Notes on Lie Algebras

Qixuan Fang

山东大学物理学院

January 8, 2023

Definition

Define commutator(交换子) as

$$[X, Y] = XY - YX \tag{1}$$

which satisfies the relations:

$$[X, X] = 0, \qquad [X, Y] = -[Y, X] \tag{2}$$

Another key is a number field $\mathcal{F}$, which coule be Real $\mathbb{R}$, Complex $\mathbb{C}$ etc. There is a basic relationship in quantum mechanics:

$$\left[x, \frac{1}{i}\frac{\partial}{\partial x}\right] = i \tag{3}$$

where we let $\hbar = 1$.In classic mechanics, the Possion bracket is real.

Lie Algebras

Consider a set of elements $X_\alpha(\alpha = 1, ..., r)$, if using $\mathcal{G}$ to represent Lie Algebras, then can write $X_\alpha \in \mathcal{G}$, if the set is to form $\mathcal{G}$. 下面的公理是很重要的：

定理 (Axiom 1.)

The commutator of any two elements is linear combination of the elements in the Lie Algebras:

$$[X_\mu, X_\nu] = \sum_\tau c^\tau_{\mu\nu} X_\tau \tag{4}$$

Lie Algebras

定理 (Axiom 2.)

The double commutators of three elements satisfy the Jacobi identity:

$$[X_\mu, [X_\nu, X_\tau]] + [X_\nu, [X_\tau, X_\mu]] + [X_\tau, [X_\mu, X_\nu]] = 0 \tag{5}$$

系数 $c^\tau_{\mu\nu}$ 称为 Lie structure constants(李结构常数). They define the Lie algebras. They satisfy

$$c^\tau_{\mu\nu} = -c^\tau_{\nu\mu} \tag{6}$$

也就是说系数构成一个反称张量. And there is

$$c^\alpha_{\mu\nu} c^\beta_{\alpha\tau} + c^\alpha_{\nu\tau} c^\beta_{\alpha\mu} + c^\alpha_{\tau\mu} c^\beta_{\alpha\nu} = 0 \tag{7}$$

Lie Algebras

r 个元素 X_α, define on a r-dim linear space $\mathcal{L}$. Now give the specific definition of this:

定义

A vetcor space $\mathcal{L}$ over a number field $\mathcal{F}$, with an operation $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$, denote $[X, Y]$ is the commutator of elements X and Y, is called Lie algebras over $\mathcal{F}$ if the following axioms are satisfied:

1. The operation is bilinear(双线性的).
2. $[X, X] = 0$ for any X in $\mathcal{L}$.
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, if X, Y and Z in $\mathcal{L}$.

Lie Algebras

Bilinear is, i.e. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[X, bY + cZ] = b[X, Y] + c[X, Z]$. 同时和 $[X, X] = 0$ 保证了交换的反对称性. 利用 $\mathcal{L}$ 的封闭性给出了 Axiom 1. And the property 3 gives the Axiom 2.

If the number field $\mathcal{F}$ is real, then the Lie algebras is called real. And it is called complex if the number field is complex. Real Lie algebras have real structure constants, and the structure constants of complex Lie algebras could be real or complex.

Lie Algebras

例 (Algebra of $SO(3)$)

The algebras:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2 \tag{8}$$

is a real Lie algebras with three elements. This is the angular momentum algebra in three dimensions, $so(3)$.

Often using L to represent the angular momentum.

Lie Algebras

例 (Algebra of $SO(2,1)$)

The algebras:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2 \tag{9}$$

is also a real Lie algebras with three elements. This is the Lorentz algebra in $2+1$ dimensions, $so(2,1)$.

Change of Basis

Let the $X_\mu(\mu = 1, ..., r)$ be the basis of $\mathcal{L}$. And the changing of basis:

$$X'_\mu = a_\mu^\nu X_\nu \tag{10}$$

Get the nwe commutator between new basis:

$$[X'_\mu, X'_\nu] = c'^{\tau}_{\mu\nu} X'_\tau \tag{11}$$

new structure constants can be solved by the equation:

$$c'^{\tau}_{\mu\nu} a_\tau^\kappa = a_\mu^\alpha a_\nu^\beta c^\kappa_{\alpha\beta} \tag{12}$$

Change of Basis

当变换系数为实数时，这样的变换是十分简单的，称为正常变换.

例

The transformation:

$$X_1' = \sqrt{2}X_1, \quad X_2' = \sqrt{2}X_2, \quad X_3' = X_3 \tag{13}$$

changes commutation of Lie algebra $so(3)$ into:

$$[X_1', X_2'] = 2X_3', \quad [X_2', X_3'] = X_1', \quad [X_3', X_1'] = X_2 \tag{14}$$

Change of Basis

在基变换下具有相同的交换关系的李代数称为 isomorphic(同构代数):

定义 (Isomorphic)

Two Lie algebras g, g' over $\mathcal{F}$ are isomorphic if there exists a vector space isomorphism: $\phi : g \to g'$ satisfying $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all X and Y in g.

Isomorphic algebras can be denoted by $\sim$.

例

The Lie algebras $so(3)$ and $su(2)$ are isomorphic:

$$so(3) \sim su(2) \tag{15}$$

Complex Extensions

考虑对 $so(3)$ 做复变换:

$$J_1 = iX_1, \quad J_2 = iX_2, \quad J_3 = iX_3 \tag{16}$$

The commutations now are:

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \tag{17}$$

The algebras composed of elements J_1, J_2 and J_3 is the "angular momentum algebra" in most quantum mechanics books.

Complex Extensions

Consider a linear combination of elements A, B, of a real Lie algebra g, with complex coeffcients, and defines $[A + iB, C] = [A, C] + i[B, C]$ 这样获得了实李代数的复拓展.Through changing the basis with complex changes, it's easy to get a complex extension of a real Lie algebra g. In some cases, different real Lie algebras have the same complex extensions.

Complex Extensions

例

The real Lie algebra $so(2,1)$ and $so(3)$ have the same complex extension.By making changes of basis:

$$Y_1 = X_1, \quad Y_2 = -iX_2, \quad Y_3 = -iX_3 \tag{18}$$

then there is:

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2 \tag{19}$$

This is the same commutation relations of the real Lie algebra $so(3)$.

Lie Subalgebras

A subset of elements Y_β, 对于原先定义的交换子是闭合的:

$$X_\alpha \in g; \quad Y_\beta \in g'; \quad g \supset g' \tag{20}$$

称为子代数. The symbol $\supset$ is used to indicate that g' is the subalgebra of g.And there is:

$$[Y_\mu, Y_\nu] = c^\tau_{\mu\nu} Y_\tau \tag{21}$$

例

Consider that X_3 forms a subalgebra $so(2)$ of $so(3)$ since:

$$[X_3, X_3] = 0 \tag{22}$$

Abelian Algebras

阿贝尔代数, $\mathcal{A}$, 要求对于所有元素:

$$[X_\mu, X_\nu] = 0, \qquad \forall\ X_\mu, X_\nu \in \mathcal{A} \tag{23}$$

例

The algebra $so(2) \ni X_3$ is Abelian,since:

$$[X_3, X_3] = 0$$

任何一个李代数显然都有平凡阿贝尔子代数，即由单元素 X_μ 组成的 $[X_\mu, X_\mu] = 0$.

Abelian Algebras

另一个非平凡的例子是二维 translation algebra(平移代数), $t(2) \ni X_1, X_2$.

例

The algebra $t(2)$ with commutation relations:

$$[X_1, X_2] = 0, \quad [X_1, X_1] = 0, \quad [X_2, X_2] = 0 \tag{24}$$

is Abelian.

