Transverse momentum resummation and matching to fixed order

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Part I. TMD evolution and its perturbative expansion

1 Evolution of the TMDs

In this section I will show how to solve the renormalisation-group equation (RGE) and the rapidity-evolution equation (often referred to as Collins-Soper (CS) equation) of a TMD distribution F. The distribution F can be either a PDF or a FF and can be associated to either to a quark or to the gluon: the structure of the solution of the evolution equations is exactly the same. In fact, the solution equation only depends on whether we are evolving a quark or a gluon while it does not distinguish between PDFs and FFs. In the impact-parameter space, F is a function of the transverse-momentum fraction x, of the bidimensional impact-parameter vector $\mathbf{b}_T$, of the renormalisation scale μ, and of the rapidity scale ζ, *i.e.* $F \equiv F(x, \mathbf{b}_T, \mu, \zeta)$. Since the evolution equations govern the behaviour of F w.r.t. the scale μ and ζ, in order to simplify the notation in this section I will drop the dependence on x and $\mathbf{b}_T$, *i.e.* $F \equiv F(\mu, \zeta)$.[1]

The solution of the evolution equation allows one to express the distribution F at some final scales (μ, ζ) in terms of the same distribution at the initial scales (μ_0, ζ_0). It will turn out that this is accomplished by computing the evolution kernel $R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right]$, such that:

$$F(\mu, \zeta) = R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{1.1}$$

The evolution kernel R can be expressed in terms of perturbatively computable quantities. A collateral aspect that will be discussed in below is the independence from the path that connects the initial and final scales (μ_0, ζ_0) and (μ, ζ). This in turn concerns the resummation of large logarithms that is required to ensure that the perturbative convergence is not spoiled.

The RGE and the CS equation read:

$$\begin{aligned} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} &= K(\mu)\,, \\ \frac{\partial \ln F}{\partial \ln \mu} &= \gamma(\mu, \zeta)\,, \end{aligned} \tag{1.2}$$

where γ and K are the anomalous dimensions of the evolution in μ and $\sqrt{\zeta}$, respectively. The equations above can be solved as follows. The first equation gives:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\right] F(\mu, \zeta_0)\,. \tag{1.3}$$

The factor $F(\mu, \zeta_0)$ can be evolved in μ using the second equation:

$$F(\mu, \zeta_0) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,, \tag{1.4}$$

such that:

$$F(\mu, \zeta) = \exp\left[K(\mu) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_0)\right] F(\mu_0, \zeta_0)\,. \tag{1.5}$$

This equation has exactly the structure of Eq. (1.1). We now need to express the argument of the exponent in terms of perturbatively computable quantities.

In order to do so, we use the fact that the rapidity anomalous dimension K needs to be renormalised and thus it obeys its own RGE, that reads:

$$\frac{\partial K}{\partial \ln \mu} = -\gamma_K(\alpha_s(\mu))\,. \tag{1.6}$$

γ_K is said cusp anomalous dimension and obeys the perturbative expansion:

$$\gamma_K(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} \gamma_K^{(n)}\,, \tag{1.7}$$

where $\gamma_K^{(n)}$ are numerical coefficients. The values up to $n = 3$ can be read from Eq. (59) of Ref. [3]. They coincide with those reported in Eq. (D.6) of Ref. [2] up to a factor two due to a different normalisation of the

[1] Notice that, despite the variables x and $\mathbf{b}_T$ will not appear explicitly, the symbol $\otimes$ indicates the Mellin convolution integral w.r.t. x while b_T indicates the length of the vector $\mathbf{b}_T$.

rapidity anomalous dimension K whose derivative w.r.t. ζ is exactly γ_K. In addition, the cusp anomalous dimension for quarks and gluon are equal up to a factor C_F in the quark case and C_A in the gluon case.

Eq. (1.6) can be easily solved obtaining:

$$K(\mu) = K(\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu'))\,. \tag{1.8}$$

We anticipate that in the $\overline{\text{MS}}$ scheme, there exists a particular scale $\mu_b = 2e^{-\gamma_E}/b_T$ such that K admits the following perturbative expansion:

$$K(\mu_b) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^{n+1} K^{(n,0)}\,, \tag{1.9}$$

where $K^{(n,0)}$ are numerical coefficients. Therefore, if $\mu_0 \simeq \mu_b$ the first term in the r.h.s. of Eq. (1.8) is free of large logs and thus its perturbative expansion, that reads:

$$K(\mu_0) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{n+1} \sum_{m=0}^{n} K^{(n,m)} \ln^m\left(\frac{\mu_0}{\mu_b}\right)\,, \tag{1.10}$$

is reliable. The second term in the r.h.s. instead takes care, through the evolution of α_s, of resumming large logarithms in the case in which $\mu \gg \mu_0$. The coefficients $K^{(n,m)}$ up to $n = 3$ are reported in Eq. (D.9) of Ref. [2] and up to $n = 2$ in Eq. (69) of Ref. [3]. They differ by a factor -2 due to a different definition of K. In addition, the logarithmic expansion is done in terms of $\ln(\mu_0/\mu_b)$ in Ref. [3] and in terms of $\ln(\mu_0^2/\mu_b^2)$ in Ref. [2]. Therefore, each coefficient differs by an additional factor 2^m, where m is the power of the logarithm that multiplies the coefficient itself.

A further important property of the anomalous dimensions can be derived by considering the fact that the crossed double derivarives of F must be equal:

$$\frac{\partial}{\partial \ln \mu} \frac{\partial \ln F}{\partial \ln \sqrt{\zeta}} = \frac{\partial}{\partial \ln \sqrt{\zeta}} \frac{\partial \ln F}{\partial \ln \mu}\,. \tag{1.11}$$

Using Eqs. (1.2) and (1.6) leads to the following differential equation:

$$\frac{\partial \gamma}{\partial \ln \sqrt{\zeta}} = -\gamma_K(\alpha_s(\mu))\,, \tag{1.12}$$

whose solution is:

$$\gamma(\mu, \zeta) = \gamma(\mu, \mu^2) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu}\,. \tag{1.13}$$

It turns out that $\gamma(\mu, \mu^2)$ has a purely perturbative expansion:

$$\gamma(\mu, \mu^2) \equiv \gamma_F(\alpha_s(\mu)) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1} \gamma_F^{(n)}\,, \tag{1.14}$$

where $\gamma_F^{(n)}$ are again numerical coefficients that are given in Eq. (58) of Ref. [3] and Eq. (D.7) of Ref. [2] (Eq. (D.8) reports the coefficients for the gluon anomalous dimension). The two sets of coefficients differ by a minus sign due to the different definition of the constant (non-log) term of the RGE anomalous dimension. Therefore:

$$\gamma(\mu, \zeta) = \gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu)) \ln \frac{\sqrt{\zeta}}{\mu}\,. \tag{1.15}$$

Finally, plugging Eqs. (1.8) and (1.15) into Eq. (1.5), one gets:

$$F(\mu, \zeta) = \exp\left\{K(\mu_0) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'}\right]\right\} F(\mu_0, \zeta_0)\,. \tag{1.16}$$

Comparing Eq. (1.16) to Eq. (1.1) allows one to give an explicit expression to the evolution kernel:

$$R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0)\right] = \exp\left\{K(\mu_0) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu')) \ln \frac{\sqrt{\zeta}}{\mu'}\right]\right\}\,. \tag{1.17}$$

Eq. (1.16) has been obtained evolving the TMD F first in the ζ direction (Eq. (1.3)) and then in the μ direction (Eq. (1.4)). However, it is easy to verify that exchanging the order of the evolutions leads to the exact same result, Eq. (1.16). In particular, the following relation holds:

$$R\left[(\mu,\zeta) \leftarrow (\mu_0,\zeta)\right] R\left[(\mu_0,\zeta) \leftarrow (\mu_0,\zeta_0)\right] = R\left[(\mu,\zeta) \leftarrow (\mu,\zeta_0)\right] R\left[(\mu,\zeta_0) \leftarrow (\mu_0,\zeta_0)\right] = R\left[(\mu,\zeta) \leftarrow (\mu_0,\zeta_0)\right] . \tag{1.18}$$

This is a direct consequence of the independence of evolution kernel R in Eq. (1.17) from the path $\mathcal{P}$ followed to connect the points (μ_0,ζ_0) to the point (μ,ζ):

$$R\left[(\mu,\zeta) \underset{\mathcal{P}}{\leftarrow} (\mu_0,\zeta_0)\right] \equiv R\left[(\mu,\zeta) \leftarrow (\mu_0,\zeta_0)\right] . \tag{1.19}$$

