Multilevel Monte Carlo for Stochastic PDEs

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0.1 Introduction

Stochastic partial differential equations (SPDEs) commonly arise in the mathematical modelling of systems whose evolution is significantly influenced by random fluctuations. They have been employed in models of turbulence, population dynamics, neurophysiology, finance, fluid dynamics and climate modelling [3] [1]. Solving SPDEs is often inherently challenging however. Very few cases have analytical solutions, and numerical solutions have their own obstacles. Additional dimensionality imposes a higher computational cost, one may have to sample over many realisations of the noise, and some SPDEs are highly singular.

The Monte Carlo method is a natural avenue of approach when trying to solve SPDEs. Repeated samples of finite-difference or finite-element derived quantities may be obtained, and an estimate of a desired quantity of interest therefore computed. They can be exhoribitantly costly however. To achieve an accuracy of ϵ in one's estimate, the standard Monte Carlo estimator requires $O(\epsilon^{-2})$ samples be computed. To improve on this a number of variance reduction techniques have been introduced, their aim being to achieve an equivalent accuracy at a reduced cost.

The Multilevel Monte Carlo (MLMC) method is one such technique. Its tenet is to sample from a blend of cheap, less accurate samples and expensive, more accurate samples, and then combine those to obtain one's estimate. Provided this is implemented apprpriately, this can result in a cost reduction when compared to standard Monte Carlo.

This dissertation investigates applying the MLMC to solve SPDEs. In particular, this dissertation has the following goals:

- Quantify the cost reductions achievable with MLMC methods, using MC methods as a baseline.
- Explore variations in MLMC implementations across different SPDEs.
- Present performance improvements achievable with high-performance implementations on GPUs

This dissertation will have the following structure. Section ?? will cover the relevant prerequisites. It will give an overview of the MLMC method as it pertains to SPDEs, before illustrating with a simple example. A review of the relevant literature will then be given. Finally, some relevant definitions will be covered. Section ?? will then begin by introducing the implemented MLMC process. Much of the section is

given over to validation of quantities, with particular focus given to the stochastic heat equation. Finally we go onto investigating results.

0.2 Prerequisites

0.2.1 Multilevel Monte Carlo

The MLMC method, introduced by Giles [2], is a variance reduction technique designed to substantially reduce the computational cost of estimating expectations arising from stochastic systems compared to the standard Monte Carlo method.

Given some quantity P, the standard Monte Carlo estimate of $\mathbb{E}[P]$ is given by the average of N independent samples of P:

$$\mathbb{E}[P] \approx \frac{1}{N}\sum_{n=1}^{N} P^{(n)} = \hat{P}_{MC}$$

The variance of this estimate is $N^{-}1\mathbb{V}[P]$, therefore the standard error is $O(N^{\frac{1}{2}})$ and consequently to achieve an accuracy of ε, $N = O(\varepsilon^{-2})$ samples are required.

The treatment above assumes that each Monte Carlo sample $P^{(n)}$ is an exact evaluation of the quantity P. In Practice, however, such evaluations are rarely possible. Typically the quantity of interest P is a functional of some unknown solution $u(x,t)$ and u is governed by some differential equation. In this dissertation, this will be the case, as $u(x,t)$ will be governed by an SPDE. Since analytic solutions of SPDEs are seldom available, numerical methods such as finite-difference or finite-element schemes must be employed. Consequently, each sample $P_h^{(n)}$ is subject not only to statistical sampling error, but also to a bias or discretisation error stemming from the finite numerical resolution of the underlying mesh, characterised by a mesh size h. A common measure of the accuracy of Monte Carlo estimates is the Mean Square Error (MSE). It can be shown that the MSE accounts for both the bias in the estimator and its variance. The Root Mean Square Error is also commonly employed.

$$\text{MSE} \equiv \mathbb{E}\left[(P_h - \mathbb{E}[P])^2\right] = \underbrace{(\mathbb{E}[P_h] - \mathbb{E}[P])^2}_{\text{Bias}^2} + \underbrace{\mathbb{V}[\hat{P}]}_{\text{Variance}}. \tag{1}$$

This distinction is fundamental. Simply increasing the number of Monte Carlo samples cannot reduce discretisation bias; only mesh refinement (reducing h) can achieve this. Conversely, reducing variance requires larger N, i.e. more samples. The motivation behind the Multilevel Monte Carlo method is to balance these competing requirements efficiently. To achieve this, MLMC leverages a heirarchy of discretisation to simultaneously control both bias and variance at significantly reduced computational cost.

MLMC introduces multiple discretisation levels, denotes by mesh sizes $h_\ell = M^{-\ell} h_0$ for $\ell = 0, 1, ..., L$ with $M \geq 2$. At each level ℓ, let P_ℓ denote the numerical approximation of quantity P. Then, exploiting the linearity of expectation, one can obtain the following telescoping sum:

$$\mathbb{E}\left[P_L\right] = \mathbb{E}\left[P_0\right] + \sum_{\ell=1}^{L} \mathbb{E}\left[P_\ell - P_{\ell-1}\right] \tag{2}$$

By obtaining a Monte Carlo estimate for each term, equation (2) brings us to the MLMC estimator:

$$\hat{P}_{MLMC} = \frac{1}{N_0} \sum_{n=1}^{N_0} P_0^{(n)} + \sum_{\ell=1}^{L} \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \left(P_\ell^{(n)} - P_{\ell-1}^{(n)} \right) \tag{3}$$

Bibliography

[1] Diego Alonso-Orán, Aythami Bethencourt de León, Darryl D Holm, and So Takao. Modelling the climate and weather of a 2d lagrangian-averaged euler–boussinesq equation with transport noise. *Journal of Statistical Physics*, 179(5):1267–1303, 2020.

[2] Michael B Giles. Multilevel monte carlo path simulation. *Operations research*, 56(3):607–617, 2008.

[3] Étienne Pardoux et al. *Stochastic partial differential equations: An introduction*. Springer, 2021.