1 Evolution of the TMDs

In these notes I will show how to solve the renormalisation-group equation (RGE) and the rapidity-evolution equation (often referred to as Collins-Soper (CS) equation) of a TMD distribution F. The distribution F can be either a PDF or a FF and can be associated to either to a quark or to the gluon: the structure of the solution of the evolution equations is exactly the same. In the impact-parameter space, F is a function of the transverse-momentum fraction x, of the bidimensional impact-parameter vector $\mathbf{b}_T$, of the renormalisation scale μ, and of the rapidity scale ζ, *i.e.* $F \equiv F(x, \mathbf{b}_T, \mu, \zeta)$. Since the evolution equations govern the behaviour of F w.r.t. the scale μ and ζ, in order to simplify the notation I will drop the dependence on x and $\mathbf{b}_T$, *i.e.* $F \equiv F(\mu, \zeta)$.[1]

The goal of the solution of the evolution equation is that of expressing the distribution F at some final scales (μ, ζ) in terms of the same distribution at the initial scales (μ_0, ζ_0). It will turn out that this is accomplished by computing the evolution kernel $R\left[(\mu_0, \zeta_0) \to (\mu, \zeta)\right]$, such that:

$$F(\mu, \zeta) = R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{1}$$

The purpose of these notes is to provide the explicit expression of the evolution kernel R in terms of perturbatively computable quantities. A collateral aspect that will be discussed is these notes is the independence from the path that connects the initial and final scales (μ_0, ζ_0) and (μ, ζ). This in turn concerns the resummation of large logarithms that is required to ensure that the perturbative convergence is not spoiled.

The RGE and the CS equation read:

$$\begin{aligned} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} &= K(\mu)\,, \\ \frac{\partial \ln F}{\partial \ln \mu} &= \gamma(\mu, \zeta)\,, \end{aligned} \tag{2}$$

where γ and K are the anomalous dimensions of the evolution in μ and $\sqrt{\zeta}$, respectively, that will be discussed in more detail below. The equations above can be solved as follows. The first equation gives:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\right] F(\mu, \zeta_0)\,. \tag{3}$$

The factor $F(\mu, \zeta_0)$ can be evolved in μ using the second equation to give:

$$F(\mu, \zeta_0) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,, \tag{4}$$

such that:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{5}$$

This equation has exactly the structure of Eq. (1). We now need to express the argument of the exponent in terms of perturbatively computable quantities.

In order to do so, we use the fact that the rapidity anomalous dimension K needs to be renormalised and thus it obeys its own RGE, that reads:

$$\frac{\partial K}{\partial \ln \mu} = -\gamma_K(\alpha_s(\mu))\,. \tag{6}$$

γ_K is said cusp anomalous dimension and obeys the perturbative expansion:

$$\gamma_K(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \gamma_K^{(n)}\,, \tag{7}$$

[1] Notice that, despite the variables x and $\mathbf{b}_T$ will no longer appear, the symbol $\otimes$ indicates the Mellin convolution integral w.r.t. x while b_T indicates the length of the vector $\mathbf{b}_T$.

where $\gamma_K^{(n)}$ are numerical coefficients. Their value up to $n = 3$ can be read from Eq. (59) of Ref. [3]. They coincide with those reported in Eq. (D.6) of Ref. [2] up to a factor two due to a different normalisation of the rapidity anomalous dimension K whose derivative w.r.t. ζ is exactly γ_K. In addition, the cusp anomalous dimension for quarks and gluon are equal up to a factor C_F in the quark case and C_A in the gluon case.

Eq. (6) can be easily solved obtaining:

$$K(\mu) = K(\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) \,. \tag{8}$$

We anticipate that in the $\overline{\mathrm{MS}}$ scheme, there exists a particular scale $\mu_b = 2e^{-\gamma_E}/b_T$ such that K admits the following perturbative expansion:

$$K(\mu_b) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^{n+1} K^{(n,0)} \,, \tag{9}$$

where $K^{(n,0)}$ are numerical coefficients. Therefore, if $\mu_0 \simeq \mu_b$ the first term in the r.h.s. of Eq. (8) is free of large logs and thus its perturbative expansion, that reads:

$$K(\mu_0) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^{n+1} \sum_{m=0}^{n} K^{(n,m)} \ln^m \left(\frac{\mu_0}{\mu_b} \right) \,, \tag{10}$$

