Theory of Relativity
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4 Electromagnetism

This lecture will discuss the relativistic formulation of electromagnetism. This will introduce a number of ideas that we will apply later in developing and applying General Relativity. (Electromagnetism, like gravity, is a field theory.) Our guiding principle is to derive tensorial equations. This makes it possible to state the theory in a form that is independent of the coordinate system. We will see that a consistent theory of electromagnetism follows by saying that there exists a pure force depending linearly on velocity that depends on a certain property of a particle, namely its charge q. Even if one had no prior knowledge of electromagnetism, one could derive the complete theory in a few lines based on this simple assumption.

4.1 The electromagnetic force on a moving charge

Classically, the force on a charge q is

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}).$$

This suggests that we write down a tensorial equation for a pure force that depends linearly on velocity. The equation must be tensorial, so it must relate the four-force $\mathbf{F}$ to the four-velocity $\mathbf{U}$:

$$F_\mu = \frac{q}{c} E_{\mu\nu} U^\nu.$$

The quantity q is some property of the particle, that tells us how big the force is (it is the particles charge).[1] The quantity $E_{\mu\nu}$ is a second rank tensor that is required to make a four-force out of the four-velocity U^ν. We will see that this will turn out to represent the electromagnetic field.

The requirement that the force is pure is

$$F_\mu U^\mu = 0.$$

This requires

$$\frac{q}{c} E_{\mu\nu} U^\mu U^\nu = 0, \tag{1}$$

[1] The fact that the charge is divided by c is a consequence of our choice of units of the electromagnetic field. I will use the SI that are familiar to you.

and this must be satisfied for **all** U^μ. The requirement (1) can only be satisfied for all U^μ if the tensor $E_{\mu\nu}$ is *antisymmetric*

$$E_{\mu\nu} = -E_{\nu\mu}.$$

We can construct a contravariant electromagnetic field tensor by raising indices

$$E^{\mu\nu} = g^{\mu\lambda} g^{\nu\kappa} E_{\lambda\kappa}.$$

Since $g^{\mu\nu}$ is symmetric and $E_{\mu\nu}$ is antisymmetric, $E^{\mu\nu}$ is also antisymmetric.

4.2 The electromagnetic field equations

In Maxwell's theory, derivatives of the fields are related to charges and currents, suggesting that we write

$$\frac{\partial E^{\mu\nu}}{\partial x^\mu} = E^{\mu\nu}{}_{,\mu} = kJ^\nu \tag{2}$$

where J^ν is a four-current density. (k is a constant, related to our choice of units.)

If we differentiate equation (2) again,

$$\frac{\partial^2 E^{\mu\nu}}{\partial x^\mu \partial x^\nu} = E^{\mu\nu}{}_{,\mu\nu} = kJ^\nu{}_{,\nu}\,. \tag{3}$$

Since $E^{\mu\nu}$ is antisymmetric, we can group the terms on the left hand side like this:

$$\frac{\partial}{\partial x^\mu \partial x^\nu}\left(E^{\mu\nu} + E^{\nu\mu}\right) = 0$$

The left hand side of equation (3) is obviously zero, hence

$$J^\nu{}_{,\nu} = 0. \tag{4}$$

We will see in a little while that equation (4) represents conservation of charge.

Do we have a viable theory yet? The field equations of the theory (equation 2) are

$$E^{\mu\nu}{}_{,\mu} = kJ^\nu$$

but, there are **six** independent fields $E^{\mu\nu}$ and only **four** field equations. Evidently our theory is underdetermined as it stands. This suggests that $E^{\mu\nu}$ is constructed from a four-vector potential

$$A_\mu.$$

If we write

$$E_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{5}$$

then $E_{\mu\nu}$ is antisymmetric by construction.

Starting from

$$kJ^{\nu} = E^{\mu\nu}{}_{,\mu}$$

lower the index ν

$$g_{\lambda\nu} kJ^{\nu} = kJ_{\lambda} = E^{\mu}_{\lambda,\mu} = g^{\mu\kappa} E_{\kappa\lambda,\mu}.$$

Hence

$$kJ_{\lambda} = g^{\mu\kappa} \left[A_{\lambda,\kappa\mu} - A_{\kappa,\lambda\mu} \right]. \tag{6}$$

Alternatively, we can express the field equations entirely in terms of the electromagnetic field tensor $E^{\mu\nu}$. In this case we require two field equations,

$$E^{\mu\nu}{}_{,\mu} = kJ^{\nu},$$

$$E_{\mu\nu,\sigma} + E_{\sigma\mu,\nu} + E_{\nu\sigma,\mu} = 0.$$

The constant k can be found demanding consistency with the standard Maxwell equations. In SI units $k = \mu_0$.

