Theory of Relativity
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1 Introduction to Special Relativity

1.1 What is Relativity?

Our aim in developing relativity is to develop a mathematical way of expressing physical laws. We need, therefore, to find physical principles to help us formulate physical laws mathematically, and we also need the mathematical machinery itself. Our requirements in developing relativity are:

- State physical laws in a way which is invariant under certain coordinate transformations.

- Develop a coordinate free language to express physical laws.

The above two points are central to the development of Special and General Relativity:

Special Relativity:

- Deals with coordinate transformations between inertial frames.

- Coordinate transformations are kinematical.

General Relativity

- Deals with coordinate transformations between non-inertial frames.

- Coordinate transformations are dynamical.

- Need equations to describe the dynamics, so need a dynamical theory.

Evidently General Relativity ('GR'), as its name implies, is more general than Special Relativity ('SR'). The mathematics required for GR are quite complicated and the steps towards the full dynamical theory are non-trivial.

Although you will have met SR before, students often have more difficulty with SR problems sets than with the superficially more sophisticated (but often rather mechanical) GR problem sets. Kinematic calculations in SR seem to be a particular difficulty. Do not underestimate the conceptual difficulties associated with SR. If you do not understand SR, then you do not really understand GR, even if you can solve GR problems.

Relativity Principle: The laws of physics are the same in all inertial frames

This is one of two postulates of SR. The *Relativity Principle* looks deceptively simple, but has far reaching consequences. As we will see soon it requires to abandon the seemingly intuitive ideas of absolute space and universal time.

I have already used a number of terms here without actually properly defining them: *(absolute) space, (universal) time, reference/inertial frame, coordinates.* As we will see during this course doing this is actually not easy.

1.2 Reference frames and coordinates

Consider two "rigid" orthogonal Cartesian coordinate systems with axis (x, y, z) and (x', y', z') and assume that a "universal" time t exists for both coordinate systems. The coordinate system S' is moving with constant velocity v_0 in the positive x-direction of coordinate system S. The path of an object moving with constant velocity u in the positive x-direction in S

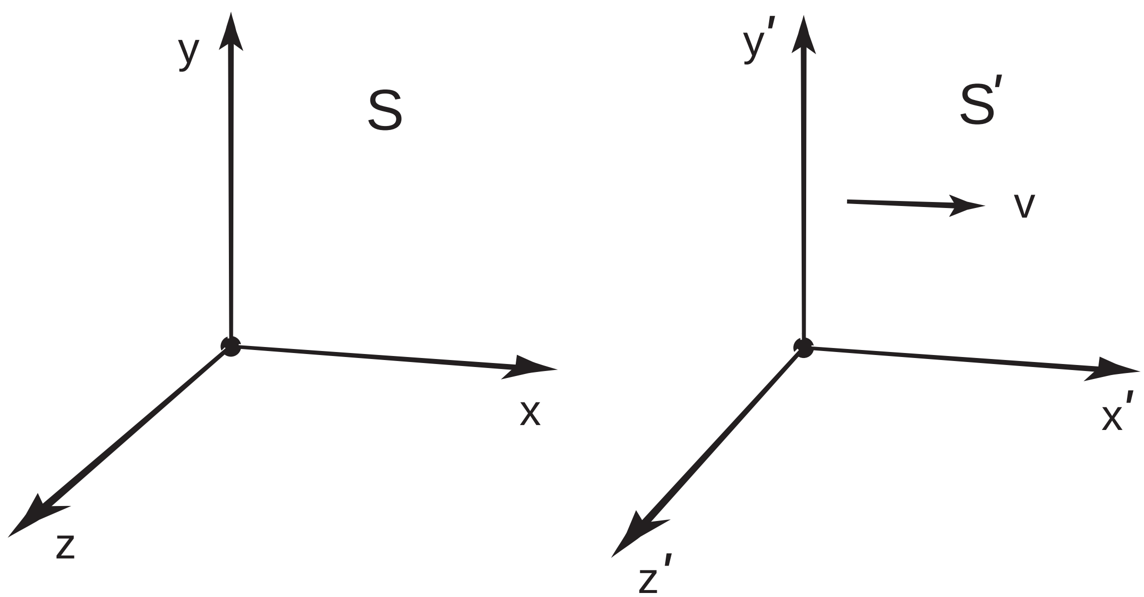

Figure 1:

is described by the equation $x = ut$. This description of the path is obviously coordinate-dependent. The path can be equally well described as $x' = (u - v)t$. Describing physical laws in such a coordinate dependent way becomes quickly very cumbersome. As already stated early on in the Introduction the aim of Relativity is to formulate physical laws in a coordinate independent way.

