# 1 Structure of the observables

Let us start from Eq. (2.6) of Ref. [1], that is the fully differential cross section for lepton-pair production in the region in which the TMD factorisation applies, *i.e.* $q_T \ll Q$. After some minor manipulations, it reads:

$$\frac{d\sigma}{dQdydq_T} = \frac{16\pi\alpha^2 q_T}{3N_c Q^3} H(Q,\mu) \sum_q C_q(Q) \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \overline{F}_q(x_1, \mathbf{b}; \mu, \zeta) \overline{F}_{\bar{q}}(x_2, \mathbf{b}; \mu, \zeta) \,, \tag{1}$$

where $Q$, $y$, and $q_T$ are the invariant mass, the rapidity, and the transverse momentum of the lepton pair, respectively, while $N_c = 3$ is the number of colours, $\alpha$ is the electromagnetic coupling, $H$ is the appropriate QCD hard factor that can be perturbatively computed, and $C_q$ are the effective electroweak charges. In addition, the variables $x_1$ and $x_2$ are functions of $Q$ and $y$ and are given by:

$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y} \,, \tag{2}$$

being $\sqrt{s}$ the centre-of-mass energy of the collision. In Eq. (1) we are using the short-hand notation:

$$\overline{F}_q(x, \mathbf{b}; \mu, \zeta) \equiv x F_q(x, \mathbf{b}; \mu, \zeta) \,, \tag{3}$$

that is convenient for the implementation. The scales $\mu$ and $\zeta$ are introduced through TMD factorisation to factorise collinear and rapidity divergences. As usual, despite they are arbitrary scales, they are typically chosen $\mu = \sqrt{\zeta} = Q$. Therefore, for all practical purposes their presence is fictitious.

The computation-intensive part of Eq.(1) has the form of the integral:

$$I_{ij}(x_1, x_2, q_T; \mu, \zeta) = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \overline{F}_i(x_1, \mathbf{b}; \mu, \zeta) \overline{F}_j(x_2, \mathbf{b}; \mu, \zeta) \,. \tag{4}$$

where $\overline{F}_{i(j)}$ are combinations of evolved TMD PDFs. At this stage, for convenience, $i$ and $j$ do not coincide with $q$ and $\bar{q}$ but they are linked through a simple linear transformation. The integral over the bidimensional impact parameter $\mathbf{b}$ has to be taken. However, $\overline{F}_{i(j)}$ only depend on the absolute value of $\mathbf{b}$, therefore Eq. (4) can be written as:

$$I_{ij}(x_1, x_2, q_T; \mu, \zeta) = \frac{1}{2} \int_0^\infty db\, b J_0(bq_T) \overline{F}_i(x_1, b; \mu, \zeta) \overline{F}_j(x_2, b; \mu, \zeta) \,. \tag{5}$$

where $J_0$ is the zero-th order Bessel function of the first kind whose integral representation is:

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{ix\cos(\theta)} \,. \tag{6}$$

The single evolved TMD PDF $\overline{F}_i$ at the final scales $\mu$ and $\zeta$ is obtained by multiplying the same TMD PDF at the initial scales $\mu_0$ and $\zeta_0$ by a single evolution factor $R_q$([1]). that is:

$$\overline{F}_i(x, b; \mu, \zeta) = R_q(\mu_0, \zeta_0 \to \mu, \zeta; b) \overline{F}_i(x, b; \mu_0, \zeta_0) \,. \tag{7}$$

The initial scale TMD PDFs at small values $b$ can be written as:

$$\overline{F}_i(x, b; \mu_0, \zeta_0) = \sum_{j=g,q(\bar{q})} x \int_x^1 \frac{dy}{y} C_{ij}(y; \mu_0, \zeta_0) f_j\left(\frac{x}{y}, \mu_0\right) \,, \tag{8}$$

where $f_j$ are the collinear PDFs (including the gluon) and $C_{ij}$ are the so-called matching functions that are perturbatively computable and are currently known to NNLO, *i.e.* $\mathcal{O}(\alpha_s^2)$. If we define:

$$\overline{f}_i(x, \mu_0) = x f_i(x, \mu_0) \,, \tag{9}$$

[1]Note that in Eq. (1) the gluon TMD PDF $\overline{F}_g$ is not involved. If also the gluon TMD PDF was involved, it would evolve by means of a different evolution factor $R_g$.

