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In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

1 Evolution equation

In general, the evolution equation for GPDs reads:

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-\infty}^{+\infty}\frac{dx'}{|2\xi|}P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{1.1}$$

The GPD f and the evolution kernel P may in general be a vector and a matrix in flavour space. For now we will just be concerned with the integral in the r.h.s. of Eq. (1.1) regardless of the flavour structure. The support of the evolution kernel $P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 1.1. Knowing the support of the evolution kernel, Eq. (1.1)

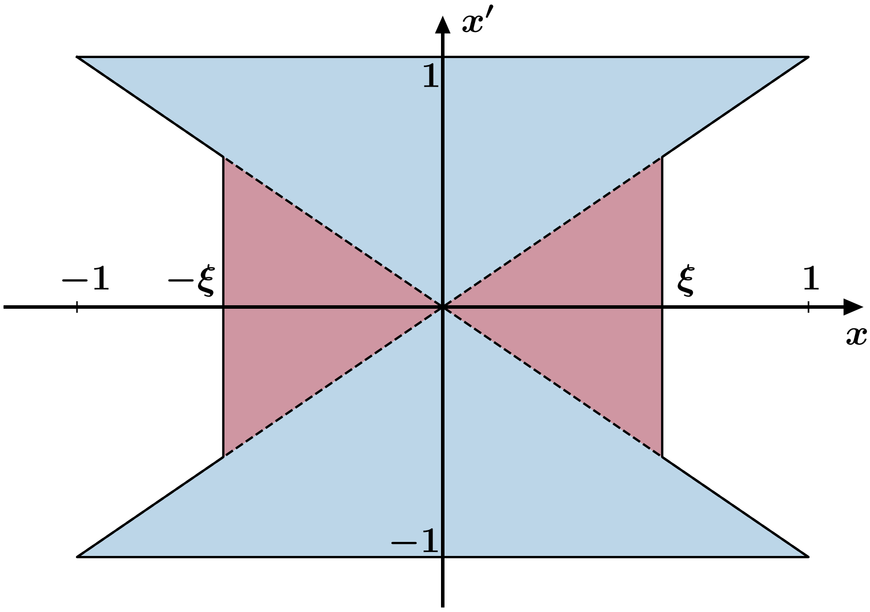

Fig. 1.1: Support domain of the evolution kernel $P\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

can be split as follows:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}f(\pm x,\xi) &= \int_{|x|}^{1}\frac{dx'}{x'}\left[\frac{x'}{|2\xi|}P\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{|2\xi|}P\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\\ &+ \theta\left(1-\left|\frac{x}{\xi}\right|\right)\int_{0}^{|x|}dx'\left[\frac{1}{|2\xi|}P\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{1}{|2\xi|}P\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,.\end{aligned} \tag{1.2}$$

where we have used the symmetry property of the evolution kernels $P(y,y')=P(-y,-y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned}f^{\pm}(x,\xi) &= f(x,\xi)\mp f(-x,\xi)\,,\\ P^{\pm}(y,y') &= P(y,y')\mp P(-y,y')\,,\end{aligned} \tag{1.3}$$

[1] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x)=-\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign.

so that the evolution equation for $f^{\pm}$ can be split as:

$$\begin{aligned} \mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) &= \int_{|x|}^1\frac{dx'}{x'}\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)+\theta\left(1-\left|\frac{x}{\xi}\right|\right)\int_0^{|x|}dx'\frac{1}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi) \\ &= I^{\pm,\mathrm{DGLAP}}(\xi,x)+I^{\pm,\mathrm{ERBL}}(\xi,x)\,. \end{aligned} \tag{1.4}$$

The first term in the second line of the equation above, $I^{\pm,\mathrm{DGLAP}}$, corresponds to integrating over the blue regions in Fig. 1.1, while the second term, $I^{\pm,\mathrm{ERBL}}$, results from the integration over the red regions. As indicated by the subscripts, $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ define the so-called DGLAP and ERBL regions in x relative ξ. Specifically, the presence of the θ-function in $I^{\pm,\mathrm{ERBL}}$ is such that for $x>\xi$ this term drops leaving only the DGLAP-like term $I^{\pm,\mathrm{DGLAP}}$. For $x\leq\xi$, instead, $I^{\pm,\mathrm{ERBL}}$ kicks in and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). Crucially, in the limits $\xi\to 0$ and $\xi\to\pm1$ one should and does recover the DGLAP and ERBL equations, respectively.