Newtown's laws of mechanics are an example for such a coordinate independent physical law. Newton's three laws of mechanics can be stated as follows.

1. Free particles move with constant velocity.
2. The force on a particle equals the product of mass and acceleration.
3. The forces of action and reaction are equal and opposite.

They are valid in inertial frames (defined as reference frames in which the first law holds). The first law (Galileo's law of inertia) is obviously a special case of the second. Newton's laws implicitely assume the existence of a universal time. Newton also postulated the existence of absolute space detfined as the reference system where the centre of mass of the solar system is at rest. Newton's laws actually do not require the concept of absolute space. They are equally valid in all inertial frames. Newton's laws thus obey the Relativity Principle. Newton's laws are invariant under coordinate transformations which connect inertial frames. We have performed such a transformation at the beginning of this section and you have encountered them in your mechanics course as Galilean transformations. As we will see later in SR Newton's laws can be made invariant under a more general class of coordinate transformation – the Lorentz transformations – if we give up on the idea of a universal time.

It was the incompatibility of Maxwell's equations and of observations with regard to the velocity of light with the ideas of absolute space and universal time which led Einstein to introduce the Theory of Special Relativity.

1.3 Maxwell's equation, the velocity of light and the (infamous) aether

1.3.1 Maxwell's equation and the aether

Let us recapitulate Maxwell's equations:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \tfrac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \wedge \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \wedge \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

In free space we can set $\mathbf{J} = \rho = 0$ and we then get the more obviously symmetrical looking equations,

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \wedge \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \wedge \mathbf{B} &= \mu_0 \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t} \quad .
\end{aligned}
$$

Taking the curl of, say, the third equation and applying the relation

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E} \quad ,$$

we derive the equations for electromagnetic waves

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad ,$$

and similarly

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

These have the form of a wave equation with a propagation speed c

$$
\begin{aligned}
\nabla^2 \mathbf{E} &= \tfrac{1}{c^2} \, \tfrac{\partial^2 \mathbf{E}}{\partial t^2} \\
c &= \tfrac{1}{\sqrt{\mu_0 \epsilon_0}}
\end{aligned}
$$

Now the constants μ_0 and ϵ_0 are properties of the ‘vacuum’,

$\mu_0 =$ permeability of a vacuum $4\pi \times 10^{-7}$ Hm^{-1},

$\epsilon_0 =$ permittivity of a vacuum 8.85×10^{-12} Fm^{-1}.

This relation between the constants ϵ_0 and μ_0 and the speed of light was one of the most startling consequences of Maxwell’s theory. But what do we mean by a ‘vacuum’? Does it define an absolute frame of rest? Before Einstein introduced SR it was believed that this absolute frame was of (quasi)-substantial nature. This appeared to fit well with the idea of the existence of an ’aether’ a space-filling material which ’transmits’ otherwise incomprehensible actions (see the book by Rindler for a more detailed discussion). The existence of the ’aether’ defining a preferred reference frame together with the assumption of a universal time leads to the prediction of an ’aether wind’. The velocity of light should be different in reference frames moving relative to the preferred reference frame of the ’aether’. Experimenters in the 2nd half of the 19th century tried but failed to measure such an effect despite the fact that their equipment was actually sufficiently accurate to do such a measurement. The most famous of these experiment is the Michelson-Morley Experiment.

1.3.2 The Michelson-Morley experiment

The Michelson-Morley experiment is described in detail in almost all textbooks on SR including that by Rindler. I give here a very brief account. In 1887 Michelson and Morley splitted a beam of light and sent it along two orthogonal path of equal length to detect the differences in travel time expected if the velocity of light depends on the relative velocity of the source to an aether due to the motion of the earth. For simplicity consider the situation where one of the two arms of the interferometer is orientated along the direction of an aether drift of velocity v as shown in Fig. 2.