Eq. (8) can be written as:

$$\overline{F}_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\overline{f}_i\left(\frac{x}{y},\mu_0\right) . \tag{10}$$

Putting Eqs. (7) and (10), one finds:

$$\overline{F}_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\overline{f}_i\left(\frac{x}{y},\mu_0\right) . \tag{11}$$

Matching and the evolution are affected by non-perturbative effects that become relevant at large $b$. In order to account for such effects, one usually introduces a phenomenological function $f_{\rm NP}$. In the traditional approach (CSS [2]) the $b$-space TMDs get a multiplicative correction that does not depend on the flavour. In addition, the perturbative content of the TMDs is smoothly damped away at large $b$ by introducing the so-called $b_*$-prescription:

$$\overline{F}_i(x,b;\mu,\zeta) \to \overline{F}_i(x,b_*;\mu,\zeta) f_{\rm NP}(x,b,\zeta) , \tag{12}$$

where $b_* \equiv b_*(b)$ is a monotonic function of the impact parameter $b$ such that:

$$\lim_{b\to 0} b_*(b) = b_{\rm min} \quad \text{and} \quad \lim_{b\to\infty} b_*(b) = b_{\rm max} , \tag{13}$$

being $b_{\rm min}$ and $b_{\rm max}$ constant values both in the perturbative region. Including the non-perturbative function, Eq. (5) becomes:

$$\begin{aligned} I_{ij}(x_1,x_2,q_T;\mu,\zeta) &= \int_0^\infty db\, J_0(bq_T)\left[\frac{b}{2}\mathcal{L}_{ij}(x_1,x_2,b_*(b);\mu,\zeta) f_{\rm NP}(x_1,b,\zeta) f_{\rm NP}(x_2,b,\zeta)\right] \\ &= \frac{1}{q_T}\int_0^\infty d\bar{b}\, J_0(\bar{b})\left[\frac{\bar{b}}{2q_T}\mathcal{L}_{ij}\left(x_1,x_2,b_*\left(\frac{\bar{b}}{q_T}\right);\mu,\zeta\right) f_{\rm NP}\left(x_1,\frac{\bar{b}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{\bar{b}}{q_T},\zeta\right)\right] , \end{aligned} \tag{14}$$

with:

$$\mathcal{L}_{ij}(x_1,x_2,b_*;\mu,\zeta) \equiv \overline{F}_i(x_1,b_*;\mu,\zeta)\overline{F}_j(x_2,b_*;\mu,\zeta) . \tag{15}$$

Eq. (14) is a Hankel tranform and can be efficiently computed using the so-called Ogata quadrature [3]. Effectively, the computation of the integral in Eq. (4) is achieved through a weighted sum:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) \simeq \frac{1}{q_T}\sum_{n=1}^N \frac{w_n^{(0)} z_n^{(0)}}{2q_T}\mathcal{L}_{ij}\left(x_1,x_2,b_*\left(\frac{z_n^{(0)}}{q_T}\right);\mu,\zeta\right) f_{\rm NP}\left(x_1,\frac{z_n^{(0)}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{z_n^{(0)}}{q_T},\zeta\right) , \tag{16}$$

where the unscaled coordinates $z_n^{(0)}$ and the weights $w_n^{(0)}$ can be precomputed in terms of Bessel functions and one single parameter (see Ref. [3] for more details, specifically Eqs. (5.1) and (5.2))[2]. Based on the (empirically verified) assumption that the absolute value of each term in the sum in the r.h.s. of Eq. (16) is smaller than that of the preceding one, the truncation number $N$ is chosen dynamically in such a way that the $(N+1)$-th term is smaller in absolute value than a user-defined cutoff relatively to the sum of the preceding $N$ terms. Eq. (16) factors out the non-perturbative part of the calculation represented by $f_{\rm NP}$ from the perturbative content.