Another important piece of information comes from the fact that, for small values of b_T, the TMD F can be matched onto the respective collinear distribution f (a PDF or a FF) through the perturbative coefficients C[2]:

$$F(\mu,\zeta) = C(\mu,\zeta) \otimes f(\mu) , \tag{1.20}$$

so that:

$$F(\mu,\zeta) = \exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\} C(\mu_0,\zeta_0) \otimes f(\mu_0) . \tag{1.21}$$

Exactly as in the case of K, for $\mu_0 = \sqrt{\zeta_0} = \mu_b$ the matching function has the expansion:

$$C(\mu_b,\mu_b^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n C^{(n,0)} , \tag{1.22}$$

where the coefficients $C^{(n,0)}$ are functions of x only. In order to be able to compute the function C for generic values of the scales μ and ζ, evolution equations can be derived. Deriving Eq. (1.20) with respect to μ and ζ one gets:

$$\begin{aligned}
\frac{\partial F}{\partial \ln\sqrt{\zeta}} &= \frac{\partial C}{\partial \ln\sqrt{\zeta}} \otimes f(\mu) , \\
\frac{\partial F}{\partial \ln\mu} &= \frac{\partial C}{\partial \ln\mu} \otimes f(\mu) + C(\mu,\zeta) \otimes \frac{\partial f}{\partial \ln\mu} = \left[\frac{\partial C}{\partial \ln\mu} + C(\mu,\zeta)\otimes 2P(\mu)\right] \otimes f(\mu) .
\end{aligned} \tag{1.23}$$

In the r.h.s. of the second equation I have used the DGLAP equation:

$$\frac{\partial f}{\partial \ln\mu} = 2P(\mu) \otimes f(\mu) . \tag{1.24}$$

One can also take the derivative of Eq. (1.16) and the result is:

$$\begin{aligned}
\frac{\partial F}{\partial \ln\sqrt{\zeta}} &= \left[K(\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right] F(\mu,\zeta) = \left[K(\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right] C(\mu,\zeta) \otimes f(\mu) , \\
\frac{\partial F}{\partial \ln\mu} &= \left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right] F(\mu,\zeta) = \left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right] C(\mu,\zeta) \otimes f(\mu) .
\end{aligned} \tag{1.25}$$

Equating Eq. (1.23) and Eq. (1.25) and dropping the distribution f, the evolution equations for C are:

$$\begin{aligned}
\frac{\partial C}{\partial \ln\sqrt{\zeta}} &= \left[K(\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\alpha_s(\mu'))\right] C(\mu,\zeta) , \\
\frac{\partial C}{\partial \ln\mu} &= \left\{\left[\gamma_F(\alpha_s(\mu)) - \gamma_K(\alpha_s(\mu))\ln\frac{\sqrt{\zeta}}{\mu}\right]\Delta(1-x) - 2P(\mu)\right\} \otimes C(\mu,\zeta) ,
\end{aligned} \tag{1.26}$$

where the Δ is a matrix in flavour space whose components are defined as:

$$\Delta_{ij}(1-x) = \delta_{ij}\delta(1-x) , \tag{1.27}$$

[2] A sum over flavours is understood. As a matter of fact, the matching function C has to be regarded as a matrix in flavour space multipling a vector of collinear PDFs/FFs.

being i and j flavour indices. The equations above can be solved to determine the evolution of the matching function C. The solution can eventually be expanded if initial and final scales are not too far apart. In particular, if $\mu_0 = \sqrt{\zeta_0} \simeq \mu_b$ in Eq. (1.16), the matching function C can be reliably expanded as:

$$C(\mu_0, \mu_0^2) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^n \sum_{m=0}^{2n} C^{(n,m)} \ln^m \left(\frac{\mu_0}{\mu_b} \right) . \tag{1.28}$$

The coefficient functions $C^{(n,m)}$ have been computed for both PDFs and FFs in SCET in Ref. [2]. The same functions have also been computed in Ref. [1] and reported in Ref. [3]. The authors of the latter paper have verified the equality of the two sets of functions.

In order to use Eq. (1.16) in phenomenological applications, one needs to define the values of both the initial and final pairs of scales (μ_0, ζ_0) and (μ, ζ). The initial scales are usually identified with μ_b up to a small factor C_i, *i.e.* $(\mu_0, \zeta_0) = (C_i \mu_b, C_i^2 \mu_b^2)$, with $\mu_b = 2e^{-\gamma_E}/b_T$. This is advantageous because possible logarithms that appear in the perturbative expansion of $K(\mu_0)$ and $C(\mu_0, \zeta_0)$ in Eq. (1.21) have the form $\ln(\mu_0/\mu_b) = \ln(\sqrt{\zeta_0}/\mu_b) = \ln C_i$ and thus are small enough not to spoil their convergence. Variations of C_i around unity can be possibly used to estimate the impact of higher-order corrections in the TMD evolution and matching.

The natural choice for the final scales is to identify them with the hard scale of the process, say Q. This choice has to match the renormalisation scale μ used in the hard factor H of the process under consideration. Therefore, choosing $(\mu, \zeta) = (C_f Q, Q^2)$, with C_f being a modest factor, can be used to estimate higher-order corrections.

2 Non-perturbative component

In the previous section, I have considered the evolution of TMDs and thus I concentrated on their dependence on the renormalisation and rapidity scales μ and ζ, leaving aside the dependence on b_T. Nonetheless, the computation of the rapidity evolution kernel in the $\overline{\text{MS}}$ scheme has led to the introduction of the scale:

$$\mu_b = \frac{2e^{-\gamma_E}}{b_T} , \tag{2.1}$$

This scale, within a modest factor, provides a natural choice for the evolution initial scales μ_0 and $\sqrt{\zeta_0}$ that prevents the appearance of large logarithms. Crucially, the strong coupling α_s has to be computed in the vicinity of the scale μ_b. Therefore, if the impact parameters b_T becomes large, $\alpha_s(\mu_b)$ may potentially become very large invalidating any perturbative calculation. Since the computation of q_T dependent observables requires Fourier transforming TMDs, they need to be accessed also at large values of b_T where the perturbative computation of the previous section is not valid. To overcome this limitation, it is customary to introduce an *arbitrary* scale $b_{\max}$ that denotes the maximum value of b_T at which one trusts perturbation theory. The value has to be such that:

$$\alpha_s \left(\frac{2e^{-\gamma_E}}{b_{\max}} \right) \ll 1 . \tag{2.2}$$

The one introduces a monotonic function b_* of b_T with the following behaviour:

$$\begin{array}{lll} b_*(b_T) \simeq b_T & \text{for} & b_T \to 0 , \\ b_*(b_T) \to b_{\max} & \text{for} & b_T \to \infty . \end{array} \tag{2.3}$$

A common choice is:

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}} . \tag{2.4}$$

Now, one writes:

$$F(x, b_T, \mu, \zeta) = \left[\frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)} \right] F(x, b_*(b_T), \mu, \zeta) \equiv f_{\text{NP}}(x, b_T, \zeta) F(x, b_*(b_T), \mu, \zeta) . \tag{2.5}$$

This separation is advantageous because, due to the behaviour of $b_*(b_T)$, $F(x, b_*(b_T), \mu, \zeta)$ can be computed in perturbation theory, while $f_{\text{NP}}(x, b_T, \zeta)$ embodies the non-perturbative dependence. It is important to stress that this separation is arbitrary and dependes of the particular choice of b_* and $b_{\max}$. Therefore, for any particular choice, only the combination in Eq. (2.5) is meaningful and it is misleading to refer to f_{NP} as to the non-perturbative part of the TMDs in a universal sense. The reason why the function f_{NP} does not depend on the renormalisation scale μ is that this dependence cancels in the ratio in Eq. (2.5). To be more specific, if one chooses $\mu_0 = \mu_b = 2e^{-\gamma_E} b_T$ and uses Eq. (1.16), one finds:

$$
\begin{gathered}
f_{\rm NP}(x, b_T, \zeta) = \frac{F(x, b_T, \mu, \zeta)}{F(x, b_*(b_T), \mu, \zeta)} = \\
\exp\left\{K(\mu_b)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_b}} - K(\mu_{b_*})\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_{b_*}}} + \int_{\mu_b}^{\mu_{b_*}}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}\frac{F(\mu_b, \zeta_b)}{F(\mu_{b_*}, \zeta_{b_*})}\,,
\end{gathered}
\tag{2.6}
$$

with $\mu_{b_*} = \sqrt{\zeta_{b_*}} = 2e^{-\gamma_E}/b_*(b_T)$. It is thus apparent that the dependence on μ cancels.