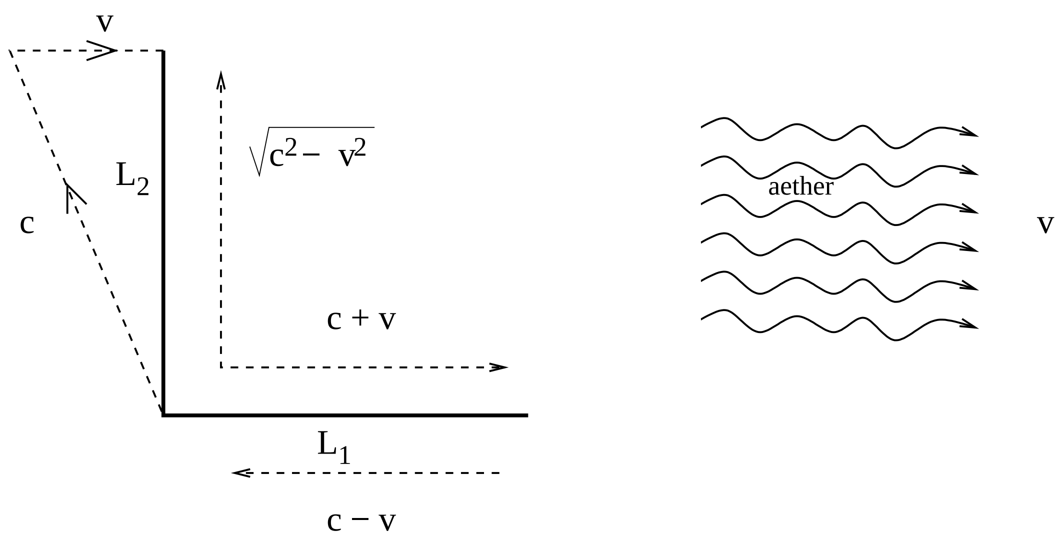

Figure 2:

The light travel times along the two arms are given by,

$$\begin{aligned} T_1 &= \tfrac{L_1}{c+v} + \tfrac{L_1}{c-v} = \tfrac{2L_1}{c(1-v^2/c^2)}, \\ T_2 &= \tfrac{L_2}{(c^2-v^2)^{1/2}} + \tfrac{L_2}{(c^2-v^2)^{1/2}} = \tfrac{2L_2}{c(1-v^2/c^2)^{1/2}}. \end{aligned}$$

They are different by a factor $(1 - v^2/c^2)^{1/2}$. Michelson and Morley performed their experiment in many orientations and at different times of the year but could not detect any effect despite the fact that the sensitivity of their apparatus should have been sufficient.

The obvious interpretation of the Michelson-Morley experiment is that the *velocity of light is the same in all inertial frames.* As we will see soon this is actually the second postulate of SR, but was obviously hard to swallow for physicists of the time.

1.3.3 Maxwell's equation and the Relativity Principle

Einstein was deeply bothered by the obvious contradiction of the existence of a seemingly preferred reference frame for Maxwell equations and the relativity principle. If we deny the existence of an absolute frame of rest, then how do we formulate a theory of electromagnetism? How do Maxwell's equations appear in frames moving with respect to each other? Do we change the value c – in which case what happens to the values of ϵ_0 and μ_0? We can get a strong clue about Einstein's thinking from the *title* of his famous 1905 paper on Special Relativity. The first paragraph is reproduced here:

> ON THE ELECTRODYNAMICS OF MOVING BODIES
> By A. EINSTEIN
>
> It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case.

You see that Einstein's paper is not called 'Transformations between Inertial Frames', or 'A Theory in which the Speed of Light is Assumed to be a Universal Constant'. Electrodynamics is at the heart of Einstein's thinking – Einstein realized that Maxwell's equations of electromagnetism *required* Special Relativity.

Einstein solves all of these problems at a stroke by saying that Maxwell's equations take the same mathematical form in all inertial frames.[1] The speed of light c should thus be the same in all inertial frames (and the null result of the Michelson-Morley experiment no surprise anymore). The theory of Special Relativity (including amazing conclusions such as $E = mc^2$) follows from a generalization of this simple and theoretically compelling assumption.

[1] I will discuss Maxwell's equations in detail in a later lecture

Maxwell's equations therefore require special relativity. For Einstein, the Michelson-Morley experiment might thus well have been a side issue. Einstein could 'see' Special Relativity lurking in Maxwell's equation.

