Contents

1 Structure of the integrals

The general structure of the quantities to be computed in APFEL is the combination of terms I that have the form of Mellin convolutions between an operator O and distribution function d, that is:

$$I(x) = x\int_0^1 dz \int_0^1 dy\, O(z) d(y) \delta(x - yz) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x\int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right) . \tag{1.1}$$

However, very often, typically in the presence of mass effects, the integration phase space is modified and the convolution in eq. (1.1), limiting ourselves to the leftmost identity, is generalised as:

$$I(x,\eta) = x\int_{x/\eta}^1 \frac{dz}{z} O(z,\eta) d\left(\frac{x}{\eta z}\right) . \tag{1.2}$$

where $\eta \leq 1$, with $\eta = 1$ reproducing eq. (1.1). However, for purposes that will become clear later, we want to write the integral in eq. (1.2) in the form of eq. (1.1), that is in such a way that lower bound of the integral is not the rescaled variable x/η but the physical Bjorken x. To do so, one needs to perform the change of variable $y = \eta z$, so that:

$$I(x,\eta) = \int_x^\eta dy\, O\left(\frac{y}{\eta},\eta\right) \frac{x}{y} d\left(\frac{x}{y}\right) . \tag{1.3}$$

In order to precompute the expensive part of the integral in eq. (1.5), we use the standard interpolation formula to the distribution d:

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right) , \tag{1.4}$$

where α runs over the node of a give grid in x space and the weights w_α are typically polynomials of degree k (*i.e.* Laplace interpolants). Now we use eq. (1.4) in eq. (1.5) and at the same time we limit the computation of the integral I to the point x_β of the grid used in eq. (1.4). This way we get:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^\eta dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.5}$$

where we have defined $\overline{d}_\alpha = x_\alpha d(x_\alpha)$. In the particular case of the Laplace interpolants (in the APFEL procedure), one can show that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d\,, \tag{1.6}$$

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k})\,, \tag{1.7}$$

and thus eq. (1.5) can be adjusted as:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_c^d dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.8}$$

Finally, we change back the integration variable $z = y/\eta$ so that eq. (1.8) becomes:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz\, O(z,\eta)\; w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)\right]}_{\Gamma_{\alpha\beta}} . \tag{1.9}$$

The quantity in squared brackets is the interesting bit.

The expressions for the operator O that we have to deal with have this general form:

$$O(z,\eta) = R(z,\eta) + \sum_i \left[P^{(i)}(z,\eta)\right]_+ S^{(i)}(z,\eta) + L(\eta)\delta(1-z)\,, \tag{1.10}$$

where R and $S^{(i)}$ are a regular functions in $z=1$, that is:

$$R(1,\eta) = \lim_{z\to 1} R(z,\eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1,\eta) = \lim_{z\to 1} S^{(i)}(z,\eta) = J^{(i)}(\eta)\,, \tag{1.11}$$

being K, $J^{(i)}$, and L a finite function of η. The functions $P^{(i)}$ are instead plus-prescripted functions whose behaviour in $z=1$ is singular in the limit $\eta \to 1$. Notice that the sum over i in eq. (1.10) depends on the perturbative order of the expressions. Plugging eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in eq. (1.9) and making use of the definition of plus-prescription, we obtain:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \eta\int_{c/\eta}^{d/\eta} dz \left\{ R\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) + \sum_i P^{(i)}(z,\eta)\left[S^{(i)}\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\ &+ \eta\left[L(\eta) - \sum_i S^{(i)}(1,\eta)\int_0^{c/\eta} dz P^{(i)}(z,\eta)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,,\end{aligned} \tag{1.12}$$

that can be further manipulated changing the integration variable back, defining $\eta z = y$:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \int_c^d dy \left\{ R\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) + \sum_i P^{(i)}\left(\frac{y}{\eta},\eta\right)\left[S^{(i)}\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) - S^{(i)}(1,\eta) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\} \\ &+ \eta\left[L(\eta) + \sum_i S^{(i)}(1,\eta) Q^{(i)}\left(\frac{c}{\eta},\eta\right)\right] w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\,.\end{aligned} \tag{1.13}$$

where we have defined:

$$Q^{(i)}(a,\eta) \equiv -\int_0^a dz P^{(i)}\left(z,\eta\right). \tag{1.14}$$

These integrals can, most of the times, be computed analytically.