Direct Sum

Consider two commuting algebras $X_\alpha \in g_1,\ \ Y_\beta \in g_2$, satisfying:

$$[X_\mu, X_\nu] = c^\tau_{\mu\nu} X_\tau, \quad [Y_\mu, Y_\nu] = c^\tau_{\mu\nu} Y_\tau, \quad [X_\mu, Y_\nu] = 0 \tag{25}$$

满足上面的交换关系可以记为 $g_1 \cap g_2 = 0$. The sets of elements $X_\alpha,\ \ Y_\beta$ forms an algebra g, called direct sum:

$$g = g_1 \oplus g_2 \tag{26}$$

一些情况下，可以把一些李代数写成另一些李代数直和的形式，例如：
the algebra $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in so(4)$, satisfying commutation relations:

Direct Sum

$$[X_1, X_2] = X_3; \quad [X_2, X_3] = X_1; \quad [X_3, X_1] = X_2$$

$$[Y_1, Y_2] = X_3; \quad [Y_2, Y_3] = X_1; \quad [Y_3, Y_1] = X_2$$

$$[X_\alpha, Y_\alpha] = 0, \ (\alpha = 1, 2, 3);$$

$$[X_1, Y_2] = Y_3; \quad [X_1, Y_3] = -Y_2; \quad [X_2, Y_1] = -Y_3$$

$$[X_2, Y_3] = Y_1; \quad [X_3, Y_1] = Y_2; \quad [X_3, Y_2] = -Y_1 \tag{27}$$

By the change of basis:

$$J_i = \frac{X_i + Y_i}{2}, \quad K_i = \frac{X_i - Y_i}{2} \quad (i = 1, 2, 3) \tag{28}$$

Direct Sum

Then the algebras can be brought to the form:

$$[J_\alpha, J_\beta] = \varepsilon_{\alpha\beta\gamma} J_\gamma, \quad [K_\alpha, K_\beta] = \varepsilon_{\alpha\beta\gamma} K_\gamma, \quad [J_\alpha, K_\beta] = 0, \quad (\alpha, \beta, \gamma = 1, 2, 3) \tag{29}$$

其中 ε_{ijk} 是克罗内克符号:

$$\varepsilon_{ijk} = \begin{cases} 1, & ijk = 123,\ 231,\ 312; \\ -1, & ijk = 132,\ 213,\ 321; \\ 0, & others. \end{cases}$$

这里将 $so(4)$ 写成了两个 $so(3)$ 代数直和的形式.

Direct Sum

Consider the algebra $X_1, X_2, X_3,\ Y_1,\ Y_2,\ Y_3 \in so(3,1)$, satisfying the commutation relations:

$$
\begin{aligned}
&[X_1, X_2] = X_3; \quad [X_2, X_3] = X_1; \quad [X_3, X_1] = X_2 \\
&[\,Y_1,\ Y_2] = -X_3; \quad [\,Y_2,\ Y_3] = -X_1; \quad [\,Y_3,\ Y_1] = -X_2 \\
&[X_\alpha,\ Y_\alpha] = 0, \ \ (\alpha = 1, 2, 3); \\
&[X_1,\ Y_2] = \ Y_3; \quad [X_1,\ Y_3] = -\,Y_2; \quad [X_2,\ Y_1] = -\,Y_3 \\
&[X_2,\ Y_3] = \ Y_1; \quad [X_3,\ Y_1] = \ Y_2; \quad [X_3,\ Y_2] = -\,Y_1
\end{aligned} \tag{30}
$$

这个代数无法写成实李代数的直和，但有时可以对实代数做复拓展.

Direct Sum

Taking the combination:

$$J_j = \frac{X_j + iY_j}{2}, \quad K_j = \frac{X_j - iY_j}{2} \quad (j = 1, 2, 3) \tag{31}$$

Then the $J_j,\ K_j$ also satisfying the relationship in (30). The algebras $so(4)$ and $so(3,1)$ have the same complex extension, and can be split in the same fashion.

例

The algebra $so(4)$ is isomorphic to the direct sum:

$$so(4) \sim so(3) \oplus so(3) \sim su(2) \oplus su(2) \sim sp(2) \oplus sp(2) \tag{32}$$

Invariant Subalgebras

考虑代数 g and its subalgebra g', $X_\mu \in g$, $Y_\nu \in g'$, $g' \subset g$. Since g' is a subalgebra, it satisfies:

$$[Y_\mu, Y_\nu] = c'^{\tau}_{\mu\nu} Y_\tau \tag{33}$$

If, in addition,

$$[Y_\mu, X_\nu] = c^{\tau}_{\mu\nu} Y_\tau \tag{34}$$

then g' is called invariant algebra(ideal, 不变子代数) of g.

Invariant Subalgebras

作为例子，考虑欧几里得代数 $e(2)$，包含三个元素 X_1, X_2, X_3 满足：

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = 0 \tag{35}$$

例

$X_2, X_3 \in g'$ is an Abelian ideal of g, $g = e(2)$, $X_1, X_2, X_3 \in g$.

Semisimple Algebras

An algebra which has no Abelian ideals is called semisimple(半单的).

例

The algebra $so(3)$ is semisimple.

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$

例

The algebra $e(2)$ is non-semisimple.

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = 0$$

Semisimple Algebras

Obviously g itself (and 0, 0 子空间) are ideals of g, 称为平凡的. An algebra is called simple if it contains no ideals except g and 0. A simple Lie algebra is necessarily semisimple, 反之不一定成立. An additional condition for semisimplicity is that the algebra g not be Abelian, $[g, g] \neq 0$. 这说明了单李代数和半单李代数都必须包含超过一个元素.The algebras $so(2) \sim u(1)$ with commutation relation

$$[X_3, X_3] = 0$$

cannot be classified as simple or semisimple, although $so(2)$ is often included in the classification of semisimple Lie algebras.

Semidirect Sum

If an algebra g has two subalgebras g_1, g_2 such that

$$[g_1, g_1] \in g_1, \quad [g_2, g_2] \in g_2, \quad [g_1, g_2] \in g_1 \tag{36}$$

then the algebra g is said to be the semidirect sum(半直和) of g_1 and g_2. Clearly g_1 is an ideal of g. Note that g_1 does not to be an ideal of g_2. 通常可以通过给出不变子代数和剩余子代数的直和形式写出半直和:

$$g = g_1 \oplus_s g_2 \tag{37}$$

Semidirect Sum

例

The Euclidean algebrae $e(2)$, composed of three elements, $X_1,\ X_2,\ X_3$, is the semidirect sum of the rotation algebra(旋转代数) in two dimensions, $so(2)$, composed of the single element, X_1, and the translation algebra(平移代数) in two dimensions, $t(2)$, composed of two commuting elements, $X_2,\ X_3$.

$$e(2) = so(2) \oplus_s t(2) \tag{38}$$

Metric Tensor

With the Lie structure constants one can form a tensor, called metric tensor(度量张量),

$$g_{\mu\nu} = g_{\nu\mu} = c^{\tau}_{\mu\lambda} c^{\lambda}_{\nu\tau} \tag{39}$$

这个度规张量也称为 Killing metric tensor. Cartan's criterion for deciding if a Lie algebra is semisimple is:

定理

A Lie algebra g is semisimple if, and only if,

$$\mathrm{Det}|g_{\mu\nu}| \neq 0 \tag{40}$$

Metric Tensor

In other words, an inverse $g^{\mu\nu}$ of the metric tensor $g_{\mu\nu}$ exists:

$$g^{\mu\tau} g_{\tau\nu} = \delta^{\mu}_{\nu} \tag{41}$$

where:

$$\delta^{\mu}_{\nu} = \begin{cases} 1, & \mu = \nu; \\ 0, & \mu \neq \nu. \end{cases}$$

Metric Tensor

例

The algebra $so(3)$ is semisimple. The metric tensor of $so(3)$ is:

$$g_{\mu\nu} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{42}$$

also written in compact form $g_{\mu\nu} = -2\delta_{\mu\nu}$. The determinant of the metric tensor is

$$\mathrm{Det}|g_{\mu\nu}| = -8 \tag{43}$$

Metric Tensor

例

The algebra $so(2,1)$ is semisimple. The metric tensor of $so(2,1)$ is:

$$g_{\mu\nu} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{44}$$

The determinant of the metric tensor is again

$$\mathrm{Det}|g_{\mu\nu}| = -8 \tag{45}$$

Metric Tensor

例

The algebra $e(2)$ is non-semisimple. The metric tensor of $e(2)$ is:

$$g_{\mu\nu} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{46}$$

In this case

$$\mathrm{Det}|g_{\mu\nu}| = 0 \tag{47}$$

Compact and Non-Compact Algebras

A real semisimple Lie algebra is compact(紧的) if its metric tensor is negative definite(负定的).

