gamma \mathbf{u})$$

we can write the equation of motion of a particle as

$$\begin{aligned}
\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = \gamma \frac{d\mathbf{P}}{dt} \quad &= \quad \gamma \left(c\frac{dm}{dt},\ \frac{d\mathbf{p}}{dt} \right) \\
&= \quad \gamma \left(\frac{1}{c}\frac{dE}{dt},\ \frac{d\mathbf{p}}{dt} \right) \\
&= \quad \gamma \left(\frac{1}{c}\frac{dE}{dt},\ \mathbf{f} \right).
\end{aligned}$$

where $\mathbf{f}$ is the relativistic three-force.

Now, from our definitions of the four-force and four-velocity:

$$\begin{aligned}
&\text{Four} - \text{force} \qquad \mathbf{F} \quad = \quad \gamma \left(c\tfrac{dm}{dt},\ \mathbf{f} \right) \\
&\text{Four} - \text{velocity} \quad \mathbf{U} \quad = \quad \gamma \left(c, \mathbf{u} \right)
\end{aligned}$$

we can form the dot product,

$$\begin{aligned}
\mathbf{U} \cdot \mathbf{F} \quad &= \quad g_{\mu\nu} U^\mu F^\nu \\
&= \quad \gamma^2 c^2 \frac{dm}{dt} - \gamma^2 \mathbf{f} \cdot \mathbf{u}.
\end{aligned}$$

$\mathbf{U} \cdot \mathbf{F}$ is an invariant under any coordinate transformation and so we can evaluate it in the instantaneous rest frame of the particle (in which $m = m_0$ and $\mathbf{u} = 0$). Thus,

$$\mathbf{U} \cdot \mathbf{F} = c^2 \frac{dm_o}{d\tau}.$$

We see that in Special Relativity, the action of a force can alter the rest mass of a particle.

A force that preserves the rest mass of the particle must have

$$\mathbf{U} \cdot \mathbf{F} = 0.$$

Such forces are called *pure forces*.

So far, our discussion of Special Relativity has been entirely kinematical. To discuss forces we need dynamics, and this requires new physics. At the time that Special Relativity was invented only two forces were known, electromagnetism and gravity. Electromagnetism, as I mentioned in the Introduction, actually led to the development of Special Relativity. I will therefore start to discuss electromagnetism in some detail in the next lecture. That will complete our discussion of Special Relativity and we will then turn our attention to gravity. We will see that understanding gravity requires a generalization of the ideas of relativity theory, culminating in the full dynamical theory of General Relativity.