1 Structure of the observables

Let us start from Eq. (2.6) of Ref. [1], that is the fully differential cross section for lepton-pair production in the region in which the TMD factorisation applies, *i.e.* $q_T \ll Q$. After some minor manipulations, it reads:

$$\frac{d\sigma}{dQdydq_T} = \frac{16\pi\alpha^2 q_T}{9Q^3} H(Q,\mu) \sum_q C_q(Q) \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \overline{F}_q(x_1,\mathbf{b};\mu,\zeta)\overline{F}_{\bar{q}}(x_2,\mathbf{b};\mu,\zeta)\,, \tag{1}$$

where Q, y, and q_T are the invariant mass, the rapidity, and the transverse momentum of the lepton pair, respectively, while α is the electromagnetic coupling, H is the appropriate QCD hard factor that can be perturbatively computed, and C_q are the effective electroweak charges. In addition, the variables x_1 and x_2 are functions of Q and y and are given by:

$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y}\,, \tag{2}$$

being $\sqrt{s}$ the centre-of-mass energy of the collision. In Eq. (1) we are using the short-hand notation:

$$\overline{F}_q(x,\mathbf{b};\mu,\zeta) \equiv xF_q(x,\mathbf{b};\mu,\zeta)\,, \tag{3}$$

that is convenient for the implementation. The scales μ and ζ are introduced as a consequence of the removal of UV and rapidity divergences in the definition of the TMDs. Despite these scales are arbitrary scales, they are typically chosen $\mu = \sqrt{\zeta} = Q$. Therefore, for all practical purposes their presence is fictitious.

The computation-intensive part of Eq.(1) has the form of the integral:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \overline{F}_i(x_1,\mathbf{b};\mu,\zeta)\overline{F}_j(x_2,\mathbf{b};\mu,\zeta)\,. \tag{4}$$

where $\overline{F}_{i(j)}$ are combinations of evolved TMD PDFs. At this stage, for convenience, i and j do not coincide with q and $\bar{q}$ but they are linked through a simple linear transformation. The integral over the bidimensional impact parameter $\mathbf{b}$ has to be taken. However, $\overline{F}_{i(j)}$ only depend on the absolute value of $\mathbf{b}$, therefore Eq. (4) can be written as:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)\overline{F}_i(x_1,b;\mu,\zeta)\overline{F}_j(x_2,b;\mu,\zeta)\,. \tag{5}$$

where J_0 is the zero-th order Bessel function of the first kind whose integral representation is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{ix\cos(\theta)}\,. \tag{6}$$

The evolved quark TMD PDF $\overline{F}_i$ at the final scales μ and ζ is obtained by multiplying the same distribution at the initial scales μ_0 and ζ_0 by a single evolution factor R_q([1]). that is:

$$\overline{F}_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b)\overline{F}_i(x,b;\mu_0,\zeta_0)\,. \tag{7}$$

The initial scale TMD PDFs at small values b can be written as:

$$\overline{F}_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} x\int_x^1 \frac{dy}{y} C_{ij}(y;\mu_0,\zeta_0) f_j\left(\frac{x}{y},\mu_0\right)\,, \tag{8}$$

where f_j are the collinear PDFs (including the gluon) and C_{ij} are the so-called matching functions that are perturbatively computable and are currently known to NNLO, *i.e.* $\mathcal{O}(\alpha_s^2)$. If we define:

$$\overline{f}_i(x,\mu_0) = xf_i(x,\mu_0)\,, \tag{9}$$

[1]Note that in Eq. (1) the gluon TMD PDF $\overline{F}_g$ is not involved. If also the gluon TMD PDF was involved, it would evolve by means of a different evolution factor R_g.

Eq. (8) can be written as:

$$\overline{F}_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\overline{f}_i\left(\frac{x}{y},\mu_0\right) . \tag{10}$$

Putting Eqs. (7) and (10), one finds:

$$\overline{F}_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\overline{f}_i\left(\frac{x}{y},\mu_0\right) . \tag{11}$$

Matching and evolution are affected by non-perturbative effects that become relevant at large b. In order to account for such effects, one usually introduces a phenomenological function $f_{\rm NP}$. In the traditional approach (CSS [2]), the b-space TMDs get a multiplicative correction that does not depend on the flavour. In addition, the perturbative content of the TMDs is smoothly damped away at large b by introducing the so-called b_*-prescription:

$$\overline{F}_i(x,b;\mu,\zeta) \to \overline{F}_i(x,b_*(b);\mu,\zeta) f_{\rm NP}(x,b,\zeta) , \tag{12}$$

where $b_* \equiv b_*(b)$ is a monotonic function of the impact parameter b such that:

$$\lim_{b\to 0} b_*(b) = b_{\rm min} \quad \text{and} \quad \lim_{b\to\infty} b_*(b) = b_{\rm max} , \tag{13}$$

being $b_{\rm min}$ and $b_{\rm max}$ constant values both in the perturbative region. Including the non-perturbative function, Eq. (5) becomes:

$$\begin{aligned} I_{ij}(x_1,x_2,q_T;\mu,\zeta) &= \int_0^\infty db\, J_0(bq_T) \left[\frac{b}{2}\overline{F}_i(x_1,b_*(b);\mu,\zeta)\overline{F}_j(x_2,b_*(b);\mu,\zeta) f_{\rm NP}(x_1,b,\zeta) f_{\rm NP}(x_2,b,\zeta)\right] \\ &= \frac{1}{q_T}\int_0^\infty d\bar{b}\, J_0(\bar{b}) \left[\frac{\bar{b}}{2q_T}\overline{F}_i\left(x_1,b_*\left(\frac{\bar{b}}{q_T}\right);\mu,\zeta\right)\overline{F}_j\left(x_2,,b_*\left(\frac{\bar{b}}{q_T}\right);\mu,\zeta\right) f_{\rm NP}\left(x_1,\frac{\bar{b}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{\bar{b}}{q_T},\zeta\right)\right] . \end{aligned} \tag{14}$$