例

The algebra $so(3)$ is compact. In its diagonal form all elements are negative.

例

The algebra $so(2,1)$ is non-compact. In its diagonal form some elements are positive.

Derived Algebras

Starting with a Lie algebra, g, with elements X_μ, it is possible to construct other algebras, called derived algebras(生成代数), by taking commutators:

$$g^{(0)} = g, \quad g^{(1)} = [g^{(0)}, g^{(0)}], \quad g^{(2)} = [g^{(1)}, g^{(1)}], \ldots \tag{48}$$

The sequence $g^{(0)}$, $g^{(1)}$, $g^{(2)}, \ldots, g^{(i)}$ is called a derived series(生成系). For example, starting with the Euclidean algebra, $e(2)$, with elements

$$g \doteqdot X_1, X_2, X_3 \tag{49}$$

Derived Algebras

satisfying the commutation relations (35), one has

$$g^{(1)} \doteqdot X_2, X_3, \quad g^{(2)} \doteqdot 0 \tag{50}$$

If, for some positive k

$$g^{(k)} \doteqdot 0 \tag{51}$$

the algebra is called solvable(可解的). 生成系记为 *Der g*.

例

The algebra $e(2)$ is solvable.

$$[e(2)]^{(2)} \doteqdot 0 \tag{52}$$

Nilpotent Algebras

Starting with a Lie algebra, g, with elements, X_μ, it is possible to construct another series(降序中心系), called descending central series, or lower central series, as

$$g^0 = g, \quad g^1 = [g, g] = g^{(1)}, \quad g^2 = [g, g^1], ..., g^i = [g, g^{i-1}] \tag{53}$$

If, for some positive k,

$$g^k = 0 \tag{54}$$

the algebra is called nilpotent(幂等的).

Nilpotent Algebras

例

The algebra $e(2)$ is not nilpotent

$$\begin{aligned} g &\doteq X_1, X_2, X_3 \\ g^1 &\doteq X_2, X_3 \\ g^2 &\doteq X_2, X_3 \\ &\ldots \end{aligned} \tag{55}$$

Invariant Casimir Operators

An operator, C , that commutes with all the elements of a Lie algebra g

$$[C, X_\mu] = 0, \quad \forall\ X_\mu \in g \tag{56}$$

称为不变卡西米尔算子. Casimir operators can be linear, quadratic(二次方), cubic(立方), . . . in the elements X_μ. 若算子可以表示为 p 个元素的积，则称算子是 p 阶的：

$$C_p = \sum_{\alpha_1 \dots \alpha_p} f^{\alpha_1 \alpha_2 \dots \alpha_p} X_{\alpha_1} X_{\alpha_2} \dots X_{\alpha_p} \tag{57}$$

Invariant Casimir Operators

If C commutes with g, so does $aC,\ C^2, \ldots$ 半单代数的二阶卡西米尔算子可由度规张量构造:

$$C_2 = g^{\mu\nu} X_\mu X_\nu = g_{\mu\nu} X^\mu X^\nu = C \tag{58}$$

证明方法即是 evaluate the commutator of C and X_τ,

$$[C, X_\tau] = g^{\mu\nu}[X_\mu X_\nu, X_\tau] = g^{\mu\nu} X_\mu [X_\nu, X_\tau] + g^{\mu\nu}[X_\mu, X_\tau] X_\nu = 0 \tag{59}$$

计算过程省略.

Invariant Casimir Operators

For a semisimple Lie algebra, indices can be raised and lowered using the metric tensor(指标升降):

$$c^{\tau}_{\mu\nu} = g^{\tau\lambda} c_{\lambda\mu\nu} \tag{60}$$

Higher order Casimir operators can be constructed in a similar fashion

$$C_p = c^{\beta_2}_{\alpha_1\beta_1} c^{\beta_3}_{\alpha_2\beta_2} \dots c^{\beta_1}_{\alpha_p\beta_p} X^{\alpha_1} X^{\alpha_2} \dots X^{\alpha_p} \tag{61}$$

Invariant Casimir Operators

For the algebra $so(3)$, the inverse of the metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{62}$$

giving

$$C = -\frac{1}{2}\left(X_1^2 + X_2^2 + X_3^2\right) \tag{63}$$

For the algebra $so(2,1)$

$$C = -\frac{1}{2}\left(X_1^2 - X_2^2 - X_3^2\right) \tag{64}$$

Invariant Casimir Operators

对于 $so(3)$，将 C 乘以 2，并对元素乘以 i，可以得到：

$$C' = 2C = J_1^2 + J_2^2 + J_3^2 \tag{65}$$

This is the usual form in which the Casimir operator of the angular momentum algebra $so(3)$ appears in quantum mechanics books, often wirte as $\boldsymbol{J^2}$.

Invariant Operators for Non-Semisimple Algebras

For non-semisimple Lie algebras, Casimir operators cannot be simply constructed. One introduces a related notion of invariant operators that commute will all elements.

例

The invariant operator of the Euclidean algebra $X_1, X_2, X_3 \in e(2)$ is

$$C = X_2^2 + X_3^2 \tag{66}$$

证明时分别计算 $[X_2^2, X_1]$ 和 $[X_3^2, X_1]$，可以得到结果是 $[C, X_\mu] = 0, \quad \forall\ X_\mu \in e(2)$.

Contractions of Lie Algebras

Let $X_1, ..., X_r$ be the elements of the Lie algebra g. For a subset $X_1, ..., X_\mu,\ \mu \leq r$, define

$$Y_i = \varepsilon X_i, \quad i = 1, .., \mu \leq r. \tag{67}$$

and express the commutation relations in terms of the Y_i

$$\begin{aligned} [Y_i, Y_j] = c_{ij}^k \varepsilon Y_k + c_{ij}^m \varepsilon^2 X_m, \quad [Y_i, X_m] = c_{im}^k Y_k + c_{im}^n \varepsilon X_n \\ [X_m, X_n] = c_{mn}^i \varepsilon^{-1} Y_i + c_{nm}^s X_s, \quad i, j, k \leq \mu, \quad \mu < m, n, s \leq r. \end{aligned} \tag{68}$$

Contractions of Lie Algebras

Now, let $\varepsilon \to 0$. If

$$c^i_{mn} = 0, \quad i \leq \mu, \quad \mu < m, n \leq r \tag{69}$$

the commutation relations

$$[Y_i, Y_j] = 0, \quad [Y_i, X_k] = c^k_{im} Y_k, \quad [X_m, X_s] = c^s_{mn} X_s \tag{70}$$

define a Lie algebra, called the Inonu contracted Lie algebra(合同李代数) g'

$$g \to_c g' \tag{71}$$

Contractions of Lie Algebras

例

The Euclidean algebra $e(2)$ is a contraction of the Lorentz algebra $so(2,1)$. The commutation relations of the complex extension of $so(2,1)$ are

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = iJ_2 \tag{72}$$

with Casimir operator

$$C = J_1^2 - J_2^2 - J_3^2 \tag{73}$$

Introducing

$$P_x = \varepsilon J_2, \quad P_y = \varepsilon J_3, \quad L_z = J_1 \tag{74}$$

Contractions of Lie Algebras

例

the commutation relations become

$$[P_x, P_y] = -i\varepsilon^2 L_z, \quad [L_z, P_y] = -iP_x, \quad [L_z, P_x] = iP_y \tag{75}$$

Letting $\varepsilon \to 0$, then obtain the commutation relations (35) of $e(2)$,

$$[P_z, P_y] = 0, \quad [L_z, P_y] = -iP_x, \quad [L_z, P_x] = iP_y \tag{76}$$

with invariant operator

$$C' = P_x^2 + P_y^2 \tag{77}$$

Also $so(3)$ contracts to $e(2)$ since $so(3)$ and $so(2,1)$ are complex

Contractions of Lie Algebras

例

The Euclidean algebra $e(3)$ is a contraction of the Lorentz algebra $so(3,1)$. The commutation relations of $so(3,1)$ written in terms of elements $L_i,\ K_i, (i=1,2,3)$, are