1.4 The Principles of Special Relativity

• Definition of an Inertial Frame

An inertial frame is a frame in which spatial relations, as determined by rigid rods at rest in the frame, are Euclidean, and in which there exists a universal time in terms of which free particles remain at rest or continue to move with constant speed along straight lines.

c.f. Newton's first law – Newton assumes Euclidean geometry and free particles are postulated to move in straight lines.

• Axioms of Special Relativity

Einstein's Special Theory of Relativity follows from the following two axioms:

1. The laws of physics are identical in all inertial frames.
2. There exists an inertial frame in which light signals in a vacuum always travel rectilinearly at constant speed c in all directions, independent of the motion of source.

Each of these axioms seems reasonable enough, but combine the two and one gets a surprising and apparently absurd conclusion:

Light signals in vacuum are propagated rectilinearly, with the same speed c at all times, in all directions, in all inertial frames.

Rindler calls this '*Einstein's law of light propagation*'.

1.5 The Lorentz Transformation

1.5.1 Coordinate transformation between inertial frames

In an inertial frame, a free particle moves in a straight line.

Imagine a freely falling clock (i.e., a physical system with repetitive events separated by equal increments of time t).

Denote the coordinates of the clock by x^μ $(x^\mu \equiv ct,\ x,\ y,\ z)$. If the clock is falling freely, then we must have

$$\left.\begin{aligned} \tfrac{dx^\mu}{dt} &= \text{constant} \\ \tfrac{d^2x^\mu}{dt^2} &= 0 \end{aligned}\right\} \qquad \text{clock follows linear trajectory.}$$

Denote the coordinates of the clock in some other inertial frame by x'^μ. The clock must, **by definition**, follow straight lines in any other **inertial** frame and so applying the chain rule of partial differentiation

$$\begin{aligned} \frac{dx'^\mu}{dt} &= \sum \tfrac{\partial x'^\mu}{\partial x^\nu}\tfrac{dx^\nu}{dt} = \text{constant}, \\ \frac{d^2x'^\mu}{dt^2} &= \sum \tfrac{\partial x'^\mu}{\partial x^\nu}\tfrac{d^2x^\nu}{dt^2} + \sum \tfrac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma}\tfrac{dx^\nu}{dt}\tfrac{dx^\sigma}{dt} = 0. \end{aligned}$$

Hence, from the second of these equations we conclude that

$$\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma} = 0 \quad ,$$

i.e. the transformation between the two inertial coordinate systems **must be linear**:

$$x'^\mu = \sum \Lambda^\mu_\nu x^\nu + B^\mu,$$

where the B^μ are (unimportant) constants and the Λ^μ_ν are a transformation matrix whose form we must uncover and which contains the physics of the theory.

Special Relativity is the study of those linear coordinate transformations between inertial frames that are consistent with Einstein's axioms.

1.5.2 The Galilean Transformation

Let us first look at the familiar **Galilean** (classical) transformation between two inertial frames S and S', with S' moving at speed v relative to S in the x-direction as shown in the figure.

The classical transformation for an event with coordinates (t, x, y, z) in S and (t', x', y', z') in S' is

