Introductory Astrophysics (PHYS08050) Notes

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Abstract

This set of notes is made with reference to the Astrophysics Coursebook by The University of Edinburgh. I have gained permission from Professor Catherine Heymans to put my modified notes in my website. It is also available in my personal website.

1 Stars

1.1 White dwarfs

- The assumption that white dwarfs can be described as ideal gas is wrong.
  - Because at $10^{10}k$, the next phase of fusion burning would commence, turning the white dwarfs Carbon and Oxygen core into Neon and Magnesium and the luminosity would rocket up.
  - This does not happen!
- The gas in a white dwarf is called **degenerate**
- Carbon nuclei (including oxygen?!) (as the end products of the triple-alpha process) and electrons inside a white dwarf are packed together at such extreme densities that electrons start to overlap with each other

1.2 Supernovae

- **Note that the sections about supernovae are directly copied out from coursebook as I have very limited knowledge!**
- The most massive stars become short-lived, very luminous, supergiants where the core temperature becomes so extreme that successive fusion **s-processes** occur
- Heavier and heavier elements are created in different shells within the star, resulting in an onion-like structure around a central Iron core
- Iron has the highest binding energy per nucleon

- No further fusion reactions will produce energy and so the core starts to contracts. Just as in the white dwarf star, electron degeneracy pressure will increase as the star shrinks, but in these massive stars even it is unable to halt the collapse.
- Large quantities of neutrons are created by the collapse as Iron nuclei are broken up by photons. Elements heavier than Iron are then produced through rapid **r-process** reactions with the new flux of neutrons. The core crushing continues and the core becomes neutron degenerate. Neutron degeneracy pressure builds up quickly causing the inner part of the core collapse to come to a sudden halt and rebound slightly.(*i.e.* It is still stopped by Neutron degeneracy pressure. What if it doesn't? Check the case for Black Holes!) This sends a giant shock wave out through the star, the end result of which is a giant supernova explosion.

1.3 Supernova 1a

- Stars often come in pairs
- They will typically start life with **different masses** and hence evolve at different rates.
- We will therefore often find binary pairings of white dwarf stars with other main sequence stars
- As the white dwarf cannibal gains in mass, its radius shrinks
- the uncertainty electron momentum increases until the electron speeds become relativistic.
- When the mass exceeds the Chandrasekhar limit with $M = 1.4M_{\odot}$,
- The white dwarf diamond core starts to collapse.
- When the compressed core finally becomes neutron degenerate the core collapse will come to a sudden halt and rebound slightly causing a supernova explosion
- Supernova 1a occurs at the precise moment when the mass exceeds the Chandrasekhar limit. It therefore looks the same, at any point in space or time,

2 Galaxies

2.1 Definition

- **Luminosity**
  - The total amount of energy an object radiates each second
- **Flux**

- The energy per second flowing through each square metre
- Defined as the **Apparent Brightness**

- **Surface Brightness**
  - The flux of light coming each unit solid angle on the sky (Refer to the back for the definition of **steradians**)

2.2 $\frac{3}{2}$ law

- We can define the formula below:

$$D_{\max} = \left(\frac{L}{4\pi F_{\min}}\right)^{\frac{1}{2}} \tag{1}$$

- We can hence define $V_{\max}$ as:

$$V_{\max} = \frac{4}{3}\pi {D_{\text{Max}}}^3 \tag{2}$$

$$= \frac{1}{6\pi^{\frac{1}{2}}}\left(\frac{L}{F_{\min}}\right)^{\frac{3}{2}} \tag{3}$$

  With a number density of n stars per unit volume[1], we have:

$$N = \frac{n}{6\pi^{\frac{1}{2}}}\left(\frac{L}{F_{\min}}\right)^{\frac{3}{2}} \tag{4}$$

  Where N is the total number of stars.

- In reality things don't work like this. The slope becomes $\frac{1}{2}$ for low fluxes. Why?
  - The likely reason is that the space is not infinite. There are fewer faint stars than our formula predicted [2]

- The reasons why the flattening off the star counts is gradual and not sharp:
  - The Milky Way star system probably does not have a hard edge, but fades out gradually
  - Stars are not all the same luminosity, so stars near the edge actually appear at a range of fluxes

2.3 Important Symbols

Below we introduce some of the most important symbols in Astrophysics.