Eqs. (1.12)-(1.13) express the full complexity of the task. However, there are a few remarks that apply is some particular cases and that reduce the complexity. In the case of the ($\overline{\text{MS}}$) splitting functions, there are two main simplications: the first is that $\eta = 1$, the second is that there is one single term in the sum over i ($i=0$) and the form of the function P is:

$$P^{(0)}(z,\eta) \to \frac{1}{1-z}\,, \tag{1.15}$$

so that:

$$Q^{(0)}(a,\eta) = \ln(1-a)\,. \tag{1.16}$$

Considering that:

$$w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}\,, \tag{1.17}$$

and that the expressions can be manipulated in such a way that the coefficient of the plus-prescripted term S is a constant, we have that:

$$\Gamma_{\beta\alpha} = \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} + \left[S\ln\left(1-c\right) + L\right]\delta_{\alpha\beta}\,. \tag{1.18}$$

The same kind of simplifications apply to the case of the Zero-Mass (ZM) coefficient functions with the only exception that the sum over i extends to more terms. In particular, since we will use exact expressions only up to $\mathcal{O}(\alpha_s)$, *i.e.* NLO, we have that:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \int_c^d dz \left\{ R\left(z\right) w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) + \frac{S^{(0)} + S^{(1)}\ln(1-z)}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right) - \delta_{\alpha\beta}\right]\right\} \\ &+ \left[S^{(0)}\ln\left(1-c\right) + \frac{1}{2}S^{(1)}\ln^2\left(1-c\right) + L\right]\delta_{\alpha\beta}\,.\end{aligned} \tag{1.19}$$

As far as the massive coefficient functions up to $\mathcal{O}(\alpha_s^2)$ things can be more complicated and we will discuss it later. In the next section we will consider the case of hadronic observables in the ZM scheme showing how to pre-compute the integral for the double-differential distributions in Drell-Yan production.

1.1 Double convolution

In the case of Drell-Yan or Semi-Inclusive DIS (SIDIS), cross sections are the result of double convolutions between partonic cross sections and a pair of non-perturbative distributions. In this case the kind of expressions we are dealing with has the following form:

$$\begin{aligned} J(x_\delta, x_\gamma) &= x_\delta x_\gamma \int_{x_\delta}^1 \frac{dy_1}{y_1} \int_{x_\gamma}^1 \frac{dy_2}{y_2} O(y_1,y_2) d^{(1)}\left(\frac{x_\delta}{y_1}\right) d^{(2)}\left(\frac{x_\gamma}{y_2}\right) \\ &= \int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2) \left[\frac{x_\delta}{y_1} d^{(1)}\left(\frac{x_\delta}{y_1}\right)\right] \left[\frac{x_\gamma}{y_2} d^{(2)}\left(\frac{x_\gamma}{y_2}\right)\right] \\ &= \sum_{\alpha=0}^{N_x} \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \overline{d}_\alpha^{(2)} \underbrace{\left[\int_{x_\delta}^1 dy_1 \int_{x_\gamma}^1 dy_2\, O(y_1,y_2)\, w_\beta^{(k)}\left(\frac{x_\delta}{y_1}\right) w_\alpha^{(k)}\left(\frac{x_\gamma}{y_2}\right)\right]}_{\Theta^{\beta\delta,\alpha\gamma}} . \end{aligned} \tag{1.20}$$

In Eq. (1.20) we assume that there are no mass corrections and thus the convolutions take the simplest form. In the case of double convolutions, the partonic cross sections O are functions of two variables, y_1 and y_2, and, as in the case of the single convolutions, they get three kinds of contributions in both these variables: local terms proportional to δ-functions, singular terms proportional to plus-prescripted functions, and regular terms. The complication here is that these contributions from the two variables y_1 and y_2 mix and thus, for example, terms local in y_1 and singular in y_2 appear. It is thus necessary to identify the general structure of the function O to see whether it is possible to decompose the double operator $\Theta^{\beta\delta,\alpha\gamma}$ into products of single operators of the kind $\Gamma^{\beta\delta}$ and $\Gamma^{\alpha\gamma}$.