Eq. (14) is a Hankel tranform and can be efficiently computed using the so-called Ogata quadrature [3]. Effectively, the computation of the integral in Eq. (4) is achieved through a weighted sum:

$$\begin{aligned} I_{ij}(x_1,x_2,q_T;\mu,\zeta) &\simeq \frac{1}{q_T}\sum_{n=1}^{N} \frac{w_n^{(0)} z_n^{(0)}}{2q_T} \overline{F}_i\left(x_1,b_*\left(\frac{z_n^{(0)}}{q_T}\right);\mu,\zeta\right)\overline{F}_j\left(x_2,b_*\left(\frac{z_n^{(0)}}{q_T}\right);\mu,\zeta\right) \\ &\times f_{\rm NP}\left(x_1,\frac{z_n^{(0)}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{z_n^{(0)}}{q_T},\zeta\right) , \end{aligned} \tag{15}$$

where the unscaled coordinates $z_n^{(0)}$ and the weights $w_n^{(0)}$ can be precomputed in terms of the zero's of the Bessel function J_0 and one single parameter (see Ref. [3] for more details, specifically Eqs. (5.1) and (5.2) or Appendix A for the relevant formula to compute the unscaled coordinates and the weights)[2]. Based on the (empirically verified) assumption that the absolute value of each term in the sum in the r.h.s. of Eq. (15) is smaller than that of the preceding one, the truncation number N is chosen dynamically in such a way that the $(N+1)$-th term is smaller in absolute value than a user-defined cutoff relatively to the sum of the preceding N terms.

Eq. (15) factors out the non-perturbative part of the calculation represented by $f_{\rm NP}$ from the perturbative content. This is done on purpose to devise a method in which the perturbative content

[2]The superscript 0 in $z_n^{(0)}$ and $w_n^{(0)}$ indicates that here we are performing a Hankel tranform that involves the Bessel function of degree zero J_0. This is useful in view of the next section in which the integration over q_T will give rise to a similar Hankel transform with J_0 replaced by J_1. Also in that case the Ogata quadrature algorithm can be applied but coordinates and weights will be different.

is precomputed and numerically convoluted with the non-perturbative functions *a posteriori*. This is convenient in view of a fit of the function $f_{\rm NP}$.

As customary in QCD, the most convenient basis for the matching in Eq. (8) is the so-called "evolution" basis (*i.e.* Σ, V, T_3, V_3, etc.). In fact, in this basis the operator matrix C_{ij} is almost diagonal with the only exception of crossing terms that couple the gluon and the singlet Σ distributions. As a consequence, this is the most convenient basis for the computation of I_{ij}. On the other hand, TMDs in Eq. (1) appear in the so-called "physical" basis (*i.e.* d, $\bar{d}$, u, $\bar{u}$, etc.). Therefore, we need to rotate $F_{i(j)}$ from the evolution basis, over which the indices i and j run, to the physical basis. This is done by means of an appropriate constant matrix T, so that:

$$\overline{F}_q(x_1, b; \mu, \zeta) = \sum_i T_{qi} F_i(x_1, b; \mu, \zeta)\,, \tag{16}$$

and similarly for $\overline{F}_{\bar{q}}$. Putting all pieces together, one can conveniently write the cross section in Eq. (1) as:

$$\frac{d\sigma}{dQdydq_T} \simeq \sum_{n=1}^{N} w_n^{(0)} \frac{z_n^{(0)}}{q_T} S\left(x_1, x_2, \frac{z_n^{(0)}}{q_T}; \mu, \zeta\right) f_{\rm NP}\left(x_1, \frac{z_n^{(0)}}{q_T}, \zeta\right) f_{\rm NP}\left(x_2, \frac{z_n^{(0)}}{q_T}, \zeta\right)\,, \tag{17}$$

with:

$$S(x_1, x_2, b; \mu, \zeta) = \frac{8\pi\alpha^2}{9Q^3} H(Q, \mu) \sum_q C_q(Q) \left[\overline{F}_q\left(x_1, b_*(b); \mu, \zeta\right)\right] \left[\overline{F}_{\bar{q}}\left(x_2, b_*(b); \mu, \zeta\right)\right]\,. \tag{18}$$

Eq. (17) allows one to precompute the weights S in such a way that the differential cross section in Eq. (1) can be computed as a simple weighted sum of the non-perturbative contribution. A misleading aspect of Eq. (18) is the fact that S has five arguments. In actual facts, S only depends on three independent variables. The reason is that μ and ζ are usually taken to be proportional to Q by a constant factor. In addition x_1 and x_2 depend on Q and y through Eq. (2). Therefore, the full dependence on the kinematics of the final state of Eq. (1) can be specified by Q, y and q_T.

2 Integrating over the final-state kinematic variables

Despite Eq. (17) provides a powerful tool for a fast computation of cross sections, it is often not sufficient to allow for a direct comparison to experimental data. The reason is that experimental measurements of differential distributions are usually delivered as integrated over finite regions of the final-state kinematic phase space. In other words, experiments measure quantities like:

$$\widetilde{\sigma} = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{y_{\rm min}}^{y_{\rm max}} dy \int_{q_{T,{\rm min}}}^{q_{T,{\rm max}}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right]\,. \tag{19}$$

As a consequence, in order to guarantee performance, we need to include the integrations above in the precomputed factors.