[1] Here, we have assumed a constant density, which is rather fishy in my opinion
[2] Why however, what happens to stars with high flux?

Symbol	**Name**	**Definition**
θ^o	Degree	
θ'	Arcminute	$\frac{1}{60}$ of a degree
θ"	Arcsecond	$\frac{1}{60}$ of an arcminute

- We have

$$1\text{rad} = \frac{180^\circ}{\pi} \tag{5}$$
$$= \frac{(180)(60)}{\pi}\text{arcminutes} \tag{6}$$
$$= \frac{(180)(3600)}{\pi}\text{arcseconds} \tag{7}$$

  Rearranging, we also obtain:

$$1\text{arcsecond} = \frac{1}{\frac{(180)(3600)}{\pi}}\text{rad} \tag{8}$$
$$= 4.848... * 10^{-6}\text{rad} \tag{9}$$
$$= 4.85\text{micro-radians} \tag{10}$$

  Meanwhile, 1 steradian (sr) is equal to:

$$\left(\frac{180}{\pi}\right)^2 \text{sq degrees} \tag{11}$$

2.4 Angular resolution

- By some complex analysis, we obtain the formula:

$$\theta = \frac{1.22\lambda}{D} \tag{12}$$

- Factors that affect angular resolution
  - Optical distortions
  - Atmospheric turbulence

2.5 Brightness of the sky

Assuming stars have constant number density n. Consider a shell of thickness dR at distance R from us. The number of stars in this shell is hence $N = 4\pi n R^2 dR$. Since each star produces a flux of

$$F = \frac{L}{4\pi R^2} \tag{13}$$

The total flux is therefore:

$$F_{\text{tot}} = NF \tag{14}$$
$$= \frac{NL}{4\pi R^2} \tag{15}$$

With the definition of surface brightness,

$$B = \frac{nLdR}{4\pi} \tag{16}$$

If we sum up the total brightness by integration, we can see that it doesn't converge, so there's something wrong with our approximations.

2.6 Dust

- The space between stars is not empty, they are filled with dust.
- Let x be the distance travelled by the dust. Consider that when $dF = 0$, $dx = 0$. With other estimations, we have:

  $$\frac{dF}{F} = -kdx \tag{17}$$

  Hence,

  $$\frac{F}{F_o} = e^{-kx} \tag{18}$$

- We define $x_e = \frac{1}{k}$, which is the distance over which the flux changes by a factor or $\frac{1}{e}$. Hence,

  $$\frac{F}{F_o} = e^{-\frac{x}{x_e}} \tag{19}$$

- For example, if every 2 kly light is reduced by a factor of e, at 6 kly, light is reduced by a factor of e^3.
- Longer wavelength light has a longer scale length as it is absorbed less.
- The size of cosmic dust grains is similar to the wavelength of visible light, which is why extinction is so sensitive to wavelength - much larger waves sail past and hardly notice the dust; much smaller waves are simply blocked.

2.7 Cosmic Distances

- The radius of the earth is denoted by R_E, which is defined as 1 A.U. (Astronomical Unit)
- Angle between direction of Sun and direction Venus is called the angle of elongation (θ_E)

- When θ_E is maximum, the line joining Venus and Earth is perpendicular to the line joining Venus and Sun [3]. We can hence measure the distance to Venus easily.[4]

2.8 Parallex

- Why it works?
  - Very distant stars show minimal angular motion compared to nearby stars as size of angular motion is inversely proportional to distance of the star , so we can assume the pattern of background stars is fixed.
- The distance at which a star has an annual parallex of 1 arcsec is defined as 1 parsec (pc)
$$D(pc) = \frac{1}{\theta"} \tag{20}$$
  For example, the nearest known star, Proxima Centauri, is at a distance of 1.30pc. The distance to Alpha Centauri is around 1.34 pc.
- It is very difficult to measure parallax accurately in Ground-based measurements. The Hipparcos space mission, however, enables us to measure measure parallaxs with an error or 1mas, for relatively bright stars.

