Contents

In this set of notes I collect the technical aspects concerning generalised parton distributions (GPDs). Since the computation GPDs introduces new kinds of convolution integrals, a strategy aimed at optimising the numerics needs to be devised.

1 Evolution equations

In general, the evolution equation for GPDs reads:

$$\mu^2\frac{d}{d\mu^2}f(x,\xi)=\int_{-\infty}^{+\infty}\frac{dx'}{2\xi}V\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)\,. \tag{1.1}$$

The GPD f and the evolution kernel V may in general be a vector and a matrix in flavour space. For now we will just be concerned with the integral in the r.h.s. of Eq. (1.2) regardless of the flavour structure. The support of the evolution kernel $V\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$ is shown in Fig. 1.1. Without loss of generality, we can assume that

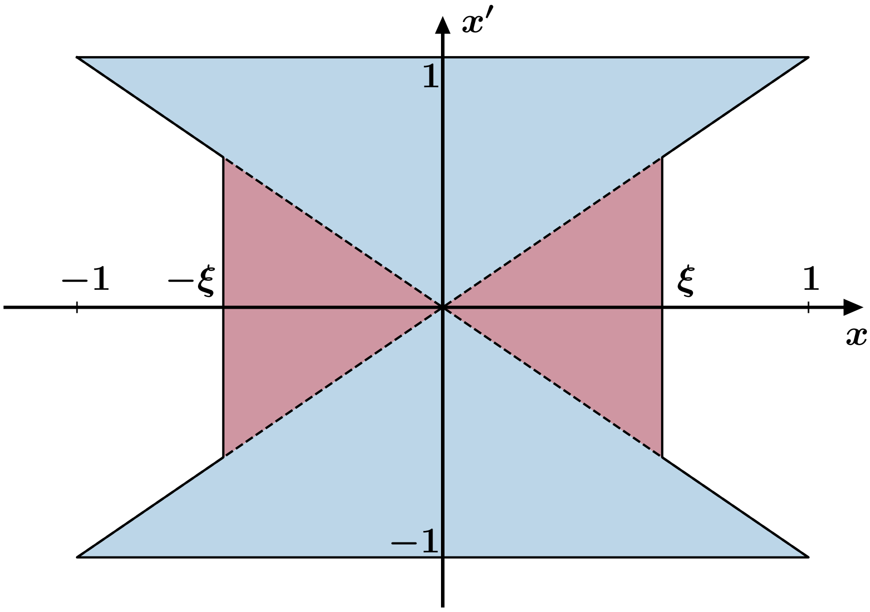

Fig. 1.1: Support domain of the evolution kernel $V\left(\frac{x}{\xi},\frac{x'}{\xi}\right)$.

$x>0$. Knowing the support of the evolution kernel, Eq. (1.2) can be split as follows:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}f(\pm x,\xi) &= \int_x^1\frac{dx'}{x'}\left[\frac{x'}{2\xi}V\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{x'}{2\xi}V\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\\ &+ \theta\left(1-\frac{x}{\xi}\right)\int_0^x dx'\left[\frac{1}{2\xi}V\left(\pm\frac{x}{\xi},\frac{x'}{\xi}\right)f(x',\xi)+\frac{1}{2\xi}V\left(\mp\frac{x}{\xi},\frac{x'}{\xi}\right)f(-x',\xi)\right]\,.\end{aligned} \tag{1.2}$$

where we have used the symmetry property $V(y,y')=V(-y,-y')$. In the unpolarised case, it is useful to define:[1]

$$\begin{aligned}f^{\pm}(x,\xi) &= f(x,\xi)\mp f(-x,\xi)\,,\\ V^{\pm}(y,y') &= V(y,y')\mp V(-y,y')\,,\end{aligned} \tag{1.3}$$

[1] Notice the seemingly unusual fact that f^+ is defined as difference and f^- as sum of GPDs computed at opposite values of x. This can be understood from the fact that, in the forward limit, $f(-x)=-\overline{f}(x)$, *i.e.* the PDF of a quark computed at $-x$ equals the PDF of the corresponding antiquark computed at x with opposite sign.

so that the evolution equation for $f^{\pm}$ can be split as:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}f^{\pm}(x,\xi) &= \int_x^1\frac{dx'}{x'}\frac{x'}{2\xi}V^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)+\theta\left(1-\frac{x}{\xi}\right)\int_0^x dx'\frac{1}{2\xi}V^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)\\ &= I^{\pm,\mathrm{DGLAP}}(\xi,x)+I^{\pm,\mathrm{ERBL}}(\xi,x)\,.\end{aligned} \tag{1.4}$$

The first term in the second line of the equation above, $I^{\pm,\mathrm{DGLAP}}$, corresponds to integrating over the blue regions in Fig. 1.1, while the second term, $I^{\pm,\mathrm{ERBL}}$, results from the integration over the red regions. As indicated by the subscripts, $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ define the so-called DGLAP and ERBL regions in x relative ξ. Specifically, the presence of the θ-function in $I^{\pm,\mathrm{ERBL}}$ is such that for $x>\xi$ this term drops leaving only the DGLAP-like term $I^{\pm,\mathrm{DGLAP}}$. For $x\leq\xi$, instead, $I^{\pm,\mathrm{ERBL}}$ kicks in and the evolution equation assumes the form of the so-called ERBL equation that describes the evolution of meson distribution amplitudes (DAs). Crucially, in the limits $\xi\to 0$ and $\xi\to 1$ one should and does recover the DGLAP and ERBL equations, respectively.