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

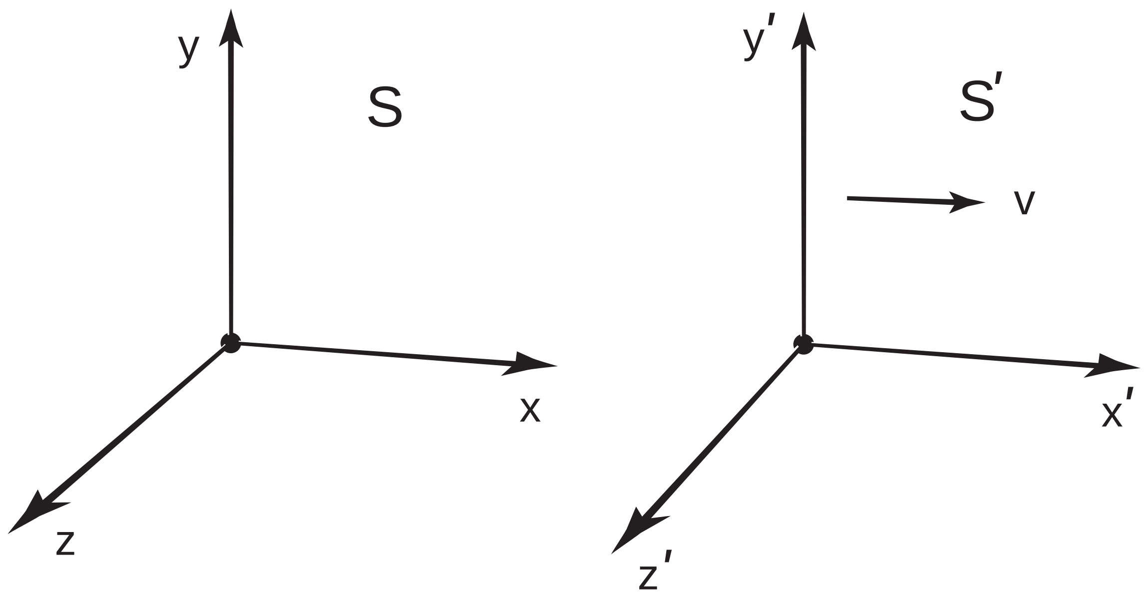

Figure 3:

The first equation follows from the concept of a universal time, which is simply taken for granted in Newtonian theory. The last two equations are 'trivial' transformation equations, since the relative motion is in the x and x' direction. The third equation follows from the 'common sense' law of velocity addition, since differentiating with respect to $t' = t$ gives

$$u' = u - v.$$

Differentiating again shows that the accelerations are invariant,

$$f' = f.$$

1.5.3 Derivation of the Lorentz Transformation

Now let us investigate transformations between frames S and S' consistent with the principles of SR. We have shown that according to the principles of SR the coordinates of an event (t', x', y', z') in the inertial frame S' must be related to the coordinates (t, x, y, z) in S by a **linear transformation**. Now as with the Galilean transformation, we have two trivial transformations

$$y' = y,$$

$$z' = z.$$

(The motion is in the x-direction and we choose the zero points of the coordinate systems in the y and z directions to coincide). Now, define the x-coordinates so that $x' = 0$ corresponds to $x = vt$, then the general **linear relation** between the coordinates

$$x' = a_1 x + a_2 y + a_3 z + a_4 t + a_5,$$

must read

$$x' = \gamma(x - vt),$$

and, reversing v

$$x = \gamma(x' + vt').$$

If we now assume a 'universal time'

$$t' = t$$

then we must have

$$\begin{aligned} \gamma &= 1, \\ x' &= x - vt, \end{aligned}$$

and we recover the **Galilean** transformation.

But if we adopt Einstein's anzatz for light propagation (imagine a pulse of light emitted at $t = 0$ propagating from the origin $x = 0$)

$$\left.\begin{aligned} x &= ct \\ x' &= ct' \end{aligned}\right\} \qquad c = \text{constant},$$

then,

$$\left.\begin{aligned} ct' &= \gamma t(c - v), \\ ct &= \gamma t'(c + v), \end{aligned}\right\}$$

and so

$$c^2 = \gamma^2(c^2 - v^2), \tag{1}$$

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}. \tag{2}$$

Hence, Einstein's law of light propagation leads to the Lorentz transformation,

$$\left.\begin{aligned} t' &= \gamma(t - vx/c^2), \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \ . \end{aligned}\right\}$$

Note that the Lorentz transformation involves an explicit transformation of time and that it mixes spatial and time coordinates for transformations between inertial frames which move relative to each other. SR requires to give up the concept of a universal time. This has also dramatic consequences for seemingly obvious concepts as events happening simultaneously and events happening at the same place. In SR simultaneity and co-locality have become "relative" and depend on the reference frame.

1.5.4 Invariant line element, Minkowski space and proper time

There is, however, an important invariant under Lorentz transformation, the four-dimensional line element ds^2.