2.9 True luminosity

- If we know the true luminosity[5] L of an object and measure its flux F, then we can deduce the distance. Recall that:
$$F = \frac{L}{4\pi D^2} \tag{21}$$
$$D = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}} \tag{22}$$
- To deduce the true luminosity, we can try to locate it on the HR diagram. However, how do we find its temperature? By Wien's displacement law [6], we can try to estimate its colour. However, we encounter some problems
  - Colours don't necessarily uniquely correspond to a particular temperature.
  - Colours are wrong due to dust extinction [7]

  Besides, the distance between us and the stars is overestimated as dust reduces the flux of the stars [8]

[3] Why?
[4] Using trigonometry, try to deduce a formula on your own!
[5] Check previous Notes!
[6] Which should be memorized at this very stage
[7] it depends on the wavelength of light
[8] Is it correct to say flux of the stars?

- Solution: Detailed **spectroscopy**
- Recall that Flux scales are the inverse square of distance. For example, if a thing is 2 times apart, the flux is 4 times less.
- Check how to convert from arcsec to angles, angles to pc and pc to meters!

2.10 Pulsating Variables as Standard Candles

You can read off the period of a Type 1 Cepheid and locate it to its luminosity in the log graph in figure 21. Afterwards, we can find its flux [9] and using (22), we can find the distance.

2.11 Supernovae as Standard Candles

Type I (in particular Type 1a) are "white dwarf bombs". this involves the conflagration of a White Dwarf exactly at its Chandrasekhar limit, the amount of energy released is always the same, so they make excellent standard candles.(See the above for details!)

2.12 Stellar Luminosity function

- The stellar luminosity function $\phi(L)$ refers to the number of stars per unit luminosity per unit volume of space
- More luminous stars can be seen further away, so plotting a graph of number of stars/luminosity is misleading. Solutions:
  - **Method 1**: Try to measure all stars. However, this is difficult.
  - **Method 2**: Recall that :

$$N(L) = \phi(L) * V_{\max}(L) \tag{23}$$

  - From $\frac{3}{2}$ law as discussed in Chapter 4, we have:

$$N(L) \propto L^{\frac{3}{2}}\phi(L) \tag{24}$$

  - After correcting for $V_{\max}$ effect, We have found out that $\phi(L) \propto L^{-\alpha}$, where $\alpha = -1.35$ by observations.

2.13 Stellar Mass Function

- We can rearrange equations and obtain:

$$n(M) = \phi(L)\frac{dL}{dM} \tag{25}$$

[9] Find out how to find the flux of a star!

Hence, with the assumption that $L \propto M^\beta$

$$n(M) \propto L^{-\alpha} M^{\beta-1} \propto M^{-\alpha\beta+\beta-1} \propto M^{\beta(1-\alpha)-1} \tag{26}$$

- Now we plug in $\alpha = -1.35$ and $\beta = 4$ and obtain $n(M) \propto M^{-2.4}$
- However, in the coursebook, rather confusingly, we use $n(M) \propto M^{-2.35}$ and $\phi(L) \propto L^{-1.35}$
- While **integrating**, remember that we are calculating $L \times \phi(L)dL$ and $m \times n(m)dm$ respectively - For More **please check Exercise 59**

2.14 Lifetime of stars

2.14.1 Motivation

- In Pleiades, $L^{-1.35}$ is consistent for all solar masses
- This isn't expected for $> 5M_\odot$
- It shows that some original high mass stars have disappeared, why?

2.14.2 Calculation

- In general, burnable fraction f varies along the mass range, with $f \propto M$. Considering other factors:

$$t_* = 10.8 \left(\frac{M_*}{M_\odot}\right)^{-2} \text{Gyr} \tag{27}$$

2.15 Lifetime of Milky way

- The fraction of all the M_* stars that are still here is $\frac{t_*}{T}$ [10]

2.16 Hydrogen gas

- In Hydrogen 1s state, there are two levels with slightly different energy levels, caused by Nuclear spin and Electron spin. The parallel state has a slightly higher energy than the anti-parallel state (think about the case of magnets)
- The total 21cm line luminosity from a region is just proportional to the number of H atoms it contains

2.17 Molecular gas

- In most of the ISM (Interstellar Medium) hydrogen is in atomic form
- In cold dense regions the starlight can be blocked out and molecules form

