5 Construction of a theory of gravity

5.1 Inertial and gravitational mass in Newtonian Gravity

In Newtonian gravity we can write the acceleration as

$$\frac{d^2\mathbf{x}}{d^2t} = -\frac{m_{\mathrm{G}}}{m_{\mathrm{I}}}\boldsymbol{\nabla}\Phi,$$

where m_{G} and m_{I} are the the gravitational and inertial mass respectively and Φ is the gravitational potential. While the ratio of charge to inertial mass is different for different particles the ratio of gravitational to inertial mass has been verified to a very high degree of accuracy to be the same for all matter. For a suitable choice of units the gravitational and inertial mass are thus the same. This *equivalence* of gravitational and inertial mass is truly remarkable and is a defining feature of General Relativity.

5.2 Guiding Principles

We need some guiding principles to construct a mathematical theory of gravity. As I have explained in the lectures, a number of lines of reasoning led Einstein to make the following assumptions. These assumptions are sufficient to define a theory of gravity (though as we will see they do not define a *unique* theory).

- **The Principle of Equivalence**

  In any gravitational field we can choose a local freely falling inertial frame in which the laws of nature take the same form as in an unaccelerated Minkowski frame in the absence of gravity.

- **The laws of Physics are covariant**

  This means the following:

  ∘ We can find a transformation from a general coordinate system x^μ to a locally inertial system ξ^μ

$$\frac{\partial \xi^\alpha}{\partial x^\mu}.$$

(These transformations will be *non-linear* in general).

○ The equations of physics *must* agree with the laws of Special Relativity in the absence of gravitation.

○ Physical laws must have a *tensorial* form, so that they can be stated in a form that is invariant under a general coordinate transformation $x^\mu \to x'^\mu$.

- **Riemannian geometry**

  We need a mathematical description of non-Cartesian geometry for spaces described by a metric
  $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

How do we proceed? We could start with Newtonian gravity, and try to make it consistent with the relativity principle.

The familiar equation of motion in Newtonian gravity (which is not Lorentz invariant) is

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi,$$

where the gravitational potential Φ is determined by Poisson's equation (the field equation of Newtonian gravity),
$$\nabla^2\Phi = 4\pi G\rho.$$

Notice that the ∇^2 operator in Poisson's equation is equivalent to the d'Alembertian operator in the limit that the speed of light tends to infinity

$$\nabla^2 \quad \equiv \quad \underset{\lim c \to \infty}{\Box^2}$$

hence in Newton's theory, the potential Φ responds *instantaneously* to a disturbance in ρ. Now, we could try replacing ∇^2 by $\Box^2$ in Poisson's equation, and writing the equation of motion of particle in a relativistically invariant form. But, if we tried this, we would not end up with a consistent theory. In fact *no* consistent theory can be constructed in which the analogue of the gravitational field is a scalar or a vector. We will see that we are led to a (rank 2) *tensor* theory of gravity. This is quite a large leap from Newtonian theory and is motivated by non-Euclidean geometry. For this reason, many physicists talk about gravity as an 'inherently' geometrical force. I don't agree with this and take the view that gravity is just another field theory that has a particular geometrical interpretation. In any case, whatever ones philosophical leanings, there is no question that gravity and geometry are closely intertwined and so it is to non-Euclidean geometry that we now turn.

5.3 Non-Euclidean geometry

All of you have a working knowledge of non-Euclidean geometry. Even the ancient Greeks knew about non-Euclidean geometry. I will show you what I mean with a simple example.

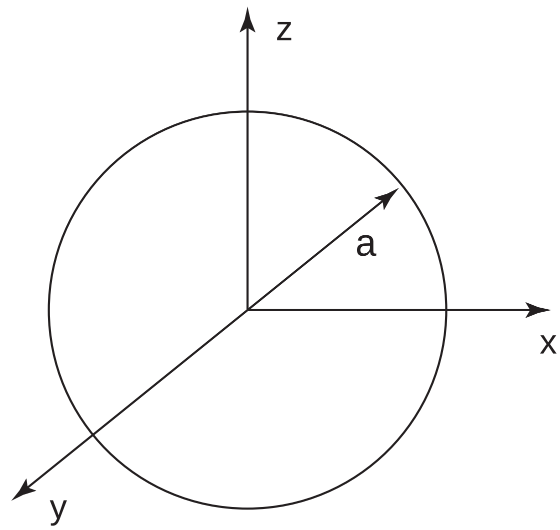

Figure 1: Sphere of radius a. The surface of the sphere is a two-dimensional surface (a two-sphere) embedded in three dimensional Euclidean space.

