Transversity Distributions

1 Perturbative evolution

In this section we discuss the structure of the DGLAP evolution equations for the transversity distributions. The same structure holds for both PDFs and FFs. Therefore we first discuss the structure of the evolution equations in terms of distributions in the so-called "evolution" basis and then report the splitting functions up to $\mathcal{O}(\alpha_s^2)$, *i.e.* next-to-leading order (NLO), separately for PDFs and FFs. Contrary to unpolarised and longitudinally polarised collinear distributions, no transversely polarised gluon distribution exists. This simplifies the structure of the evolution equations that, when written in the evolution basis, are completely decoupled.

As a first step we define the evolution basis. Given a set of quark distributions in the more familiar "physical" basis, *i.e.* $\{\bar{t},\bar{b},\bar{c},\bar{s},\bar{u},\bar{d},d,u,s,c,b,t\}$, and defining $q^{\pm}\equiv q\pm\bar{q}$, the "evolution" basis is defined as follows:

$$\begin{array}{rcl}
\Sigma &=& \sum_q q^+\,,\\
V &=& \sum_q q^-\,,\\
T_3 &=& u^+-d^+\,,\\
V_3 &=& u^--d^-\,,\\
T_8 &=& u^++d^+-2s^+\,,\\
V_8 &=& u^-+d^--2s^-\,,\\
T_{15} &=& u^++d^++s^+-3c^+\,,\\
V_{15} &=& u^-+d^-+s^--3c^-\,,\\
T_{24} &=& u^++d^++s^++c^+-4b^+\,,\\
V_{24} &=& u^-+d^-+s^-+c^--4b^-\,,\\
T_{35} &=& u^++d^++s^++c^++b^+-5t^+\,,\\
V_{35} &=& u^-+d^-+s^-+c^-+b^--5t^-\,.
\end{array} \tag{1.1}$$

It is possible to show that in this basis the general form of the DGLAP evolution equations for the transversity distribution reduces to the following set of *decoupled* integro-differential equation:

$$\begin{array}{rcl}
\mu^2\dfrac{d\Sigma}{d\mu^2} &=& P_{qq}\otimes\Sigma\,,\\[2ex]
\mu^2\dfrac{dV}{d\mu^2} &=& P^V\otimes V\,,\\[2ex]
\mu^2\dfrac{dT_i}{d\mu^2} &=& P^+\otimes T_i\,,\\[2ex]
\mu^2\dfrac{dV_i}{d\mu^2} &=& P^-\otimes V_i\,,
\end{array} \tag{1.2}$$

with $i=3,8,15,24,35$ and where the Mellin convolution symbol $\otimes$ is defined as:

$$f(x)\otimes g(x)\equiv\int_x^1\frac{dy}{y}f(y)g\left(\frac{x}{y}\right)=\int_x^1\frac{dy}{y}f\left(\frac{x}{y}\right)g(y)\,. \tag{1.3}$$

The splitting functions P_{qq}, P^V, P^+, and P^- are usually decomposed as follows:

$$\begin{array}{l}
P^{\pm}\equiv P_{qq}^V\pm P_{q\bar{q}}^V\,,\\[2ex]
P_{qq}\equiv P^++n_f(P_{qq}^S+P_{q\bar{q}}^S)\,,\quad ,\\[2ex]
P^V\equiv P^-+n_f(P_{qq}^S-P_{q\bar{q}}^S)\,,
\end{array} \tag{1.4}$$

where n_f is the number of active flavours at a given scale μ and the splitting functions P_{qq}^V, $P_{q\bar{q}}^V$, P_{qq}^S, $P_{q\bar{q}}^S$ have the usual perturbative expansion:

$$P(x,\mu) = \sum_{n=0} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} P^{(n)}(x)\,. \tag{1.5}$$

Given the expansion above, one can show that at $\mathcal{O}(\alpha_s)$, *i.e.* leading order, all coefficients but $P_{qq}^{V,(0)}$ vanish. It is then easy to see that:

$$P_{qq}^{(0)} = P^{V,(0)} = P^{+,(0)} = P^{-,(0)} = P_{qq}^{V,(0)}\,. \tag{1.6}$$

This means that the evolution equations in Eq. (1.2) have all the same evolution kernel.