As customary in QCD, the most convenient basis for the matching in Eq. (8) is the so-called "evolution" basis (*i.e.* $\Sigma$, $V$, $T_3$, $V_3$, etc.). In fact, in this basis the operators matrix $C_{ij}$ is almost diagonal with the only exception of crossing terms that couple the gluon and the singlet $\Sigma$ distributions. As a consequence, this is the most convenient basis for the computation of $I_{ij}$. On the other hand, TMDs in Eq. (1) are in the so-called "physical" basis (*i.e.* $d$, $\bar{d}$, $u$, $\bar{u}$, etc.). Therefore, we need to rotate $I_{ij}$

[2] The superscript 0 in $z_n^{(0)}$ and $w_n^{(0)}$ indicates that here we are performing a Hankel tranform that involves the Bessel function of degree zero $J_0$. This is useful in view of the next section in which the integration over $q_T$ will give rise to a similar Hankel transform with $J_0$ replaced by $J_1$. Also in that case the Ogata quadrature algorithm can be applied.

from the evolution basis, over which the indices $i$ and $j$ run, to the physical basis. This is done by means of an appropriate constant matrix $T$, so that:

$$I_{q\bar{q}}(x_1, x_2, q_T; \mu, \zeta) = \sum_{ij} T_{qi} T_{\bar{q}j} I_{ij}(x_1, x_2, q_T; \mu, \zeta)\,. \tag{17}$$

Putting all pieces together, one can conveniently write the cross section in Eq. (1) as:

$$\frac{d\sigma}{dQdydq_T} = \sum_{n=1}^{N} w_n^{(0)} S\left(x_1, x_2, \frac{z_n^{(0)}}{q_T}; \mu, \zeta\right) f_{\rm NP}\left(x_1, \frac{z_n^{(0)}}{q_T}, \zeta\right) f_{\rm NP}\left(x_2, \frac{z_n^{(0)}}{q_T}, \zeta\right)\,, \tag{18}$$

with:

$$\begin{aligned} S(x_1, x_2, b; \mu, \zeta) &= \frac{16\pi\alpha^2}{3N_c Q^3} H(Q,\mu) \frac{b}{2} \sum_q C_q(Q) \sum_{ij} T_{qi} T_{\bar{q}j} \mathcal{L}_{ij}\left(x_1, x_2, b_*(b); \mu, \zeta\right) \\ &= \frac{16\pi\alpha^2}{3N_c Q^3} H(Q,\mu) \frac{b}{2} \sum_q C_q(Q) \left[\overline{F}_q\left(x_1, b_*(b); \mu, \zeta\right)\right] \left[\overline{F}_{\bar{q}}\left(x_2, b_*(b); \mu, \zeta\right)\right]\,, \end{aligned} \tag{19}$$

with:

$$\overline{F}_q(x_1, b; \mu, \zeta) = \sum_i T_{qi} \overline{F}_i(x_1, b; \mu, \zeta)\,. \tag{20}$$

Eq. (18) allows one to precompute the weights $S$ in such a way that the differential cross section in Eq. (1) can be computed as a simple weighted sum of the non-perturbative contribution. A misleading aspect of Eq. (19) is the fact that $S$ has five arguments. In actual facts, $S$ only depends on three independent variables. The reason is that $\mu$ and $\zeta$ are usually taken to be proportional to $Q$ by a constant factor. In addition $x_1$ and $x_2$ depend on $Q$ and $y$ through Eq. (2). Therefore, as expected, the full dependence on the kinematics of the final state of Eq. (1) can be specified by $Q$, $y$ and $q_T$ or alternatively by $x_1$, $x_2$ and $q_T$.

# 2 Integrating over the final-state kinematic variables

Despite Eq. (18) provides a powerful tool for a fast computation of cross sections, it is often not sufficient to allow for a direct comparison to experimental data. The reason is that experimental measurements of differential distributions are usually delivered as integrated over finite regions of the final-state kinematic phase space. In other words, experiments measure quantities like:

$$\widetilde{\sigma} = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ \int_{y_{\rm min}}^{y_{\rm max}} dy \int_{q_{T,\rm min}}^{q_{T,\rm max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right]\,. \tag{21}$$

As a consequence, in order to guarantee performance, we need to include the integrations above in the precomputed factors.