Let us imagine a usual Cartesian coordinate system x, y, z defining a Euclidean three-dimensional space with line element

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{1}$$

Now, suppose that we have a sphere of radius a with its centre at the origin of our coordinate system as shown in Figure 1. We will now ask the following question: what is the line element (*i.e.* the analogue of equation 1) on the surface of the sphere?

The equation defining the sphere is

$$x^2 + y^2 + z^2 = a^2.$$

So, differentiating this equation

$$2xdx + 2ydy + 2zdz = 0, \tag{2}$$

and so we can write an equation for dz,

$$dz = -\frac{(xdx + ydy)}{z} = \frac{-(xdx + ydy)}{\left[a^2 - (x^2 + y^2)\right]^{1/2}}.$$

Equation (2) thus provides a constraint on dz that keeps us on the surface of the sphere if we are displaced by small amounts dx and dy from an arbitrary point on the sphere. Substituting for dz in equation (1) gives us the interval for such constrained displacements (in other words, the metric for the surface of the sphere)

$$ds^2 = dx^2 + dy^2 + \frac{(xdx + ydy)^2}{[a^2 - (x^2 + y^2)]}.$$

Now make the substitutions

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}$$

and after a little algebra we get

$$ds^2 = \frac{a^2 dr^2}{(a^2 - r^2)} + r^2 d\theta^2, \tag{3}$$

(which, in the parlance of modern field theory, contains a 'hidden symmetry', namely our freedom to choose an arbitrary point on the sphere as the origin $r = 0$).

[Note that this looks different to the metric we would write down using spherical polars

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2,$$

but (3) is a valid metric nonetheless.]

It is worth your while spending a little time to understand the metric (3) and its construction. Think about how the coordinates r and θ relate to points on the spherical surface. If you don't understand this simple example, you don't have a hope of understanding the more complicated example that follows. As a simple test of your understanding, let's calculate the surface area of the sphere from the metric (3),

$$\begin{aligned} A &= 2\int_0^{2\pi}\int_0^a \frac{a}{(a^2 - r^2)^{1/2}} r dr d\theta \\ &= 4\pi a^2, \end{aligned}$$

i.e. the usual result for the surface area of a sphere. (Make sure that you understand where the factor of two comes from in front of the integrals).

We can make an analogous construction to find the metric for a **three-sphere** embedded in **four dimensional** Euclidean space (*i.e.* we want to find the metric on the surface of a four dimensional 'sphere').

The metric of the four dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{4}$$

and, in analogy with the example above, the equation defining a three-sphere is

$$x^2 + y^2 + z^2 + w^2 = a^2.$$

Differentiating as before gives

$$2xdx + 2ydy + 2zdz + 2wdw = 0,$$

and so substituting for dw in (4) gives the equation for the metric

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{[a^2 - (x^2 + y^2 + z^2)]}.$$

Transforming to spherical polar coordinates

$$\begin{aligned} x &= r \sin\theta \sin\phi, \\ y &= r \sin\theta \cos\phi, \\ z &= r \cos\theta, \end{aligned}$$

we get the metric

$$ds^2 = \frac{a^2}{(a^2 - r^2)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{5}$$

Notice that in the limit $a \to \infty$, the metric tends to the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

which is simply the metric of ordinary Euclidean three dimensional space

$$ds^2 = dx^2 + dy^2 + dz^2,$$

rewritten in spherical polar coordinates. The metric (5) therefore describes a *non-Euclidean* three dimensional space. Furthermore, the space is *finite* with a volume

$$\begin{aligned} V &= 2\int_0^a \int_0^{\pi/2} \int_0^{2\pi} \frac{ar^2 dr}{(a^2 - r^2)^{1/2}} \sin\theta d\theta d\phi \\ &= 2\pi^2 a^3. \end{aligned}$$

We can generate a metric for an *infinite* non-Euclidean three dimensional space by making the substitution $a = ib$, *i.e.* choosing the 'radius' of the space to be pure imaginary. The metric (5) then becomes

$$ds^2 = \frac{dr^2}{(1 + r^2/b^2)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{6}$$