2.1 Integrating over q_T

The integration over bins in q_T can be carried out analytically exploiting the following property of Bessel's function:

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x)\,, \tag{20}$$

that leads to:

$$\int dx\, x J_0(x) = x J_1(x) \quad \Rightarrow \quad \int_{x_1}^{x_2} dx\, x J_0(x) = x_2 J_1(x_2) - x_1 J_1(x_1)\,. \tag{21}$$

To see it, we observe that the differential cross section in Eq. (1) has the following structure:

$$\frac{d\sigma}{dQdydq_T} \propto \int_0^\infty db\, q_T J_0(bq_T)\dots \tag{22}$$

where the ellipses indicate terms that do not depend on q_T. Therefore, using Eq. (21) we find:

$$\begin{aligned}\int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right] \propto \int_0^\infty db \int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\dots = \\ \int_0^\infty \frac{db}{b^2}\int_{bq_{T,\min}}^{bq_{T,\max}} dx\, xJ_0(x)\dots = \int_0^\infty \frac{db}{b}\left[q_{T,\max}J_1(bq_{T,\max}) - q_{T,\min}J_1(bq_{T,\min})\right]\dots\,.\end{aligned} \tag{23}$$

Therefore, defining:

$$K(q_T) \equiv \int dq_T \left[\frac{d\sigma}{dQdydq_T}\right] \tag{24}$$

as the indefinite integral over q_T of the cross section in Eq. (1), we have that:

$$\int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right] = K(Q,y,q_{T,\max}) - K(Q,y,q_{T,\min})\,, \tag{25}$$

with:

$$\begin{aligned}K(Q,y,q_T) = & \frac{8\pi\alpha^2 q_T}{9Q^3}H(Q,\mu) \\ \times & \int_0^\infty db\, J_1(bq_T)\sum_q C_q(Q)\overline{F}_q(x_1,b;\mu,\zeta)\overline{F}_{\bar{q}}(x_2,b;\mu,\zeta)f_{\rm NP}(x_1,b,\zeta)f_{\rm NP}(x_2,b,\zeta)\,,\end{aligned} \tag{26}$$

that can be computed using the Ogata quadrature as:

$$K(Q,y,q_T) \simeq \sum_{n=1}^N w_n^{(1)} S\left(x_1,x_2,\frac{z_n^{(1)}}{q_T};\mu,\zeta\right) f_{\rm NP}\left(x_1,\frac{z_n^{(1)}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{z_n^{(1)}}{q_T},\zeta\right)\,, \tag{27}$$

with S defined in Eq. (18). The unscaled coordinates $z_n^{(1)}$ and the weights $w_n^{(1)}$ can again be precomputed and stored in terms of the zero's of the Bessel function J_1. Eq. (25) reduces the integration in q_T to a calculation completely analogous to the unintegrated cross section. This is particularly convenient because it avoids the computation a numerical integration.

2.1.1 Leptonic cuts

In the presence of cuts on the final state leptons, the analytic integration over q_T discussed above cannot be performed. The reason is that the implementation of these cuts effectively introduces a q_T-dependent function $\mathcal{P}$([3]) in the integral:

$$\frac{d\sigma}{dQdydq_T} \propto \int_0^\infty db\, q_T J_0(bq_T)\mathcal{P}(q_T)\dots\,, \tag{28}$$

that prevents the direct use of Eq. (21). However, integrating by parts, we can write([4]):

$$\int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\mathcal{P}(q_T) = \frac{1}{b}\left[q_T J_1(bq_T)\mathcal{P}(q_T) - \int_0^{q_T} d\bar{q}_T\, \bar{q}_T J_1(b\bar{q}_T)\mathcal{P}'(\bar{q}_T)\right]\Bigg|_{q_{T,\min}}^{q_{T,\max}}\,. \tag{29}$$

[3] In fact, $\mathcal{P}$ also depends on the invariant mass Q and the rapidity y of the lepton pair that also need to be integrated over.

[4] Notice that the lower bound of the integral in the r.h.s., that is 0 here, is actually arbitrary because the final result does not depend on it.

Now, if we assume that $\mathcal{P}$ is a slowly-varying function of q_T (*i.e.* $\mathcal{P}'$ is small), we could, in first approximation, neglect the second term in the r.h.s. of the equation above. Unfortunately, despite $\mathcal{P}$ is an actual slowly-varying function of q_T, the contribution of the integral in the r.h.s. is still large, particularly at large q_T. This is mostly due to the fact that the integral over b of $J_1(bq_T)$, particularly for large values of q_T, is numerically large.

Since $\mathcal{P}$ is a slowly-varying function of q_T of the typical bins, we can approximate the integral over the bins in q_T as:

$$\begin{aligned}\int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\mathcal{P}(q_T) &\simeq \mathcal{P}\left(\frac{q_{T,\max}+q_{T,\min}}{2}\right)\int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\\ &= \mathcal{P}\left(\frac{q_{T,\max}+q_{T,\min}}{2}\right)\frac{1}{b}\left[q_{T,\max}J_1(bq_{T,\max})-q_{T,\min}J_1(bq_{T,\min})\right].\end{aligned} \tag{30}$$

Unfortunately, this structure is inconvenient because it mixes different bin bounds and prevents a recursive computation. However, we can try to go further and, assuming that the bin width is small enough, we can expand $\mathcal{P}$ is the following ways:

$$\begin{aligned}\mathcal{P}\left(\frac{q_{T,\max}+q_{T,\min}}{2}\right) &= \mathcal{P}\left(q_{T,\min}+\Delta q_T\right) = \mathcal{P}\left(q_{T,\min}\right)+\mathcal{P}'\left(q_{T,\min}\right)\Delta q_T + \mathcal{O}\left(\Delta q_T^2\right),\\ \mathcal{P}\left(\frac{q_{T,\max}+q_{T,\min}}{2}\right) &= \mathcal{P}\left(q_{T,\max}-\Delta q_T\right) = \mathcal{P}\left(q_{T,\max}\right)-\mathcal{P}'\left(q_{T,\max}\right)\Delta q_T + \mathcal{O}\left(\Delta q_T^2\right),\end{aligned} \tag{31}$$

with:

$$\Delta q_T = \frac{q_{T,\max}-q_{T,\min}}{2}\,. \tag{32}$$

Therefore:

$$\begin{aligned}b\int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\mathcal{P}(q_T) &\simeq q_{T,\max}J_1(bq_{T,\max})\left[\mathcal{P}\left(q_{T,\max}\right)-\mathcal{P}'\left(q_{T,\max}\right)\Delta q_T\right]\\ &- q_{T,\min}J_1(bq_{T,\min})\left[\mathcal{P}\left(q_{T,\min}\right)+\mathcal{P}'\left(q_{T,\min}\right)\Delta q_T\right].\end{aligned} \tag{33}$$

The advantage of this approximation as compared to Eq. (30) is the fact that each single term depends on one single q_T-bin-bound rather than on a combination of two consecutive bounds.