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \quad [L_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}L_k \tag{78}$$

Defining $P_i = \lambda^{-1}K_i(i=1,2,3)$ and letting $lambda \to \infty$, they become

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \quad [L_i, P_j] = i\varepsilon_{ijk}P_k, \quad [P_i, P_j] = 0 \tag{79}$$

Contractions of Lie Algebras

例

These are the commutation relations of the algebra $e(3)$ composed of the three components P_1, P_2, P_3 of the momentum and the three components L_1, L_2, L_3 of the angular momentum in three dimensions. The algebra e(3) is the semidirect sum of $so(3)$ and $t(3)$,

$$e(3) = t(3) \oplus_s so(3) \tag{80}$$

Contractions of Lie Algebras

例

The Poincare'algebra $p(4)$ is a contraction of the de Sitter algebra $so(3,2)$. The commutation relations of $so(3,2)$ written in terms of the 10 elements $M_{ab} = -M_{ba}(a, b = 1, ..., 5)$ are

$$[M_{ab}, M_{cd}] = -(g_{bc}M_{ad} - g_{ac}M_{bd} + g_{ad}M_{bc} - g_{bd}M_{ac}) \tag{81}$$

其中 $g_{ab} = (-,-,-,+,+)$.Defining $P_\mu = \varepsilon M_{5\mu}$, and letting $\varepsilon \to 0$, obtain the commutation relations of $p(4)$, the Poincare'algebra in four dimensions

Contractions of Lie Algebras

例

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\mu] = -g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu, \quad \mu, \nu, \lambda = 1, 2, 3, 4 \tag{82}$$

The algebra $p(4)$ composed of the four components, P_μ, of the four-momentum and the six components, $M_{\mu\nu}$, of the angular momentum and boost, is the semidirect sum of $so(3,1)$ and $t(3,1)$

$$p(4) = t(3,1) \oplus_s so(3,1) \tag{83}$$

Also, $so(4,1)$, with metric $g_{ab} = (-,-,-,+,-)$, contracts to $p(4)$.

Algebras with One Element

An algebra with one element, which means $r = 1$, there is

$$[X, X] = 0$$

There is no doubt that this algebra is Abelian.

例

The algebras $so(2) \sim u(1)$ are examples of this.

Algebras with Two Elements

考虑两个元素的代数，即 $r = 2$ 的情况，有

$$(a)\ [X_1, X_2] = 0\ and\ (b)\ [X_1, X_2] = X_1 \tag{84}$$

两种情况.In case (a), the algebra is Abelian. And in case (b), X_1 is an Abelian ideal.

例

二维平移代数 $t(2)$ 即是满足 case (a) 的情况.

Algebras with Three Elements

对于 $r=3$ 的情况，三个元素的交换关系存在四种可能:

$$(a)\ [X_1,X_2]=[X_2,X_3]=[X_3,X_1]=0 \tag{85}$$

$$(b)\ [X_1,X_2]=X_3,\quad [X_1,X_3]=[X_2,X_3]=0\quad \text{or}$$

$$[X_1,X_3]=X_2,\quad [X_1,X_2]=[X_2,X_3]=0 \tag{86}$$

$$(c)\ [X_1,X_2]=0,\quad [X_3,X_1]=\alpha X_1+\beta X_2,\quad [X_3,X_2]=\gamma X_1+\delta X_2$$

$$\text{where}\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\ \text{is non-singular} \tag{87}$$

Algebras with Three Elements

$$(d)\ [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$
$$[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2 \tag{88}$$

In case (a), the algebra is Abelian.

例

1. 三维平移代数 $t(3)$ 满足 (a).
2. 二维欧几里得代数 $e(2)$ 满足 (c),where $\alpha = 0,\ \beta = 1,\ \gamma = -1,\ \delta = 0.$
3. 旋转代数 $so(3)$ 和 $so(2,1)$ 满足 (d).

Cartan–Weyl Form of a (Complex) Semisimple Lie Algebra

这里介绍 Cartan 和 Weyl 对于半单李代数的分类. 首先将元素改写成如下形式:

$$X_\mu = (H_i, E_\alpha), \quad i = 1, ..., l \tag{89}$$

The elements H_i form the maximal Abelian subalgebra(最大阿贝尔子代数组), often called the Cartan subalgebra

$$[H_i, H_k] = 0, \quad i, k = 1, ..., l \tag{90}$$

The number of elements in the Cartan subalgebra, l , is called the rank of the algebra. The commutation relations of H_i with E_α are

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad i = 1, ..., l \tag{91}$$

Cartan–Weyl Form of a (Complex) Semisimple Lie Algebra

对于 E 的交换关系满足：

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \ (\text{if } \alpha + \beta \neq 0), \quad [E_\alpha, E_{-\alpha}] = a^i H_i \tag{92}$$

The α_i's are called roots and $N_{\alpha\beta}$ is a normalization. This form of the Lie algebra is called the Cartan–Weyl form.

Graphical Representation of Root Vectors

将 α_i 看作是定义在一个 l 维权重空间上的系数向量的分量，其中标量积记为：

$$(\alpha, \beta) = \alpha^i \beta_i = \alpha_i \beta^i \tag{93}$$

满足如下的规则：

Rules

1. If α is a root, so $-\alpha$ is.
2. If α, β are roots, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
3. If α, β are roots, $\beta - 2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is a root.

Graphical Representation of Root Vectors

From these, it follows that the angle φ between roots

$$\cos\varphi = \frac{(\alpha,\alpha)}{\sqrt{(\alpha,\alpha)(\beta,\beta)}} \tag{94}$$

can take values:

$$\cos^2\varphi = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1; \quad \varphi = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2} \tag{95}$$

可以利用根图来表示其取值情况.

Graphical Representation of Root Vectors

对于 $l = 1$，根图是一条直线，只有一种可能方式.The algebra, called A_1 by Cartan and $so(3) \sim su(2)$ by physicists, has three elements, $r = 3$.

对于 $l = 2$，根图是一个平面，有如下几种可能：

1. $\varphi = \frac{\pi}{6}$. This algebra called G_2, has 14 elements, $r = 14$.
2. $\varphi = \frac{\pi}{4}$. This algebra called B_2 or $so(5)$, has 10 elements, $r = 10$.
3. $\varphi = \frac{\pi}{3}$. This algebra called A_2 or $su(3)$, has 8 elements, $r = 8$.
4. $\varphi = \frac{\pi}{2}$. This algebra called D_2 or $so(4)$, has $r = 6$ elements. It can be seen as the directsum of two A_1 algebras, $D_2 \sim A_1 \oplus A_1$ or $so(4) \sim so(3) \oplus so(3)$.

Groups of Transformations

一个非空集合和定义在集合上的二元运算构成了群 G. First giving four axioms of group:

Axioms

1. 封闭性: if $A, B \in G$, then $AB \in G$.
2. 结合律: $(AB)C = A(BC)$.
3. 单位元: $\exists I \in G$, then $AI = IA \in G$.
4. 逆元: for each $A \in G$, $\exists A^{-1} \in G$, then $AA^{-1} = A^{-1}A = I$.

变换群可以分成离散群和连续群，以及根据元素的个数又分为有限群和无限群. 对于连续群，其参数个数称为群的阶. 对于离散群，群元个数称为群的阶.

Groups of Matrices

方阵形成的群是很重要的例子. 很明显矩阵及其乘法运算可以满足上述公理. 接下来介绍一些连续矩阵群.

例

1. 一般线性群. These are the most general linear transformations. They are denoted by

$$GL(n,C), \quad r=2n^2; \qquad GL(n,R), \quad r=n^2 \tag{96}$$

这分别给出了实数域和复数域上 n 维空间的一般线性群及描述该群所需的实参数个数.

Groups of Matrices

例

2. 特殊线性群.