## 2.1 Integrating over $q_T$

The integration over bins in $q_T$ can be carried out analytically exploiting the following property of Bessel's function:

$$\int dx\, x J_0(x) = x J_1(x) \quad \Rightarrow \quad \int_{x_1}^{x_2} dx\, x J_0(x) = x_2 J_1(x_2) - x_1 J_1(x_1)\,. \tag{22}$$

To see it, we observe that the differential cross section in Eq. (1) has the following general structure:

$$\frac{d\sigma}{dQdydq_T} \propto \int_0^\infty db\, q_T J_0(bq_T)\dots \tag{23}$$

where the ellipses indicate terms that do not depend on $q_T$. Therefore, using Eq. (22) we find:

$$
\begin{aligned}
&\int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right] \propto \int_0^\infty db \int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T J_0(bq_T)\cdots = \\
&\int_0^\infty \frac{db}{b^2} \int_{bq_{T,\min}}^{bq_{T,\max}} dx\, x J_0(x)\cdots = \int_0^\infty \frac{db}{b}\left[q_{T,\max}J_1(bq_{T,\max}) - q_{T,\min}J_1(bq_{T,\min})\right]\dots .
\end{aligned}
\tag{24}
$$

Therefore, defining:

$$
K(q_T) \equiv \int dq_T \left[\frac{d\sigma}{dQdydq_T}\right] \tag{25}
$$

as the indefinite integral over $q_T$ of the cross section in Eq. (1), we have that:

$$
\int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right] = K(Q,y,q_{T,\max}) - K(Q,y,q_{T,\min})\,, \tag{26}
$$

with:

$$
\begin{aligned}
K(Q,y,q_T) &= \frac{16\pi\alpha^2 q_T}{3N_c Q^3} H(Q,\mu) \\
&\times \sum_q C_q(Q)\frac{1}{2}\int_0^\infty db\, J_1(bq_T)\overline{F}_q(x_1,b;\mu,\zeta)\overline{F}_{\bar{q}}(x_2,b;\mu,\zeta) f_{\rm NP}(x_1,b,\zeta) f_{\rm NP}(x_2,b,\zeta)\,,
\end{aligned}
\tag{27}
$$

that can be computed using the Ogata quadrature as:

$$
K(Q,y,q_T) \simeq \sum_{n=1}^{N} w_n^{(1)} P\left(x_1,x_2,\frac{z_n^{(1)}}{q_T};\mu,\zeta\right) f_{\rm NP}\left(x_1,\frac{z_n^{(1)}}{q_T},\zeta\right) f_{\rm NP}\left(x_2,\frac{z_n^{(1)}}{q_T},\zeta\right)\,, \tag{28}
$$

where:

$$
\begin{aligned}
P(x_1,x_2,b;\mu,\zeta) &= \frac{1}{b} S(x_1,x_2,b;\mu,\zeta) \\
&= \frac{16\pi\alpha^2}{3N_c Q^3} H(Q,\mu)\frac{1}{2}\sum_q C_q(Q)\left[\overline{F}_q\left(x_1,b_*(b);\mu,\zeta\right)\right]\left[\overline{F}_{\bar{q}}\left(x_2,b_*(b);\mu,\zeta\right)\right]\,,
\end{aligned}
\tag{29}
$$

with $S$ defined in Eq. (19). The unscaled coordinates $z_n^{(1)}$ and the weights $w_n^{(1)}$ can again be precomputed and stored. Eq. (26) reduces the integration in $q_T$ to a calculation completely analogous to the unintegrated cross section. This is particularly convenient because it avoids the computation a numerical integration.