is reliable. The second term in the r.h.s. instead takes care, through the evolution of α_s, of resumming large logarithms in the case in which $\mu \gg \mu_0$. The coefficients $K^{(n,m)}$ up to $n = 3$ are reported in Eq. (D.9) of Ref. [2] and up to $n = 2$ in Eq. (69) of Ref. [3]. They differ by a factor -2 due to a different definition of K. In addition, the logarithmic expansion is done in terms of $\ln(\mu_0/\mu_b)$ in Ref. [3] and in terms of $\ln(\mu_0^2/\mu_b^2)$ in Ref. [2]. Therefore, each coefficient differs by an additional factor 2^m, where m is the power of the logarithm that multiplies the coefficient itself.

A further important property of the anomalous dimensions can be derived by considering the fact that the crossed double derivarives of F must be equal:

$$\frac{\partial}{\partial \ln \mu} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} = \frac{\partial}{\partial \ln \sqrt{\zeta}} \frac{\partial \ln F}{\partial \ln \mu} \,. \tag{11}$$

Using Eqs. (2) and (6) leads to the following differential equation:

$$\frac{\partial \gamma}{\partial \ln \sqrt{\zeta}} = -\gamma_K(\alpha_s(\mu)) \,, \tag{12}$$

whose solution is:

$$\gamma(\mu, \zeta) = \gamma(\mu, \mu^2) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu} \,. \tag{13}$$

It turns out that $\gamma(\mu, \mu^2)$ has a purely perturbative expansion:

$$\gamma(\mu, \mu^2) \equiv \gamma_F(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} \gamma_F^{(n)} \,, \tag{14}$$

where $\gamma_F^{(n)}$ are again numerical coefficients that are given in Eq. (58) of Ref. [3] and Eq. (D.7) of Ref. [2] (Eq. (D.8) reports the coefficients for the gluon anomalous dimension). The two sets of coefficients differ by a minus sign due to the different definition of the constant (non-log) term of the RGE anomalous dimension. Therefore:

$$\gamma(\mu, \zeta) = \gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu} \,. \tag{15}$$

Finally, plugging Eqs. (8) and (15) into Eq. (5), one gets:

$$F(\mu,\zeta) = \exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\} F(\mu_0,\zeta_0)\,. \quad (16)$$

Comparing Eq. (16) to Eq. (1) allows one to give an explicit expression to the evolution kernel:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right] = \exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}\,. \quad (17)$$

Eq. (16) has been obtained evolving the TMD F first in the ζ direction (Eq. (3)) and then in the μ direction (Eq. (4)). However, it is easy to verify that exchanging the order of the evolutions leads to the exact same result, Eq. (16). In particular, the following relation holds:

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta)\right]R\left[(\mu_0,\zeta)\leftarrow(\mu_0,\zeta_0)\right] = R\left[(\mu,\zeta)\leftarrow(\mu,\zeta_0)\right]R\left[(\mu,\zeta_0)\leftarrow(\mu_0,\zeta_0)\right] = R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]\,. \quad (18)$$

This is a direct consequence of the independence of evolution kernel R in Eq. (17) from the path $\mathcal{P}$ followed to connect the points (μ_0,ζ_0) to the point (μ,ζ):

$$R\left[(\mu,\zeta)\underset{\mathcal{P}}{\leftarrow}(\mu_0,\zeta_0)\right] \equiv R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right]\,. \quad (19)$$

Another important piece of information comes from the fact that, for small values of b_T, the TMD F can be matched onto the respective collinear distribution f (a PDF or a FF) through the perturbative coefficients C[2]:

$$F(\mu,\zeta) = C(\mu,\zeta)\otimes f(\mu)\,, \quad (20)$$

so that:

$$F(\mu,\zeta) = \exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\} C(\mu_0,\zeta_0)\otimes f(\mu_0)\,. \quad (21)$$

Exactly as in the case of K, for $\mu = \sqrt{\zeta} = \mu_b$ the matching function has the expansion:

$$C(\mu_b,\mu_b^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n C^{(n,0)}\,, \quad (22)$$

where the coefficients $C^{(n,0)}$ are functions of x only. In order to be able to compute the function C for generic values of the scales μ and ζ, evolution equations can be derived. Deriving Eq. (20) with respect to μ and ζ one gets:

$$\begin{aligned}
\frac{\partial F}{\partial\ln\sqrt{\zeta}} &= \frac{\partial C}{\partial\ln\sqrt{\zeta}}\otimes f(\mu)\,,\\
\frac{\partial F}{\partial\ln\mu} &= \frac{\partial C}{\partial\ln\mu}\otimes f(\mu) + C(\mu,\zeta)\otimes\frac{\partial f}{\partial\ln\mu} = \left[\frac{\partial C}{\partial\ln\mu} + C(\mu,\zeta)\otimes 2P(\mu)\right]\otimes f(\mu)\,.
\end{aligned} \quad (23)$$