If, on the general linear transformation, the condition

$$\mathrm{Det}|A| = \pm 1 \tag{97}$$

is imposed, the group is called special linear group, denoted by

$$SL(n, C), \quad r = 2(n^2 - 1); \qquad SL(n, R), \quad r = n^2 - 1 \tag{98}$$

Groups of Matrices

例

3. 酉群. Imposing the condition

$$A^{\dagger}A = I \tag{99}$$

one obtains the unitary groups

$$U(n, C) = U(n), \quad r = n^2; \qquad U(p, q, C) = U(p, q), \quad r = n^2 \tag{100}$$

其中不变量为:

$$U(n): \sum_{i=1}^{n} z_i z_i^*; \qquad U(p, q): -\sum_{i=1}^{p} z_i z_i^* + \sum_{j=p+1}^{p+q} z_j z_j^* \tag{101}$$

Groups of Matrices

例

4. 特殊酉群. The combination of the special condition with the unitary condition

$$A^\dagger A = I, \qquad \mathrm{Det}|A| = \pm 1 \tag{102}$$

给出了特殊酉群:

$$SU(n, C) = SU(n), \quad r = n^2 - 1; \qquad SU(p, q, C) = SU(p, q), \quad r = n^2 - 1 \tag{103}$$

For special unitary groups, there is an (anomalous) case, denoted by $SU^*(2n)$

Groups of Matrices

例

$$SU^*(2n) \quad r = (2n)^2 - 1 \tag{104}$$

通过如下矩阵定义：

$$A = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix} \tag{105}$$

其中 A_1, A_2 都是 $n \times n$ 的复矩阵，满足 $\mathrm{Tr}A_1 + \mathrm{Tr}A_1^* = 0.$

Groups of Matrices

例

5. 正交群. 通过正交条件定义:

$$A^T A = I \tag{106}$$

常用正交群是定义在实数域上的，因此 $O(n, R)$ 一般记为 $O(n)$.

$$O(n, C), \quad r = n(n-1); \qquad O(n, R) = O(n), \quad r = \frac{1}{2}n(n-1) \tag{107}$$

其中不变量为:

$$O(n, C) : \sum_{i=1}^{n} z_i^2 \qquad O(n, R) = O(n) : \sum_{i=1}^{n} x_i^2 \tag{108}$$

Groups of Matrices

例

另外，利用 p, q 可以表示为：

$$O(p,q,C),\quad r=n(n-1);\qquad O(p,q,R),\quad r=\frac{1}{2}n(n-1) \tag{109}$$

其中不变量为：

$$O(p,q,C):\ -\sum_{i=1}^{p} z_i^2+\sum_{j=1}^{p+q} z_j^2\qquad O(p,q,R):\ -\sum_{i=1}^{p} x_i^2+\sum_{j=1}^{p+q} x_j^2 \tag{110}$$

Groups of Matrices

例

6. 特殊正交群. 两个条件:

$$A^T A = I, \quad \mathrm{Det}|A| = \pm 1 \tag{111}$$

给出了特殊正交群:

$$\begin{aligned} &SO(n, C), \quad r = n(n-1); \qquad SO(n, R), \quad r = \frac{1}{2}n(n-1); \\ &SO(p, q, C), \quad r = n(n-1); \qquad SO(p, q, R), \quad r = \frac{1}{2}n(n-1) \end{aligned} \tag{112}$$

Also here there is an (anomalous) case, called $SO^*(2n)$

Groups of Matrices

例

$SO^*(2n)$ 通过矩阵描述:

$$A = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix} \tag{113}$$

其中 A_1, A_2 是 $n \times n$ 的复矩阵，满足 $A_1 = -A_1^T$，$A_2 = A_2^\dagger$.

Groups of Matrices

例

7. 对称群. 将向量 $\boldsymbol{x}$ 和 $\boldsymbol{y}$ 分成两部分

$\boldsymbol{x} = (x_1, ..., x_n; x'_1, ..., x'_n)$, $\boldsymbol{y} = (y_1, ..., y_n; y'_1, ..., y'_n)$. 对称群:

$$Sp(2n, C), \quad r = 2n(n+1); \qquad Sp(2n, R), \quad r = n(2n+1) \tag{114}$$

其中不变量为:

$$\sum_{i=1}^{n} (x_i y'_i - y_i x'_i) \tag{115}$$

向量可以是实的或者复的.

Groups of Matrices

例

If the unitary condition is imposed

$$A^\dagger A = I \tag{116}$$

the group is called unitary symplectic

$$USp(2n, C) = Sp(2n), \quad r = n(2n+1) \tag{117}$$

and is often denoted by $Sp(2n)$.

The Rotation Group in Two Dimensions, $SO(2)$

As a first example we consider the rotation group in two dimensions $SO(2) = SO(2, R)$. Under a general linear real transformation the two coordinates x, y transform as

$$x' = a_{11}x + a_{12}y$$
$$y' = a_{21}x + a_{22}y \tag{118}$$

The corresponding group, $GL(2, R)$, is a four parameter group. 不变量是 $x^2 + y^2$.

$$(a_{11}^2 + a_{21}^2)x^2 + (a_{12}^2 + a_{22}^2)y^2 + 2(a_{11}a_{12} + a_{21}a_{22})xy = x^2 + y^2 \tag{119}$$

The Rotation Group in Two Dimensions, $SO(2)$

等式两边系数应当相等：

$$a_{11}^2 + a_{22}^2 = 1 \qquad a_{11}a_{12} + a_{21}a_{22} = 0 \qquad a_{21}^2 + a_{22}^2 = 1 \tag{120}$$

四个参数被三个约束条件限制，仅存在一个自由参数.

例

The group $SO(2)$ is a one parameter group. The parameter can be chosen as the angle of rotation, φ,

$$\begin{aligned} x' &= \cos\varphi x - \sin\varphi y \\ y' &= \sin\varphi x + \cos\varphi y \end{aligned} \tag{121}$$

The Lorentz Group in One Plus One Dimension, $SO(1,1)$

A group closely related to the rotation group is the Lorentz group $SO(1,1) = SO(1,1;R)$. The general linear real transformation in space-time, x, t can be written

$$\begin{aligned} x' &= a_{11}x + a_{12}t \\ t' &= a_{21}x + a_{22}t \end{aligned} \tag{122}$$

不变量是 $x^2 - t^2$. 同上，给出三个约束条件，仅有一个自由参数.

The Lorentz Group in One Plus One Dimension, $SO(1,1)$

例

The group $SO(1,1)$ is a one parameter group. A convenient parametrization is in term of the boost, θ,

$$\begin{aligned} x' &= \cosh\theta x + \sinh\theta t \\ t' &= \sinh\theta x + \cosh\theta t \end{aligned} \tag{123}$$

p, q 给出了不变量中正负号个数，根据惯性定律，这也是一组不变量.

The Rotation Group in Three Dimensions, $SO(3)$

考虑三维旋转群 $SO(3) = SO(3, R)$，在三维一般线性变换群 $GL(3, R)$ 作用下，坐标 x，y，z 变换写成：

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{21}x + a_{22}y + a_{23}z \\ z' &= a_{31}x + a_{32}y + a_{33}z \end{aligned} \tag{124}$$

给出了 9 个参数，不变量给出的约束条件是：

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \tag{125}$$

给出 6 个约束条件，则仅有三个自由参数.

