Theory of Relativity
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10 The Gravitational Field Equations

10.1 Derivation of the Einstein field equations

We are now in a position to tackle the problem of deriving a set of field equations for gravity. Let's recall a few results from previous lectures:

- We showed that in a weak gravitational field

$$g_{00} = \left(1 + \frac{2\phi}{c^2}\right),$$

- We have constructed an energy-momentum tensor

$$T^{\mu\nu},\ T_{\mu\nu},$$

  with a T_{00} component which, for 'dust', measures the mass density

$$T_{00} = \gamma^2 \rho_0 c^2.$$

- The gravitational field equation of Newtonian gravity is

$$\nabla^2 \phi = 4\pi G \rho,$$

  i.e., for a weak gravitational field

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00}. \tag{1}$$

- The last equation suggests that the field equations are tensorial equations of rank 2, of the form

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

  where the tensor $G_{\mu\nu}$ must be consistent with the Newtonian limit (1).

But what is the form of $G_{\mu\nu}$? We have a few clues:

- $G_{\mu\nu}$ must involve second derivatives of the metric tensor so that in the weak field limit $G_{00} \to \nabla^2 g_{00}$.
- The vanishing of a gravitational field is equivalent to the vanishing of the curvature tensor, which as we know involves the second derivatives of $g_{\mu\nu}$. This suggests that $G_{\mu\nu}$ is related to rank two tensors constructed from the curvature tensor.
- $T_{\mu\nu}$ is symmetric and so $G_{\mu\nu}$ must be symmetric.
- $T_{\mu\nu}$ is conserved, $T_{\mu\nu;\nu} = 0$, hence $G_{\mu\nu}$ must also be conserved $G_{\mu\nu;\nu} = 0$.

As a guess, let's write down a rank 2 tensor constructed from the curvature tensor:

$$G_{\mu\nu} \;=\; \underset{\nwarrow}{\alpha}\; \overset{\substack{\text{Ricci}\\ \text{tensor}\\ \downarrow}}{R_{\mu\nu}} \;+\; \underset{\nearrow}{\beta}\; g_{\mu\nu}\; \overset{\substack{\text{Curvature}\\ \text{Scalar}\\ \downarrow}}{R}$$

constants

This guess automatically satisfies the first three conditions. The tensor $G_{\mu\nu}$ involves just the metric tensor and its first two derivatives. The tensor $G_{\mu\nu}$ is symmetric and it vanishes if all components of the curvature tensor vanish.

Now let's make use of the condition that $G_{\mu\nu}$ satisfies a conservation equation. Raise an index

$$\begin{aligned} G^{\kappa}_{\nu} &= g^{\kappa\mu} G_{\mu\nu} = \alpha g^{\kappa\mu} R_{\mu\nu} + \beta g^{\kappa\mu} g_{\mu\nu} R \\ &= \alpha R^{\kappa}_{\nu} + \beta \delta^{\kappa}_{\nu} R, \end{aligned}$$

and take the covariant derivative,

$$G^{\kappa}_{\nu;\kappa} = \alpha R^{\kappa}_{\nu;\kappa} + \beta \delta^{\kappa}_{\nu} R_{;\kappa}.$$

Now recall the Bianchi identity

$$\left(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta} R \right)_{;\mu} = 0,$$

this gives

$$G^{\kappa}_{\nu;\kappa} = \left(\frac{1}{2}\alpha + \beta \right) R_{;\nu},$$

and so if $G_{\mu\nu}$ satisfies a conservation equation

$$G^{\kappa}_{\nu;\kappa} = 0, \quad \text{either} \quad \begin{cases} \beta = -\frac{1}{2}\alpha, \\ \\ R_{;\nu} = 0. \end{cases}$$

We can easily rule out the idea that $R_{;\nu}$ is always equal to zero starting from the field equations (2).

$$\begin{aligned} G^{\kappa}_{\kappa} &= \frac{8\pi G}{c^4} T^{\kappa}_{\kappa} \\ &= \alpha R^{\kappa}_{\kappa} + \beta \delta^{\kappa}_{\kappa} R \\ &= \alpha R + 4\beta R \\ &= \alpha' R, \qquad\qquad \text{where } \alpha' = \alpha + 4\beta. \end{aligned}$$

So, taking the covariant derivative of this last relation

$$\alpha' R_{;\nu} = \frac{8\pi G}{c^4} T^{\kappa}_{\kappa;\nu} = 0, \tag{3}$$

But (3) requires that

$$\frac{\partial T^{\kappa}_{\kappa}}{\partial x^{\nu}} = 0$$ always, which cannot be satisfied in general since it requires $\partial \rho_0 / \partial x = 0$.

We must therefore select the solution

$$\beta = -\frac{1}{2}\alpha$$

and so the field equations must look like

$$\alpha \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{4}$$

To fix the constant α, we can look at the weak field limit of equations (4).