2.2 On the position of the peak of the q_T distribution

It is interesting at this point to take a short detour to discuss the position of the peak on the distribution in q_T of the cross section in Eq. (1). The peak can be located by setting the derivative in q_T of the cross section equal to zero. To do so, we use another property of Bessel's functions:

$$\frac{dJ_0(x)}{dx} = -J_1(x)\,. \tag{34}$$

Using this relation, it is easy to see that:

$$\begin{aligned}&0 = \frac{d}{dq_T}\left[\frac{d\sigma}{dQdydq_T}\right] =\\ &\frac{8\pi\alpha^2}{9Q^3}H(Q,\mu)\int_0^\infty db\, b\left[J_0(bq_T)-bq_TJ_1(bq_T)\right]\sum_q C_q(Q)\overline{F}_q(x_1,b_*(b);\mu,\zeta)\overline{F}_{\bar{q}}(x_2,b_*(b);\mu,\zeta)\\ &\times f_{\rm NP}(x_1,b,\zeta)f_{\rm NP}(x_2,b,\zeta)\,,\end{aligned} \tag{35}$$

that is equivalent to require that:

$$\int_0^\infty db\, b\left[J_0(bq_T) - bq_T J_1(bq_T)\right]\sum_q C_q(Q)\overline{F}_q(x_1, b_*(b);\mu,\zeta)\overline{F}_{\bar{q}}(x_2, b_*(b);\mu,\zeta) f_{\rm NP}(x_1, b,\zeta) f_{\rm NP}(x_2, b,\zeta) = 0\,. \tag{36}$$

The integral above can be solved numerically using the technique discussed above and the value of q_T that satisfies this equation represents the position of the peak of the q_T distribution.

2.3 Integrating over Q and y

As a final step, we need to perform the integrals over Q and y defined in Eq. (19). To compute these integrals we can only rely on numerical methods. Having reduced the integration in q_T to the difference of the two terms in the r.h.s. of Eq. (25), we can concentrate on integrating the function K over Q and y for a fixed value of q_T:

$$\widetilde{K}(q_T) = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{y_{\rm min}}^{y_{\rm max}} dy\, K(Q, y, q_T)\,, \tag{37}$$

such that:

$$\widetilde{\sigma} = \widetilde{K}(q_{T,\rm max}) - \widetilde{K}(q_{T,\rm min})\,. \tag{38}$$

To this purpose, it is convenient to make explicit the dependence of x_1 and x_2 on Q and y using Eq. (2). In addition, for the sake of simplicity we will identify the scales μ and $\sqrt{\zeta}$ with Q (possible scale variations can be easily reinstated at a later stage) and thus drop one of the arguments from the TMD distributions $\overline{F}$ and from the hard factor H. This yields:

$$\begin{aligned}
\widetilde{K}(q_T) &= \frac{8\pi q_T}{9}\int_0^\infty db\, J_1(bq_T)\int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{e^{y_{\rm min}}}^{e^{y_{\rm max}}}\frac{d\xi}{\xi} \\
&\times \frac{1}{Q^3}\alpha^2(Q)H(Q)\sum_q C_q(Q)\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*(b);Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*(b);Q\right) \\
&\times f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\xi, b;Q\right) f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b;Q\right)\,,
\end{aligned} \tag{39}$$

where we have performed the change of variable $e^y = \xi$. Now we define one grid in ξ, $\{\xi_\alpha\}$ with $\alpha = 0,\dots,N_\xi$, and one grid in Q, $\{Q_\tau\}$ with $\tau = 0,\dots,N_Q$, each of which with a set of interpolating functions $\mathcal{I}$ associated. In addition, the grids are such that: $\xi_0 = e^{y_{\rm min}}$ and $\xi_{N_\xi} = e^{y_{\rm max}}$, and $Q_0 = Q_{\rm min}$ and $Q_{N_Q} = Q_{\rm max}$. More details on the interpolation procedure are presented in Appendix B. This allows us to interpolate the pair of functions $f_{\rm NP}$ in Eq. (39) for generic values of ξ and Q as:

$$f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\xi, b;Q\right) f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b;Q\right) \simeq \sum_{\alpha=0}^{N_\xi}\sum_{\tau=0}^{N_Q}\mathcal{I}_\alpha(\xi)\mathcal{I}_\tau(Q) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, b;Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, b;Q_\tau\right)\,. \tag{40}$$

Plugging the equation above into Eq. (39) we obtain:

$$\begin{aligned}
\widetilde{K}(q_T) &\simeq \frac{8\pi q_T}{9}\int_0^\infty db\, J_1(bq_T)\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi}\left[\int_{Q_{\rm min}}^{Q_{\rm max}} dQ\, \mathcal{I}_\tau(Q)\frac{1}{Q^3}\alpha^2(Q)H(Q)\right. \\
&\times \left.\int_{e^{y_{\rm min}}}^{e^{y_{\rm max}}} d\xi\, \mathcal{I}_\alpha(\xi)\frac{1}{\xi}\sum_q C_q(Q)\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*(b);Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*(b);Q\right)\right] \\
&\times f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, b;Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, b;Q_\tau\right)\,.
\end{aligned} \tag{41}$$