The Rotation Group in Three Dimensions, $SO(3)$

例

The group $SO(3)$ is a three parameter group. A convenient parametrization is in terms of Euler angles, φ, θ, ψ:

$$\begin{pmatrix} \cos\varphi\cos\theta\cos\psi - \sin\varphi\sin\psi & -\cos\varphi\cos\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta\cos\psi + \cos\varphi\sin\psi & -\sin\varphi\cos\theta\sin\psi + \cos\varphi\cos\psi & -\sin\varphi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix} \tag{126}$$

The Special Unitary Group in Two Dimensions, $SU(2)$

This group is denoted by $SU(2) = SU(2, C)$. Under a general linear complex transformation, $GL(2, C)$, the complex quantities, u, v, called a spinor, transform as

$$\begin{aligned} u' &= a_{11} u + a_{12} v \\ v' &= a_{21} u + a_{22} v \end{aligned} \tag{127}$$

This is a eight parameter group. Call the matrix of the transformation A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A^\dagger = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} \tag{128}$$

The Special Unitary Group in Two Dimensions, $SU(2)$

根据酉性，$A^\dagger A = I$，给出四个约束条件：

$$\begin{aligned} a_{11}^* a_{11} + a_{21}^* a_{21} &= 1 & \quad a_{11}^* a_{12} + a_{21}^* a_{22} &= 0 \\ a_{12}^* a_{11} + a_{22}^* a_{21} &= 0 & \quad a_{12}^* a_{12} + a_{22}^* a_{22} &= 1 \end{aligned} \tag{129}$$

这给出了酉群 $U(2)$ 的四个参数，考虑到行列式为 1 的条件：

$$a_{11} a_{22} - a_{12} a_{21} = 1 \tag{130}$$

得到了 $SU(2)$ 群的约束条件，显然其有三个自由参数.

The Special Unitary Group in Two Dimensions, $SU(2)$

例

The group $SU(2)$ is a three parameter group. This group can be parametrized as

$$
\begin{aligned}
u' &= a_{11}u + a_{12}v \\
v' &= -a_{12}^*u + a_{11}^*v
\end{aligned} \tag{131}
$$

其中

$$
a_{11}a_{11}^* + a_{12}a_{12}^* = 1 \tag{132}
$$

Relation Between $SO(3)$ and $SU(2)$

Both $SO(3)$ and $SU(2)$ are three parameter groups. Consider the following combination of the complex spinor $u,\ v$

$$x_1 = u^2, \quad x_2 = uv, \quad x_3 = v^2 \tag{133}$$

These combinations transform as

$$\begin{aligned} x_1' &= u'^2 = a_{11}^2 x_1 + 2a_{11}a_{12}x_2 + a_{12}x_3 \\ x_2' &= u'v' = -a_{11}a_{12}^* x_1 + (a_{11}a_{11}^* - a_{12}a_{12}^*)x_2 + a_{11}a_{12}x_3 \\ x_3' &= v'^2 = a_{12}^{*2} x_1 - 2a_{11}^* a_{12}^* x_2 + a_{11}^{*2} x_3 \end{aligned} \tag{134}$$

By introducing the coordinates x, y, z

Relation Between $SO(3)$ and $SU(2)$

$$x = \frac{x_1 - x_3}{2}, \qquad y = \frac{x_1 + x_3}{2i}, \qquad z = x_2 \tag{135}$$

one can see that they transform as

$$\begin{aligned}
x' &= \frac{1}{2}(a_{11}^2 - a_{12}^{*2} - a_{12}^2 + a_{11}^{*2})x + \frac{i}{2}(a_{11}^2 - a_{12}^{*2} + a_{12}^2 - a_{11}^{*2})y \\
&+ (a_{11}a_{12} + a_{11}^{*}a_{12}^{*})z \\
y' &= \frac{-i}{2}(a_{11}^2 - a_{12}^{*2} - a_{12}^2 + a_{11}^{*2})x + \frac{1}{2}(a_{11}^2 - a_{12}^{*2} + a_{12}^2 - a_{11}^{*2})y \\
&- i(a_{11}a_{12} + a_{11}^{*}a_{12}^{*})z \\
z' &= -(a_{11}a_{12} + a_{11}^{*}a_{12}^{*})x + i(a_{11}^{*}a_{12} - a_{11}a_{12}^{*})y + (a_{11}a_{11}^{*} - a_{12}a_{12}^{*})z
\end{aligned} \tag{136}$$

Relation Between $SO(3)$ and $SU(2)$

This is a real orthogonal transformation in three dimensions, satisfying

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \tag{137}$$

Thus $SU(2)$ and $SO(3)$ are related by a change of variables. In order to elucidate the correspondence between $SU(2)$ and $SO(3)$, we consider a rotation of an angle α around the z-axis. 取 $a_{11} = e^{\frac{i\alpha}{2}}$，$a_{12} = 0$，相应群的旋转矩阵：

$$SU(2): \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix} \qquad SO(3): \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{138}$$

Relation Between $SO(3)$ and $SU(2)$

考虑对三个角 α, β, γ 轮换，得到旋转矩阵:

$$SU(2): \begin{pmatrix} \cos\frac{\beta}{2} e^{\frac{i}{2}(\alpha+\gamma)} & \sin\frac{\beta}{2} e^{-\frac{i}{2}(\alpha-\gamma)} \\ -\sin\frac{\beta}{2} e^{\frac{i}{2}(\alpha-\gamma)} & \cos\frac{\beta}{2} e^{-\frac{i}{2}(\alpha+\gamma)} \end{pmatrix} \qquad SO(3): \; R(\alpha,\beta,\gamma) \tag{139}$$

其中 $R(\alpha,\beta,\gamma)$ 由式 (126) 给出. For no rotation, $R(0,0,0)$, the correspondence is

$$SU(2): \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad SO(3): \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{140}$$

Relation Between $SO(3)$ and $SU(2)$

while for rotation of 2π, $R(2\pi, 0, 0)$, the correspondence is

$$SU(2): \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad SO(3): \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{141}$$

由上可以建立一个 $2 \to 1$ 的一个对应关系，也就是说 $SU(2)$ 到 $SO(3)$ 建立一个同态映射 (homomorphic mapping).

Translation Group, $T(n)$

变换子群和一般线性群及其子群的组合构成了一些常用变换，这些变换群也是李群，但对应的李代数是非半单的，下面举一些例子.

Translations in n-dimensions form a group. Under a translation $\boldsymbol{a}$, the new coordinates are

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{a}; \qquad x_i' = x_i + a, \quad i = 1, ..., n \tag{142}$$

The translation group is a n parameter group.

Affine Group, $A(n)$

满足 $\mathrm{Det}|A| \neq 0$ 的一般线性群和平移群构成 affine group(仿射群), $A(n)$

$$\mathbf{x}' = A\mathbf{x} + \mathbf{a}, \quad x_i' = \sum_k A_{ik}x_k + a_i, \quad i = 1, ..., n \tag{143}$$

可以记为平移群和一般线性群的半直积，且矩阵表示是

$(n+1) \times (n+1)$ 的矩阵：

$$A(n) = T(n) \otimes_s GL(n) \quad A(n) : \begin{pmatrix} A & \mathbf{a} \\ 0 & 1 \end{pmatrix} \tag{144}$$

The number of parameters of $A(n)$ for real transformations is $n^2 + n$.

Euclidean Group, $E(n)$

Rotations plus translations in an n-dimensional space form a group, called the Euclidean group, $E(n)$. A vector $\boldsymbol{x}$ transforms under $E(n)$ as

$$\mathbf{x}' = R\mathbf{x} + \mathbf{a}, \qquad x_i' = \sum_k R_{ik} x_k + a_i, \quad i = 1, ..., n \tag{145}$$

where R_{ik} is the rotation matrix and a_i are the components of the translation vector. The Euclidean group is the semidirect product of $SO(n)$ and $T(n)$.

$$E(n) = T(n) \otimes_s SO(n) \tag{146}$$

$n = 3$ is a case of particular interest.

Euclidean Group, $E(n)$

The Lie algebra $e(n)$ associated with $E(n)$ are the semidirect sums

$$e(n) = t(n) \oplus_s so(n) \tag{147}$$

$e(2)$ is a good example. 因为 $E(n)$ 是 $A(n)$ 子群，$E(n)$ 的矩阵表示也满足 (144) 给出的形式. 其参数个数为 $\frac{n(n-1)}{2} + n$.