In frame S, the propagation of light obeys

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0.$$

From the Lorentz transformation,

$$\begin{aligned} dt' &= \gamma dt - \tfrac{v\gamma}{c^2}dx, \\ dx' &= \gamma dx - v\gamma dt, \end{aligned}$$

and so,

$$\begin{aligned} ds^2 = c^2dt'^2 - dx'^2 &= c^2\gamma^2dt^2 + \tfrac{v^2\gamma^2}{c^2}dx^2 - 2v\gamma^2dxdt \\ &\quad -\gamma^2dx^2 - v^2\gamma^2dt^2 + 2v\gamma^2dxdt \\ &= c^2(1 - v^2/c^2)\gamma^2dt^2 + (v^2/c^2 - 1)\gamma^2dx^2 \\ &= c^2dt^2 - dx^2 \quad . \end{aligned}$$

Hence

$$ds^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

This motivates the concept of a 4-dimensional space-time, called Minkowski space. Note that for any fixed t the spatial part of the geometry of Minkowski space is Euclidean.

The invariant line element is often expressed in the form of the *proper time* τ,

$$ds^2 = c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

Note that for objects at rest in an inertial frame proper time and coordinate time are identical.

1.6 Elementary (but profound) consequences of the Lorentz transformation

1.6.1 Length Contraction

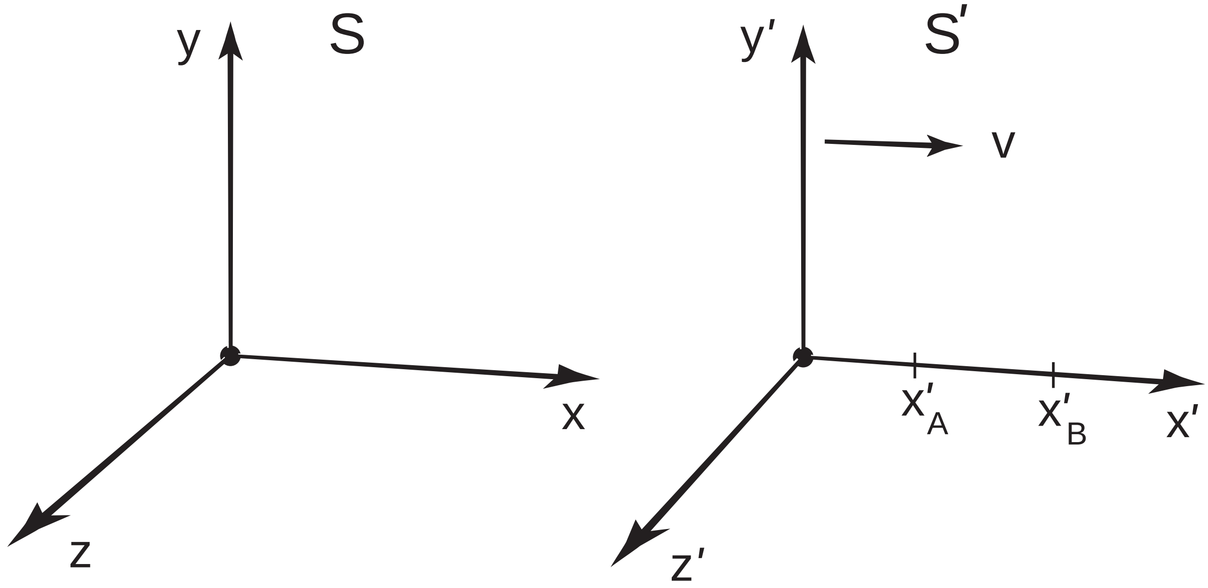

Figure 4:

Consider a rod in frame S' of proper length

$$\ell_0 = x'_B - x'_A \quad .$$

We want to apply the Lorentz transformation formulae and see what length an observer in frame S assigns to the rod. Applying the Lorentz transformation,

$$\begin{aligned} x'_A &= \gamma\left(x_A - vt_A\right), \\ x'_B &= \gamma\left(x_B - vt_B\right), \end{aligned}$$

relates the coordinates of the ends of the rod in S' to the coordinates in S. The observer in S measures the length of the rod at a *fixed* time $t = t_A = t_B$ as

$$\ell = x_B - x_A = \frac{1}{\gamma}\left(x'_B - x'_A\right) = \frac{\ell_0}{\gamma}.$$

Hence the rod in S appears contracted by an amount

$$\ell = \ell_0 \left(1 - v^2/c^2\right)^{1/2} .$$

1.6.2 Time Dilation

Suppose we have a clock at x'_A in S' and we record two successive events separated by time T_0.