3 Extraction of the singular behaviour for $q_T \to 0$

In this section we derive the expansion of the different ingredients of TMDs in order to finally extract the singular behaviour of a cross section involving TMDs for $q_T \to 0$.

3.1 Expansion of the evolution kernel

In this subsection we work out the perturbative expansion of the evolution kernel R up to $\mathcal{O}(\alpha_s^2)$. This expansion is useful to extract the (singular) asymptotic behaviour of a cross section computed at fixed order in pQCD as q_T tends to zero. Since TMDs always appear in pairs in the computation of a cross section and that the evolution kernel is universal, it is convenient to compute the expansion of R^2. To do so, we start from Eq. (1.17) where we set $\mu_0 = \sqrt{\zeta_0} = \mu_b = 2e^{-\gamma_E}/b_T$ and $\mu = \sqrt{\zeta} = Q$, possible variations can be reinstated at a later stage:

$$
R^2 = \exp\left\{K(\mu_b)\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left[\gamma_F(\alpha_s(\mu')) - \frac{1}{2}\gamma_K(\alpha_s(\mu'))\ln\frac{Q^2}{\mu'^2}\right]\right\}\,. \tag{3.1}
$$

Now we use the expansions in Eqs. (1.7), (1.9) and (1.14):

$$
R^2 = \exp\left\{\left[\sum_{n=0}^{\infty} a_s^{n+1}(\mu_b)K^{(n,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}a_s^{n+1}(\mu')\left(\gamma_F^{(n)} - \frac{1}{2}\gamma_K^{(n)}\ln\frac{Q^2}{\mu'^2}\right)\right]\right\}\,, \tag{3.2}
$$

where we have defined:

$$
a_s = \frac{\alpha_s}{4\pi}\,. \tag{3.3}
$$

In order to carry out the perturbative expansion we need to write the argument of the exponential in terms of a common value of α_s computed at some hard scale: the natural choice is $\alpha_s(Q)$. This can be achieved by using the RGE for a_s at leading order:

$$
\mu^2\frac{da_s}{d\mu^2} = -\beta_0 a_s^2(\mu)\,, \tag{3.4}
$$

whose solution is:

$$
a_s(\mu) = \frac{a_s(Q)}{1 + a_s(Q)\beta_0\ln(\mu^2/Q^2)} \simeq a_s(Q)\left[1 + a_s(Q)\beta_0\ln(Q^2/\mu^2) + \mathcal{O}(a_s^2)\right]\,. \tag{3.5}
$$

Plugging the expansion in the r.h.s. of Eq. (3.5) into Eq. (3.2) and retaining only terms up to $\mathcal{O}(a_s^2)$, one finds:

$$
\begin{aligned}
R^2 \;=\;& \exp\left\{a_s(Q)\left[K^{(0,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)} - \frac{1}{2}\gamma_K^{(0)}\ln\frac{Q^2}{\mu'^2}\right)\right]\right.\\
&+\; a_s^2(Q)\beta_0\left[K^{(0,0)}\ln^2\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(0)}\ln\frac{Q^2}{\mu'^2} - \frac{1}{2}\gamma_K^{(0)}\ln^2\frac{Q^2}{\mu'^2}\right)\right]\\
&+\; \left.a_s^2(Q)\left[K^{(1,0)}\ln\frac{Q^2}{\mu_b^2} + \int_{\mu_b^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}\left(\gamma_F^{(1)} - \frac{1}{2}\gamma_K^{(1)}\ln\frac{Q^2}{\mu'^2}\right)\right] + \mathcal{O}(a_s^3)\right\}.
\end{aligned}
\tag{3.6}
$$

The final step before carrying out the expansion is computing the integrals:

$$\int_{\mu_b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_{\ln(\mu_b^2)}^{\ln Q^2} d\ln\mu'^2 \ln^k\left(\frac{Q^2}{\mu'^2}\right) = \int_0^{\ln(Q^2/\mu_b^2)} dx\, x^k = \frac{1}{k+1}\ln^{k+1}\left(\frac{Q^2}{\mu_b^2}\right) . \tag{3.7}$$

so that:

$$\begin{aligned} R^2 \;\simeq\; & \exp\left\{a_s(Q)\left[\left(K^{(0,0)} + \gamma_F^{(0)}\right)L - \frac{1}{4}\gamma_K^{(0)}L^2\right]\right. \\ & + \left. a_s^2(Q)\Big[\left(K^{(1,0)} + \gamma_F^{(1)}\right)L + \left(\beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)}\right)L^2 - \frac{1}{6}\beta_0\gamma_K^{(0)}L^3\Big]\right\} , \end{aligned} \tag{3.8}$$

where we have defined:

$$L \equiv \ln\frac{Q^2}{\mu_b^2} , \tag{3.9}$$

Eq. (3.6) can be conveniently written as:

$$R^2 = \exp\left\{\sum_{n=1}^{2} a_s^n(Q)\sum_{k=1}^{n+1} S^{(n,k)}L^k\right\} , \tag{3.10}$$

with:

$$\begin{aligned} & S^{(1,1)} = K^{(0,0)} + \gamma_F^{(0)} , \quad S^{(1,2)} = -\frac{1}{4}\gamma_K^{(0)} , \\ & S^{(2,1)} = K^{(1,0)} + \gamma_F^{(1)} , \quad S^{(2,2)} = \beta_0 K^{(0,0)} + \frac{1}{2}\beta_0\gamma_F^{(0)} - \frac{1}{4}\gamma_K^{(1)} , \quad S^{(2,3)} = -\frac{1}{6}\beta_0\gamma_K^{(0)} . \end{aligned} \tag{3.11}$$

Eq. (3.10) can be easily expanded up to order a_s^2 as:

$$\begin{aligned} R^2 &= 1 + a_s(Q)\sum_{k=1}^{2} S^{(1,k)}L^k + a_s^2(Q)\left[\sum_{k=1}^{3} S^{(2,k)}L^k + \frac{1}{2}\left(\sum_{k=1}^{2} S^{(1,k)}L^k\right)^2\right] + \mathcal{O}(a_s^3) \\ &= 1 + a_s(Q)\sum_{k=1}^{2} S^{(1,k)}L^k + a_s^2(Q)\sum_{k=1}^{4}\widetilde{S}^{(2,k)}L^k + \mathcal{O}(a_s^3) , \end{aligned} \tag{3.12}$$

with:

$$\begin{aligned} & \widetilde{S}^{(2,1)} = S^{(2,1)} , \qquad \widetilde{S}^{(2,2)} = S^{(2,2)} + \frac{1}{2}\left[S^{(1,1)}\right]^2 , \\ & \widetilde{S}^{(2,3)} = S^{(2,3)} + S^{(1,1)}S^{(1,2)} , \qquad \widetilde{S}^{(2,4)} = \frac{1}{2}\left[S^{(1,2)}\right]^2 . \end{aligned} \tag{3.13}$$

3.2 Expansion of the DGLAP evolution

A further step towards the extraction of the singular behaviour of the resummed cross section in the limit $q_T \to 0$ requires expanding the solution of the DGLAP equation that also resums logs as that in Eq. (3.9). The solution of the DGLAP equation in Eq. (1.24) that evolves PDFs from the scale $\mu_0 = \mu$ to the scale Q can be written as[3]:

$$f(Q) = \Gamma(Q, \mu_b) \otimes f(\mu_b) , \tag{3.14}$$

where the evolution operator Γ obeys the equation:

$$\frac{\partial\Gamma(Q, \mu_b)}{\partial\ln Q^2} = P(Q) \otimes \Gamma(Q, \mu_b) , \tag{3.15}$$