Poincare'Group, $P(n)$

洛伦兹变换和平移变换组成庞加莱群 $P(n)$，向量变换表示为：

$$\mathbf{x}' = L\mathbf{x} + \mathbf{a}, \qquad x'_\mu = \sum_\nu L_\mu^\nu x_\nu + a_\mu, \quad \mu = 1, ..., n \tag{148}$$

这个群可以写成 $SO(p, q)$ 和 $T(p, q)$ 的半直积，其中 $p + q = n$

$$P(n) = SO(p, q) \otimes_s T(p, q) \tag{149}$$

其中 $n = 4$ 最为常用：

$$P(4) = SO(3, 1) \otimes_s T(3, 1) \tag{150}$$

这个群也被称为非齐次 Lorentz 群，记为 $ISO(3, 1) = P(4)$.

Poincare'Group, $P(n)$

从庞加莱变换，通过合同代数很容易构造伽利略变换 (Galilean transformations)

$$\mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a}, \qquad x_i' = \sum_k A_{ik}x_k + v_i t + a_i, \quad i = 1, ..., n \tag{151}$$

其中 v_i 是速度矢量的分量.

$P(n)$ 的矩阵表示也是满足 (144) 的形式.

Dilatation Group, $D(1)$ and Special Conformal Group, $C(n)$

伸缩变换构成单参数群, called the dilatation group

$$D(1): \ x'^{\mu} = \rho x^{\mu} \tag{152}$$

如下的非线性变换构成了特殊共形群:

$$x'^{\mu} = \frac{1}{\sigma(x)}(x^{\mu} + c^{\mu} x^2), \qquad \sigma(x) = 1 + 2c^{\nu} x_{\nu} + c^2 x^2 \tag{153}$$

在四维情况下，群 $C(4)$ 有四个参数 $c_{\mu}(\mu = 0, 1, 2, 3)$.

General Conformal Group, $GC(n)$

The set of Lorentz transformations plus translations plus dilatations plus special conformal transformations form a group, the General Conformal Group, $GC(n)$, or simply the Conformal Group. 在四维共形群 $GC(4)$ 中共有 15 个参数，其中庞加莱群 $P(4)$ 有 10 个，$D(1)$ 有 1 个，$C(4)$ 有 4 个.

群 $GL(4)$ 与群 $SO(4,2)$ 是同构的. 可以给出一个六维空间，在上面存在线性的共形群. 另外，李群 $SO(4,2)$ 对应的李代数 $so(4,2)$ 可以分别表示:

General Conformal Group, $GC(n)$

$$\begin{aligned} M_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu, \quad SO(3,1); \qquad P_\mu = \partial_\mu, \quad T(3,1); \\ K_\mu &= 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, \quad C(4); \qquad D = x^\nu\partial_\nu, \quad D(1). \end{aligned} \tag{154}$$

其中 $\mu,\nu = 0,1,2,3.$ Conformal transformations can be witten as linear transformations in a six-dimensional space with coordinates $\eta^\mu = kx^\mu, k, \lambda = kx^2$. Dilatations and special conformal transformations acting in this space are

$$\begin{aligned} &D(1): \quad \eta'^\mu = \eta^\mu, \quad k' = \rho^{-1}k, \quad \lambda' = \rho\lambda \\ &C(4): \quad \eta'^\mu = \eta^\mu + c^\mu\lambda, \quad k' = -2c_\nu\eta^\nu + k + c^2\lambda, \quad \lambda = \lambda' \end{aligned} \tag{155}$$

The Exponential Map

Let the Lie algebra be g and the corresponding Lie group G. The relation is

$$\text{Lie algebra} \qquad X_i \in g, \quad (i = 1, .., r) \tag{156}$$

$$\text{Lie Group} \qquad \exp\left(\sum_{i=1}^{r} \alpha_i X_i\right) \tag{157}$$

其中 α_i 是群的参数，求和上限为群的阶数 (这里的 α_i 和前面的根向量意义不同).

The Exponential Map

This relationship is called an exponential map and denoted by

$$g \xrightarrow{exp} G \tag{158}$$

例

The Lie group $SO(3)$ is

$$A(\alpha_1, \alpha_2, \alpha_3) = e^{\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3} \tag{159}$$

Definition of Exp

考虑指数形式的幂级数展开. 对于一阶代数，只有一个元素 X 和一个参数 α

$$e^{\alpha X} = 1 + \alpha X + \frac{1}{2!}\alpha^2 X^2 + \ldots = \sum_{p=0}^{\infty} \frac{(\alpha X)^p}{p!} \tag{160}$$

对于无穷小群，只保留到一阶项:

$$e^{\alpha X} \xrightarrow[\alpha\to 0]{1} +\alpha X \tag{161}$$

对于更高阶的代数，指数展开要包括非交换项，矩阵形式更方便.

Matrix Exponentials

Let A be a $n \times n$ matrix. Then

$$e^A = I + A + \frac{1}{2!}A^2 + ... \tag{162}$$

Some properties of matrix exponentials are:

1. 若矩阵元 $|a_{ij}|$ 有上界，则 e^A 收敛，即群是紧的.
2. If A and B commute, then
$$e^{A+B} = e^A e^B \tag{163}$$
3. If B can be inverted, then
$$Be^A B^{-1} = e^{BAB^{-1}} \tag{164}$$

Matrix Exponentials

4. If $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A, then $e^{\lambda_1}, ..., e^{\lambda_n}$ are eigenvalues of e^A.
5. The exponential series satisfies

$$e^{A*} = (e^A)^* \quad (e^{A^T}) = (e^A)^T \quad e^{A^\dagger} = (e^A)^\dagger \quad e^{-A} = (e^A)^{-1} \tag{165}$$

6. The determinant of e^A is $e^{\mathrm{Tr}A}$.
7. 若 A 反称, e^A 是正交的. 若 A 是反厄米的, e^A 是酉的.
8. Campbell–Hausdorff 公式:

$$e^{-A}Be^{A} = B + \frac{1}{1!}[B, A] + \frac{1}{2!}[[B, A], A] + ... \tag{166}$$

More on Exponential Maps

The exponential map produces a particular parametrization(参数化) of the group，和恒等元相关联.

例 (Lie group $SO(3)$)

Denote by α_1 the angle of rotation about x, α_2 about y and α_3 about z. The rotation matrix $A(\alpha_1, \alpha_2, \alpha_3)$ in terms of these angles is

More on Exponential Maps

例 (Lie group $SO(3)$)

$$
\begin{pmatrix}
\cos\alpha_2\cos\alpha_3 & -\sin\alpha_1\sin\alpha_2\cos\alpha_3+\cos\alpha_1\sin\alpha_3 \\
-\cos\alpha_2\sin\alpha_3 & \sin\alpha_1\sin\alpha_2\sin\alpha_3+\cos\alpha_1\cos\alpha_3 \\
-sin\alpha_2 & -\sin\alpha_1\cos\alpha_2 \\
 & \\
 & \cos\alpha_1\sin\alpha_2\cos\alpha_3+\sin\alpha_1\sin\alpha_3 \\
 & -\cos\alpha_1\sin\alpha_2\sin\alpha_3+\sin\alpha_1\cos\alpha_2 \\
 & \cos\alpha_1\cos\alpha_2
\end{pmatrix} \tag{167}
$$

其中 $-\pi \leq \alpha_1 \leq \pi$, $-\pi \leq \alpha_2 \leq \pi$, $-\frac{\pi}{2} \leq \alpha_1 \leq \frac{\pi}{2}$. This matrix can be obtained from the exponential map (159) with

More on Exponential Maps

例 (Lie group $SO(3)$)

$$X_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_{\alpha_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{168}$$

satisfying the commutation relations of the Lie algebra $so(3)$

$$[X_{\alpha_i}, X_{\alpha_j}] = \varepsilon_{ijk} X_{\alpha_k} \tag{169}$$

where ε_{ijk} is the antisymmetric rank-3 tensor.

More on Exponential Maps

例 (Lie group $SO(3)$)

The elements of the Lie algebra $X_{\alpha_i}(i = 1, 2, 3)$ are called generators of the group. There are other parametrizations. For example, in the Euler angle parametrization, an operator

$$R(\varphi, \theta, \psi) = R_{z'}(\psi) R_u(\theta) R_z(\varphi) = e^{-i\psi J_{z'}} e^{-i\theta J_u} e^{-i\varphi J_z} \tag{170}$$

By a series of transformations R can be brought to the form

$$R(\varphi, \theta, \psi) = e^{-i\varphi J_x} e^{-i\theta J_y} e^{-i\psi J_z} \tag{171}$$

More on Exponential Maps

例 (Lie group $SO(3)$)

where J_x, J_y, J_z satisfy the commutation relations

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y$$

Although this expression is useful in practical calculations, it is not an exponential map, since it is not connected with the identity element, and therefore it is not a parametrization of the group.

