1 Structure of the integrals

The general structure of the quantities to be computed in APFEL is the combination of terms I that have the form of Mellin convolutions between an operator O and distribution function d, that is:

$$I(x) = x\int_0^1 dz \int_0^1 dy\, O(z) d(y) \delta(x - yz) = x\int_x^1 \frac{dz}{z} O\left(\frac{x}{z}\right) d(z) = x\int_x^1 \frac{dz}{z} O(z) d\left(\frac{x}{z}\right) . \tag{1.1}$$

However, very often, typically in the presence of mass effects, the integration phase space is modified and the convolution in eq. (1.1), limiting ourselves to the leftmost identity, is generalised as:

$$I(x,\eta) = x\int_{x/\eta}^1 \frac{dz}{z} O(z,\eta) d\left(\frac{x}{\eta z}\right) . \tag{1.2}$$

where $\eta \leq 1$, with $\eta = 1$ reproducing eq. (1.1). However, for purposes that will become clear later, we want to write the integral in eq. (1.2) in the form of eq. (1.1), that is in such a way that lower bound of the integral is not the rescaled variable x/η but the physical Bjorken x. To do so, one needs to perform the change of variable $y = \eta z$, so that:

$$I(x,\eta) = \int_x^\eta dy\, O\left(\frac{y}{\eta},\eta\right) \frac{x}{y} d\left(\frac{x}{y}\right) . \tag{1.3}$$

In order to precompute the expensive part of the integral in eq. (1.5), we use the standard interpolation formula to the distribution d:

$$\frac{x}{y} d\left(\frac{x}{y}\right) = \sum_{\alpha=0}^{N_x} x_\alpha d(x_\alpha) w_\alpha^{(k)}\left(\frac{x}{y}\right) , \tag{1.4}$$

where α runs over the node of a give grid in x space and the weights w_α are typically polynomials of degree k (*i.e.* Laplace interpolants). Now we use eq. (1.4) in eq. (1.5) and at the same time we limit the computation of the integral I to the point x_β of the grid used in eq. (1.4). This way we get:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_{x_\beta}^\eta dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.5}$$

where we have defined $\overline{d}_\alpha = x_\alpha d(x_\alpha)$. In the particular case of the Laplace interpolants (in the APFEL procedure), one can show that:

$$w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) \neq 0 \quad \text{for} \quad c < y < d\,, \tag{1.6}$$

with:

$$c = \max(x_\beta, x_\beta/x_{\alpha+1}) \quad \text{and} \quad d = \min(\eta, x_\beta/x_{\alpha-k})\,, \tag{1.7}$$

and thus eq. (1.5) can be adjusted as:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \int_c^d dy\, O\left(\frac{y}{\eta},\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{y}\right) . \tag{1.8}$$

Finally, we change back the integration variable $z = y/\eta$ so that eq. (1.8) becomes:

$$I(x_\beta,\eta) = \sum_{\alpha=0}^{N_x} \overline{d}_\alpha \underbrace{\left[\eta \int_{c/\eta}^{d/\eta} dz\, O\left(z,\eta\right) w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)\right]}_{\Gamma_{\alpha\beta}} . \tag{1.9}$$

The quantity in squared brackets is the interesting bit.

The expressions for the operator O that we have to deal with have this general form:

$$O(z,\eta) = R(z,\eta) + \sum_{i=0}^{n} \left[\frac{\ln^i(1-z)}{1-z}\right]_+ S^{(i)}(z,\eta) + L(\eta)\delta(1-z)\,, \tag{1.10}$$

where R and $S^{(i)}$ is a regular function in $z = 1$, that is:

$$R(1,\eta) = \lim_{z\to 1} R(z,\eta) = K(\eta) \quad \text{and} \quad S^{(i)}(1,\eta) = \lim_{z\to 1} S^{(i)}(z,\eta) = J^{(i)}(\eta)\,, \tag{1.11}$$

being K, $J^{(i)}$, and L a finite function of η. Notice that the sum over i in eq. (1.10) is between zero and n where n is typically not bigger than two as it depends on the perturbative order of the expressions. Plugging eq. (1.10) into the definition of $\Gamma_{\alpha\beta}$ in eq. (1.9) and making use of the definition of plus-prescription, we obtain:

$$\begin{aligned}\Gamma_{\beta\alpha} &= \eta\int_{c/\eta}^{d/\eta} dz\left\{R\left(z,\eta\right)w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)+\sum_{i=0}^{n}\frac{\ln^i(1-z)}{1-z}\left[S^{(i)}\left(z,\eta\right)w_\alpha^{(k)}\left(\frac{x_\beta}{\eta z}\right)-S^{(i)}(1,\eta)w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right)\right]\right\}\\ &+ \eta\left[\sum_{i=0}^{n}\frac{1}{(i+1)!}S^{(i)}(1,\eta)\ln^{i+1}\left(1-\frac{c}{\eta}\right)+L(\eta)\right]w_\alpha^{(k)}\left(\frac{x_\beta}{\eta}\right).\end{aligned} \tag{1.12}$$

Eq. (1.14) expresses the full complexity of the task. However, there are a few remarks that apply is some particular cases and that reduce the complexity. In the case of the ($\overline{\text{MS}}$) splitting functions, there are two main simplications: the first is that $\eta = 1$, the second is that $n = 0$ in the sums. Considering that:

$$w_\alpha^{(k)}(x_\beta) = \delta_{\alpha\beta}\,, \tag{1.13}$$

and that the expressions can be manipulated in such a way that the coefficient of the plus-prescripted term S is a constant, we have that:

$$\Gamma_{\beta\alpha} = \int_c^d dz\left\{R\left(z\right)w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right)+\frac{S^{(0)}}{1-z}\left[w_\alpha^{(k)}\left(\frac{x_\beta}{z}\right)-\delta_{\alpha\beta}\right]\right\}+\left[S^{(0)}\ln\left(1-c\right)+L\right]\delta_{\alpha\beta}\,. \tag{1.14}$$