[10] I personally don't quite understand this statement

2.18 Dust

- The energy of starlight that dust particles are absorbing in the visible light regime has to come out somewhere else
- Flat-slab approximation:

$$P_{\text{abs}} = A \times \frac{L}{4\pi D^2} \tag{28}$$

$$P_{\text{emit}} = A\sigma T^4 \tag{29}$$

2.19 Galaxy Types

Spirals	**Ellipticals**	**Irregulars**
Flat disc with spiral arms	No disc	Disc
Central bulge and spheroidal halo	Single spheroidal component	Sometimes bulge/halo
Blue and red stars	Red stars only	Blue and red stars
Lots of gas	No gas	Lots of gas

- **Spirals** mainly form stars along the spiral arms
- **Ellipticals** stopped making stars a long time ago since the massive blue stars have all disappeared)[11]
- The reason is that they have no gas, which is the material needed to make new stars
- **Irregulars** don't have gas because:
  - Gas has not "settled down" into a disc
  - The disc has been disrupted by collision with another galaxy

2.20 Spiral Discs

- The brightness I as a function of separation from the centre R is given by:

$$I(R) = I_0 e^{\frac{-R}{R_d}} \tag{30}$$

- Nearly all spiral discs have pretty much the same value of I_0, but different values of R_d
- With $\int_0^\infty x^n e^{-x} dx = n!$, the reader can try to prove that the total light present is given by:

$$F_{\text{tot}} = 2\pi {R_d}^2 I_0 \tag{31}$$

[11]Since blue stars have a shorter lifespan

2.21 Ellipticals

$$I(R) = I_0 \exp[\frac{-R}{R_e}]^{\frac{1}{4}} \tag{32}$$

2.22 Dark Matter

- The predicted rotation curve still disagrees with the observations badly
- The expected rotation curve is flat
  - For increasing radius the velocity remains constant
- How does dark matter depend on radius?

$$V^2 = \frac{GM}{R^2} \tag{33}$$

- Therefore $M_{\text{dark}}(< R) \propto R$
  - The $(< R)$ part is an obvious consequence of Birkhoff's theorem from General Relativity
- Considering the density (ρ_{dark}):

$$\rho_{\text{dark}} = \frac{M_{\text{dark}}}{R^3} \tag{34}$$

$$\propto \frac{1}{R^2} \tag{35}$$

- Therefore dark Matter is concentrated near the center
- A large galaxy like the Milky Way has about $10^{11}M_\odot$

  in stars, and $10^{12}M_\odot$ in dark matter

2.23 Random star motions and interactions in spheroids

- Elliptical galaxies do not rotate.
- What is the chance of star collisions?

2.23.1 Probability

- Given by the ratio of $\pi {R_{\text{sep}}}^2$ and πR^2, **where $R = 2R_*$ and R_* is the radius of the stars**

$$p = \left(\frac{R}{R_{\text{sep}}}\right)^2 \tag{36}$$

$$= \left(\frac{2R_*}{R_{\text{sep}}}\right)^2 \tag{37}$$

The second line only applies if they have the same size

Time between collisions

- Consider a cylinder covering the path of the stars
- As a convention, we let σ as the cross-sectional area of the cylinder
- Consider the number density of the stars is n, then the number of stars in the cylinder:
$$N = n\sigma v t_{\text{collision}} \tag{38}$$
We set $N = 1$ and hence:
$$t_{\text{coll}} = \frac{1}{n\sigma v} \tag{39}$$
- After substitution, we find out $t_{\text{coll}} >$ age of universe Gravitational Radius R_{grav} The radius in which the kinetic and potential energy of the passing star are equal in magnitude is defined as the R_{grav}. For $R_{\text{star}} < R_{\text{grav}}$, its path is deflected by the other star. Assuming both stars have the same mass m: For potential energy:
$$U = \frac{-Gm^2}{R} \tag{40}$$
For kinetic energy:
$$K = \frac{mv^2}{2} \tag{41}$$
Taking their magnitudes:
$$\frac{Gm^2}{R} = \frac{mv^2}{2} \tag{42}$$
$$2Gm = Rv^2 \tag{43}$$
$$R_{\text{grav}} = \frac{2Gm}{v^2} \tag{44}$$

Galaxy collisions

- Stars almost never collide, but galaxies do
- Galaxy interactions make strange non-symmetric disturbed structures as they pass by each other

2.24 Active Galactic Nuclei

2.24.1 Luminosity

- All galaxies except irregulars are brightest at the centre. However a minority have an especially bright central spot
- Example: Quasars
- Some galaxies can have nuclear luminosities which are around 10% of the luminosity of an entire normal galaxy of stars like the Milky Way
- Do exercise 73

2.24.2 Board emission lines

- AGN emission lines are very broad. The $H\alpha$ line at 656.3nm is about 20nm across.
- The broadening of the emission lines must be due to internal gas motions in the nucleus
- See the notes for Doppler Shift!