More on Exponential Maps

例 (Lie group $SU(2)$)

The matrix parametrization $A(\alpha_1, \alpha_2, \alpha_3)$ of $SU(2)$ in terms of the angles $\alpha_1, \alpha_2, \alpha_3$ is

$$\begin{pmatrix} (\cos\alpha_1\cos\alpha_2 + i\sin\alpha_1\sin\alpha_2)e^{i\alpha_3} & -\cos\alpha_1\sin\alpha_2 + i\sin\alpha_1\cos\alpha_2 \\ \cos\alpha_1\sin\alpha_2 + i\sin\alpha_1\cos\alpha_2 & (\cos\alpha_1\cos\alpha_2 - i\sin\alpha_1\sin\alpha_2)e^{-i\alpha_3} \end{pmatrix} \tag{172}$$

其中 $-\pi \le \alpha_1 \le \pi,\ -\pi \le \alpha_2 \le \pi,\ 0 \le \alpha_1 \le \pi.$

More on Exponential Maps

例 (Lie group $SU(2)$)

This matrix can be obtained from the exponential map with

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{173}$$

satisfying the commutation relations of $su(2) \sim so(3)$.

Infinitesimal Transformations

无穷小变换可以通过展开指数变换到一阶来获得.

例 (Infinitesimal $SO(3)$ rotation around z)

This infinitesimal rotation is obtained from $A(\alpha_1, \alpha_2, \alpha_3)$ by letting $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = \varepsilon$. One obtains as $\varepsilon \to 0$,

$$A(0,0,\varepsilon) = \begin{pmatrix} 1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{174}$$

Infinitesimal Transformations

例 (Infinitesimal $SO(3)$ rotation around z)

By acting with $A(0,0,\varepsilon)$ on a vector with components x, y, z, one obtains

$$A(0,0,\varepsilon)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+\varepsilon y \\ -\varepsilon x + y \\ z \end{pmatrix} \tag{175}$$

Definitions

考虑由元素 $X_i(i=1,...,r)$ 组成的代数 g 及其指数映射得到的群 $G(\exp\left(\sum_i \alpha_i X_i\right))$. 拓扑空间 Γ 中的点记为 γ. Group G transforms any point γ of Γ into another point γ'. 群 G 称为 Γ 上的拓扑变换群，若满足:

1. 对任何 $G_\alpha \in G$ 存在一个从 Γ 到 Γ 同态 $\gamma \to G\gamma$.
2. 群单位元 $e \in G$ 对应单位同态.
3. 映射 $G \times \Gamma \to \Gamma: \quad \gamma \to G\gamma$ 是连续映射.
4. 对 $\forall$ $G_1, G_2 \in G$, $\gamma \in \Gamma$，有 $G_1(G_2\gamma) = (G_1 G_2)\gamma$.

Definitions

若对任意一组 $\gamma_1, \gamma_2 \in \Gamma$，存在一个元素 G 使得 $\gamma_2 = G\gamma_1$，则称群 G 在 Γ 上的作用是传递的. G acts effectively on Γ if e is the only element that leaves each $\gamma \in \Gamma$ fixed. The space Γ is called a homogeneous space(齐次空间).

若 G 是一个解析李群，H 是 G 的一个紧子群，那么商空间 G/H 称为 globally symmetric Riemannian space, also called a coset space.

Definitions

Construct Coset Spaces

给出代数 g 及其子代数 h，将 g 分为

$$g = h \oplus p \tag{176}$$

The algebra h is called the stability algebra(稳定代数), g/h the factor algebra(因子代数), and p the remainder(余数), not closed(闭合) with respect to commutation. The number of elements of p gives the topological dimension of the space, and the dimension of the maximal abelian subalgebra of p gives the rank of the space.

Construct Coset Spaces

The decomposition (183) has a counterpart in the Lie group, called a coset decomposition. Let

$$G = e^{g}, \qquad H = e^{h}, \qquad P = e^{p} \tag{177}$$

Then

$$G = HP, \qquad G = PH \tag{178}$$

分别称为左右陪集. G/H 商空间是陪集的参数空间.

Construct Coset Spaces

例

The Riemannian space

$$U(6)/U(5) \otimes U(1) \tag{179}$$

has 10 variables (5 complex variables).

例

The Riemannian space

$$SO(3)/SO(2) \tag{180}$$

has 2 real variables.

Construct Coset Spaces

例

The Riemannian space

$$U(5,1)/U(5)\otimes U(1) \tag{181}$$

has 10 variables (5 complex variables).

例

The Riemannian space

$$SO(3,1)/SO(3) \tag{182}$$

has 3 real variables.

Construct Coset Spaces

The spaces $U(n)/U(n-1) \otimes U(1)$ are useful when describing systems of bosons.

The spaces $SO(n)/SO(n-1)$ are useful in quantum mechanics.

The spaces $SO(n,1)/SO(n)$ are particularly useful in relativistic quantum mechanics.

Two other important Riemannian spaces related to those of Table 5.2 are $U(n,1)/U(n) \otimes U(1)$ and $SO(n,1)/SO(n)$. Their properties are listed in Table 5.3.

Construct Coset Spaces

Definitions and Independent Casimir Operators

回顾之前的卡西米尔算子及其阶数的定义 (56) 和 (57):

$$[C, X_\mu] = 0, \quad \forall\ X_\mu \in g; \quad C_p = \sum_{\alpha_1 \dots \alpha_p} f^{\alpha_1 \alpha_2 \dots \alpha_p} X_{\alpha_1} X_{\alpha_2} \dots X_{\alpha_p}$$

其位于 g 的包络代数 $T(g)$ 中.

The number of independent Casimir operators, C , of a Lie algebra g, is equal to the rank l of g, and hence equal to the number of labels that characterize the irreducible representations of g.

Casimir Operators of $u(n)$

$u(n)$ 代数有 n 个独立卡西米尔算子，阶数 $1, 2, ..., n$

$$C_1, C_2, ..., C_n \tag{183}$$

之前通过结构常数 $c^{\gamma}_{\alpha\beta}$ 给出了卡西米尔算子的构造. 对代数 $u(n)$, 若元素记为 $E_{ij}(i, j = 1, ..., n)$, p 阶卡西米尔算子可以写成

$$C_p = E_{i_1 i_2} E_{i_2 i_3} ... E - i_{p-1} i_p E_{i_p i_1}, \qquad p = 1, 2, ..., n \tag{184}$$

特别地，

$$C_i = E_{i_1 i_1}, \quad \text{or} \quad \sum_{i=1}^{n} E_{ii} \tag{185}$$

Casimir Operators of $su(n)$

代数 $su(n)$ 的卡西米尔算子阶数是 $2, 3, ..., n$

$$C_2, C_3, ..., C_n \tag{186}$$

和 $u(n)$ 类似，但不存在 C_1. $su(n)$ 的元素可以通过保持 $u(n)$ 非对角元素不变，改变对角元为

$$\tilde{E}_{ii} = E_{ii} - \frac{1}{n}\sum_{j=1}^{n} E_{jj} \tag{187}$$

and deleting $\tilde{E}_{nn}$. $su(n)$ 的线性卡西米尔算子：

$$C_1 = \sum_{i=1}^{n} \tilde{E}_{ii} = 0 \tag{188}$$

Casimir Operators of $so(n)$, n=Odd

The independent Casimir operators of $so(n)$, n=Odd are

$$C_2, C_4, C_6, ..., C_{2\nu-2}, C_{2\nu}; \qquad \nu = \frac{n-1}{2} \tag{189}$$

These operators are all of even order.

Definitions and Independent Casimir Operators

Definitions and Independent Casimir Operators

Reference