In the r.h.s. of the second equation I have used the DGLAP equation:

$$\frac{\partial f}{\partial\ln\mu} = 2P(\mu)\otimes f(\mu)\,. \quad (24)$$

One can also take the derivative of Eq. (16) and the result is:

$$\begin{aligned}
\frac{\partial F}{\partial\ln\sqrt{\zeta}} &= \left[K(\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]F(\mu,\zeta) = \left[K(\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right]C(\mu,\zeta)\otimes f(\mu)\,,\\
\frac{\partial F}{\partial\ln\mu} &= \left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]F(\mu,\zeta) = \left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]C(\mu,\zeta)\otimes f(\mu)\,.
\end{aligned} \quad (25)$$

[2] A sum over flavours is understood. As a matter of fact, the matching function C has to be regarded as a matrix in flavour space multipling a vector of collinear PDFs/FFs.

Equating Eq. (23) and Eq. (25) and dropping the distribution f, the evolution equations for C are:

$$
\begin{aligned}
\frac{\partial C}{\partial \ln \sqrt{\zeta}} &= \left[K(\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right] C(\mu,\zeta)\,, \\
\frac{\partial C}{\partial \ln \mu} &= \left\{\left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]\delta(1-x) - 2P(\mu)\right\}\otimes C(\mu,\zeta)\,.
\end{aligned}
\tag{26}
$$

These equations can be solved to determine the evolution of the matching function C. The solution can eventually be expanded if initial and final scales are not too far apart. In particular, if $\mu_0 = \sqrt{\zeta_0} \simeq \mu_b$ in Eq. (16), the matching function C can be reliably expanded as:

$$
C(\mu_0,\mu_0^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^n \sum_{m=0}^{2n} C^{(n,m)}\ln^m\left(\frac{\mu_0}{\mu_b}\right)\,. \tag{27}
$$

The coefficient functions $C^{(n,m)}$ have been computed for both PDFs and FFs in SCET in Ref. [2]. The same functions have also been computed in Ref. [1] and reported in Ref. [3]. The authors of the latter paper have verified the equality of the two sets of functions.

In order to use Eq. (16) in phenomenological applications, one needs to define the values of both the initial and final pairs of scales (μ_0,ζ_0) and (μ,ζ). The initial scales are usually identified with μ_b up to a small factor C_i, *i.e.* $(\mu_0,\zeta_0) = (C_i\mu_b, C_f^2\mu_b^2)$, with $\mu_b = 2e^{-\gamma_E}/b_T$. This is advantageous because possible logarithms that appear in the perturbative expansion of $K(\mu_0)$ and $C(\mu_0,\zeta_0)$ in Eq. (21) have the form $\ln(\mu_0/\mu_b) = \ln(\sqrt{\zeta_0}/\mu_b) = \ln C_i$ and thus are small enough not to spoil their convergence. Variations of C_i around unity can be possibly used to estimate the impact of higher-order corrections in the TMD evolution and matching.

The natural choice for the final scales is to identify them with the hard scale of the process, say Q. This choice has to match the renormalisation scale μ used in the hard factor H of the process under consideration. Therefore, choosing $(\mu,\zeta) = (C_fQ, Q^2)$, with C_f being a modest factor, can be used to estimate higher-order corrections.

References

[1] S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, Eur. Phys. J. C **72** (2012) 2195 doi:10.1140/epjc/s10052-012-2195-7 [arXiv:1209.0158 [hep-ph]].

[2] M. G. Echevarria, I. Scimemi and A. Vladimirov, JHEP **1609** (2016) 004 doi:10.1007/JHEP09(2016)004 [arXiv:1604.07869 [hep-ph]].

[3] J. Collins and T. C. Rogers, Phys. Rev. D **96** (2017) no.5, 054011 doi:10.1103/PhysRevD.96.054011 [arXiv:1705.07167 [hep-ph]].