## 2.2 On the position of the peak of the $q_T$ distribution

It is interesting at this point to take a short detour to discuss the postion of the peak on the distribution in $q_T$ of the cross section in Eq. (1). The peak can be located by setting the derivative in $q_T$ of the cross section equal to zero. To do so, we use another property of Bessel's functions:

$$
\frac{dJ_0(x)}{dx} = -J_1(x)\,. \tag{30}
$$

Using this relation, it is easy to see that:

$$
\begin{aligned}
&0 = \frac{d}{dq_T}\left[\frac{d\sigma}{dQdydq_T}\right] = \\
&\frac{16\pi\alpha^2}{3N_cQ^3}H(Q,\mu)\sum_q C_q(Q)\frac{1}{2}\int_0^\infty db\, b\left[J_0(bq_T) - bq_TJ_1(bq_T)\right]\overline{F}_q(x_1,b_*(b);\mu,\zeta)\overline{F}_{\bar{q}}(x_2,b_*(b);\mu,\zeta) \\
&\times f_{\rm NP}(x_1,b,\zeta)f_{\rm NP}(x_2,b,\zeta)\,,
\end{aligned}
\tag{31}
$$

that is equivalent to:

$$
\sum_q C_q(Q)\int_0^\infty db\, b\left[J_0(bq_T) - bq_TJ_1(bq_T)\right]\overline{F}_q(x_1,b_*(b);\mu,\zeta)\overline{F}_{\bar{q}}(x_2,b_*(b);\mu,\zeta)f_{\rm NP}(x_1,b,\zeta)f_{\rm NP}(x_2,b,\zeta) = 0\,.
\tag{32}
$$

The integral above can be solved numerically using the technique discussed above and the value of $q_T$ that satisfies this equation can be found.

## 2.3 Integrating over $Q$ and $y$

As a final step we need to perform the integrals over $Q$ and $y$ defined in Eq. (34). To compute these integrals we can only rely on numerical methods. Having reduced the integration in $q_T$ to the difference of the two terms in the r.h.s. of Eq. (26), we can concentrate on integrating the functions $K$ over $Q$ and $y$ for a fixed value of $q_T$:

$$
\widetilde{K}(q_T) = \int_{Q_{\rm min}}^{Q_{\rm max}} dQ\int_{y_{\rm min}}^{y_{\rm max}} dy\, K(Q,y,q_T)\,,
\tag{33}
$$

such that:

$$
\widetilde{\sigma} = \widetilde{K}(q_{T,\rm max}) - \widetilde{K}(q_{T,\rm min})\,.
\tag{34}
$$

To this purpose, it is convenient to make explicit the dependence of $x_{1,2}$ on $Q$ and $y$ using Eq. (2). In addition, for the sake of simplicity we will identify the scales $\mu$ and $\sqrt{\zeta}$ with $Q$ (possible scale variations can be easily reinstated at a later stage) and thus drop one of the arguments from the TMD distributions $\overline{F}$ and from the hard factor $H$. This yields:

$$
\begin{aligned}
\widetilde{K}(q_T) &= \int_0^\infty db\, J_1(bq_T)\frac{16\pi q_T}{3N_c}\int_{Q_{\rm min}}^{Q_{\rm max}} dQ\,\frac{\alpha^2(Q)}{Q^3}H(Q)\sum_q C_q(Q)\frac{1}{2} \\
&\times \int_{e^{y_{\rm min}}}^{e^{y_{\rm max}}} d\xi\,\frac{1}{\xi}\,\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi,b_*(b);Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi},b_*(b);Q\right) \\
&\times f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\xi,b;Q\right)f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi},b;Q\right)\,.
\end{aligned}
\tag{35}
$$

Now we define a bidimensional grid in $\xi$, $\{\xi_\alpha\}$ with $\alpha = 0,\dots,N_\xi$, and in $Q$, $\{Q_\tau\}$ with $\tau = 0,\dots,N_Q$, each of which a set of interpolating functions $w$ associated. This allows us to interpolate the pair of functions $f_{\rm NP}$ in Eq. (35) for generic values of $\xi$ and $Q$ as:

$$
f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\xi,b;Q\right)f_{\rm NP}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi},b;Q\right) \simeq \sum_{\alpha=0}^{N_\xi}\sum_{\tau=0}^{N_Q} w_\alpha(\xi)w_\tau(Q)f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha,b;Q_\tau\right)f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha},b;Q_\tau\right)\,.
\tag{36}
$$