4.3 Electromagnetic field equations in the Lorentz gauge

Suppose we add an arbitrary vector to A_{μ}

$$A'_{\mu} = A_{\mu} + B_{\mu}$$

we will recover the **same** electromagnetic fields

$$\begin{aligned} A'_{\mu,\nu} - A'_{\nu,\mu} &= A_{\mu,\nu} - A_{\nu,\mu} + B_{\mu,\nu} - B_{\nu,\mu} \\ &= A_{\mu,\nu} - A_{\nu,\mu} \end{aligned}$$

provided that

$$\frac{\partial B_{\mu}}{\partial x^{\nu}} = \frac{\partial B_{\nu}}{\partial x^{\mu}}. \tag{7}$$

Equation (6) can be satisfied if $\mathbf{B}$ is the gradient of a scalar function Ψ,

$$B_{\mu} = \frac{\partial \Psi}{\partial x^{\mu}}.$$

Hence there is what is called a **gauge** freedom – we are free to add the gradient of any arbitrary scalar function Ψ to A_{μ},

$$A'_{\mu} = A_{\mu} + \frac{\partial \Psi}{\partial x^{\mu}},$$

and we will recover the **same** electromagnetic fields.

In the field equations

$$kJ_\lambda = g^{\mu\kappa} \left[A_{\lambda,\kappa\mu} - A_{\kappa,\lambda\mu} \right]$$

the second term on the right-hand side is $A^\mu{}_{,\lambda\mu}$. To zero this term, choose a scalar function Ψ so that

$$A^\mu{}_{,\mu} = 0.$$

With this condition (called the Lorentz gauge condition) the electromagnetic field equations then take a very simple form,

$$g^{\mu\kappa} A_{\lambda,\kappa\mu} = kJ_\lambda$$

which we can write as

$$\Box^2 A_\lambda = kJ_\lambda$$

where $\Box^2$ is the d'Alembertian operator,

$$\Box^2 = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \tag{8}$$

In the absence of currents

$$\Box^2 A_\mu = 0,$$

and so A_μ has wave solutions travelling at the speed of light, as do the components of $E_{\mu\nu}$, since

$$\Box^2 E_{\mu\nu} = 0$$

in the absence of currents.

4.4 The four-current density

The four-current density **J** must be given by the product of a scalar quantity times a four velocity (compare our discussion of the four momentum **P**):

$$\begin{aligned} J^\mu &= \underbrace{\rho_0}_{\text{local charge density in rest frame}} U^\mu \\ &= \rho_0 \gamma (c, \mathbf{u}) \\ &= (c\rho,\ \underbrace{\mathbf{j}}_{\text{relativistic three current density}}) \end{aligned}$$

You see that the 0 component, $c\rho$ is the charge density in the frame of observation, since by length contraction the proper volume in that frame is reduced by a factor $1/\gamma$.

The condition $J^{\nu}{}_{,\nu}$ derived in equation (4) reads

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

and expresses charge conservation. This equation looks the same as the non-relativistic equation of charge continuity, but with the *relativistic* forms for ρ and $\mathbf{j}$.

4.5 Identification of $E_{\mu\nu}$ with electric and magnetic fields

We still have to identify the components of the electro-magnetic field tensor with the familiar electric and magnetic 3-vector fields $\mathbf{E}$ and $\mathbf{B}$ as observed in some Cartesian inertial frame S. If we identify the four vector potential A^{μ} with the usual electrostatic potential ϕ and three vector potential $\mathbf{A}$ as,

$$A^{\mu} = (\frac{\phi}{c}, \mathbf{A}),$$

then the Lorentz gauge condition becomes

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0.$$

In this gauge the field equations take the form

$$\Box^2 \mathbf{A} = \mu_0 \mathbf{j}, \qquad \Box^2 \phi = \frac{\rho}{\epsilon_0}.$$

The electric and magnetic fields can then be written in terms of the scalar and vector potential as

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \qquad \mathbf{E} = \boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{9}$$

From equations (5) and (9) we can now derive how the components of the electromagnetic field tensor related to the electric and magnetic fields E and B.