[3] A possible summation over the flavour indices is understood

with the splitting functions P having the perturbative expansion:

$$P(Q) = \sum_{n=0}^{\infty} a_s^{n+1}(Q) P^{(n)} \,. \tag{3.16}$$

The coefficients $P^{(n)}$ are functions of x (or z) only and thus (in $\overline{\mathrm{MS}}$) P depends on Q only through the coupling a_s. We now need to expand Γ in powers of a_s up to $\mathcal{O}(a_s^2)$. To this end, we assume that Γ obeys the expansion:

$$\Gamma(Q,\mu_b) = \Delta(1-x) + \sum_{n=1}^{\infty} a_s^n(Q) \sum_{k=1}^{n} L^k \Gamma^{(n,k)} \,, \tag{3.17}$$

where the coefficients $\Gamma^{(n,k)}$ are functions of x only that we need to determine up to $n = 2$. Now Eq. (3.15) can be solved iteratively. At $\mathcal{O}(a_s)$, considering that:

$$\frac{da_s}{d\ln Q^2} = \mathcal{O}(a_s^2) \,, \tag{3.18}$$

this gives:

$$\Gamma^{(1,1)} = P^{(0)} \,. \tag{3.19}$$

At $\mathcal{O}(a_s^2)$, this gives:

$$-\beta_0 P^{(0)} + \Gamma^{(2,1)} + 2L\Gamma^{(2,2)} = P^{(1)} + LP^{(0)} \otimes P^{(0)} \,, \tag{3.20}$$

so that:

$$\begin{aligned} \Gamma^{(2,1)} &= P^{(1)} + \beta_0 P^{(0)} \,, \\ \Gamma^{(2,2)} &= \frac{1}{2} P^{(0)} \otimes P^{(0)} \,. \end{aligned} \tag{3.21}$$

Putting all pieces together we find:

$$\Gamma(Q,\mu_b) = \Delta(1-x) + a_s(Q)LP^{(0)} + a_s^2(Q)\left[L\left(P^{(1)} + \beta_0 P^{(0)}\right) + \frac{1}{2}L^2 P^{(0)} \otimes P^{(0)}\right] + \mathcal{O}(a_s^3) \,. \tag{3.22}$$

However, the quantity we are interested in is the inverse of the operator Γ, that evolves the PDFs from the scale Q to μ_b:

$$\begin{aligned} \Gamma(\mu_b, Q) &= \Gamma^{-1}(Q,\mu_b) \\ &= \Delta(1-x) - a_s(Q)LP^{(0)} - a_s^2(Q)\left[L\left(P^{(1)} + \beta_0 P^{(0)}\right) - \frac{1}{2}L^2 P^{(0)} \otimes P^{(0)}\right] + \mathcal{O}(a_s^3) \,. \end{aligned} \tag{3.23}$$

3.3 Expansion of the matching coefficients at the hard scale

The final step is the expansion of the matching coefficients C defined in Eq. (1.20) for $\mu_0 = \mu$ around the scale Q up to order a_s^2. To do this, we use the expansion in Eq. (1.22), in which a_s is computed at μ_b, and combine it with Eq. (3.5). This yields:

$$C(\mu_b, \mu_b^2) = \Delta(1-x) + a_s(Q)C^{(1,0)} + a_s^2(Q)\left[C^{(2,0)} + L\beta_0 C^{(1,0)}\right] + \mathcal{O}(a_s^3) \,. \tag{3.24}$$

3.4 Expansion of the single low-scale TMD

Before going into the more convoluted (but more relevant) case of a cross section, it is useful to compute the expansion of the low-scale TMD, *i.e.* the convolution between matching coefficients and collinear distributions at the scale $\mu_0 = \mu_b$. This can be achieved combining Eqs. (3.23) and (3.24):

$$\begin{aligned} C(\mu_b,\mu_b^2) \otimes f(\mu_b) &= C(\mu_b,\mu_b^2) \otimes \Gamma(\mu_b, Q) \otimes f(Q) \\ &= \Big\{\Delta(1-x) + a_s(Q)\left[C^{(1,0)} - LP^{(0)}\right] \\ &+ a_s^2(Q)\Big[C^{(2,0)} + L\left(-P^{(1)} - C^{(1,0)} \otimes P^{(0)} + \beta_0 C^{(1,0)} - \beta_0 P^{(0)}\right) \\ &+ \frac{1}{2}L^2 P^{(0)} \otimes P^{(0)}\Big]\Big\} \otimes f(Q) + \mathcal{O}(a_s^3) \,. \end{aligned} \tag{3.25}$$

3.5 Expansion of the cross section

We are finally in the position to gather all pieces and write down the perturbative expansion of a cross section involving TMDs relevant in the limit $q_T \to 0$. The only additional information required is the hard factor H that also admits a perturbative expansion. It is now opportune to reinstate the flavour indices. In general, the hard factor H has a flavour structure, meaning that it carries a pair of flavour indices. Choosing $\mu = Q$, the expansion of the hard factor reads:

$$H_{ij}(Q) = \sum_{n=0}^{\infty} a_s^n(Q) H_{ij}^{(n)} = H_{ij}^{(0)} + a_s(Q) H_{ij}^{(1)} + a_s^2(Q) H_{ij}^{(2)} + \mathcal{O}(a_s^3)\,, \tag{3.26}$$

where $H_{ij}^{(n)}$ are numerical factors.

A physical cross section differential in q_T can finally be computed as:

$$\frac{d\sigma}{dq_T^2} \propto \sum_{ij} H_{ij}(Q) \int d^2\mathbf{b}_T e^{i\mathbf{b}_T\cdot\mathbf{q}_T} F_i(x, \mathbf{b}_T, Q, Q^2) D_j(z, \mathbf{b}_T, Q, Q^2) \tag{3.27}$$

where F_i and D_j are two different TMDs distributions (*e.g.* a PDF and a FF) indexed by a flavour index. Assuming that the flavour indices i and j run *either* on quarks *or* on the gluon and writing explicitly all perturbative factors, one has:

$$\begin{aligned}
\frac{d\sigma}{dq_T^2} &\propto \sum_{ij} H_{ij}(Q) \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T\cdot\mathbf{q}_T} R^2\left[(\mu_b, \mu_b^2) \to (Q, Q^2)\right] \\
&\times \sum_k \left[\mathcal{C}_{ik}(x; \mu_b, \mu_b^2) \otimes f_k(x, \mu_b)\right] \sum_l \left[\mathbb{C}_{jl}(z; \mu_b, \mu_b^2) \otimes d_l(z, \mu_b)\right] \\
&= \sum_{n=0}^{\infty} a_s^n(Q) \sum_{p=0}^{2n} I_p(q_T, Q) \sum_{ij,kl} B_{ij,kl}^{(n,p)}(x, z) \underset{x}{\otimes} f_k(x, Q) \underset{z}{\otimes} d_l(z, Q)\,.
\end{aligned} \tag{3.28}$$

In the equation above we have defined:

$$I_p(q_T, Q) \equiv \int \frac{d^2\mathbf{b}_T}{4\pi} e^{i\mathbf{b}_T\cdot\mathbf{q}_T} \ln^p\left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}}\right) = \frac{1}{2}\int_0^\infty db_T\, b_T J_0(b_T q_T) \ln^p\left(\frac{b_T^2 Q^2}{4e^{-2\gamma_E}}\right)\,. \tag{3.29}$$

Results for I_p have been computed up to $p = 4$ in Eq. (136) of Appendix B of Ref. [7]. Specifically, and including the trivial tranform with $p = 0$, they read:

$$\begin{aligned}
I_0(q_T, Q) &= \delta(q_T)\,, \\
I_1(q_T, Q) &= -\frac{1}{q_T^2}\,, \\
I_2(q_T, Q) &= -\frac{2}{q_T^2} \ln\left(\frac{Q^2}{q_T^2}\right)\,, \\
I_3(q_T, Q) &= -\frac{3}{q_T^2} \ln^2\left(\frac{Q^2}{q_T^2}\right)\,, \\
I_4(q_T, Q) &= -\frac{4}{q_T^2} \left[\ln^3\left(\frac{Q^2}{q_T^2}\right) - 4\zeta_3\right]\,.
\end{aligned} \tag{3.30}$$

The functions above are compute under the assumption that there is no non-perturbative component. Upon this assumption, that we will relax below, all terms with $p = 0$ will be proportional to $\delta(q_T)$. Analogous terms are *not* included in the fixed-order calculation. Therefore, when matching the resummed calculation to the fixed-order one, one should remove from Eq. (3.28) all terms with $p = 0$. Importantly, this means removing the full $\mathcal{O}(1)$ terms such that leading-order term for $q_T > 0$ is $\mathcal{O}(a_s)$.

