Proof:

Let $\{\hat{f}_k : \mathbb{R}^n \to [0,\infty)^n\}_{k=1}^{\infty}$ be a monotonically increasing sequence of elementary functions, i.e. $|\hat{f}_k(\mathbb{R}^n)| < \infty$. Then let $B_i := \{x \in \mathbb{R}^n \mid \hat{f}_k(x) = \alpha_i\} \in \mathcal{B}(\mathbb{R}^n)$ such that:

$$
\begin{aligned}
&(1) \quad \hat{f}_k(x) = \sum\nolimits_{i=0}^{n} \alpha_i \mathbb{I}_{B_i}(x) \\
&(2) \quad \lim_{x\to\infty} \hat{f}_k = f
\end{aligned}
$$

Note that $\int_\Omega \mathbb{I}_A(\boldsymbol{\epsilon}) p(\boldsymbol{\epsilon}) d^n\boldsymbol{\epsilon} = P(\boldsymbol{\epsilon} \in B)$ for any $B \in \mathcal{B}(\mathbb{R}^n)$. By replacing $\mathbf{z}$ with $g(\boldsymbol{\epsilon}, \mathbf{x}; \psi))$ we obtain the following calculation:

$$
\begin{aligned}
\mathbb{E}_{q_\psi(\mathbf{z}|x^l)}[\hat{f}_k(\mathbf{z})] &= \int_\Omega \hat{f}_k(\mathbf{z}) q_\psi(\mathbf{z}|x^l) d^n\mathbf{z} \\
&= \int_\Omega \sum\nolimits_{i=0}^{n} \alpha_i \mathbb{I}_{B_i}(\mathbf{z}) q_\psi(\mathbf{z}|x^l) d^n\mathbf{z} \\
&= \sum\nolimits_{i=0}^{n} \alpha_i \int_\Omega \mathbb{I}_{B_i}(\mathbf{z}) q_\psi(\mathbf{z}|x^l) d^n\mathbf{z} \\
&= \sum\nolimits_{i=0}^{n} \alpha_i P(\mathbf{z} \in B_i \mid \mathbf{x} = x^l) && (1) \\
&= \sum\nolimits_{i=0}^{n} \alpha_i P(g(\boldsymbol{\epsilon}, x^l; \psi) \in B_i) && (2) \\
&= \sum\nolimits_{i=0}^{n} \alpha_i P(\boldsymbol{\epsilon} \in g^{-1}(B_i)) && (3) \\
&= \sum\nolimits_{i=0}^{n} \alpha_i \int_\Omega \mathbb{I}_{g^{-1}(B_i)}(\boldsymbol{\epsilon}) p(\boldsymbol{\epsilon}) d^n\boldsymbol{\epsilon} && (4) \\
&= \int_\Omega \sum\nolimits_{i=0}^{n} \alpha_i \mathbb{I}_{B_i}(g(\boldsymbol{\epsilon}, x^l; \psi)) p(\boldsymbol{\epsilon}) d^n\boldsymbol{\epsilon} && (5) \\
&= \mathbb{E}_{p(\boldsymbol{\epsilon})}[\hat{f}_k(g(\boldsymbol{\epsilon}, x^l; \psi))]
\end{aligned}
$$

In step (2) we took advantage that g expresses already the dependence of $\mathbf{z}$ on the value x^l. From step (4) to step (5) we applied the transformation.