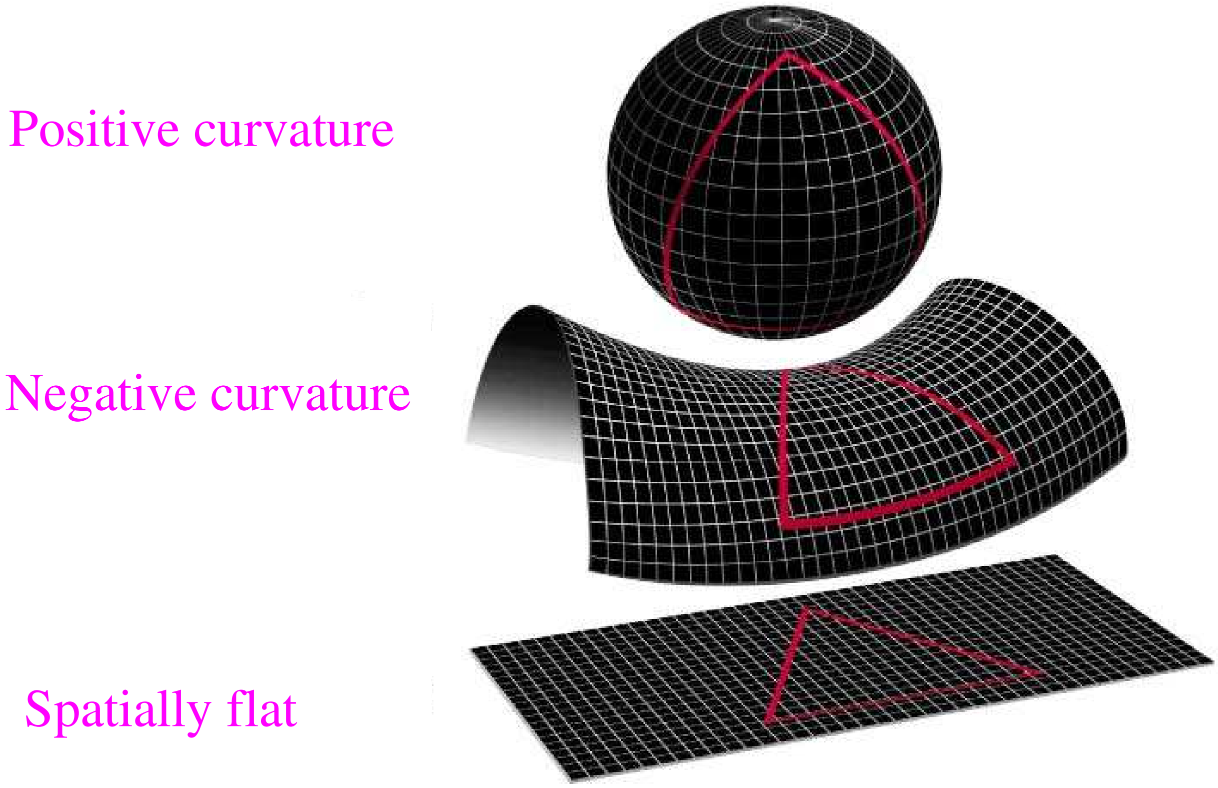

Figure 2: Positively curved, negatively curved and spatially flat surfaces.

The metrics (5) and (6) describe positively curved and negatively curved three-dimensional spaces, while the special case $a \to \infty$ corresponds to spatially flat Euclidean space. You can view these three-dimensional metrics as extensions to three-dimensions of the corresponding two-dimensional surfaces shown in Figure 2. In this section, we have derived the metrics for positively and negatively curved three-dimensional spaces by embedding the spaces within a 4-dimensional Euclidean space. However, you can view this as a mathematical device (or pictorial aid). The metric of three-dimension space is a valid quantity in its own right and one is not obliged to embed the three-dimensional space within a 4-dimensional Euclidean space.

5.4 Gauss's hypothesis

In a sufficiently small region, we can construct a locally Euclidean coordinate system $(\xi_1,\ \xi_2)$ such that the distance between two neighbouring points is

$$ds^2 = d\xi_1^2 + d\xi_2^2. \tag{6}$$

If a surface is not Euclidean, it will not be possible to cover any finite part of it with a Euclidean system ξ_1, ξ_2.