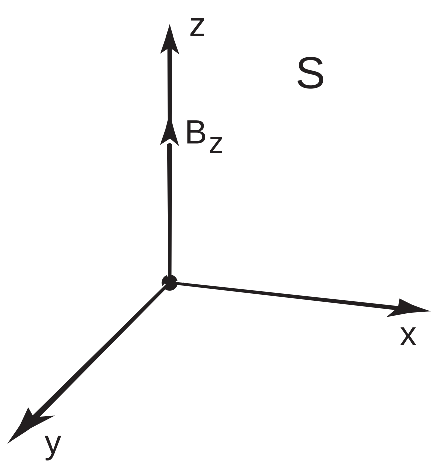
I will write down the answer:

$$E_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix}$$

$$E^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix}$$

Note that these identifications also follow from the rotation properties of a second rank tensor. The triplets $(E_1,\ E_2,\ E_3)$ and $(B_1,\ B_2,\ B_3)$ behave as three-vectors under spatial rotations.

4.6 Transformation of the electromagnetic field between inertial frames



In this section, we will work through a specific problem. Suppose that in frame S, we have a magnetic field B oriented along the z axis as shown in the figure. How will the electromagnetic field look in a frame S' moving at speed v relative to the x-axis?

According to the transformation rules for tensors, the rank two electromagnetic field tensor transforms as

$$E'_{\lambda\kappa} = E_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\lambda}} \frac{\partial x^{\nu}}{\partial x'^{\kappa}}$$

In matrix notation we can write this as

$$\mathbf{E}' = \mathbf{L}^T\ \mathbf{E}\ \mathbf{L}$$

where $\mathbf{L}$ is the transformation matrix:

$$\mathbf{L} = \begin{pmatrix} \partial x^0/\partial x'^0 & , & \partial x^0/\partial x'^1 & , & \partial x^0/\partial x'^2 \ldots \\ \partial x^1/\partial x'^0 & , & \partial x^1/\partial x'^1 & , & \partial x^1/\partial x'^1 \ldots \\ \partial x^2/\partial x'^0 & , & \partial x^2/\partial x'^1 & , & \partial x^2/\partial x'^2 \ldots \\ \vdots & & \vdots & & \vdots \end{pmatrix}$$

From the Lorentz transformation

$$\begin{aligned} x &= \gamma\left(x' + vt'\right), \\ t &= \gamma\left(t' + vx'/c^2\right), \end{aligned}$$

The transformation matrix is

$$\mathbf{L} = \begin{pmatrix} \gamma & v\gamma/c & 0 & 0 \\ v\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, from the identification of the components electromagnetic field tensor given in Section 4.5, we can write down what it looks like in the frame S

$$E_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, it is a straightforward exercise to show that in frame S' the fields transform to

$$\begin{aligned} E_2' &= -\gamma B v, \\ B_3' &= \gamma B. \end{aligned}$$

The first of these equations tells us that a moving magnetic field generates an electric field. In limit of small velocities $v \ll c$, the electric field in S' is $E_2' = -vB$. This is just Faraday's law,

$$dV = (\mathbf{v} \wedge \mathbf{B})d\mathbf{r}$$

$$\Rightarrow \mathbf{E} = -\mathbf{v} \wedge \mathbf{B}.$$

This example illustrates the beautiful symmetry (or reciprocity) that exists between electric and magnetic fields in Special Relativity, and which was alluded to in the introductory paragraph of Einstein's 1905 paper. The electric and magnetic fields form components of a single entity, a rank two tensor $E_{\mu\nu}$. The values of the components of the electric field and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ differ in different inertial frames. However, within an inertial frame, the triplets of numbers defining $\mathbf{E}$ and $\mathbf{B}$ transform as distinct vectors under spatial rotations and so it makes sense to talk about electric and magnetic fields within an inertial frame.

4.7 Summary

We have shown that the field equations of electromagnetism take the following form

$$\Box^2 A_\mu = \mu_0 J_\mu$$

$$A^\mu{}_{,\mu} = 0 \qquad\qquad \text{Lorentz gauge}$$

Both equations are required, since the second specifies the gauge.

The four-current density is

$$\mathbf{J} = (\rho c, \mathbf{j})$$

and the electromagnetic field tensor is related to the potential $\mathbf{A}$ by

$$\begin{cases} E_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \\ \\ E^{\mu\nu}{}_{,\mu} = \mu_0 J^\nu \end{cases}$$

It is straightforward to show that these equations give the Maxwell equations in their more familiar form

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \boldsymbol{\nabla} \wedge \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t},$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \wedge \mathbf{B} = \mu_0 \mathbf{J} + \tfrac{1}{c^2}\tfrac{\partial \mathbf{E}}{\partial t}.$$

By re-writing Max-well equations in tensorial/covariant form we have found a coordinate independent formulation of electromagnetism.