Plugging the equation above into Eq. (35) we obtain:

$$\begin{aligned}\widetilde{K}(q_T) &\simeq \int_0^\infty db\, J_1(bq_T) \sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi}\left[\frac{16\pi q_T}{3N_c}\int_{Q_{\min}}^{Q_{\max}} dQ\, w_\tau(Q)\,\frac{\alpha^2(Q)}{Q^3}H(Q)\sum_q C_q(Q)\frac{1}{2}\right.\\ &\times \left.\int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\, w_\alpha(\xi)\,\frac{1}{\xi}\,\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*(b); Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*(b); Q\right)\right] \\ &\times f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, b; Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, b; Q_\tau\right)\,. \end{aligned} \tag{37}$$

Finally, the integration over $b$ can be performed using the Ogata quadrature as discussed above, so that:

$$\begin{aligned}\widetilde{K}(q_T) &\simeq \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi}\left[w_n^{(1)}\frac{16\pi q_T}{3N_c}\int_{Q_{\min}}^{Q_{\max}} dQ\, w_\tau(Q)\,\frac{\alpha^2(Q)}{Q^3}H(Q)\sum_q C_q(Q)\frac{1}{2}\right.\\ &\times \left.\int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\, w_\alpha(\xi)\,\frac{1}{\xi}\,\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*\left(\frac{z_n}{q_T}\right); Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*\left(\frac{z_n}{q_T}\right); Q\right)\right] \\ &\times f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, \frac{z_n}{q_T}; Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, \frac{z_n}{q_T}; Q_\tau\right)\,. \end{aligned} \tag{38}$$

In conclusion, if we define:

$$\begin{aligned} W_{n\alpha\tau} &\equiv w_n^{(1)}\frac{16\pi q_T}{3N_c}\int_{Q_{\min}}^{Q_{\max}} dQ\, w_\tau(Q)\,\frac{\alpha^2(Q)}{Q^3}H(Q)\sum_q C_q(Q)\frac{1}{2} \\ &\times \int_{e^{y_{\min}}}^{e^{y_{\max}}} d\xi\, w_\alpha(\xi)\,\frac{1}{\xi}\,\overline{F}_q\left(\frac{Q}{\sqrt{s}}\xi, b_*\left(\frac{z_n}{q_T}\right); Q\right)\overline{F}_{\bar{q}}\left(\frac{Q}{\sqrt{s}}\frac{1}{\xi}, b_*\left(\frac{z_n}{q_T}\right); Q\right)\,, \end{aligned} \tag{39}$$

the quantity $\widetilde{K}(q_T)$ can be computed as:

$$\widetilde{K}(q_T) \simeq \sum_{n=1}^{N}\sum_{\tau=0}^{N_Q}\sum_{\alpha=0}^{N_\xi} W_{n\alpha\tau} f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\xi_\alpha, \frac{z_n}{q_T}; Q_\tau\right) f_{\rm NP}\left(\frac{Q_\tau}{\sqrt{s}}\frac{1}{\xi_\alpha}, \frac{z_n}{q_T}; Q_\tau\right)\,. \tag{40}$$

The advantage of Eq. (40) is that the weights $W_{n\alpha\tau}$ can be precomputed once and for all and used to fit the function $f_{\rm NP}$.