The times recorded in S are:

$$
\begin{aligned}
t_1 &= \gamma\,(t'_1 + v x'_A/c^2)\,, \\
t_2 &= \gamma\,(t'_1 + T_0 + v x'_A/c^2)\,.
\end{aligned}
$$

Subtracting

$$
T = t_2 - t_1 = \gamma T_0 = \frac{T_0}{(1 - v^2/c^2)^{1/2}},
$$

hence, the moving clock goes slower by a factor of $(1 - v^2/c^2)^{1/2}$ (time dilation).

Note that an **ideal clock** is one that is unaffected by acceleration – external forces act identically on all parts of the clock (example is a muon). The **instantaneous rate** of an ideal clock depends on its **instantaneous speed** .

Rossi-Hall experiment: Unlike length contraction time dilation is actually rather straightforward to verify experimentally. In 1941 Rossi and Hall timed the decay times of relativistically moving muons and found them to be dilated as expected according to SR. Note that without the relativistic time dilation the lifetime of muons produced at the top of the atmosphere by cosmic rays would actually be about a factor ten too short to reach the surface of the Earth as observed.

1.6.3 A heuristic derivation of the (radial) relativistic Doppler effect

Waves received from an moving source have a different frequency depending on whether the source is approaching the receiver or moving away from it. This classical Doppler effect also exists in SR but for velocities close to the velocity light relativistic time dilation has to be taken into account. The book of Rindler gives a detailed heuristic derivation based on that of the classic Doppler effect and relativistic time dilation. I give a short summary here. Consider a light source travelling through an inertial frame S with instantaneous velocity u and radial velocity component u_r relative to an observer O at the origin. Let the time between successive pulses be $d\tau$ as measured by a comoving clock at the source, and therefore $d\tau\gamma(u)$ in S (due to the time dilation). The second pulse is emitted later (by $\gamma d\tau$) and has to travel farther (by $\gamma d\tau u_r$). Consequently these pulses arrive at O a time $d\tau\gamma + d\tau\gamma u_r/c$ apart. But $d\tau$ and dt are inversely proportional to the frequency ν_0 of the source and its frequency ν as observed by O,

$$\frac{\nu_0}{\nu} = \frac{1 + u_r/c}{(1 - u^2/c^2)^{1/2}}.$$

When the motion of the source is purely radial this reduces to

$$\frac{\nu_0}{\nu} = (\frac{1 + u/c}{1 - u/c})^{1/2}.$$

1.7 A more mathematical description of Lorentz transformations

A Lorentz transformation is a transformation from one system of space-time coordinates x^α to another system $x'^\alpha[\equiv (x'_1, x'_2, x'_3, x'_4), \equiv (ct',\ x',\ y',\ z')]$, such that

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha \quad ,$$

where

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}, \qquad (*)$$

$$\eta_{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 1, 2, 3, \\ 1 & \alpha = \beta = 0, \\ 0 & \alpha \neq \beta. \end{cases}$$

This is the unique, non-singular coordinate transformation that leaves the proper-time interval, $d\tau$ invariant.

Let us now prove this:

Write the line element ds^2 as,

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

.

The transformations are linear,

$$dx'^\alpha = \Lambda^\alpha_\gamma dx^\gamma$$

and so,

$$\begin{aligned}\eta_{\alpha\beta}dx'^{\alpha}dx'^{\beta} &= \eta_{\alpha\beta}\Lambda^{\alpha}_{\gamma}dx^{\gamma}\Lambda^{\beta}_{\delta}dx^{\delta} \\ &= \eta_{\alpha\beta}\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\delta}dx^{\gamma}dx^{\delta} \\ &= \eta_{\gamma\delta}dx^{\gamma}dx^{\delta} \\ &= c^2 d\tau^2 \quad ,\end{aligned}$$

hence

$$d\tau'^2 = d\tau^2.$$

The set of all Lorentz transformations as defined by (*) is called the *inhomogenous Lorentz group* or *Poincaré group.* Those with $a^{\alpha} = 0$ are called the the *homogenous Lorentz group.*

Those with $|\Lambda| = 1,\ \Lambda^0_0 \geq 1$, which can be converted to the identity δ^{α}_{β} by a continuous variation of parameters are called the *proper homogenous Lorentz group.*

As we will see later in the course Lorentz transformations can be viewed as rotations in Minkowski space .