

1 Structure of the integrals

The general structure of the quantities to be computed in APFEL is the combination of terms I that have the form of Mellin convolutions between an operator O and distribution function d , that is:

$$I(x) = x \int_0^1 dz \int_0^1 dy O(z) d(y) \delta(x - yz) = x \int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x \int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right). \quad (1.1)$$

However, very often, typically in the presence of mass effects, the integration phase space is modified and the convolution in eq. (1.1), limiting ourselves to the leftmost identity, is generalised as:

$$I(x, \eta) = x \int_{x/\eta}^1 \frac{dz}{z} O(z, \eta) d\left(\frac{x}{\eta z}\right). \quad (1.2)$$

where $\eta \leq 1$, with $\eta = 1$ reproducing eq. (1.1). However, for purposes that will become clear later, we want to write the integral in eq. (1.2) in the form of eq. (1.1), that is in such a way that lower bound of the integral is not the rescaled variable x/η but the physical Bjorken x . To do so, one needs to perform the change of variable $y = \eta z$, so that:

$$I(x, \eta) = \int_x^\eta dy O\left(\frac{y}{\eta}, \eta\right) \frac{x}{y} d\left(\frac{x}{y}\right). \quad (1.3)$$

In order to precompute the expensive part of the integral in eq. (1.5), we use the standard interpolation formula to the distribution d :

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right), \quad (1.4)$$

where α runs over the node of a give grid in x space and the weights w_α are typically polynomials of degree k (*i.e.* Laplace interpolants). Now we use eq. (1.4) in eq. (1.5) and at the same time we limit the computation of the integral I to the point x_β of the grid used in eq. (1.4). This way we get:

$$I(x_\beta, \eta) = \sum_{\alpha=0}^{N_x} \bar{d}_\alpha \int_{x_\beta}^\eta dy O\left(\frac{y}{\eta}, \eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right). \quad (1.5)$$

where we have defined $\bar{d}_\alpha = x_\alpha d(x_\alpha)$. In the particular case of the Laplace interpolants (in the APFEL procedure), one can show that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d, \quad (1.6)$$

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k}), \quad (1.7)$$

and thus eq. (1.5) can be adjusted as:

$$I(x_\beta, \eta) = \sum_{\alpha=0}^{N_x} \bar{d}_\alpha \int_c^d dy O\left(\frac{y}{\eta}, \eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right). \quad (1.8)$$

Finally, we change back the integration variable $z = y/\eta$ so that eq. (1.8) becomes:

$$I(x_\beta, \eta) = \sum_{\alpha=0}^{N_x} \bar{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz O(z, \eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) \right]}_{\Gamma_{\alpha\beta}}. \quad (1.9)$$

The quantity in squared brackets is the interesting bit.

The expressions for the operator O that we have to deal with have this general form:

$$O(z, \eta) = R(z, \eta) + \sum_{i=0}^n \left[\frac{\ln^i(1-z)}{1-z} \right]_+ S^{(i)}(z, \eta) + L(\eta) \delta(1-z), \quad (1.10)$$

where R and $S^{(i)}$ is a regular function in $z = 1$, that is:

$$R(1, \eta) = \lim_{z \rightarrow 1} R(z, \eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1, \eta) = \lim_{z \rightarrow 1} S^{(i)}(z, \eta) = J^{(i)}(\eta), \quad (1.11)$$

being K , $J^{(i)}$, and L a finite function of η . Notice that the sum over i in eq. (1.10) is between zero and n where n is typically not bigger than two as it depends on the perturbative order of the expressions. Plugging eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in eq. (1.9) and making use of the definition of plus-prescription, we obtain:

$$\begin{aligned} \Gamma_{\beta\alpha} = & \eta \int_{c/\eta}^{d/\eta} dz \left\{ R(z, \eta) w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{\eta z} \right) + \sum_{i=0}^n \frac{\ln^i(1-z)}{1-z} \left[S^{(i)}(z, \eta) w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{\eta z} \right) - S^{(i)}(1, \eta) w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{\eta} \right) \right] \right\} \\ & + \eta \left[\sum_{i=0}^n \frac{1}{(i+1)!} S^{(i)}(1, \eta) \ln^{i+1} \left(1 - \frac{c}{\eta} \right) + L(\eta) \right] w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{\eta} \right). \end{aligned} \quad (1.12)$$

Eq. (1.12) expresses the full complexity of the task. However, there are a few remarks that apply to some particular cases and that reduce the complexity. In the case of the ($\overline{\text{MS}}$) splitting functions, there are two main simplifications: the first is that $\eta = 1$, the second is that $n = 0$ in the sums. Considering that:

$$w_{\alpha}^{(k)}(x_{\beta}) = \delta_{\alpha\beta}, \quad (1.13)$$

and that the expressions can be manipulated in such a way that the coefficient of the plus-prescribed term S is a constant, we have that:

$$\Gamma_{\beta\alpha} = \int_c^d dz \left\{ R(z) w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{z} \right) + \frac{S}{1-z} \left[w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{z} \right) - \delta_{\alpha\beta} \right] \right\} + [S \ln(1-c) + L] \delta_{\alpha\beta}. \quad (1.14)$$

The same kind of simplifications apply to the case of the Zero-Mass (ZM) coefficient functions with the only exception that the sum over i extends to n 's larger than zero. In particular, since we will use exact expressions only up to $\mathcal{O}(\alpha_s)$, *i.e.* NLO, we have $n = 1$ so that:

$$\begin{aligned} \Gamma_{\beta\alpha} = & \int_c^d dz \left\{ R(z) w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{z} \right) + \frac{S^{(0)} + S^{(1)} \ln(1-z)}{1-z} \left[w_{\alpha}^{(k)} \left(\frac{x_{\beta}}{z} \right) - \delta_{\alpha\beta} \right] \right\} \\ & + \left[S^{(0)} \ln(1-c) + \frac{1}{2} S^{(1)} \ln^2(1-c) + L \right] \delta_{\alpha\beta}. \end{aligned} \quad (1.15)$$

As far as the massive coefficient functions up to $\mathcal{O}(\alpha_s^2)$ things can be more complicated and we will discuss later. In the next section we will consider the case of hadronic observables in the ZM scheme showing how to pre-compute the integral for the double-differential distributions in Drell-Yan production.