Multi-Level Monte Carlo Methods for SPDEs

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Motivating Example: Stochastic Heat Equation

SPDE: additive noise model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma \xi(x, t)$$

where:

- $u(x, t)$ is the solution (random field)
- $\xi(x, t)$ is space-time white noise
- σ controls noise amplitude

Why study this?

- Arises in physical systems with thermal or environmental randomness
- Captures uncertainty and variability in diffusion processes

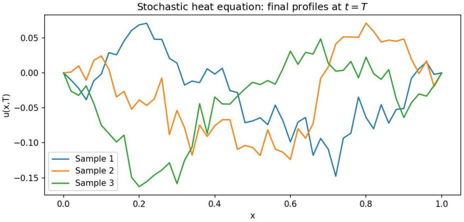

Illustration: sample profiles of $u(x, T)$ at final time T.

MLMC Approach to Solving the Stochastic Heat Equation

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- To estimate expectations like $\mathbb{E}[u(x,T)]$ or $\mathbb{E}\left[\int_0^1 u(x,T)^2\,dx\right]$, we can use standard Monte Carlo (MC):
  - Simulate many independent sample paths of the SPDE
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- For the stochastic heat equation, this means solving on multiple space-time grids and carefully coupling the noise

Make Titles Informative

Standard Monte Carlo:

$$\mathbb{E}[P] \approx \frac{1}{N} \sum_{i=1}^{N} P^{(i)}$$

Cost: $N \cdot (\text{cost per sample}) \sim \varepsilon^{-2} \cdot \varepsilon^{-2}$

$$= \boxed{O(\varepsilon^{-4})}$$

Multi-Level Monte Carlo:

$$\mathbb{E}[P_L] = \sum_{\ell=0}^{L} \mathbb{E}[P_\ell - P_{\ell-1}]$$

$$\approx \sum_{\ell=0}^{L} \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \left(P_\ell^{(i)} - P_{\ell-1}^{(i)}\right)$$

What Are SPDEs? And Why Are They Hard?

- **Stochastic partial differential equations (SPDEs)** model the evolution of systems under spatial and temporal uncertainty:

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(u, x, t) + \text{noise}$$

  where the noise may be spatially correlated, time-dependent, or both (e.g. space-time white noise).
- **Analytical difficulties** arise because many SPDEs lack classical solutions:
  - Noise is often modelled as a generalised function (e.g. white noise), requiring weak or mild formulations.
  - Nonlinearities and multiplicative noise may lead to ill-posedness or singular behaviour.
- **Example: The Dean–Kawasaki equation** (a parabolic SPDE modelling stochastic particle densities):

$$\partial_t \rho = \frac{1}{2}\Delta\rho + \nabla \cdot (\rho \nabla V * \rho) + \frac{1}{\sqrt{N}}\nabla \cdot (\sqrt{\rho}\,\xi)$$

MLMC for SPDEs: Method and Complexity

- Let u be the solution to an SPDE and let $Q(u) \in \mathbb{R}$ be a functional of interest, e.g.

$$Q(u) = \int_D u(x,T)^2 \, dx$$

- Introduce a sequence of approximations $Q_\ell := Q(u_\ell)$, where u_ℓ is the numerical solution on grid level ℓ, with mesh size $h_\ell \sim 2^{-\ell}$
- The MLMC estimator:

$$\mathbb{E}[Q_L] = \sum_{\ell=0}^{L} \mathbb{E}[Q_\ell - Q_{\ell-1}] \quad \text{with } Q_{-1} := 0$$

$$\hat{Q}_{\mathrm{MLMC}} := \sum_{\ell=0}^{L} \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \left(Q_\ell^{(i)} - Q_{\ell-1}^{(i)} \right)$$

- Sample costs $C_\ell \sim h_\ell^{-\gamma}$, variances $V_\ell := \mathbb{V}[Q_\ell - Q_{\ell-1}] \sim h_\ell^{\beta}$, and bias $\sim h_L^{\alpha}$
- Under conditions $\beta > \gamma$, the MLMC estimator achieves:

MLMC for SPDEs: Method and Variance

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- Let u solve an SPDE and $Q(u) \in \mathbb{R}$ be a quantity of interest (e.g., energy functional).
- Define approximations $Q_\ell := Q(u_\ell)$, where u_ℓ is the numerical solution at level ℓ.
- The MLMC identity:

$$\mathbb{E}[Q_L] = \sum_{\ell=0}^{L} \mathbb{E}[Q_\ell - Q_{\ell-1}], \quad Q_{-1} := 0$$

- Each term is estimated via Monte Carlo:

$$Y_\ell := \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \left(Q_\ell^{(i)} - Q_{\ell-1}^{(i)} \right)$$

- The total MLMC estimator is $\sum_{\ell=0}^{L} Y_\ell$, with variance:

$$\mathbb{V}[\hat{Q}_{\mathrm{MLMC}}] = \sum^{L} \frac{V_\ell}{N_\ell}, \quad V_\ell := \mathbb{V}[Q_\ell - Q_{\ell-1}]$$