For convenience, we define the parameter:

$$\kappa(x)=\frac{\xi}{x}\,, \tag{1.5}$$

so that:

$$\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)=\frac{1}{2\kappa}\frac{x'}{x}P^{\pm}\left(\frac{1}{\kappa},\frac{1}{\kappa}\frac{x'}{x}\right)\equiv\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)\,, \tag{1.6}$$

where the last equality effectively defines the function:

$$\mathcal{P}^{\pm}(\kappa,y)=\frac{1}{2\kappa y}P^{\pm}\left(\frac{1}{\kappa},\frac{1}{\kappa y}\right)\,. \tag{1.7}$$

Plugging this definition into the first integral in the r.h.s. of Eq. (1.4) gives:

$$I^{\pm,\mathrm{DGLAP}}(\xi,x)=\int_{|x|}^1\frac{dx'}{x'}\frac{x'}{|2\xi|}P^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)=\int_{|x|}^1\frac{dx'}{x'}\mathcal{P}^{\pm}\left(\kappa,\frac{x}{x'}\right)f^{\pm}(x',\xi)\equiv\mathcal{P}^{\pm}\left(\kappa,x\right)\otimes f^{\pm}(x,\xi)\,. \tag{1.8}$$

Therefore, $I^{\pm,\mathrm{DGLAP}}$ has the form of a "standard" Mellin convolution that, up to minor modifications due to the fact that κ depends on the physical x, is easily handled by APFEL. Assuming a grid in x indexed by α or β, we have:

$$x_\beta I^{\pm,\mathrm{DGLAP}}(\xi,x_\beta)=\sum_\alpha\mathcal{P}^{\pm,\mathrm{DGLAP}}_{\beta\alpha}(\xi)f^{\pm}_\alpha(\xi)\,, \tag{1.9}$$

with:

$$f^{\pm}_\alpha(\xi)=x_\alpha f^{\pm}(x_\alpha,\xi)\,, \tag{1.10}$$

and:

$$\mathcal{P}^{\pm,\mathrm{DGLAP}}_{\beta\alpha}(\xi)=\int_c^d dx'\,\mathcal{P}^{\pm}\left(\kappa_\beta,x'\right)w^{(k)}_\alpha\left(\frac{x_\beta}{x'}\right)\,, \tag{1.11}$$

where $\kappa_\beta=\kappa(x_\beta)=\xi/x_\beta$ and $\{w^{(k)}_\alpha\}$ is a set of Lagrange interpolating functions of degree k and the integration bounds are:

$$c=\max(x_\beta,x_\beta/x_{\alpha+1})\quad\text{and}\quad c=\min(1,x_\beta/x_{\alpha-k})\,. \tag{1.12}$$

Now we need to treat $I^{\pm,\mathrm{ERBL}}$ in Eq. (1.4). The structure of this term is rather unusual for APFEL because the pre-computation of convolution integrals is usually done on logarithmically-spaced grids in x and integrating down to zero might be problematic. However, contrary to forward distributions, GPDs are generally well-behaved at $x=0$ and thus it is not strictly necessary to reach this point in the integral.[2] Upon this assumption, we find:

$$x_\beta I^{\pm,\mathrm{ERBL}}(\xi,x_\beta)=\sum_\alpha\mathcal{P}^{\pm,\mathrm{ERBL}}_{\beta\alpha}(\xi)f^{\pm}_\alpha(\xi)\,, \tag{1.13}$$

with:

$$\mathcal{P}^{\pm,\mathrm{ERBL}}_{\beta\alpha}(\xi)=\theta\left(1-\frac{1}{|\kappa_\beta|}\right)x_\beta\int_\epsilon^{x_\beta}dx'\,\mathcal{P}^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\frac{1}{x'^2}w^{(k)}_\alpha(x')\,, \tag{1.14}$$

[2] We will need to verify this conjecture numerically.

where ϵ is a small number. Since the interpolating functions $w_\alpha^{(k)}$ are such that:

$$w_\alpha^{(k)}(x) \neq 0 \quad \text{for} \quad x_{\alpha-k} < x < x_{\alpha+1}\,, \tag{1.15}$$

the integral above can be computed more efficiently as:

$$\mathcal{P}^{\pm,\mathrm{ERBL}}_{\beta\alpha}(\xi) = \theta\left(1-\frac{1}{|\kappa_\beta|}\right) x_\beta \int_a^b dx'\,\mathcal{P}^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)\frac{1}{x'^2} w_\alpha^{(k)}(x')\,, \tag{1.16}$$

with:

$$a = \max(\epsilon, x_{\alpha-k}) \quad \text{and} \quad b = \min(x_\beta, x_{\alpha+1})\,. \tag{1.17}$$

Finally, summing $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ and multiplying by a factor x_β, the evolution equation in Eq. (1.1) can be approximated on an grid in x as:

$$\mu^2\frac{d}{d\mu^2} f_\beta^{\pm}(\xi) = \sum_\alpha \left[\mathcal{P}^{\pm,\mathrm{DGLAP}}_{\beta\alpha}(\xi) + \mathcal{P}^{\pm,\mathrm{ERBL}}_{\beta\alpha}(\xi)\right] f_\alpha^{\pm}(\xi) \tag{1.18}$$

This is a system of coupled differential equation that can be solved numerically using the fourth-order Runge-Kutta algorithm as implemented in APFEL.