In the case of the NLO corrections to SIDIS, the general structure of the function O can be inferred looking at Eqs. (C.2)-(C.7) of Ref. [1]:

$$\begin{aligned} O(y_1,y_2) &= \mathrm{LL}\,\delta(1-y_1)\delta(1-y_2) + \mathrm{LS}\,\delta(1-y_1)\left[\frac{\ln(1-y_2)}{1-y_2}\right]_+ + \delta(1-y_1)\,\mathrm{LR}(y_2) \\ &+ \mathrm{SL}\left[\frac{\ln(1-y_1)}{1-y_1}\right]_+ \delta(1-y_2) + \mathrm{SS}\left[\frac{1}{1-y_1}\right]_+ \left[\frac{1}{1-y_2}\right]_+ + \left[\frac{1}{1-y_1}\right]_+ \mathrm{SR}(y_2) \\ &+ \mathrm{RL}(y_1)\,\delta(1-y_2) + \mathrm{RS}(y_1)\left[\frac{1}{1-y_2}\right]_+ + \sum_i K_i \mathrm{R}_i^{(1)}(y_1)\mathrm{R}_i^{(2)}(y_2) . \end{aligned} \tag{1.21}$$

It is clear that in Eq. (1.21) all terms factorise into a part only depending on y_1 and a part only depending on y_2. This is crucial to use the same technology developed above for the single convolutions. Plugging Eq. (1.21) into Eq. (1.20), one finds that:

$$\begin{aligned} \Theta^{\beta\delta,\alpha\gamma} &= \mathrm{LL}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{LS}\,\Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{S1}}_{\alpha\gamma} + \Gamma^{\mathrm{L}}_{\beta\delta}\Gamma^{\mathrm{LR}}_{\alpha\gamma} \\ &+ \mathrm{SL}\,\Gamma^{\mathrm{S1}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \mathrm{SS}\,\Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \Gamma^{\mathrm{S0}}_{\beta\delta}\Gamma^{\mathrm{SR}}_{\alpha\gamma} \\ &+ \Gamma^{\mathrm{RL}}_{\beta\delta}\Gamma^{\mathrm{L}}_{\alpha\gamma} + \Gamma^{\mathrm{RS}}_{\beta\delta}\Gamma^{\mathrm{S0}}_{\alpha\gamma} + \sum_i K_i \Gamma^{R_i^{(1)}}_{\beta\delta}\Gamma^{R_i^{(2)}}_{\alpha\gamma} \end{aligned} \tag{1.22}$$

with:

$$\begin{aligned}
\Gamma^{\mathrm{L}}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, \delta(1-z) w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) = w_\beta^{(k)}(x_\alpha) = \delta_{\alpha\beta} \\
\Gamma^{\mathrm{S}n}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz \frac{\ln^n(1-z)}{1-z}\left[w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right) - \delta_{\alpha\beta}\right] + \frac{1}{(n+1)!}\ln^{n+1}(1-c_{\alpha\beta})\,\delta_{\alpha\beta} \\
\Gamma^{f}_{\alpha\beta} &= \int_{c_{\alpha\beta}}^{d_{\alpha\beta}} dz\, f(z)\, w_\beta^{(k)}\left(\frac{x_\alpha}{z}\right)
\end{aligned} \tag{1.23}$$

where f is a regular function and the integration bounds are defined as:

$$c_{\alpha\beta} = \max(x_\alpha, x_\alpha/x_{\beta+1}) \quad \text{and} \quad d_{\alpha\beta} = \min(1, x_\alpha/x_{\beta-k})\,. \tag{1.24}$$