For non-relativistic matter,

$$T_{ij} \ll T_{00} \quad \begin{pmatrix} T_{00} & \sim & \rho_0 c^2 \\ T_{ij} & \sim & \rho_0 v^2 \end{pmatrix}$$

and for weak gravitational fields

$$g_{\mu\nu} \approx \eta_{\mu\nu},$$

i.e. the metric tensor is nearly that of Minkowski space-time. The field equations look like

$$\alpha \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{8\pi G}{c^4} T_{\mu\nu},$$

and so if $T_{ij} \approx 0$, the spatial components on the left hand side must be approximately zero, *i.e.*

$$\begin{aligned} R_{ij} &\approx \frac{1}{2} g_{ij} R, \\ &\approx \frac{1}{2} \eta_{ij} R. \end{aligned}$$

The definition of the curvature scalar is

$$\begin{aligned} R &= g^{\mu\kappa} R_{\mu\kappa} \\ &\approx R_{00} - R_{ii} \\ &= R_{00} + \frac{3}{2} R, \qquad \text{using } R_{ij} \approx \frac{1}{2}\eta_{ij} R. \end{aligned}$$

Hence

$$R \approx -2R_{00},$$

and since

$$\begin{aligned} G_{\mu\nu} &= \alpha\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right), \\ G_{00} &= \alpha\left(R_{00} - \frac{1}{2} g_{00} R\right), \\ &\approx \alpha\left(R_{00} + \frac{1}{2} 2R_{00}\right) \\ &= 2\alpha R_{00}. \end{aligned}$$

The curvature tensor is, by definition,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right].$$

In a weak static field, this expression simplifies significantly because

- all time derivatives vanish,
- products of Γ's can be ignored (they are second order in $g_{\mu\nu}$).
- we know that in the weak field limit,

$$g_{00} \approx (1 + 2\phi).$$

so, since

$$R_{\mu\nu} = g^{\lambda\kappa} R_{\lambda\mu\kappa\nu},$$

the 00 component is

$$R_{00} = g^{\lambda\kappa} R_{\lambda 0 \kappa 0} \approx R_{0000} - R_{i0i0}.$$

But

$$R_{0000} = 0, \quad R_{i0j0} = \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j},$$

and so for a static weak gravitational field,

$$G_{00} = 2\alpha R_{00} = -\alpha \nabla^2 g_{00},$$

i.e.

$$-\alpha \nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00}.$$

But, in a weak gravitational field the field equations must take the form

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00},$$

hence

$$\alpha = -1.$$

Having fixed the constant α, this means that the field equations must have the form

$$\boxed{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) = -\tfrac{8\pi G}{c^4} T_{\mu\nu}.}$$

These are Einstein's field equations of General Relativity.

10.2 Sign Conventions

There is no accepted system of sign conventions in General Relativity. Different books use different sign conventions for the metric tensor, curvature tensors and for the field equations. We can summarize these sign conventions in terms of three sign factors $S1$, $S2$ and $S3$. These are defined as follows:

$$\begin{aligned}
\eta^{\mu\nu} &= [S1]\,(-1\ +1,\ +1,\ +1) \\
R^{\mu}_{\alpha\beta\gamma} &= [S2]\left(\tfrac{\partial \Gamma^{\mu}_{\alpha\gamma}}{\partial x^{\beta}} - \tfrac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - -\Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha}\right) \\
G_{\mu\nu} &= [S3]\,\tfrac{8\pi G}{c^4} T_{\mu\nu} \\
R_{\mu\nu} &= [S2]\,[S3]\,R^{\alpha}_{\mu\alpha\nu}
\end{aligned}$$

I have chosen to use a convention in which all three sign factors are *negative*. This because of the way that I learned General Relativity, but it is exactly the opposite of the convention used by, for example, Misner, Thorne and Wheeler. My convention is similar to d'Inverno and Rindler, except that I define the curvature tensor with the opposite sign. Here is a summary of the sign conventions used in various books:

	Us	d'Inverno/Rindler	Misner, Thorne, Wheeler	Weinberg
[$S1$]	-	-	+	+
[$S2$]	-	+	+	-
[$S3$]	-	-	+	-

10.3 Are the field equations unique?

In Section 10.1 we derived the field equations

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{5}$$

I now want to ask whether these field equations are unique. In fact, shortly after Einstein derived these field field equations, he proposed a modification. We have seen that energy-momentum conservation in GR is expressed by

$$T_{\mu\nu;\nu} = 0,$$

and we constructed the right hand side of equation (1) from the curvature tensor to satisfy this relation. But you will recall that the covariant derivative of the metric tensor is equal to zero

$$g_{\mu\nu;\nu} = 0,$$

and so it follows that a *consistent* set of field equations is

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu}\right) = -\frac{8\pi G}{c^4}T_{\mu\nu} \qquad (6),$$

where Λ is some new universal constant of Nature. The term Λ is called the *cosmological constant.* Einstein introduced this term because he was unable to construct static models of the Universe from his standard field equations (1). What he found (and we will discuss this in detail in later lectures) was that the standard field equations predicted a Universe that was either expanding or contracting. Einstein did this work in about 1916, when people thought that our Milky Way Galaxy represented the whole Universe, which Einstein represented as a uniform distribution of 'fixed stars'. By introducing Λ, Einstein constructed static models of the Universe (which as we will see are actually unstable). Later, in 1929, Hubble discovered the expansion of the Universe by measuring distances and redshifts to nearby galaxies. The Universe was proved to be expanding and the need for a cosmological constant disappeared. Einstein is reputed to have said that the introduction of the cosmological constant was his 'biggest blunder'.