2.24.3 Multi-wavelength spectral energy distribution

- AGNs radiate strongly at all sorts of wavelengths
- The feature which dominates the energy output for most AGN output however is the "Big Blue Bump", which peaks in the UV
- There are secondary peaks of emission in the mid-infrared (at a wavelength of around 10μm), and in the X-rays.
- **UV**
  - It peaks around $\lambda = 0.03\mu m$ suggesting $T \approx 10^5 K$
  - However, the UV bump looks broader than a single blackbody
  - It is variable
- **IR**
  - It peaks around $\lambda = 10\mu m$ suggesting $T \approx 10^5 K$
  - The IR bump is almost certainly due to dust re-radiation, but the dust is hotter than the dust we normally see in the interstellar medium in normal galaxies, which is it at $T \approx 50K$
- **X-rays**
  - It is almost certainly not due to blackbody radiation, but if the emission is due to hot gas in some way, it must have a wide range of temperatures, $10^{7-9}K$.

2.25 Variability of AGNs

- Vary erratically, cannot be used like Cepheids
- There is some disturbance transmission mechanism (some sort of waves). The transmittion time:

$$\Delta t = \frac{R}{v_d} \tag{45}$$

- Waves are transmitted by particle collisions

- If we equate the Thermal Energy:

$$E = \frac{3kT}{2} \tag{46}$$

with the Kinetic Energy:

$$K = \frac{mv^2}{2} \tag{47}$$

- We obtain:

$$\frac{3kT}{2} = \frac{mv^2}{2} \tag{48}$$

$$3kT = mv^2 \tag{49}$$

$$v = \sqrt{\frac{3kT}{m}} \tag{50}$$

- Assume $T = 10^5 k$ and we obtain 49.8 kms^{-1}**, and according to Astrophysicists**[12]**, 31.7 $\approx$ 49.8, so we can see it probably comes from sound waves** [13]

2.26 Motion around AGN

- Recall our equation:

$$v^2 = \frac{GM}{R} \tag{51}$$

- Most of the time we only see a single unresolved spot
- However, for some of the closest AGN, we can see glowing gas which is ionised by the AGN at large enough distances to visually separate from the central spot
- Need to correct the angle since movements along the plane of sky (as opposed to plane of sight) don't induce Doppler's shift
  - Solution: Observe the image on the sky, if it looks like a ellipse

2.27 Emission line widths and time lag

- Some gases emit lines, they are ionised by the time-varying UV light
- Time lag t_{lag} exists since light need time to travel from centre of AGN to the gas

$$R_{\text{Broad line region}} = ct_{\text{lag}} \tag{52}$$

[12]The assumptions in all the calculations are too approximate for my liking!

[13]31.7 kms^{-1} is obtained in **Exercise 77**

Afterwards, we can obtain V_{BLR} by estimating the line width. The mass M can be estimated by:

$$M = \frac{R_{\mathrm{BLR}} V_{\mathrm{BLR}}^2}{G} \tag{53}$$

2.28 AGN masses and luminosity

- Stars range over about a factor of 10^9 (1 billion) in luminosity, but only about a factor of 500 in mass
- Similarly, AGNs range over about a factor of 10^6 (1 million) in luminosity, but only about a factor of 1000 in mass
- In AGNs, there is little correlation between mass and luminosity, as opposed to stars

2.29 Dark masses

- By measuring the width of absorption lines in the light from a galaxy, we can estimate the speed of random motions of the stars in that galaxy [14]
- If we make such a measurement at a series of different locations within the galaxy, we can see that stellar velocities start to increase rapidly towards the centre of most galaxies
- From the methods introduced above, we can deduce the mass in the centre of AGN, which is far less than the stellar mass estimated from starlight
- We deduce the "missing mass" is dark mass

2.30 Power from black hole accretion

- If a small mass Δm starts from a distance and falls onto a large mass M, it gains kinetic energy:

$$E = \frac{GM\Delta m}{R} \tag{54}$$

- For gases however, friction and collisions randomise the energy and turn it into heat
- For a black hole:

$$R_{\text{Event Horizon}} = \frac{2GM}{c^2} \tag{55}$$

- By substitution,

$$E = \frac{GM\Delta m}{\frac{2GM}{c^2}} \tag{56}$$

$$= \frac{1}{2}\Delta m c^2 \tag{57}$$

[14] How is it linked to "broad emission lines" in the section above?