The coefficients $B^{(n,k)}$ in Eq. (3.28) can be determined by using the expansions worked out above in Eqs. (3.12), (3.25), and (3.26). The leading order is obtained by just retaining the $\mathcal{O}(a_s^0)$ terms in all the expansions. This gives:

$$B^{(0,0)}_{ij,kl}(x,z) = H^{(0)}_{ij}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\,. \tag{3.31}$$

The $\mathcal{O}(a_s)$ terms are obtained by combining the leading-order once of all expansions but one. Finally organising the terms in powers of L, this yields:

$$\begin{aligned}
B^{(1,0)}_{ij,kl}(x,z) &= H^{(1)}_{ij}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z) + H^{(0)}_{ij}\delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z) + H^{(0)}_{ij}\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z)\,,\\
B^{(1,1)}_{ij,kl}(x,z) &= H^{(0)}_{ij}S^{(1,1)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z) - H^{(0)}_{ij}\delta_{ik}\delta(1-x)\mathbb{P}^{(0)}_{jl}(z) - H^{(0)}_{ij}\mathcal{P}^{(0)}_{ik}(x)\delta_{jl}\delta(1-z)\,,\\
B^{(1,2)}_{ij,kl}(x,z) &= H^{(0)}_{ij}S^{(1,2)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\,.
\end{aligned} \tag{3.32}$$

Finally, the $\mathcal{O}(a_s^2)$ terms are:

$$\begin{aligned}
B^{(2,0)}_{ij,kl}(x,z) &= H^{(2)}_{ij}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\,,\\
&+ H^{(1)}_{ij}\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right]\\
&+ H^{(0)}_{ij}\mathcal{C}^{(2,0)}_{ik}(x)\delta_{jl}\delta(1-z) + H^{(0)}_{ij}\mathcal{C}^{(1,0)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) + H^{(0)}_{ij}\delta_{ik}\delta(1-x)\mathbb{C}^{(2,0)}_{jl}(z)\,,\\
B^{(2,1)}_{ij,kl}(x,z) &= \left(H^{(0)}_{ij}\widetilde{S}^{(2,1)} + H^{(1)}_{ij}S^{(1,1)}\right)\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\\
&- H^{(1)}_{ij}\left[\mathcal{P}^{(0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}^{(0)}_{jl}(z)\right]\\
&+ H^{(0)}_{ij}S^{(1,1)}\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right]\\
&+ H^{(0)}_{ij}\left(-\mathcal{P}^{(1)}(x) - \sum_\alpha \mathcal{C}^{(1,0)}_{i\alpha}(x)\otimes\mathcal{P}^{(0)}_{\alpha k}(x) + \beta_0\mathcal{C}^{(1,0)}_{ik}(x) - \beta_0\mathcal{P}^{(0)}_{ik}(x)\right)\delta_{jl}\delta(1-z)\\
&- H^{(0)}_{ij}\mathcal{P}^{(0)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) - H^{(0)}_{ij}\mathcal{C}^{(1,0)}_{ik}(x)\mathbb{P}^{(0)}_{jl}(z)\\
&+ H^{(0)}_{ij}\delta_{jl}\delta(1-z)\left(-\mathbb{P}^{(1)}(z) - \sum_\beta \mathbb{C}^{(1,0)}_{j\beta}(z)\otimes\mathbb{P}^{(0)}_{\beta l}(z) + \beta_0\mathbb{C}^{(1,0)}_{jl}(z) - \beta_0\mathbb{P}^{(0)}_{jl}(z)\right)\,,\\
B^{(2,2)}_{ij,kl}(x,z) &= \left(H^{(0)}_{ij}\widetilde{S}^{(2,2)} + H^{(1)}_{ij}S^{(1,2)}\right)\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\\
&+ H^{(0)}_{ij}\widetilde{S}^{(1,2)}\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right]\\
&- H^{(0)}_{ij}S^{(1,1)}\left[\mathcal{P}^{(0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}^{(0)}_{jl}(z)\right]\\
&+ H^{(0)}_{ij}\frac{1}{2}\sum_\alpha\mathcal{P}^{(0)}_{i\alpha}(x)\otimes\mathcal{P}^{(0)}_{\alpha k}(x)\delta_{jl}\delta(1-z) + H^{(0)}_{ij}\mathcal{P}^{(0)}_{ik}(x)\mathbb{P}^{(0)}_{jl}(z) + H^{(0)}_{ij}\delta_{ik}\delta(1-x)\frac{1}{2}\sum_\beta\mathbb{P}^{(0)}_{j\beta}(z)\otimes\mathbb{P}^{(0)}_{\beta l}(z)\,,\\
B^{(2,3)}_{ij,kl}(x,z) &= H^{(0)}_{ij}\widetilde{S}^{(2,3)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z) - H^{(0)}_{ij}S^{(1,2)}\left[\mathcal{P}^{(0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}^{(0)}_{jl}(z)\right]\,,\\
B^{(2,4)}_{ij,kl}(x,z) &= H^{(0)}_{ij}\widetilde{S}^{(2,4)}\delta_{ik}\delta(1-x)\delta_{jl}\delta(1-z)\,.
\end{aligned} \tag{3.33}$$

4 Non-perturbative effect on the asymptotic cross section

As discussed in Sect. 2, TMDs have a non-perturbative component that can be parameterised as in Eq. (2.5). This implies two main ingredients: the introduction of a function $b_*(b_T)$ that prevents the perturbative component to enter the non-perturbative regime and a function $f_{\rm NP}$ that parametrises the actual non-perturbative component. These changes leave almost unchanged the derivation of the asymptotic behavior of a cross section for $q_T \to 0$. The only thing that needs to be changed are the functions I_p defined in Eq. (3.29), that have to be replaced with:

$$\widetilde{I}_p(x,z,q_T,Q) \equiv \frac{1}{2}\int_0^\infty db_T\, b_T J_0(b_T q_T) f_{\rm NP}^{(1)}(x,b_T,Q^2) f_{\rm NP}^{(2)}(z,b_T,Q^2) \ln^p\left(\frac{b_*^2(b_T)Q^2}{4e^{-2\gamma_E}}\right), \tag{4.1}$$

where $f_{\rm NP}^{(1)}$ and $f_{\rm NP}^{(2)}$ are the non-perturbative functions associated to the two TMDs involved in the cross sections and where we are assuming $\mu = \sqrt{\zeta} = Q$ and $\mu_0 = \sqrt{\zeta_0} = \mu_b$. In general, the integral in Eq. (4.1) cannot be evaluated analytically but one can use the Ogata quadrature method to compute it numerically. In the case $k = 0$ Eq. (4.1) has an intersting consequence. Specifically, contrary to the case in which no non-perturbative contribution is introduced, $\widetilde{I}_0$ is different from zero also for $q_T > 0$. Therefore, also the terms in the expansion in Eq. (3.28) that are independent of L contribute to the cross section differential in q_T. Clearly, any non-perturbative contribution is expected to give a substantial contribution only in the vicinity of $q_T = 0$.

In this respect, one should check that the introduction of the non-perturbative components does not affect the large-q_T region. This is amounts to show that:

$$\widetilde{I}_p(x,z,q_T,Q) \underset{q_T \gtrsim Q}{\sim} I_p(q_T,Q)\,. \tag{4.2}$$

Using the asymptotic expansion of the Hankel transform one can show that this is actually the case. Alternatively, this can be shown numerically on a case-by-case base. In particular, we can compute the integral in Eq. (4.1) using the Ogata quadrature method and compare the results to those in Eq. (3.30). For the comparison we use Eq. (2.4) and choose as non-perturbative functions $f_{\rm NP}$ the following x-independent form:

$$f_{\rm NP}^{(1)}(b_T,\zeta) = f_{\rm NP}^{(2)}(b_T,\zeta) = \exp\left[\left(-g_1 - g_2 \ln\left(\frac{\sqrt{\zeta}}{2Q_0}\right)\right)\frac{b_T^2}{2}\right], \tag{4.3}$$

with $g_1 = 0.02$, $g_1 = 0.5$, and $Q_0 = 1.6$ GeV. In Fig. 4.1, I compare the integral in Eq. (4.1) for $p = 0,1,2,3$ (solid lines) to the expressions in Eq. (3.30) (dashed lines) at $Q = 10$ GeV as functions of q_T. Notice that there is no black dashed line is not present because it corresponds to $\delta(q_T)$. It is clear that the relation in Eq. (4.2) is fulfilled but the differences between the two sets of curves remain sizeable up to large values of q_T, particularly for larger values of p. Therefore, when performing the matching between resummed and fixed-order cross sections, it is very important to include non-perturbative effects in the expanded calculation in order to ensure a proper cancellation with the resummed calculation in the large-q_T region.