For convenience, we define the parameter:

$$\kappa=\frac{\xi}{x}\,, \tag{1.5}$$

so that:

$$\frac{x'}{2\xi}V^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)=\frac{1}{2\kappa}\frac{x'}{x}V^{\pm}\left(\frac{1}{\kappa},\frac{1}{\kappa}\frac{x'}{x}\right)\equiv\mathcal{V}^{\pm}\left(\kappa,\frac{x}{x'}\right)\,, \tag{1.6}$$

where the last equality effectively defines the function $\mathcal{V}^{\pm}(\kappa,y)$. Plugging this definition into the first integral in the r.h.s. of Eq. (1.4) gives:

$$I^{\pm,\mathrm{DGLAP}}(\xi,x)=\int_x^1\frac{dx'}{x'}\frac{x'}{2\xi}V^{\pm}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)f^{\pm}(x',\xi)=\int_x^1\frac{dx'}{x'}\mathcal{V}^{\pm}\left(\kappa,\frac{x}{x'}\right)f^{\pm}(x',\xi)\equiv\mathcal{V}^{\pm}\left(\kappa,x\right)\otimes f^{\pm}(x,\xi)\,. \tag{1.7}$$

Therefore, $I^{\pm,\mathrm{DGLAP}}$ has the form of a "standard" Mellin convolution that, up to minor modifications due to the fact that κ depends on x, is easily handled by APFEL. Assuming a grid in x indexed by α or β, we have:

$$x_\beta I^{\pm,\mathrm{DGLAP}}(\xi,x_\beta)=\sum_\alpha\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{DGLAP}}(\xi)f_\alpha^{\pm}(\xi)\,, \tag{1.8}$$

with:

$$f_\alpha^{\pm}(\xi)=x_\alpha f^{\pm}(x_\alpha,\xi)\,, \tag{1.9}$$

and:

$$\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{DGLAP}}(\xi)=\int_c^d dx'\,\mathcal{V}^{\pm}\left(\kappa_\beta,x'\right)w_\alpha^{(k)}\left(\frac{x_\beta}{x'}\right)\,, \tag{1.10}$$

where $\kappa_\beta=\xi/x_\beta$ and $\{w_\alpha^{(k)}\}$ is a set of Lagrange interpolating functions of degree k and the integration bounds are:

$$c=\max(x_\beta,x_\beta/x_{\alpha+1})\quad\text{and}\quad c=\min(1,x_\beta/x_{\alpha-k})\,. \tag{1.11}$$

Now we need to treat $I^{\pm,\mathrm{ERBL}}$ in Eq. (1.4). The structure of this term is rather unusual for APFEL because the pre-computation of convolution integrals is usually done on logarithmically-spaced grids in x and integrating down to zero might be problematic. However, contrary to forward distributions, GPDs are generally well-behaved at $x=0$ and thus it is not strictly necessary to reach this point in the integral.[2] Upon this assumption, we find: to compute:

$$x_\beta I^{\pm,\mathrm{ERBL}}(\xi,x_\beta)=\sum_\alpha\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi)f_\alpha^{\pm}(\xi)\,, \tag{1.12}$$

with:

$$\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi)=\theta\left(1-\frac{1}{\kappa_\beta}\right)\int_\epsilon^{x_\beta}dx'\,\frac{x_\beta}{x'^2}\mathcal{V}^{\pm}\left(\kappa_\beta,\frac{x_\beta}{x'}\right)w_\alpha^{(k)}(x')\,, \tag{1.13}$$

where ϵ is a small number. Since the interpolating functions $w_\alpha^{(k)}$ are such that:

$$w_\alpha^{(k)}(x)\neq 0\quad\text{for}\quad x_{\alpha-k}<x<x_{\alpha+1}\,, \tag{1.14}$$

[2] We will verify this conjecture numerically.

the integral above can be computed more efficiently as:

$$\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi) = \theta\left(1-\frac{1}{\kappa_\beta}\right)\int_a^b dx'\,\frac{x_\beta}{x'^2}\mathcal{V}^{\pm}\left(\kappa_\beta, \frac{x_\beta}{x'}\right) w_\alpha^{(k)}(x')\,, \tag{1.15}$$

with:

$$a = \max(\epsilon, x_{\alpha-k}) \quad \text{and} \quad b = \min(x_\beta, x_{\alpha+1})\,. \tag{1.16}$$

Finally, summing $I^{\pm,\mathrm{DGLAP}}$ and $I^{\pm,\mathrm{ERBL}}$ and multiplying by a factor x_β, the evolution equation in Eq. (1.2) can be approximated on an grid in x as:

$$\mu^2\frac{d}{d\mu^2}f_\beta^{\pm}(\xi) = \sum_\alpha\left[\mathcal{V}_{\beta\alpha}^{\pm,\mathrm{DGLAP}}(\xi) + \mathcal{V}_{\beta\alpha}^{\pm,\mathrm{ERBL}}(\xi)\right]f_\alpha^{\pm}(\xi) \tag{1.17}$$

This is a system of coupled differential equation that can be solved numerically using the fourth-order Runge-Kutta algorithm as implemented in APFEL.

References

[1] M. Diehl, Phys. Rept. **388** (2003) 41 doi:10.1016/j.physrep.2003.08.002, 10.3204/DESY-THESIS-2003-018 [hep-ph/0307382].