- The efficiency, μ, is 0.5, which is much higher than $\mu_{\text{nuc}} = 0.007$
- This is obviously, a fallacy
- **Incorrect Assumptions**
  - Radial free fall is unlikely
  - At larger distances, the accreting material will always have some angular momentum, and end up forming a rotating disc around the black hole
  - Friction between neighbouring radial annuli then allows the material to slowly spiral inwards, forming a gradually heated accretion disc
  - Efficiency μ is reduced by the following 2 effects:
    * Effective radius is **not** the Event Horizon. For non-rotating black-holes, the ISCO is located at:

$$r_{\text{ms}} = 6\frac{GM}{c^2} \tag{58}$$
$$= 3R_{\text{EH}} \tag{59}$$

    * I.e. There is no stable orbit $< 3R_{\text{EH}}$
    * And therefore the thermal energy gained by the gas decreases
    * Some energy (half) is converted to rotational energy, as:

$$\Delta K = \frac{dK}{dR}\Delta R \tag{60}$$
$$= -\frac{GM\Delta m}{2R^2}\Delta R \tag{61}$$

$$\Delta U = \frac{dU}{dR}\Delta R \tag{62}$$
$$= -\frac{GM\Delta m}{R^2}\Delta R \tag{63}$$

$$\frac{\Delta K}{\Delta U} = \frac{-\frac{GM\Delta m}{2R^2}\Delta R}{-\frac{GM\Delta m}{R^2}\Delta R} \tag{64}$$
$$= \frac{1}{2} \tag{65}$$

Combining with other effects, we have $\mu \approx 0.1$

2.31 Eddington Luminosity

- A photon has energy $E = \frac{hc}{\lambda}$ and momentum $p = \frac{h}{\lambda} = \frac{E}{c}$. So you can visualize momentum flux being $\frac{S}{c}$, where S is the radiation flux
- Most of that flux may pass straight through, but some of it will scatter on the electrons inside atoms. That scattering produces a force on the electrons, which drag the atoms with them. The scattering process has a cross-section $\sigma_e = 6.651029m^2$
- The **rate** [15] of momentum transfer (or in other words, radiation force on each atom) is therefore:
$$\frac{dp}{dt} = \frac{S\sigma_e}{c} \tag{66}$$
- Combining with $L = \frac{S}{4\pi R^2}$ we obtain:
$$F_{\text{radiation}} = \frac{L\sigma_e}{4\pi R^2 c} \tag{67}$$
- Assuming all the gas are hydrogen and each hydrogen atom composes of 1 proton, which has mass of m_p. As $m_e << m_p$, we have:
$$F_{\text{gravitation}} = \frac{GMm_p}{R^2} \tag{68}$$
- At the limiting luminosity, the outward radiation force is equal to the inwards gravitational force, *i.e.* $F_{\text{radiation}} = F_{\text{gravitation}}$. Combining everything gives us:
$$L_{\text{Max}} = \frac{4\pi G m_p c}{\sigma_e} \times M \tag{69}$$

3 Cosmology

3.1 Big Bang

- Consider a galaxy on the edge of a sphere in the universe centered on us
- By Birkhoff's theorem
  - This galaxy is effected gravitationally only by the mass within the sphere
  - The mass within a sphere acts like a point mass at the centre
- Hence the energy of the galaxy:
$$E = \frac{mV^2}{2} - \frac{GMm}{r} \tag{70}$$

[15] This implies differentiation

- **If $E < 0$, the universe re-collaspes in a big crunch**
- **If $E > 0$, the universe continues to expand**

3.2 Hubble constant

- It changes with radius (see below)
- An important equation:

$$z = \frac{H_0 D}{c} \tag{71}$$

This is an obvious result from Doppler's shift. Recall that $z = \frac{\Delta\lambda}{\lambda}$, $V = H_0 r$ and $\frac{\Delta\lambda}{\lambda} = \frac{v_r}{v_w}$. Substitute everything and you'll get it.