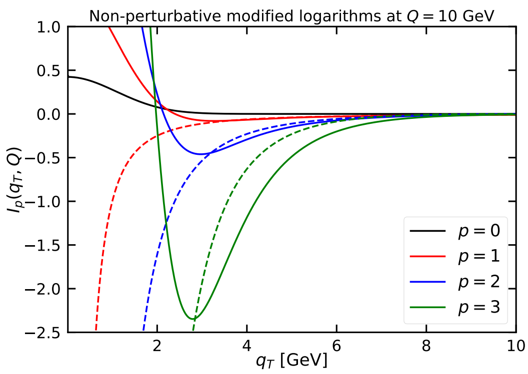

Fig. 4.1: Comparison of the integral in Eq. (4.1) for $p = 0,1,2,3$ (solid lines) to the expressions in Eq. (3.30) (dashed lines) as functions of q_T at $Q = 10$ GeV. The black dashed line is not present because it corresponds to $\delta(q_T)$.

Moreover, by design, the inclusion of the non-perturbative effects modifies very significantly also the low-q_T region. The modification is such to prevent the cancellation between the resummed and the fixed order at small

q_T. A possible solution is to use yet another definition of the functions I_p in the expanded calculation. These functions have to be such to tend to those in Eq. (3.30) for small values of q_T but converge to the definition in Eq. (4.1) more rapidly as q_T increases. To do this, one may exploit the fact that power-suppressed contributions do not affect the logarithmic expansion of the resummed calculation to combine the two definitions as follows:

$$\hat{I}_p(x, z, q_T, Q) = \left[1 - \left(\frac{q_T}{Q}\right)^S\right] I_p(q_T, Q) + \left(\frac{q_T}{Q}\right)^S \widetilde{I}_p(x, z, q_T, Q)\,, \tag{4.4}$$

where the exponent S can be adjusted to make the transition from one regime to the other more or less strong. Clearly, this definition should not be pushed to values of q_T much above Q. As an example, Fig. 4.2 shows the shape of $\hat{I}_1$ defined in Eq. (4.4) for $S = 0.2$ along with its components given in Eqs. (3.30) and (4.1). This prescription seems to do the job.

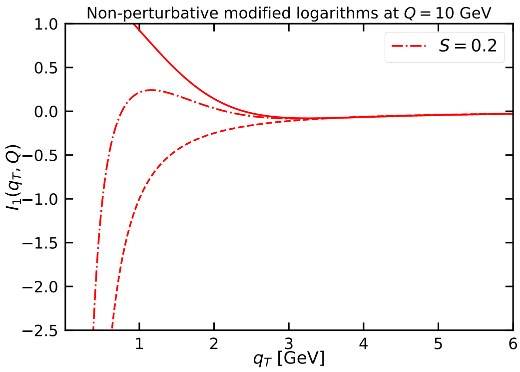

Fig. 4.2: Behaviour of $\hat{I}_1$ defined in Eq. (4.4) for $S = 0.2$ compared to the single components defined in Comparison of the integral in Eqs. (3.30) and (4.1).

Part II. Semi-inclusive deep-inelastic-scattering

5 Fixed order and asymptotic limit

In order to validate the results above, it is opportune to compare the $\mathcal{O}(a_s)$ expressions to those present in the literature. To this end, we write explicitly the expression for the cross section differential in q_T for $q_T > 0$, *i.e.* without the $\delta(q_T)$ terms, and with no non-perturbative effect. Considering that, $S^{(1,1)} = 6C_F$, $S^{(1,2)} = -2C_F$, and $H_{ij}^{(0)} = e_i^2\delta_{ij}$, this yields:

$$\begin{aligned}
\frac{d\sigma}{dxdydzdq_T^2} &\propto a_s(Q)\sum_{ij,kl}\left[-B_{ij,kl}^{(1,1)}(x,z)\frac{1}{q_T^2} - B_{ij,kl}^{(1,2)}(x,z)\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\right]\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q)\\
&= \frac{a_s(Q)}{q_T^2}\sum_{ij,kl} e_i^2\delta_{ij}\Bigg[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\\
&+ \mathcal{P}_{ik}^{(0)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(0)}(z)\Bigg]\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q)\\
&= \frac{a_s(Q)}{q_T^2}\sum_i e_i^2\Bigg[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x,Q)d_i(z,Q)\\
&+ \left(\sum_k \mathcal{P}_{ik}^{(0)}(x)\underset{x}{\otimes} f_k(x,Q)\right)d_i(z,Q) + f_i(x,Q)\left(\sum_l \mathbb{P}_{il}^{(0)}(z)\underset{z}{\otimes} d_l(z,Q)\right)\Bigg] + \mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.1}$$

This result, up to pre-factors that will be made explicit below, nicely agrees with that of, *e.g.*, Refs. [8, 10, 9].

In order to check that the matching is actually removing the double counting terms, it is instructive to derive Eq. (5.1) extracting the asymptote from the fixed-order computation at $\mathcal{O}(a_s)$. We take the expressions for the coefficient functions from Eqs. (106)-(109) of Appendix B of Ref. [10] or from Eqs. (4.6)-(4.20) of Ref. [11]. Referring to the second reference, some simplifications apply. First, we consider cross sections with unpolarised projectiles ($\lambda_e = 0$) on unpolarised targets ($S_\perp^\mu = 0$) and integrated over the azimuthal angles ϕ_H and ϕ_S. By doing so and after a simple manipulation, the cross section simplifies greatly and can be written in terms of structure functions as:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{2\pi\alpha^2}{xyQ^2}\left[Y_+ F_{UU,T} + 2(1-y)F_{UU,L}\right] = \frac{2\pi\alpha^2}{xyQ^2}Y_+\left[F_{UU,2} - \frac{y^2}{Y_+}F_{UU,L}\right]\,, \tag{5.2}$$

with:

$$Y_+ \equiv 1 + (1-y)^2\,, \tag{5.3}$$

and where we have defined the structure function:

$$F_{UU,2} \equiv F_{UU,T} + F_{UU,L}\,. \tag{5.4}$$

Notice that, as compared to Ref. [11], we have factored out from the structure functions a factor $1/(\pi z^2)$[4] so that they factorize as:

$$\begin{aligned}
F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)\Bigg[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\\
&+ \hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\Bigg] + \mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.5}$$

[4] The factor z^2 is the consequence of the fact that we are writing the cross section differential in q_T^2 that is the transverse momentum of the exchanged photon while in Ref. [11] the cross section is differentila in p_T^2 that is the transverse momentum of the of the outgoing hadrons. Since $p_T = zq_T$, the factor z^2 cancels.

with $S = 2, L$ and where the sum over i runs over the active quark and antiquark flavours and e_i is the electric charge of the i-th flavour. The explicit expressions for the coefficient functions are:

$$\begin{aligned}
\hat{B}_{qq}^{2,\mathrm{FO}}(x,z,q_T) &= 2C_F\left[(1-x)(1-z)+4xz+\frac{1+x^2z^2}{xz}\frac{Q^2}{q_T^2}\right],\\
\hat{B}_{qq}^{L,\mathrm{FO}}(x,z,q_T) &= 8C_F xz\,,\\
\hat{B}_{qg}^{2,\mathrm{FO}}(x,z,q_T) &= 2T_R\left[[x^2+(1-x)^2][z^2+(1-z)^2]\frac{1-x}{xz^2}\frac{Q^2}{q_T^2}+8x(1-x)\right],\\
\hat{B}_{qg}^{L,\mathrm{FO}}(x,z,q_T) &= 16T_R x(1-x)\,.\\
\hat{B}_{gq}^{2,\mathrm{FO}}(x,z,q_T) &= 2C_F\left[(1-x)z+4x(1-z)+\frac{1+x^2(1-z)^2}{xz}\frac{1-z}{z}\frac{Q^2}{q_T^2}\right],\\
\hat{B}_{gq}^{L,\mathrm{FO}}(x,z,q_T) &= 8C_F x(1-z)\,,
\end{aligned} \tag{5.6}$$