Finally, the integration over b can be performed using the Ogata quadrature as discussed above, so that:

$$
\begin{aligned}
\widetilde{K}(q_T) \simeq & \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi}\left[\frac{8\pi}{9}w_n^{(1)}\int_{Q_{\min}}^{Q_{\max}} dQ\,\mathcal{I}_\tau(Q)\,\frac{1}{Q^3}\alpha^2(Q)H(Q)\right.\\
\times & \left.\int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\,\mathcal{I}_\alpha(\xi)\,\frac{1}{\xi}\sum_q C_q(Q)\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*\left(\frac{z_n}{q_T}\right);Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*\left(\frac{z_n}{q_T}\right);Q\right)\right] \\
\times & f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, \frac{z_n}{q_T};Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, \frac{z_n}{q_T};Q_\tau\right).
\end{aligned}
\tag{42}
$$

In conclusion, if we define:

$$
\begin{aligned}
W_{n\tau\alpha}(q_T) \equiv & \; w_n^{(1)}\frac{8\pi}{9}\int_{Q_{\min}}^{Q_{\max}} dQ\,\mathcal{I}_\tau(Q)\,\frac{\alpha^2(Q)}{Q^3}H(Q)\\
\times & \int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\,\mathcal{I}_\alpha(\xi)\,\frac{1}{\xi}\sum_q C_q(Q)\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*\left(\frac{z_n}{q_T}\right);Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*\left(\frac{z_n}{q_T}\right);Q\right),
\end{aligned}
\tag{43}
$$

the quantity $\widetilde{K}(q_T)$ can be computed as:

$$
\widetilde{K}(q_T) \simeq \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi} W_{n\tau\alpha}(q_T) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, \frac{z_n}{q_T};Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, \frac{z_n}{q_T};Q_\tau\right). \tag{44}
$$

The advantage of Eq. (44) is that the weights $W_{n\alpha\tau}$, that clearly depend on q_T but also on the intervals $[Q_{\min} : Q_{\max}]$ and $[y_{\min} : y_{\max}]$, can be precomputed once and for all for each of the experimental points included in a fit and used to determine the function $f_{\rm NP}$. This provides a fast tool for the computation of predictions that makes the extraction of the non-perturbative part of the TMDs much easier.

2.4 Cross section differential in x_F

In some cases, the Drell-Yan differential cross section may be presented as differential in the invariant mass of the lepton pair Q and, instead of the rapidity y, of the Feynman variable x_F defined as:

$$
x_F = \frac{Q}{\sqrt{s}}\left(e^y - e^{-y}\right) = \frac{2Q}{\sqrt{s}}\sinh y = x_1 - x_2\,, \tag{45}
$$

so that:

$$
\frac{dx_F}{dy} = \frac{2Q}{\sqrt{s}}\cosh y = x_1 + x_2\,. \tag{46}
$$

Therefore:

$$
\frac{d\sigma}{dQdx_Fdq_T} = \frac{dy}{dx_F}\frac{d\sigma}{dQdydq_T} = \frac{\sqrt{s}}{2Q\cosh y}\frac{d\sigma}{dQdydq_T} = \frac{1}{x_1+x_2}\frac{d\sigma}{dQdydq_T} \tag{47}
$$

with:

$$
y(x_F, Q) = \sinh^{-1}\left(\frac{x_F\sqrt{s}}{2Q}\right) = \ln\left[\frac{\sqrt{s}}{2Q}\left(x_F + \sqrt{x_F^2 + \frac{4Q^2}{s}}\right)\right], \tag{48}
$$

so that:

$$
x_1 = \frac{1}{2}\left(x_F + \sqrt{x_F^2 + \frac{4Q^2}{s}}\right) \quad \text{and} \quad x_2 = \frac{Q^2}{sx_1}\,. \tag{49}
$$

Therefore, we can compute the integral:

$$\widetilde{I}(q_T) = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{x_{F,\min}}^{x_{F,\max}} dx_F\, I(Q, x_F, q_T)\,, \tag{50}$$

where I is the primitive in q_T of the cross section differential in x_F:

$$I(Q, x_F, q_T) = \int dq_T \left[\frac{d\sigma}{dQ dx_F dq_T}\right]\,, \tag{51}$$

following the same steps of Sect. 2.3. This leads to:

$$\widetilde{I}(q_T) \quad \simeq \quad \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_x} \overline{W}_{n\tau\alpha}(q_T) f_{\rm NP}\left(x_{1,\alpha\tau}, \frac{z_n}{q_T}; Q_\tau\right) f_{\rm NP}\left(x_{2,\alpha\tau}, \frac{z_n}{q_T}; Q_\tau\right)\,, \tag{52}$$

with:

$$\begin{aligned}\overline{W}_{n\tau\alpha}(q_T) \quad &\equiv \quad w_n^{(1)} \frac{8\pi}{9} \int_{Q_{\min}}^{Q_{\max}} dQ\, \mathcal{I}_\tau(Q)\, \frac{1}{Q^3} \alpha^2(Q) H(Q) \\ &\times \quad \int_{x_{F,\min}}^{x_{F,\max}} dx_F\, \mathcal{I}_\alpha(x_F)\, \frac{1}{x_1 + x_2} \sum_q C_q(Q) \overline{F}_q\left(x_1, b_*\left(\frac{z_n}{q_T}\right); Q\right) \overline{F}_{\bar{q}}\left(x_2, b_*\left(\frac{z_n}{q_T}\right); Q\right)\,,\end{aligned} \tag{53}$$

where x_1 and x_2 are functions of x_F and Q through Eq. (49). In addition, we have defined a grid in x_F, $\{x_{F,\alpha}\}$ with $\alpha = 0, \dots, N_x$, that allowed us to define $x_{1(2),\alpha\tau} \equiv x_{1(2)}(x_{F,\alpha}, Q_\tau)$.