For a general coordinate system, $(x_1,\ x_2)$,

$$\begin{aligned}
d\xi_1 &= \left(\frac{\partial \xi_1}{\partial x_1}\right) dx_1 + \left(\frac{\partial \xi_1}{\partial x_2}\right) dx_2, \\
d\xi_2 &= \left(\frac{\partial \xi_2}{\partial x_1}\right) dx_1 + \left(\frac{\partial \xi_2}{\partial x_2}\right) dx_2.
\end{aligned}$$

So, substituting into (6),

$$ds^2 = g_{11}\left(x_1, x_2\right) dx_1^2 + 2 g_{12}\left(x_1, x_2\right) dx_1 dx_2 + g_{22}\left(x_1, x_2\right) dx_2^2$$

where

$$\begin{aligned}
g_{11} &= \left(\frac{\partial \xi_1}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_1}\right)^2, \\
g_{12} &= \left(\frac{\partial \xi_1}{\partial x_1}\right)\left(\frac{\partial \xi_1}{\partial x_2}\right) + \left(\frac{\partial \xi_2}{\partial x_1}\right)\left(\frac{\partial \xi_2}{\partial x_2}\right), \\
g_{22} &= \left(\frac{\partial \xi_1}{\partial x_2}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_2}\right)^2.
\end{aligned}$$

Transforming from coordinates $(x_1, x_2) \to (x_1', x_2')$ will change the form of the metric functions g_{ij}, *e.g.*, for the component g_{11}

$$\begin{aligned}
g_{11}' &= \left(\frac{\partial \xi_1}{\partial x_1'}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_1'}\right)^2 \\
&= \left(\frac{\partial \xi_1}{\partial x_1}\frac{\partial x_1}{\partial x_1'} + \frac{\partial \xi_1}{\partial x_2}\frac{\partial x_2}{\partial x_1'}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_1}\frac{\partial x_1}{\partial x_1'} + \frac{\partial \xi_2}{\partial x_2}\frac{\partial x_2}{\partial x_1'}\right)^2 \\
&= g_{11}\left(\frac{\partial x_1}{\partial x_1'}\right)^2 + 2 g_{12}\frac{\partial x_1}{\partial x_1'}\ \frac{\partial x_2}{\partial x_1'} + g_{22}\left(\frac{\partial x_2}{\partial x_1'}\right)^2.
\end{aligned}$$

Can we find a quantity that depends on g_{ij} and its derivatives and provides a measure of derivation from Euclidean geometry, independent of the coordinate system?

This problem was solved by Gauss in 1827 for two dimensional surfaces. Gauss discovered what is now known as the *Gaussian curvature*:

$$\begin{aligned} K(x_1,\ x_2) \ &= \ \frac{1}{2g}\left[2\frac{\partial^2 g_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2}\right] - \frac{g_{22}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x_1}\right)\left(2\frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right) - \left(\frac{\partial g_{11}}{\partial x_2}\right)^2\right] \\ &+ \ \frac{g_{12}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x_1}\right)\left(\frac{\partial g_{22}}{\partial x_2}\right) - 2\left(\frac{\partial g_{11}}{\partial x_2}\right)\left(\frac{\partial g_{22}}{\partial x_2}\right) + \left(2\frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right)\left(2\frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right)\right] \\ &- \ \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x_2}\right)\left(2\frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right) - \left(\frac{\partial g_{22}}{\partial x_1}\right)^2\right], \end{aligned}$$

where

$$g(x_1, x_2) = \det|g_{ij}| = g_{11}g_{22} - g_{12}^2.$$

If we evaluate this expression for the surface of a sphere, we find that the Gaussian curvature is $K = 1/a^2$. It is no real surprise that the curvature of a highly symmetric space such as a spherical surface is related to the radius a. But Gauss's formulae gives a general relation for expressing the curvature of any *arbitrary* surface at any point. Where does Gauss's formidably complicated formula come from? We will find out in later lectures – not only that, but we will generalize Gauss's analysis to any number of dimensions.

In fact, in a space of D dimensions, there are

$$D(D+1)/2$$

independent metric functions ($g_{\mu\nu}$ is symmetric). The freedom to choose the functional forms of the D coordinates (which of course does not change the nature of the space itself) allows us to impose D functional relations on $g_{\mu\nu}$. This means that the space is characterised by

$$C = \frac{D(D+1)}{2} - D = \frac{D(D-1)}{2}$$

functions, so

$$D = 2 \rightarrow C = 1$$

$$D = 3 \rightarrow C = 3$$

$$D = 4 \rightarrow C = 6$$

For $D = 2$, there is *one* function that expresses the 'character' or 'inner' property of the space – and this is the Gaussian curvature K. For three dimensions, we need to find *three* functions and so the problem is much more complicated. This formidable mathematical problem was solved by Riemann and we will need to understand some of the mathematics underlying Riemannian geometry before we can construct a consistent theory of gravity.

Before we do that, however, we can make a remarkable amount of progress by developing a formalism to describe freely falling particles and studying the motion of a particle in a *weak* gravitational field.