These expressions are enough to compute the SIDIS cross section at $\mathcal{O}(a_s)$ in the region $q_T \lesssim Q$. In order to match Eq. (5.1), one has to take the limit $q_T/Q \to 0$ and retain in the coefficient functions only the terms enhanced as $\ln(Q^2/q_T^2)$. This automatically means that $F_{UU,L}$ does not contribute in this limit because:

$$F_{UU,L} \underset{q_T/Q\to 0}{\longrightarrow} 0\,. \tag{5.7}$$

Another crucial observation is that the δ-function in Eq. (5.5) can be expanded as follows[5]:

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-x)(1-z)}{xz}\right) \underset{q_T^2/Q^2\to 0}{\longrightarrow} \ln\left(\frac{Q^2}{q_T^2}\right)\delta(1-x)\delta(1-z)+\frac{x\delta(1-z)}{(1-x)_+}+\frac{z\delta(1-x)}{(1-z)_+}\,, \tag{5.8}$$

so that:

$$\begin{aligned}
F_{UU,2} \underset{q_T/Q\to 0}{\longrightarrow}\; & a_s\frac{x}{q_T^2}\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\left[\hat{B}_{qq}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\right.\\
+\; & \left.\hat{B}_{qg}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}_{gq}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\right]+\mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.9}$$

with:

$$\begin{aligned}
\hat{B}_{qq}^{2,\mathrm{asy}}(x,z,q_T) &= 2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)+\frac{1+x^2}{(1-x)_+}\delta(1-z)+\delta(1-x)\frac{1+z^2}{(1-z)_+}\right]\\
&= 2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)-3\right]\delta(1-x)\delta(1-z)+\mathcal{P}_{qq}^{(0)}(x)\delta(1-z)+\delta(1-x)\mathbb{P}_{qq}^{(0)}(z)\,,\\
\hat{B}_{qg}^{2,\mathrm{asy}}(x,z,q_T) &= 2T_R\left[x^2+(1-x)^2\right]\delta(1-z)=\mathcal{P}_{qg}^{(0)}(x)\delta(1-z)\,,\\
\hat{B}_{gq}^{2,\mathrm{asy}}(x,z,q_T) &= \delta(1-x)2C_F\left[\frac{1+(1-z)^2}{z}\right]=\delta(1-x)\mathbb{P}_{qg}^{(0)}(z)\,.
\end{aligned} \tag{5.10}$$

It is thus easy to see that we can rewrite Eq. (5.9) as:

$$\begin{aligned}
F_{UU,2} \underset{q_T/Q\to 0}{\longrightarrow}\; & a_s\frac{x}{q_T^2}\sum_i e_i^2\left[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x)\,d_i(z)\right.\\
+\; & \left.\left(\sum_{k=q,g}\mathcal{P}_{qk}^{(0)}(x)\otimes f_k(x)\right)d_i(z)+f_i(x)\left(\sum_{k=q,g}\mathbb{P}_{qk}^{(0)}(z)\otimes d_k(z)\right)\right]+\mathcal{O}(a_s^2)\,.
\end{aligned} \tag{5.11}$$

[5] The proof of this relation is given in Appendix A.

Eq. (5.11) agrees with Eq. (5.1). This confirms that the expansion of the resummed calculation, as well as the asymptotic limit of the fixed order, removes the double-counting terms when doing the matching.

In order to provide a version of Eq. (5.5) that can be readily implemented, we need to perform one of the integrals making use of the δ-function. We integrate over $\bar{x}$ so that we write:

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)=\frac{\bar{z}\bar{x}_0^2}{1-\bar{z}}\delta(\bar{x}-\bar{x}_0)\,, \tag{5.12}$$

with:

$$\bar{x}_0=\frac{1-\bar{z}}{1-\bar{z}\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.13}$$

This allows us to write:

$$\begin{aligned}F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}\bar{x}_0\bigg[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)\\ &+ \hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_g\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_g\left(\frac{z}{\bar{z}}\right)\bigg]+\mathcal{O}(a_s^2)\,,\end{aligned} \tag{5.14}$$

with:

$$z_{\max}=\frac{1-x}{1-x\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{5.15}$$

Now we can rewrite the cross section above in such a way that it matches that at $\mathcal{O}(a_s)$ of Ref. [12]. That would allow us to confidently use the $\mathcal{O}(a_s^2)$ calculation presented right in that reference for the matching to the resummed calculation. This is made a little tricky by the different notation used in Ref. [12] and from the fact that in that paper the cross section is differential in a different set of variables. Specifically, we would like it to be differential in x, y, z, and q_T^2 while in Ref. [12] it is differential in x, Q^2, η, and p_T^2, where the last two are the rapidity and the transverse momentum of the outgoing hadron, respectively. Eq. (13) of Ref. [12], can be translated into our notation by noticing that:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta}=\frac{x}{zQ^2}\sum_{i,j}\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}f_i\left(\frac{x}{\bar{x}_0}\right)d_j\left(\frac{z}{\bar{z}}\right)\frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta}+\mathcal{O}(a_s^2)\,, \tag{5.16}$$

where we have exploited the δ-functions in Eqs. (18)-(20) to get rid of the integral over z.[6]

The $\mathcal{O}(a_s)$ partonic cross sections in Eqs. (18)-(20) of Ref. [12], setting $\varepsilon=0$, can be written as:

$$\frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta}=\frac{2\pi\alpha^2a_se_q^2\bar{x}_0}{xQ^4}Y_+\left[\underbrace{\left(F_{UU,M}^{ij}(\bar{x}_0,\bar{z})+\frac{3}{2}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right)}_{F_{UU,2}^{ij}}-\frac{y^2}{Y_+}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right]\,. \tag{5.17}$$

One can verify that $F_{UU,2}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{gq}(\bar{x}_0,\bar{z})$, and $F_{UU,L}^{gq}(\bar{x}_0,\bar{z})$ exactly correspond to $\hat{B}_{qq}^{2,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{gq}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, and $\hat{B}_{gq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$ of Eq. (5.6). It is crucial to notice that the correspondence holds only if $\bar{x}_0$ defined in Eq. (5.13) as a function of $\bar{z}$ is used. As an example, taking into account the different factors (a factor 2 for $F_{UU,M}$ and a factor 4 for $F_{UU,L}$) and using our notation for the integration variables ($y\to\bar{z}$, $\rho(z=0)\to\bar{x}_0$), we read off from Eq. (18) of Ref. [12]:

$$\begin{aligned}F_{UU,M}^{qq}(\bar{x}_0,\bar{z}) &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2+2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})}\right]\,,\\ F_{UU,L}^{qq}(\bar{x}_0,\bar{z}) &= 8C_F\bar{x}_0\bar{z}\,,\end{aligned} \tag{5.18}$$

[6] Notice that the z variable of Ref. [12] does not coincide with our definition.

so that:

$$\begin{aligned} F^{qq}_{UU,2}(\bar{x}_0,\bar{z}) &= F^{qq}_{UU,M}(\bar{x}_0,\bar{z}) + \frac{3}{2}F^{qq}_{UU,L}(\bar{x}_0,\bar{z}) \\ &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2+2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})}+3\bar{x}_0\bar{z}\right] \\ &= 2C_F\left[(1-\bar{x}_0)(1-\bar{z})+4\bar{x}_0\bar{z}+\frac{1+\bar{x}_0^2\bar{z}^2}{\bar{x}_0\bar{z}}\left(\frac{\bar{x}_0\bar{z}}{(1-\bar{x}_0)(1-\bar{z})}\right)\right]. \end{aligned} \tag{5.19}$$

Using Eq. (5.13), it is easy to see that the factor in round brackets in the last line of the equation above is equal to Q^2/q_T^2. Therefore, it reduces exactly to the first relation in Eq. (5.6). The same holds also for the two remaining partonic channels.