2.5 Flavour dependence

It may be advantageous to introduce a flavour dependence of the non-perturbative contributions to TMDs. This can be easily done by observing that the tensor $W_{n\tau\alpha}$ defined in Eq. (43) can be decomposed as[5]:

$$W_{n\tau\alpha}(q_T) = \sum_q W_{n\tau\alpha}^{(q)}(q_T)\,, \tag{54}$$

with:

$$\begin{aligned}W_{n\tau\alpha}^{(q)}(q_T) \quad &\equiv \quad w_n^{(1)} \frac{8\pi}{9} \int_{Q_{\min}}^{Q_{\max}} dQ\, \mathcal{I}_\tau(Q)\, \frac{\alpha^2(Q)}{Q^3} H(Q) C_q(Q) \\ &\times \quad \int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\, \mathcal{I}_\alpha(\xi)\, \frac{1}{\xi} \overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*\left(\frac{z_n}{q_T}\right); Q\right) \overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*\left(\frac{z_n}{q_T}\right); Q\right)\,.\end{aligned} \tag{55}$$

This allows for an independent parameterisation of the non-perturbative contribution such that Eq. (44) can be written as:

$$\widetilde{K}(q_T) \simeq \sum_q \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi} W_{n\tau\alpha}^{(q)}(q_T) f_{\rm NP}^{(q)}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, \frac{z_n}{q_T}; Q_\tau\right) f_{\rm NP}^{(q)}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, \frac{z_n}{q_T}; Q_\tau\right)\,, \tag{56}$$

where $f_{\rm NP}^{(q)}$ parametrises the non-perturbative component of the TMD with flavour q.

[5] The same procedure applies to the tensor $\overline{W}_{n\tau\alpha}$ defined in Eq. (53).

2.6 Gradient with respect to the free parameters

A very appealing implication of the computation of cross section in terms of precomputed table as in Eqs. (44) and (56) is the fact that it exposes the free parameters of the non-perturbative functions. To be more specific, the non-perturbative function $f_{\rm NP}$, on top of being a function of x, b, and ζ, depends parameterically on a set of N_p parameters $\{\theta_k\}$, $k = 1, \dots, N_p$, that are typically determined by fits to data, in other words:

$$f_{\rm NP} \equiv f_{\rm NP}\left(x, b, \zeta; \{\theta_k\}\right) . \tag{57}$$

Now, when performing a fit, it is very useful to be able to compute the derivative of the figure of merit (usually the χ^2) with respect to the parameters to be determined. In turn, this immediately implies being able to compute the derivative of the observables. Referring to Eq. (44), the relevant quantity is:

$$\frac{d\widetilde{K}}{d\theta_k} = \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi} W_{n\tau\alpha}(q_T)\left[\frac{df_{\rm NP}^{(1)}}{d\theta_k} f_{\rm NP}^{(2)} + f_{\rm NP}^{(1)}\frac{df_{\rm NP}^{(2)}}{d\theta_k}\right] , \tag{58}$$

where $f_{\rm NP}^{(1)}$ and $f_{\rm NP}^{(2)}$ refer to the non-perturbative function $f_{\rm NP}$ computed in x_1 and x_2, respectively. It is thus clear that the derivatives w.r.t. the free parameters penetrates the observable. Since in most cases the derivative of $f_{\rm NP}$ can be computed analytically, this allows one to compute the gradient of the figure of merit analytically. This potentially makes any fit much simpler.

2.7 Narrow-width approximation

A possible alternative to the numerical integration in Q when the integration region includes the Z-peak region is the so-called narrow-width approximation (NWA). In the NWA one assumes that the width of the Z boson, Γ_Z, is much smaller than its mass, M_Z. This way one can approximate the peaked behaviour of the couplings $C_q(Q)$ around $Q = M_Z$ with a δ-function, *i.e.* $C_q(Q) \sim \delta(Q^2 - M_Z^2)$. Therefore, the integration over Q can be done analytically. The exact structure of the electroweak couplings is the following:

$$C_q(Q) = e_q^2 - 2e_q V_q V_e \chi_1(Q) + (V_e^2 + A_e^2)(V_q^2 + A_q^2)\chi_2(Q) , \tag{59}$$

with:

$$\begin{aligned}
\chi_1(Q) &= \frac{1}{4\sin^2\theta_W\cos^2\theta_W}\frac{Q^2(Q^2-M_Z^2)}{(Q^2-M_Z^2)^2+M_Z^2\Gamma_Z^2} ,\\
\chi_2(Q) &= \frac{1}{16\sin^4\theta_W\cos^4\theta_W}\frac{Q^4}{(Q^2-M_Z^2)^2+M_Z^2\Gamma_Z^2} .
\end{aligned} \tag{60}$$

In the limit $\Gamma_Z/M_Z \to 0$, the leading contribution to the coupling in Eq. (59) comes from the region $Q \simeq M_Z$ and is that proportional to χ_2:

$$C_q(Q) \simeq (V_e^2 + A_e^2)(V_q^2 + A_q^2)\chi_2(Q) , \quad Q \simeq M_Z . \tag{61}$$

In addition, in this limit one can show that:

$$\frac{1}{(Q^2-M_Z^2)^2+M_Z^2\Gamma_Z^2} \to \frac{\pi}{M_Z\Gamma_Z}\delta(Q^2-M_Z^2) = \frac{\pi}{2M_Z^2\Gamma_Z}\delta(Q-M_Z) . \tag{62}$$

Therefore, considering that:

$$\Gamma_Z = \frac{\alpha M_Z}{\sin^2\theta_W\cos^2\theta_W} , \tag{63}$$

the electroweak couplings in the NWA have the following form:

$$C_q(Q) \simeq \frac{\pi M_Z(V_e^2 + A_e^2)(V_q^2 + A_q^2)}{32\alpha\sin^2\theta_W\cos^2\theta_W}\delta(Q-M_Z) = \widetilde{C}_q(Q)\delta(Q-M_Z) . \tag{64}$$