Nowadays we have a rather different view of the cosmological constant. Recall that the energy momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (\frac{p}{c^2} + \rho)u^\mu u^\nu - pg^{\mu\nu}$$

Imagine some type of 'substance' with a strange equation of state $p = -\rho c^2$. This is unlike any kind of substance that you have ever encountered because it has a negative pressure! The energy momentum tensor for this substance is

$$T_{\mu\nu} = -pg_{\mu\nu} = \rho c^2 g_{\mu\nu}.$$

Now there are two points to note about this equation. Firstly, the energy momentum tensor of this strange stuff depends only on the *metric* tensor – it is therefore a property of the vacuum itself and we can call ρ the energy density of the *vacuum*. Secondly, the form of $T_{\mu\nu}$ looks just like the cosmological constant term in (2). We can therefore view the cosmological constant as a universal constant that fixes the energy density of the vacuum,

$$\rho_{\text{vac}}c^2 = \frac{\Lambda}{\frac{8\pi G}{c^4}}.$$

How can we calculate the energy density of the vacuum? This is one of the major unsolved problems in physics. The simplest calculation, involves summing the quantum mechanical zero point energies of all of the fields known in Nature. This gives an answer about 120 orders of magnitude higher than the upper limits on Λ set by cosmology. This is probably the worst theoretical prediction in the history of physics! Nobody knows how to make sense of this result. There must exist some physical mechanism that makes the cosmological constant very small. Some physicists have thought that there must exist a mechanism that makes Λ *exactly* equal to zero. But in the last few years there has been increasing evidence that the cosmological constant is small but non-zero. The strongest evidence comes from observations of distant Type Ia supernovae that indicate that the Universe is actually accelerating rather than decelarating. Normally, one would have thought that the gravity of matter in the Universe would cause the expansion to slow down (perhaps even eventually halting the expansion and causing the Universe to collapse). But if the cosmological constant is non-zero, the negative pressure of the vacuum can cause the Universe to accelerate.

Whether the supernova observations are right or not is an area of active research. The theoretical problem of explaining the value of the cosmological constant is, as I have said, one of the great challenges of theoretical physics. My bet is that we will need a fully developed theory of quantum gravity (perhaps superstring theory) before we can understand Λ.

10.4 More on Einstein's field equations

Einstein's field equations are

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{7}$$

and in 4-dimensions we have 10 independent components of the $g_{\mu\nu}$ and so from (1) in Einstein's theory we have 10 field equations. (Compare this to Newtonian gravity where we have only one gravitational field equation). Furthermore, the Einstein field equations are *non-linear* in the $g_{\mu\nu}$ (whereas Newtonian gravity is linear in the field ϕ). Einstein's theory thus involves lots of non-linear differential equations, and so it should come as no surprise that the theory is complicated.

There is another interesting aspect of the theory. Contract the field equations (1),

$$\begin{aligned} \left(g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R\right) &= -\frac{8\pi G}{c^4} T_{\mu\nu} g^{\mu\nu} \\ \left(R - \frac{1}{2} 4R\right) &= -R = -\frac{8\pi G}{c^4} T^{\mu}_{\mu}. \end{aligned}$$

So, an entirely equivalent form of the field equations is

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu}\right). \tag{8}$$

It follows from (2) that the field equations in *empty space* are

$$\boxed{R_{\mu\nu} = 0.}$$

From this simple equation, we can immediately establish a profound result. Look at the number of field equations as a function of the number of space-time dimensions

No. of Dimensions	2	3	4
No. of field equations	3	6	10
No. of independent components of $R_{\lambda\mu\nu\kappa}$	1	6	20
	Vanishing of $R_{\mu\nu}$ guarantees vanishing of $R_{\lambda\mu\nu\kappa}$		

You see that in two or three dimensions, the field equations in empty space guarantee that the full curvature tensor must vanish. In four dimensions, however, we have 10 field equations but 20 independent components of the curvature tensor. It is therefore possible to satisfy the field equations in empty space with a non-vanishing curvature tensor. But remember that a non-vanishing curvature tensor represents a non-vanishing gravitational field. We conclude that *it is only in 4 dimensions or more that gravitational fields can exist in empty space.*