## 2.4 Narrow-width approximation

A possible alternative to the numerical integration in $Q$ when the integration region includes the $Z$-peak region is the so-called narrow-width approximation (NWA). In the NWA one assumes that the width of the $Z$ boson, $\Gamma_Z$, is much smaller than its mass $M_Z$. This way one can approximate the peaked behaviour of the couplings $C_q(Q)$ around $Q = M_Z$ with a $\delta$-function, *i.e.* $C_q(Q) \sim \delta(Q^2 - M_Z^2)$. This way the integration over $Q$ can be done analytically essentially setting $Q = M_Z$ everywhere in the expression. The exact structure of the electroweak couplings is the following:

$$C_q(Q) = e_q^2 - 2e_q V_q V_e \chi_1(Q) + (V_e^2 + A_e^2)(V_q^2 + A_q^2)\chi_2(Q)\,, \tag{41}$$

with:

$$\begin{aligned}\chi_1(Q) &= \frac{1}{4\sin^2\theta_W\cos^2\theta_W}\frac{Q^2(Q^2 - M_Z^2)}{(Q^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}\,,\\ \chi_2(Q) &= \frac{1}{16\sin^4\theta_W\cos^4\theta_W}\frac{Q^4}{(Q^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}\,.\end{aligned} \tag{42}$$

In the limit in which the width $\Gamma_Z$ is much smaller that the $Z$ mass $M_Z$ ($\Gamma_Z/M_Z \to 0$), the leading contribution to the coupling in Eq. (41) comes from the region $Q \simeq M_Z$ and is that proportional to $\chi_2$:

$$C_q(Q) \simeq (V_e^2 + A_e^2)(V_q^2 + A_q^2)\chi_2(Q)\,, \quad Q \simeq M_Z\,. \tag{43}$$

In addition, in this limit one can show that:

$$\frac{1}{(Q^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2} \to \frac{\pi}{M_Z\Gamma_Z}\delta(Q^2 - M_Z^2) = \frac{\pi}{2M_Z^2\Gamma_Z}\delta(Q - M_Z)\,. \tag{44}$$

Therefore, considering that:

$$\Gamma_Z = \frac{\alpha M_Z}{\sin^2\theta_W \cos^2\theta_W}\,, \tag{45}$$

the electroweak couplings in the NWA have the following form:

$$C_q(Q) \simeq \frac{\pi M_Z(V_e^2 + A_e^2)(V_q^2 + A_q^2)}{32\alpha \sin^2\theta_W \cos^2\theta_W}\delta(Q - M_Z)\,, \tag{46}$$

such that the differential cross section in Eq. (1) becomes:

$$\frac{d\sigma}{dQdydq_T} = \frac{2q_T\pi^2\alpha}{3N_cM_Z^2}H(M_Z, M_Z)\sum_q \frac{(V_e^2 + A_e^2)(V_q^2 + A_q^2)}{4\sin^2\theta_W\cos^2\theta_W}I_{q\bar{q}}(x_1, x_2, q_T; M_Z, M_Z^2)\delta(Q - M_Z)\,. \tag{47}$$

Integrating the cross section over $Q$ under the condition that $Q_{\min} < M_Z < Q_{\max}$ yields:

$$\frac{d\sigma}{dydq_T} = \int_{Q_{\min}}^{Q_{\max}} dQ\,\frac{d\sigma}{dQdydq_T} = \frac{2q_T\pi^2\alpha}{3N_cM_Z^2}H(M_Z, M_Z)\sum_q \frac{(V_e^2 + A_e^2)(V_q^2 + A_q^2)}{4\sin^2\theta_W\cos^2\theta_W}I_{q\bar{q}}(x_1, x_2, q_T; M_Z, M_Z^2)\,. \tag{48}$$

As a final step, one may want to let the $Z$ boson decay into leptons. At leading order in the EW sector and assuming an equal decay rate for electrons, muons, and taus, this can be done by multiplying the cross section above by three times the branching ratio for the $Z$ decaying into any pair of leptons, $\mathrm{Br}(Z \to \ell^+\ell^-)$:

$$\frac{d\sigma}{dydq_T} = \frac{2q_T\mathrm{Br}(Z \to \ell^+\ell^-)\pi^2\alpha}{N_cM_Z^2}H(M_Z, M_Z)\sum_q \frac{(V_e^2 + A_e^2)(V_q^2 + A_q^2)}{4\sin^2\theta_W\cos^2\theta_W}I_{q\bar{q}}(x_1, x_2, q_T; M_Z, M_Z^2)\,. \tag{49}$$

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