Therefore, using Eq. (64) the integral of the cross section over Q under the condition that $Q_{\min} < M_Z < Q_{\max}$ has the consequence of adjusting the couplings and of setting $Q = M_Z$ in the computation. This yields:

$$\int_{Q_{\min}}^{Q_{\max}} dQ\,\frac{d\sigma}{dQdydq_T} = \frac{16\pi\alpha^2 q_T}{9M_Z^3} H(M_Z, M_Z) \sum_q \widetilde{C}_q(M_Z) I_{q\bar{q}}(x_1, x_2, q_T; M_Z, M_Z^2)\,, \tag{65}$$

where we are also assuming that $\mu = \sqrt{\zeta} = M_Z$. As a final step, one may want to let the Z boson decay into leptons. At leading order in the EW sector and assuming an equal decay rate for electrons, muons, and tauons, this can be done by multiplying the cross section above by three times the branching ratio for the Z decaying into any pair of leptons, $3\mathrm{Br}(Z \to \ell^+\ell^-)$.

A Ogata quadrature

In this section we limit ourselves to write the formulas for the computation of the unscaled coordinates $z_n^{(\nu)}$ and weights $w_n^{(\nu)}$ required to compute the following integral:

$$I_\nu(q_T) = \int_0^\infty db J_\nu(bq_T) f(b) = \frac{1}{q_T}\int_0^\infty d\bar{b} J_\nu(\bar{b}) f\left(\frac{\bar{b}}{q_T}\right) \simeq \frac{1}{q_T}\sum_{n=1}^{\infty} w_n^{(\nu)} f\left(\frac{z_n^{(\nu)}}{q_T}\right) \qquad \nu = 0, 1, \dots\,, \tag{66}$$

using the Ogata-quadrature algorithm. More details can be found in Ref. [3]. There relevant formulas are:

$$\begin{aligned} z_n^{(\nu)} &= \frac{\pi}{h}\psi\left(\frac{h\xi_{\nu n}}{\pi}\right)\,, \\ w_n^{(\nu)} &= \pi\frac{Y_\nu(\xi_{\nu n})}{J_{\nu+1}(\xi_{\nu n})} J_\nu(z_n^{(\nu)})\psi'\left(\frac{h\xi_{\nu n}}{\pi}\right)\,. \end{aligned} \tag{67}$$

where:

- h is a free parameter of the algorithm that has to be typically small (we choose $h = 10^{-3}$),
- $\xi_{\nu n}$ are the zero's of J_ν, *i.e.* $J_\nu(\xi_{\nu n}) = 0\ \forall\, n$,
- J_ν and Y_ν are the Bessel functions of first and second kind, respectively, of degree ν,
- ψ is the following function:

  $$\psi(t) = t\tanh\left(\frac{\pi}{2}\sinh t\right) \tag{68}$$

  and its derivative:

  $$\psi'(t) = \frac{\pi t\cosh t + \sinh(\pi\sinh t)}{1 + \cosh(\pi\sinh t)}\,. \tag{69}$$

B Lagrange interpolation

Just for the record, it is useful to derive a general expression for the Lagrange interpolating functions $\mathcal{I}$ introduced in Eq. (40) and used to interpolate the non-perturbative functions f_{NP}. More, importantly, we need to understand how these functions behave upon integration.

Suppose one wants to interpolate the test function g in the point x using a set of Lagrange polynomials of degree k of. This requires a subset of $k+1$ consecutive points on an interpolation grid, say $\{x_\alpha, \dots, x_{\alpha+k}\}$. The relative position between the point x and the subset of points used for the interpolation is arbitrary. It is convenient to choose the subset of points such that $x_\alpha < x \leq x_{\alpha+k}$.[6]

[6] In fact, it is not even necessary to impose the constraint $x_\alpha < x \leq x_{\alpha+k}$. In case this relation is not fulfilled one usually refers to *extrapolation* rather than *interpolation*. If not necessary, this option is typically not convenient because it may lead to a substantial deterioration in the accuracy with which $g(x)$ is determined.

However, the ambiguity remains because there are k possible choices according to whether $x_\alpha < x \leq x_{\alpha+1}$, or $x_{\alpha+1} < x \leq x_{\alpha+2}$, and so on.

In order to determine the exact form of the interpolation functions $\mathcal{I}$, let us see how to derive eq. (40). Using the standard Lagrange interpolation procedure, we can approximate the function g in x as:

$$g(x) = \sum_{i=0}^{k} \ell_i^{(k)}(x) g(x_{\alpha+i})\,, \tag{70}$$

where $\ell_i^{(k)}$ is the i-th Lagrange polynomial of degree k which can be written as:

$$\ell_i^{(k)}(x) = \prod_{m=0,m\neq i}^{k} \frac{x - x_{\alpha+m}}{x_{\alpha+i} - x_{\alpha+m}}\,. \tag{71}$$

We now assume that:

$$x_\alpha < x \leq x_{\alpha+1}\,, \tag{72}$$

Eq. (70) becomes:

$$g(x) = \theta(x - x_\alpha)\theta(x_{\alpha+1} - x) \sum_{i=0}^{k} g(x_{\alpha+i}) \prod_{m=0,m\neq i}^{k} \frac{x - x_{\alpha+m}}{x_{\alpha+i} - x_{\alpha+m}}\,. \tag{73}$$

In order to make Eq. (73) valid for all values of α, one just has to sum over all N_x intervals of the *global* interpolation grid $\{x_0, \dots, x_{N_x}\}$, that is:

$$g(x) = \sum_{\alpha=0}^{N_x-1} \theta(x - x_\alpha)\theta(x_{\alpha+1} - x) \sum_{i=0}^{k} g(x_{\alpha+i}) \prod_{m=0,m\neq i}^{k} \frac{x - x_{\alpha+m}}{x_{\alpha+i} - x_{\alpha+m}}\,, \tag{74}$$