2 Treatment of the plus prescription

In the case of the evolution of GPDs we deal with complicated evolution kernels whose algebraic structure is hard to disentangle. Therefore, we need to devise a general strategy to treat the plus prescriptions that arise for the cancellation of soft gluons in order to make them easily implementable. The case to treat is function of this kind:

$$P(y) = \left[\frac{F(y)}{1-y}\right]_+\,, \tag{2.1}$$

where $F(y)$ is a regular function at $y=1$. The first step is to write this function as follows:

$$P(y) = R(y) + S\left(\frac{1}{1-y}\right)_+ + L\delta(1-y)\,, \tag{2.2}$$

where $R(y)$ is a regular function at $y=1$, and S and L are constants that need to be determined. To this end we compute the integral:

$$I = \int_0^1 dy\,P(y)f(y)\,, \tag{2.3}$$

where $f(y)$ is a regular function at $y=1$. Using the definition of plus distribution, we find:

$$\begin{aligned} I &= \int_0^1 dy\,\frac{F(y)}{1-y}\left[f(y)-f(1)\right] = \int_0^1 dy\,\frac{1}{1-y}\left[F(y)f(y)-F(y)f(1)-F(1)f(1)+F(1)f(1)\right] \\ &= \int_0^1 dy\,\frac{F(y)}{(1-y)_+}f(y) - f(1)\int_0^1 dz\,\frac{F(z)}{(1-z)_+} = \int_0^1 dy\left[\frac{F(y)}{(1-y)_+} + L\delta(1-y)\right]f(y)\,, \end{aligned} \tag{2.4}$$

where in the last step we have defined:

$$L = -\int_0^1 dz\,\frac{F(z)}{(1-z)_+} = -\int_0^1 dz\,\frac{F(z)-F(1)}{1-z}\,. \tag{2.5}$$

In order to find R and S, we just rewrite the last step of Eq. (2.4) as:

$$I = \int_0^1 dy\left[\frac{F(y)-F(1)}{1-y} + F(1)\left(\frac{1}{1-y}\right)_+ + L\delta(1-y)\right]f(y)\,, \tag{2.6}$$

that allows us to read off:

$$S = F(1)\,,\qquad R(y) = \frac{F(y)-S}{1-y}\,,\qquad L = -\int_0^1 dy\,R(y)\,. \tag{2.7}$$

For an incomplete integral between x and one, typical of a Mellin convolution, the result is:

$$I(x) = \int_x^1 dy\left[\frac{F(y)-S}{1-y} + S\left(\frac{1}{1-y}\right)_+ + \left(L + S\ln(1-x)\right)\delta(1-y)\right]f(y)\,. \tag{2.8}$$

3 Flavour structure

In this section we report the leading-order (LO) evolution kernels $\mathcal{P}^{\pm}$ taking into the flavour structure. The explicit expressions of the relevant LO kernels are extracted from Ref. [2]. As usual, the perturbative expansion of the evolution kernels reads:

$$\mathcal{P}^{\pm}(\kappa, y; \mu) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} \mathcal{P}^{\pm,[n]}(\kappa, y)\,. \tag{3.1}$$

At leading-order (LO) it turns out that the evolution kernels P appearing in Eq. (1.1) vanish for negative values of x'. As a consequence of the definition in Eq. (1.3), at LO one has $P^{+} = P^{-}$. The explicit form of the LO evolution kernels $\mathcal{P}$, defined in Eq. (1.7), is:

$$\begin{aligned}
\mathcal{P}^{\pm,[0]}_{qq,\mathrm{NS}}(\kappa, y) &= 2C_F \left[\frac{-1-y}{1-\kappa^2 y^2} + 2\left(\frac{1}{1-y}\right)_+ - \left(\frac{(1+\kappa)\ln(1-\kappa) + (1-\kappa)\ln(1+\kappa)}{2\kappa^2}\right)\delta(1-x)\right]\,,\\
\mathcal{P}^{\pm,[0]}_{qg}(\kappa, y) &= 4n_f T_R \frac{1}{1-\kappa^2 y^2}\left[1 - \frac{2y(1-y)}{1-\kappa^2 y^2}\right]\,,\\
\mathcal{P}^{\pm,[0]}_{gq}(\kappa, y) &= 2C_F \left[\frac{(1-y)^2}{y(1-\kappa^2 y^2)} + \frac{1}{y}\right]\,,\\
\mathcal{P}^{\pm,[0]}_{gg}(\kappa, y) &= 4C_A \left[\frac{-2 + (1+\kappa^2)y - (1-\kappa^2)y^2}{(1-\kappa^2 y^2)^2} + \frac{1}{y} + \left(\frac{1}{1-y}\right)_+\right] + \left(\frac{11}{3}C_A - \frac{4}{3}n_f T_R\right)\delta(1-x)\,.
\end{aligned} \tag{3.2}$$

References

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[2] J. Blumlein, B. Geyer and D. Robaschik, Nucl. Phys. B **560** (1999) 283 doi:10.1016/S0550-3213(99)00418-6 [hep-ph/9903520].