If one wants to include NLO corrections, one finds that the $\mathcal{O}(\alpha_s^2)$ coefficients $P_{qq}^{V,(1)}$ and $P_{q\bar{q}}^{V,(1)}$ are different from zero while $P_{qq}^{S,(1)}$ and $P_{q\bar{q}}^{S,(1)}$ vanish. This immediately implies that $P_{qq} = P^+$ and $P^V = P^-$. Therefore, at NLO, the evolution equations are fully determined by the functions $P_{qq}^{V,(0)}$, $P_{qq}^{V,(1)}$, and $P_{q\bar{q}}^{V,(1)}$. We are now in the position to discuss the specific expressions of these functions for both PDFs and FFs. A further simplification is given by the fact that the function $P_{qq}^{V,(0)}$ is the same for both PDFs and FFs. However, this is no longer the case for $P_{qq}^{V,(1)}$ and $P_{q\bar{q}}^{V,(1)}$ whose form differs between PDFs and FFs.

In order to carry out the implementation of the splitting functions in APFEL, it is necessary to make sure that the expressions of the single coefficients of the perturbative expansions have the following structure:

$$P(y) = R(y) + S\left(\frac{1}{1-x}\right)_+ + L\delta(1-y)\,, \tag{1.7}$$

where R is a regular function in $x=1$, and S and L are numerical coefficients. The plus prescription used above has the following definition upon integration with a test function f:

$$\int_0^1 dy \left(\frac{1}{1-y}\right)_+ f(y) \equiv \int_0^1 dy \frac{f(y)-f(1)}{1-y}\,. \tag{1.8}$$

An important detail to notice is that the definition above is strictly true only if the lower integration bound is equal to zero. In actual facts, this is never the case because Mellin convolutions involving plus-prescripted functions have the following structure:

$$\int_x^1 dy \left(\frac{1}{1-y}\right)_+ f(y)\,. \tag{1.9}$$

with $0 < x < 1$. This integral can be manipulated as follows:

$$\begin{aligned}
\int_x^1 dy \left(\frac{1}{1-y}\right)_+ f(y) &= \int_0^1 dy \left(\frac{1}{1-y}\right)_+ f(y) - \int_0^x dy \left(\frac{1}{1-y}\right)_+ f(y) \\
&= \int_0^1 dy \frac{f(y)-f(1)}{1-y} - \int_0^x dy \frac{f(y)}{1-y} \\
&= \int_x^1 dy \frac{f(y)-f(1)}{1-y} - f(0)\int_0^x \frac{dy}{1-y} \\
&= \int_x^1 dy \left(\frac{1}{1-y}\right)_\oplus f(y) + f(0)\ln(1-x) \\
&= \int_x^1 dy \left[\left(\frac{1}{1-y}\right)_\oplus + \ln(1-x)\delta(1-y)\right] f(y)\,,
\end{aligned} \tag{1.10}$$

where we have defined a "generalised" plus prescription that holds in its form regardless of the lower integration bound:

$$\int_x^1 dy \left(\frac{1}{1-y}\right)_\oplus f(y) \equiv \int_x^1 dy \frac{f(y)-f(1)}{1-y}\,. \tag{1.11}$$

Therefore, a Mellin-like convolution of the splitting function in Eq. (1.7) with the test function f will take the form:

$$\int_x^1 dy P(y) f(y) = \int_x^1 dy \left[R(x) + S \left(\frac{1}{1-y} \right)_{\oplus} + \left(L + S \ln(1-x) \right) \delta(1-y) \right] f(y)\,. \tag{1.12}$$

This provides a suitable expression for the implementation in APFEL. Therefore, one just needs to manipulate the expressions given in the literature to reduce them to the form of Eq. (1.7). This is typically an easy task.

Let us start with $P_{qq}^{V,(0)}$ that we take from Eq. (38) of Ref. [1].[1] After a simple manipulation, it takes the form:

$$P_{qq}^{V,(0)}(y) = 2C_F \left[-2 + 2 \left(\frac{1}{1-y} \right)_{+} + \frac{3}{2} \delta(1-y) \right]. \tag{1.13}$$

In this form it is easy to identify the elements introduced in Eq. (1.7). In particular, we find that $R(x) = -4C_F$, $S = 4C_F$, and $L = 3C_F$. As mentioned above, $P_{qq}^{V,(0)}$ is the same for PDFs and FFs, therefore the expression in Eq. (1.13) is all one needs to implement the LO evolution of both transversity PDFs and FFs