In general terms, it should always be possible to write an object of the kind of $\Theta^{\beta\delta,\alpha\gamma}$ as series of bilinear terms:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_j C_j \Gamma_j^{(1),\beta\delta}\Gamma_j^{(2),\alpha\gamma} \tag{1.25}$$

where C_j is a scalar, and $\Gamma_j^{(1),\beta\delta}$ and $\Gamma_j^{(2),\alpha\gamma}$ are single operators that can be computed using the technology discussed in the previous section. Plugging Eq. (1.25) into Eq. (1.20), one finds that:

$$J(x_\delta, x_\gamma) = \sum_j C_j f_j^{(1),\delta} f_j^{(2),\gamma}\,, \tag{1.26}$$

where we have defined:

$$f_j^{(1),\delta} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(1)} \Gamma_j^{(1),\beta\delta} \quad \text{and} \quad f_j^{(2),\gamma} \equiv \sum_{\alpha=0}^{N_x} \overline{d}_\alpha^{(2)} \Gamma_j^{(2),\alpha\gamma}\,. \tag{1.27}$$

Eq. (1.26) shows that, in the hypothesis that the double function $O(y_1, y_2)$ can be expressed in terms of a series of terms where the dependence on y_1 and y_2 factorizes[1], the double convolution in Eq. (1.20) can be expressed as a series of bilinear terms of distributions ($f_j^{(1),\delta}$ and $f_j^{(2),\gamma}$) singularly obtained as convolutions of a limited number of single operators with distributions. This is a particularly useful achievement that allows us to compute double convolutions without the need of extending the integration and the interpolation procedures with an obvious gain in accuracy and performance. As a matter of fact, the same argument can be extended to a multiple convolution of the function $O(\{y_i\})$, which again can be expressed as a series of n-linear terms, with $i = 1, \ldots, n$, with n distributions:

$$J(\{x_{\alpha_i}\}) = \sum_j C_j \prod_{i=1}^{n} f_j^{(i),\alpha_i}\,, \tag{1.28}$$

with:

$$f_j^{(i),\alpha_i} \equiv \sum_{\beta=0}^{N_x} \overline{d}_\beta^{(i)} \Gamma_j^{(i),\beta\alpha_i}\,. \tag{1.29}$$

This technology could be useful for more complicated observables, like cross sections in pp collisions with an identified hadron in the final state, that requires for example three convolutions.

The challenging part of the technology just presented resides in the "pre-processing" of the function $O(y_1, y_2)$, that is the analytical work required to disentangle the different terms. This step however has to be taken only once.

Before employing this procedure for any concrete application, it is appropriate to connect Eq. (1.22) to Eq. (1.25) by identifying number and form of the coefficients and operators involved. Specifically, assuming that the series in the last term in the r.h.s. of Eq. (1.22) has r terms, the series in Eq. (1.25) will have $8 + r$ terms, that is:

$$\Theta^{\beta\delta,\alpha\gamma} = \sum_{j=1}^{8+r} C_j \Gamma_j^{(1),\beta\delta}\Gamma_j^{(2),\alpha\gamma} \tag{1.30}$$

[1] This is the case for SIDIS and DY up to NLO but we expect this feature to hold also beyond, despite complications due to the more involved plus-prescripted functions are also expected.

with:

$$\begin{array}{lcll}
j=1 & : & C_1=\mathrm{LL}, & \Gamma_1^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_1^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=2 & : & C_2=\mathrm{LS}, & \Gamma_2^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_2^{(2),\alpha\gamma}=\Gamma^{\mathrm{S1}}_{\alpha\gamma},\\
j=3 & : & C_3=1, & \Gamma_3^{(1),\beta\delta}=\Gamma^{\mathrm{L}}_{\beta\delta}, \quad \Gamma_3^{(2),\alpha\gamma}=\Gamma^{\mathrm{LR}}_{\alpha\gamma},\\
j=4 & : & C_4=\mathrm{SL}, & \Gamma_4^{(1),\beta\delta}=\Gamma^{\mathrm{S1}}_{\beta\delta}, \quad \Gamma_4^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=5 & : & C_5=\mathrm{SS}, & \Gamma_5^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, \quad \Gamma_5^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma},\\
j=6 & : & C_6=1, & \Gamma_6^{(1),\beta\delta}=\Gamma^{\mathrm{S0}}_{\beta\delta}, \quad \Gamma_6^{(2),\alpha\gamma}=\Gamma^{\mathrm{SR}}_{\alpha\gamma},\\
j=7 & : & C_7=1, & \Gamma_7^{(1),\beta\delta}=\Gamma^{\mathrm{RL}}_{\beta\delta}, \quad \Gamma_7^{(2),\alpha\gamma}=\Gamma^{\mathrm{L}}_{\alpha\gamma},\\
j=8 & : & C_8=1, & \Gamma_8^{(1),\beta\delta}=\Gamma^{\mathrm{RS}}_{\beta\delta}, \quad \Gamma_8^{(2),\alpha\gamma}=\Gamma^{\mathrm{S0}}_{\alpha\gamma},\\
j=9 & : & C_9=K_1 & \Gamma_9^{(1),\beta\delta}=\Gamma^{R_1^{(1)}}_{\beta\delta}, \quad \Gamma_9^{(2),\alpha\gamma}=\Gamma^{R_1^{(2)}}_{\alpha\gamma},\\
\vdots & & & \\
j=8+r & : & C_{8+r}=K_r & \Gamma_{8+r}^{(1),\beta\delta}=\Gamma^{R_r^{(1)}}_{\beta\delta}, \quad \Gamma_{8+r}^{(2),\alpha\gamma}=\Gamma^{R_r^{(2)}}_{\alpha\gamma}
\end{array} \tag{1.31}$$

It should be noted that, despite the large number of terms in the series in Eq. (1.30), the number of operators to be precomputed it pretty limited. In addition, in many cases many of the terms of the series are zero so the number of contributions is greatly reduced.

We can now apply this procedure up to NLO in QCD to two specific cases: SIDIS first and DY second.

1.2 Semi-inclusive deep inelastic scattering (SIDIS)

the structure of the SIDIS observables and the expressions for the respective hard cross sections can be found in Ref. [1]. Following this paper, the SIDIS differential cross section for the exchange of a virtual photon can be written as:

$$\frac{d^3\sigma}{dx\,dy\,dz}=\frac{2\,\pi\alpha^2}{xyQ^2}\left[(1+(1-y)^2)2xF_1(x,z,Q^2)+2(1-y)F_L(x,z,Q^2)\right]\,, \tag{1.32}$$

with $Q^2=-q^2$ the (negative) virtuality of the exchanged photon, x and z the momentum fractions of PDFs and FFs, and $Q^2=xys$ the definition of the inelasticity y in terms of the squared collision energy in the center of mass s. Notice that, as compared to Ref. [1], we have absorbed into the definition of F_L a factor x as customary in the definition of the longitudinal structure function in inclusive DIS.

We now use the Callan-Gross relation:

$$F_2=2xF_1+F_L \tag{1.33}$$

to replace $2xF_1$ with F_2 in Eq. (1.32):

$$\frac{d^3\sigma}{dx\,dy\,dz}=\frac{2\,\pi\alpha^2}{xyQ^2}\left[Y_+F_2(x,z,Q^2)-y^2F_L(x,z,Q^2)\right]\,, \tag{1.34}$$

where we have defined:

$$Y_+=1+(1-y)^2\,. \tag{1.35}$$

It is also useful to write Eq. (1.34) as differential in x, Q^2, and z:

$$\frac{d^3\sigma}{dx\,dQ^2\,dz}=\frac{2\,\pi\alpha^2}{xQ^4}\left[Y_+F_2(x,z,Q)-y^2F_L(x,z,Q)\right]\,. \tag{1.36}$$