3.3 Critical Density of the Universe

- Consider the equations below:

$$\frac{mV^2}{2} = \frac{GMm}{r} \tag{72}$$

$$M = \frac{4\pi r^3}{3}\rho_{\text{crit}} \tag{73}$$

We can find ρ_{crit} by combining the equations and using $V = H_0 r$

$$\frac{mV^2}{2} = \frac{4Gm\pi r^3 \rho_{\text{crit}}}{3r} \tag{74}$$

$$3V^2 = 8G\pi r^2 \rho_{\text{crit}} \tag{75}$$

$$3{H_0}^2 r^2 = 8G\pi r^2 \rho_{\text{crit}} \tag{76}$$

$$3{H_0}^2 = 8G\pi \rho_{\text{crit}} \tag{77}$$

$$\rho_{\text{crit}} = \frac{3{H_0}^2}{8G\pi} \tag{78}$$

- As ρ_{crit} decrease with increasing r, by referring to the equation above, we can proof that H_0 changes with time

3.4 Cosmic Scale Factor

3.4.1 Definition

$$R(t) = \frac{r(t)}{r(t = \text{today})} \tag{79}$$

Note that it can be rewritten as:

$$R(t_{\text{emit}}) = \frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} \tag{80}$$

3.4.2 Redshift

$$z = \frac{\lambda_\circ - \lambda_e}{\lambda_e} \tag{81}$$

And hence:

$$R(t) = \frac{1}{1+z} \tag{82}$$

3.5 Geometry of the universe

We define the density parameter as:

$$\Omega_0 = \frac{\rho}{\rho_{\text{crit}}} \tag{83}$$

- Closed (the big crunch): $\Omega > 1$, $E < 0$
- Open(Expanding):$\Omega < 1$,$E > 0$
- Flat: $\Omega = 1$,$E = 0$

3.6 Problems of the Big Bang Model

- Flatness Problem
  - The fine-tuning of density such that it is so close to the critical density
- Horizon Problem
  - Two ends of the horizon looks the same and has the same temperature, which implies they must have been connected. By def, as our horizon is the furthest light can travel since the big bang, these two regions (in this case separated by 2 horizon scales) can not have been connected at the big bang

3.7 Guth Inflation Solution

- Alan Guth developed a theory called **Inflation** where a new form of energy from a field created roughly 10^{-30} seconds after the big bang accelerated the expansion of the Universe at speeds faster than light
- How it solved the aforementioned problems?
  - Flatness Problem:
    * The rapid acceleration would flatten out any irregularity in the geometry of the early post-big-bang Universe
  - Horizon Problem:
    * The Universe was all connected before inflation

3.8 Energy density of matter and radiation

$$\rho_{\mathrm{m}} \propto R(t)^{-3} \tag{84}$$

$$\rho_{\mathrm{r}} \propto R(t)^{-4} \tag{85}$$

- At the start of the universe, $R(t)$ was small, and the universe was dominated by radiation.
- What is the redshift at the time when energy density of matter and radiation equal? We are given:

  $$\frac{\rho_r(t = \mathrm{today})}{\rho_m(t = \mathrm{today})} = 0.0003 \tag{86}$$

  At the redshift where the energy density of matter and radiation were equal:

  $$\frac{\rho_r(t = \mathrm{equal})}{\rho_m(t = \mathrm{equal})} = 1 \tag{87}$$

  Remember from above,

  $$\rho_{\mathrm{m}} \propto R(t)^{-3} \tag{88}$$

  $$\rho_{\mathrm{r}} \propto R(t)^{-4} \tag{89}$$

  Here, $t = \mathrm{equal}$:

  $$\frac{\rho_r(t = \mathrm{equal})}{\rho_m(t = \mathrm{equal})} = \frac{k}{R(t_{\mathrm{equal}})} \tag{90}$$

  We can compare it with:

  $$\frac{\rho_r(t = \mathrm{today})}{\rho_m(t = \mathrm{today})} = \frac{k}{R(t_{\mathrm{today}})} \tag{91}$$