We now consider the NLO corrections. In order to distinguish between PDFs and FF we will use the symbols $\mathcal{P}$ and $\mathbb{P}$, respectively, for the splitting functions. We first consider the PDF splitting functions that we again take from Ref. [1]. We observe that $\mathcal{P}_{q\bar{q}}^{V,(1)}$, taken from Eq. (44) of this paper, is a purely regular functions with no plus-prescripted and δ-function terms. Therefore, it needs no manipulation:

$$\mathcal{P}_{q\bar{q}}^{V,(1)}(y) = 4C_F \left(C_F - \frac{1}{2} C_A \right) \left[-1 + y - \frac{4 S_2(y)}{1+y} \right], \tag{1.14}$$

with:

$$S_2(y) = -2\mathrm{Li}_2(-y) - 2 \ln y \ln(1+y) + \frac{1}{2} \ln^2 y - \frac{\pi^2}{6}\,. \tag{1.15}$$

The function $\mathcal{P}_{qq}^{V,(1)}$ from Eq. (43) of Ref. [1] is instead more complicated but can be recasted in the form of Eq. (1.13) as:

$$\begin{aligned}
\mathcal{P}_{qq}^{V,(1)}(y) \;=\;& \Bigg\{ 4C_F^2 \left[1 - y - \left(\frac{3}{2} + 2\ln(1-y) \right) \frac{2y \ln y}{1-y} \right] \\
+\;& 2C_F C_A \left[-\frac{143}{9} + \frac{2\pi^2}{3} + y + \left(\frac{11}{3} + \ln y \right) \frac{2y \ln y}{1-y} \right] + \frac{8}{3} n_f C_F T_R \left[-\frac{2y\ln y}{1-y} + \frac{10}{3} \right] \Bigg\} \\
+\;& \left\{ 2C_F C_A \left(\frac{134}{9} - \frac{2\pi^2}{3} \right) - \frac{80}{9} n_f C_F T_R \right\} \left(\frac{1}{1-y} \right)_{+} \\
+\;& \left\{ 4C_F^2 \left(\frac{3}{8} - \frac{\pi^2}{2} + 6\zeta(3) \right) + 2C_F C_A \left(\frac{17}{12} + \frac{11\pi^2}{9} - 6\zeta(3) \right) - \frac{8}{3} n_f C_F T_R \left(\frac{1}{4} + \frac{\pi^2}{3} \right) \right\} \delta(1-y)\,,
\end{aligned} \tag{1.16}$$

where the regular, plus-prescripted, and δ-function terms are enclosed between curly brackets.

We can now turn to consider the splitting functions for FFs. In this case we take the expressions from Ref. [2]. From Eq. (17) of this paper we immediately see that:

$$\mathbb{P}_{q\bar{q}}^{V,(1)}(y) = \mathcal{P}_{q\bar{q}}^{V,(1)}(y)\,, \tag{1.17}$$

[1] A factor 2 is introduced to account for the different expansion parameter, here $\alpha_s/4\pi$ rather than $\alpha_s/2\pi$

where $\mathcal{P}_{q\bar{q}}^{V,(1)}$ is given in Eq. (1.14). For $\mathbb{P}_{qq}^{V,(1)}$ we instead find:

$$\begin{aligned}
\mathbb{P}_{qq}^{V,(1)}(y) &= \left\{4C_F^2\left[1-y+\left(\frac{3}{2}+2\ln(1-y)-2\ln y\right)\frac{2y\ln y}{1-y}\right]\right. \\
&+ \left. 2C_FC_A\left[-\frac{143}{9}+\frac{2\pi^2}{3}+y+\left(\frac{11}{3}+\ln y\right)\frac{2y\ln y}{1-y}\right]+\frac{8}{3}n_fC_FT_R\left[-\frac{2y\ln y}{1-y}+\frac{10}{3}\right]\right\} \\
&+ \left\{2C_FC_A\left(\frac{134}{9}-\frac{2\pi^2}{3}\right)-\frac{80}{9}n_fC_FT_R\right\}\left(\frac{1}{1-y}\right)_+ \\
&+ \left\{4C_F^2\left(\frac{3}{8}-\frac{\pi^2}{2}+6\zeta(3)\right)+2C_FC_A\left(\frac{17}{12}+\frac{11\pi^2}{9}-6\zeta(3)\right)-\frac{8}{3}n_fC_FT_R\left(\frac{1}{4}+\frac{\pi^2}{3}\right)\right\}\delta(1-y)\,,
\end{aligned} \tag{1.18}$$

that is just a small difference in the regular term as compared to $\mathcal{P}_{qq}^{V,(1)}$ in Eq. (1.16).

References

[1] W. Vogelsang, Phys. Rev. D **57** (1998) 1886 doi:10.1103/PhysRevD.57.1886 [hep-ph/9706511].

[2] M. Stratmann and W. Vogelsang, Phys. Rev. D **65** (2002) 057502 doi:10.1103/PhysRevD.65.057502 [hep-ph/0108241].