The structure functions F_2 and F_L are given at NLO by:

$$\begin{aligned}
F_{2,L}(x,z,Q) &= x\sum_{q,\overline{q}}e_q^2\Big[q(x,Q)\otimes C_{qq}^{2,L}(x,z)\otimes D_q(z,Q)\\
&+ q(x,Q)\otimes C_{gq}^{2,L}(x,z)\otimes D_g(z,Q)+g(x,Q)\otimes C_{qg}^{2,L}(x,z)\otimes D_q(z,Q)\Big]\,,
\end{aligned} \tag{1.37}$$

where $\{q,g\}$ are the quark and gluon PDFs and $\{D_q,D_g\}$ are the quark and gluon FFs, e_q is the electric charge of the quark q and $\{C_{qq}^{2,L},C_{qg}^{2,L},C_{gq}^{2,L}\}$ are the relevant partonic cross sections. The partonic cross sections allow for a perturbative expansion in power of α_s:

$$C=\sum_{n=0}\left(\frac{\alpha_s}{4\pi}\right)^n C^{(n)} \tag{1.38}$$

that we truncate to NLO, *i.e.* to $n = 1$. At LO ($n = 0$) we have the simple expression:

$$C_{qq}^{2,(0)}(x,z) = \delta(1-x)\delta(1-z) \quad \text{and all others zero.} \tag{1.39}$$

At NLO ($n = 1$) we take the expressions from Appendix C of Ref. [1] being careful to take into account an additional factor two due to the difference in the expansion parameter ($\alpha_s/4\pi$ rather than $\alpha_s/2\pi$) and to combine the expressions for F_1 and F_L using Eq. (1.33) to obtain the partonic cross sections for F_2. We start with the partonic cross sections for F_L that read:

$$\begin{array}{lcl} C_{qq}^{L,(1)} &=& 8C_F xz\,, \\ C_{gq}^{L,(1)} &=& 8C_F x(1-z)\,, \\ C_{qg}^{L,(1)} &=& 8x(1-x)\,, \end{array} \tag{1.40}$$

while those for F_2 read:

$$\begin{aligned} \frac{C_{qq}^{2,(1)}}{2C_F} &= -8\delta(1-x)\delta(1-z) + 2\delta(1-x)\left(\frac{\ln(1-z)}{1-z}\right)_+ + \delta(1-x)\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\ &+ 2\left(\frac{\ln(1-x)}{1-x}\right)_+ \delta(1-z) + 2\left(\frac{1}{1-x}\right)_+\left(\frac{1}{1-z}\right)_+ - \left(\frac{1}{1-x}\right)_+(1+z) \\ &+ \left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\delta(1-z) - (1+x)\left(\frac{1}{1-z}\right)_+ + (2+6xz) \\ \frac{C_{gq}^{2,(1)}}{2C_F} &= \delta(1-x)\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right] \\ &+ \left(\frac{1}{1-x}\right)_+ \frac{1+(1-z)^2}{z} \\ &+ 2(1+3x) - 6xz - (1+x)\frac{1}{z} \\ C_{qg}^{2,(1)} &= \left[(x^2+(1-x)^2)\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\right]\delta(1-z) + (x^2+(1-x)^2)\left(\frac{1}{1-z}\right)_+ \\ &+ 2(-1+6x-6x^2) + (x^2+(1-x)^2)\frac{1}{z} \end{aligned} \tag{1.41}$$

By inspection of Eqs. (1.40) and (1.41) we can deduce the various coefficients of Eq. (1.21). In particular, as usual, F_L involves only regular functions so that all contributions are zero but the fully regular ones:

$$\begin{array}{lcl} C_{qq}^{L,(1)}(x,z) &:& K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = z\,, \\ C_{gq}^{L,(1)}(x,z) &:& K_1 = 8C_F\,, \quad R_1^{(1)}(x) = x\,, \quad R_1^{(2)}(z) = 1-z\,, \\ C_{qg}^{L,(1)}(x,z) &:& K_1 = 8\,, \quad R_1^{(1)}(x) = x(1-x)\,, \quad R_1^{(2)}(z) = 1\,. \end{array} \tag{1.42}$$