  $R(t_{\mathrm{today}}) = 1$, and k is a the same for both cases, so $k = 0.0003$, and therefore $R(t_{\mathrm{equal}}) = 0.0003$. We have defined:

  $$R(t) = \frac{1}{1+z} \tag{92}$$

  And we can easily rearrange find $z \approx 3332$

3.9 Beginning of the universe

- This era was described as "opaque" as the light from this era cannot travel out into the Universe ,**because the photons scatter off electrons**
- The particles in the early Universe are not distributed randomly and there are fluctuations due to tiny random **quantum fluctuations** in the pea-sized Universe that existed after the big bang and before $10^{-30}s$

3.10 Recombination

- In the radiation dominated Universe, the photons are so dense that the photon pressure can push the clumps apart. An ongoing battle between matter trying to clump and photon pressure pushing these apart can be observed in the first view of the Universe at **Recombination**.
- Recombination is the era where the energy density of matter now dominates and the charged particles have combined to form neutral atoms so the photons freely travel out into the expanding Universe. We detect the first radiation from this era in the form of the cosmic microwave background.

3.11 Chemical building blocks

- In the very early radiation dominated Universe, protons and neutrons move fast, collide often and fuse to form nuclei. The high density of high energy gamma ray photon however manage to blast any nuclei that form apart $\implies$ None are formed
- In a later stage, the expansion has reduced the photon energy to such an extent that they are unable to break the more stable nuclei up, while the Universe is dense and hot enough for particle collisions and nuclear fusion to occur $\implies$ Starts to form
- At the end of the nucleosynthesis period the Universe has cooled to such an extent that no heavier elements can be made as the density decreases (with the expansion) the collisions become rare and unimportant. The rest of the chemicals on our periodic table are created much later on in the history of the Universe in the cores of stars [16]

3.12 Deuterium

- Heavy Hydrogen
- Why important?
  - There is no way to produce any quantity of this element at any other point in the history of the Universe except during nucleosynthesis.
  - The cores of stars have the required temperatures for Deuterium to be created, but the stars are so dense, the chain of reactions don't stop and they use up all of their Deuterium in other reactions

3.13 Nucleosynthesis and the baryon density

- Protons, neutrons, everything in the periodic table is baryons

[16] You have learnt them before!

- We define the baryon density parameter as:

$$\Omega_b = \frac{\rho_b}{\rho_{\text{crit}}} \tag{93}$$

- From nucleosynthesis observation we measure $\Omega_b = 0.045$

3.14 Cosmic Microwave Background

- If the Universe started in a hot big bang, then the atoms would have interacted so strongly that all detailed features in their energy distribution are washed out and we would expect to see a thermal continuous blackbody spectrum from the Big Bang
- Experimental data fits extremely well with blackbody curve for $T = 2.74k$, which confirms our theory on the competition between matter and radiation in the early Universe

3.15 What is Dark Matter

- An early hypothesis was that this dark matter was cold baryonic matter bound up in dark objects such as failed faint Brown Dwarf stars
- We know from nucleosynthesis that only $\approx 5\%$ of the Universe is made up of baryons, so whatever this extra mass is it must be non-baryonic
- Our best guess is that it's **cold non-baryonic massive particles**
- Modified Newtonian Dynamics uses $F = ma^2$
  - Need a tuning parameter which changes for different galaxies
  - Goes against the Cosmological Principle that everything (including the law of physics) is homogenous and isotropic

3.16 Distance-redshift relation

- This section resulted in a Nobel Prize in 2011. They used distant supernova as standard candles to probe the Universe
- We have the formula $D = \frac{cz}{H_0}$. There is an extension that works for more distant galaxies:

$$D_L \approx \frac{c(1+z)(z - \frac{1+q}{2}z^2)}{H_0} \tag{94}$$

- Where $q = -\left(1 + \frac{\dot{H}}{H^2}\right)$ and $\dot{H}$ is the rate of change of the Hubble parameter

Concordant cosmology

- Baryonic content $\Omega_b \approx 0.05$

- The total dark and baryonic matter content $\Omega_m \approx 0.3$
- Dark energy content $\Omega_\Lambda \approx 0.7$
- The university is flat and has critical density: $\Omega_b + \Omega_m + \Omega_\Lambda = 1$