Defining $\beta = \alpha + i$, we can rearrange the equation above as:

$$g(x) = \sum_{\beta=0}^{N_x+k-1} \mathcal{I}_\beta^{(k)}(x) g(x_\beta)\,, \tag{75}$$

that leads us to the definition of the interpolating functions:

$$\mathcal{I}_\beta^{(k)}(x) = \sum_{i=0,i\leq\beta}^{k} \theta(x - x_{\beta-i})\theta(x_{\beta-i+1} - x) \prod_{m=0,m\neq i}^{k} \frac{x - x_{\beta-i+m}}{x_\beta - x_{\beta-i+m}}\,, \tag{76}$$

where the condition $i \leq \beta$ comes from the condition $\alpha \geq 0$. It is important to observe that the sum in Eq. (75) extends up to the $(N_x + k - 1)$-th node. Therefore, the original grid needs to be extended by $k - 1$ nodes. However, the range of validity of the interpolation remains that defined by the original grid, *i.e.* $x_0 \leq x \leq x_{N_x}$. Finally, it is crucial to realise that the interpolation function $\mathcal{I}_\beta^{(k)}(x)$ is different from zero only over a limited interval, specifically:

$$\mathcal{I}_\beta^{(k)}(x) \neq 0 \quad \Leftrightarrow \quad x_{\beta-k} < x < x_{\beta+1}\,. \tag{77}$$

In the rest of this document we will stick to the assumption in Eq. (72). However, before going further, it is interesting to generalise Eq. (72) to:

$$x_{\alpha+t} < x \leq x_{\alpha+t+1} \quad \text{with} \quad t = 0, \dots, k-1\,, \tag{78}$$

such that the interpolation formula becomes:

$$g(x) = \sum_{\alpha=-t}^{N_x-t-1} \theta(x - x_{\alpha+t})\theta(x_{\alpha+t+1} - x) \sum_{i=0}^{k} g(x_{\alpha+i}) \prod_{m=0,m\neq i}^{k} \frac{x - x_{\alpha+m}}{x_{\alpha+i} - x_{\alpha+m}}\,, \tag{79}$$

that can be rearranged as:

$$g(x) = \sum_{\beta=-t}^{N_x+k-t-1} \mathcal{I}_{\beta,t}^{(k)}(x) g(x_\beta)\,, \tag{80}$$

with:

$$\mathcal{I}_{\beta,t}^{(k)}(x) = \sum_{i=0,i\leq\beta}^{k} \theta(x - x_{\beta-i+t})\theta(x_{\beta-i+t+1} - x) \prod_{m=0,m\neq i}^{k} \frac{x - x_{\beta-i+m}}{x_\beta - x_{\beta-i+m}}\,, \tag{81}$$

being the "generalised" interpolation functions. The generalised interpolation functions can be used to overcome the "drawback" of requiring $k-1$ additional nodes on the interpolation grid. In practice, given the grid $\{x_0,\dots,x_{N_x}\}$, one can tune t according to the position of x on the grid. More specifically, one can choose t in such a way that $\beta+t$ in Eq. (81) never exceeds N_x.

Now suppose we want to compute the following intergral:

$$I_1 = \int_{x_0}^{x_{N_x}} dx\, g(x) f(x)\,, \tag{82}$$

where f is some other function that we don't want to interpolate. Using Eqs. (75) and (77) we finally have that:

$$I_1 = \sum_{\beta=0}^{N_x+k-1} W_\beta g(x_\beta)\,, \tag{83}$$

with:

$$W_\beta = \int_{x_{\max(0,\beta-k)}}^{x_{\min(N_x,\beta+1)}} dx\, \mathcal{I}_\beta^{(k)}(x) f(x)\,. \tag{84}$$

The equation above can be easily generalised to a bidimensional integral as:

$$I_2 = \int_{x_0}^{x_{N_x}} dx \int_{y_0}^{y_{N_y}} dy\, g(x,y) f(x,y) = \sum_{\alpha=0}^{N_x+k-1} \sum_{\beta=0}^{N_y+l-1} W_{\alpha\beta}\, g(x_\alpha, y_\beta)\,, \tag{85}$$

with:

$$W_{\alpha\beta} = \int_{x_{\max(0,\alpha-k)}}^{x_{\min(N_x,\alpha+1)}} dx \int_{y_{\max(0,\beta-k)}}^{y_{\min(N_y,\beta+1)}} dy\, \mathcal{I}_\alpha^{(k)}(x)\, \mathcal{I}_\beta^{(l)}(y)\, f(x,y)\,. \tag{86}$$

This formalism nicely applies to the integral in Q and $\xi = e^y$ discussed above in Eq. (43). In view of a numerical implementation, it is worth noticing that the functions $\mathcal{I}$ are piecewise. In particular, while these functions are continuos in correspondence of the nodes of the grid, their first derivative is not. As a consequence, the result of the numerical integrals in Eqs. (84) and (86) may be inaccurate. To overcome this problem, it is sufficient to split the integrals in sub-integrals over the intervals delimited by two consecutive nodes. Using Eq. (77), it is easy to see that, for an interpolation of degree k, one needs to do $k+1$ integrals over the intervals included between the $(\beta-k)$-th and the $(\beta+1)$-th node.

References

[1] I. Scimemi and A. Vladimirov, arXiv:1706.01473 [hep-ph].

[2] J. Collins, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **32** (2011) 1.

[3] H. Ogata, "A Numerical Integration Formula Based on the Bessel Functions," `http://www.kurims.kyoto-u.ac.jp/~okamoto/paper/Publ_RIMS_DE/41-4-40.pdf`