For F_2 the situation is more complicated but we can still identify the different contributions:

$$
\begin{aligned}
C_{qq}^{2,(1)} \;:\;& \mathrm{LL} = -16C_F\,, \quad \mathrm{LS} = 4C_F\,, \quad \mathrm{LR}(z) = 2C_F\left[\frac{1+z^2}{1-z}\ln z + (1-z) - (1+z)\ln(1-z)\right] \\
& \mathrm{SL} = 4C_F\,, \quad \mathrm{SS} = 4C_F\,, \quad \mathrm{SR}(z) = -2C_F(1+z) \\
& \mathrm{RL}(x) = 2C_F\left[-\frac{1+x^2}{1-x}\ln x + (1-x) - (1+x)\ln(1-x)\right]\,, \quad \mathrm{RS}(x) = -2C_F(1+x)\,, \\
& \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = 12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \\
C_{gq}^{2,(1)} \;:\;& \mathrm{LR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\ln\left[z(1-z)\right] + z\right]\,, \\
& \mathrm{SR}(z) = 2C_F\left[\frac{1+(1-z)^2}{z}\right]\,, \\
& \left\{K_1 = 4C_F,\, R_1^{(1)}(x) = 1+3x,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = -12C_F,\, R_2^{(1)}(x) = x,\, R_2^{(2)}(z) = z\right\}\,, \\
& \left\{K_3 = -2C_F,\, R_3^{(1)}(x) = 1+x,\, R_3^{(2)}(z) = \frac{1}{z}\right\}\,, \\
C_{qg}^{2,(1)} \;:\;& \mathrm{RL}(x) = \left[x^2 + (1-x)^2\right]\ln\left(\frac{1-x}{x}\right) + 2x(1-x)\,, \quad \mathrm{RS}(x) = x^2 + (1-x)^2\,, \\
& \left\{K_1 = 2,\, R_1^{(1)}(x) = -1 + 6x - 6x^2,\, R_1^{(2)}(z) = 1\right\}\,, \\
& \left\{K_2 = 1,\, R_2^{(1)}(x) = x^2 + (1-x)^2,\, R_2^{(2)}(z) = \frac{1}{z}\right\}\,.
\end{aligned}
\tag{1.43}
$$

Analogously, for the only LO partonic cross sections we find that:

$$
C_{qq}^{2,(0)} \;:\; \mathrm{LL} = 1\,. \tag{1.44}
$$

All the coefficients that are not mentioned are equal to zero. We can now implement explicitly Eq. (1.22). The one thing that is left to sort out is the structure of the structure functions in terms of the appropriate PDF and FF combinations. Considering Eq. (1.37), we observe that none of the coefficient functions depends on the particular quark flavour (this is a feature of the ZM scheme). Therefore, simplifying the notation, we can rewrite Eq. (1.37) as:

$$
F = C_{qq}\sum_q e_q^2\left[qD_q + \overline{q}D_{\overline{q}}\right] + C_{gq}\sum_q e_q^2\left[q + \overline{q}\right]\, D_g + C_{qg}g\sum_q e_q^2\left[D_q + D_{\overline{q}}\right]\,, \tag{1.45}
$$

where now the sums run only over the quark flavours and not the antiflavours.

1.3 Drell Yan (DY)

In this section we apply to the Drell-Yan (DY) process the same procedure followed above for SIDIS. As a matter of fact, SIDIS and DY are strictly connected in that DY can be regarded as the time-like counterpart of SIDIS. As a consequence, the structure of the relevant observables as well as the form of the expressions involved are very similar. Therefore, the application of the method described above is straightforward.

References

[1] D. de Florian, M. Stratmann and W. Vogelsang, Phys. Rev. D **57** (1998) 5811 doi:10.1103/PhysRevD.57.5811 [hep-ph/9711387].