Putting all pieces together, Eq. (5.16) can be recast as:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{zxQ^4}Y_+\left[F_{UU,2}-\frac{y^2}{Y_+}F_{UU,L}\right], \tag{5.20}$$

with:

$$\begin{aligned} F_{UU,S} &= a_s\frac{x}{Q^2}\sum_i e_i^2\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}\bar{x}_0\left[F^{qq}_{UU,L}(\bar{x}_0,\bar{z})f_i\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \left. F^{qg}_{UU,L}(\bar{x}_0,\bar{z})f_g\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)+F^{gq}_{UU,L}(\bar{x}_0,\bar{z})f_i\left(\frac{x}{\bar{x}_0}\right)d_g\left(\frac{z}{\bar{z}}\right)\right]+\mathcal{O}(a_s^2). \end{aligned} \tag{5.21}$$

Therefore, the structure of the observables is exactly the same. What is left to work out is the Jacobian to express the cross section as differential in the same variables as in Eq. (5.14). What we need is to know is how the variables Q^2, p_T, and η are related to y, z, and q_T. The relevant relations are:

$$\left\{\begin{aligned} Q^2 &= xyS_H \\ p_T^2 &= z^2q_T^2 \\ \eta &= \frac{1}{2}\ln\left(\frac{y(1-x)S_H}{q_T^2}\right) \end{aligned}\right. \quad\Longrightarrow\quad dQ^2dp_T^2d\eta = \frac{zQ^2}{y}dydzdq_T^2. \tag{5.22}$$

where S_H is the squared center-of-mass energy of the collision, so that:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{zQ^2}{y}\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{xyQ^2}Y_+\left[F_{UU,2}-\frac{y^2}{Y_+}F_{UU,L}\right], \tag{5.23}$$

exactly like in Eq. (5.2). This confirms that the computation of Ref. [12] can be matched to the resummed calculation provided that the correct Jacobian is taken into account.

6 Integrating over q_T

It is interesting to carry out the integration over q_T of Eq. (5.5) analytically. The result can then be compared to the results presented in Appendix C of Ref. [14].

$$\begin{aligned} \int_0^\infty dq_T^2F_{UU,S} &= a_sx\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\left[\hat{B}^{S,\mathrm{FO}}_{qq}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \hat{B}^{S,\mathrm{FO}}_{qg}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right) \\ &+ \left.\hat{B}^{S,\mathrm{FO}}_{gq}\left(\bar{x},\bar{z},\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\right]+\mathcal{O}(a_s^2). \end{aligned} \tag{6.1}$$

replacing q_T^2/Q^2 in Eq. (5.6) has no effect on the coefficient functions for $F_{UU,L}$. The reason is that $F_{UU,L}$ is independent of q_T and therefore integrating over q_T has essentially the effect of removing the δ-function. On the other hand, integrating $F_{UU,2}$ gives:

$$
\begin{aligned}
\hat{B}^{2,\mathrm{FO}}_{qq}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2C_F\left[(1-x)(1-z)+4xz+\frac{1+x^2z^2}{(1-x)(1-z)}\right]\,,\\
\hat{B}^{2,\mathrm{FO}}_{qg}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2T_R\left[[x^2+(1-x)^2]\left(\frac{1}{1-z}+\frac{1}{z}-2\right)+8x(1-x)\right]\,,\\
\hat{B}^{2,\mathrm{FO}}_{gq}\left(x,z,\frac{(1-x)(1-z)}{xz}\right) &= 2C_F\left[(1-x)z+4x(1-z)+\frac{1+x^2(1-z)^2}{z(1-x)}\right]\,,
\end{aligned}
\tag{6.2}
$$

Instating the plus prescription in all terms that behave as $(1-x)^{-1}$ and $(1-z)^{-1}$[7], one can see that up to "local" terms (*i.e.* those terms proportional to the δ-function) the expressions of Appendix C of Ref. [14] are reproduced. The local terms correspond to virtual corrections that thus give contributions at $q_T = 0$. As such, they should coincide with the terms proportional to $\delta(q_T)$ generated in the expansion of the resummed calculation. Specifically, they should be equal to the coefficients $B^{(n,0)}_{ij,kl}$ in the expansion in Eq. (3.28). This is evidently the case for $B^{(0,0)}_{ij,kl}$ that gives the leading-order contribution to the integrated cross section. Unfortunately, this is not true at next-to-leading order because the expressions for the local terms in Appendix C of Ref. [14] (Eqs. (C.2)-(C.4)) do not match $B^{(1,0)}_{ij,kl}$. However, when checking the expressions explicitly an interesting pattern emerges: if all the occurrences of $\ln(1-x)$ (or $\ln(1-z)$) in Eqs. (C.2)-(C.4) of Ref. [14] are replaced with $\ln(x)$ (or $\ln(z)$) the two sets of expressions coincide. The expressions reported in Ref. [14] are those computed long time ago in Ref. [15].

[7] This step has the effect of removing the uncanceled soft divergencies.

Part III. Drell-Yan production

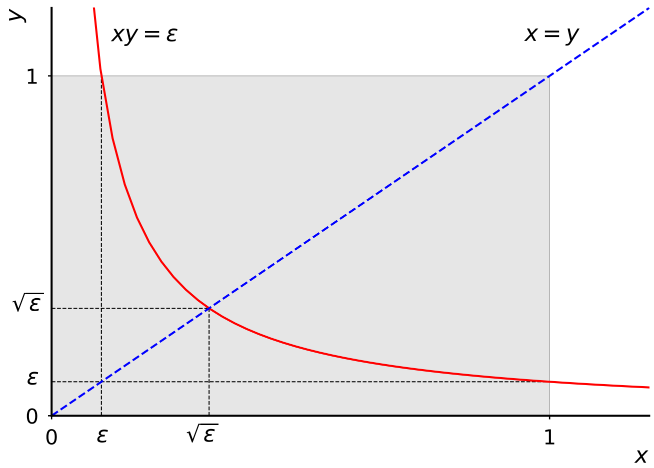

Fig. A.1: Integration region of the integral in Eq. (A.1). The integral is along the red curve defined by the δ-function.

Part IV. Appendices

A Expansion of the kinematic δ-function

In order to prove the equality in Eq. (5.8), we consider the following integral:

$$I(\varepsilon) = \int_0^1 dx \int_0^1 dy\, \delta(xy - \varepsilon) f(x, y)\,, \tag{A.1}$$

where $f(x, y)$ is a test function "well-behaved" over the integration region. Now I split the integral as follows:

$$I(\varepsilon) = \left(\int_0^1 dx \int_x^1 dy + \int_0^1 dy \int_y^1 dx \right) \delta(xy - \varepsilon) f(x, y)\,, \tag{A.2}$$

where the first term in the r.h.s. corresponds to the integral over the grey region *above* the blue line while the second over the grey region *below* the blue line in Fig. A.1.

Now I use the following equalities:

$$\delta(xy - \varepsilon) = \begin{cases} \dfrac{1}{x}\delta\left(y - \dfrac{\varepsilon}{x}\right)\theta(y - \sqrt{\varepsilon}) & \text{integral over } y, \\[2ex] \dfrac{1}{y}\delta\left(x - \dfrac{\varepsilon}{y}\right)\theta(x - \sqrt{\varepsilon}) & \text{integral over } x. \end{cases} \tag{A.3}$$

The θ-functions arise from the fact that the first integral has to be done along the upper branch on the red curve while the second along the lower branch. The two branches are joint at the point $x = y = \sqrt{\varepsilon}$ and thus the integration ranges are bounded from below by this point. Therefore, I find:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} f\left(x, \frac{\varepsilon}{x}\right) + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} f\left(\frac{\varepsilon}{y}, y\right)\,. \tag{A.4}$$

It is crucial to realise that in the first and the second integral the following conditions hold: $\varepsilon < \sqrt{\varepsilon} \leq x$ and $\varepsilon < \sqrt{\varepsilon} \leq y$, respectively. Therefore, in the limit $\varepsilon \to 0$, the arguments ε/x and ε/y of the function f will tend to zero. I now add and subtract a term proportional to $f(0,0)$ to both integrals, so that:

$$I(\varepsilon) = \int_{\sqrt{\varepsilon}}^1 \frac{dx}{x} \left[f\left(x, \frac{\varepsilon}{x}\right) - f(0,0) \right] + \int_{\sqrt{\varepsilon}}^1 \frac{dy}{y} \left[f\left(\frac{\varepsilon}{y}, y\right) - f(0,0) \right] + 2f(0,0) \underbrace{\int_{\sqrt{\varepsilon}}^1 \frac{d\xi}{\xi}}_{-\ln\sqrt{\varepsilon}}\,. \tag{A.5}$$

Finally, I take the limit for $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \int_0^1 \frac{dx}{x}\left[f(x,0) - f(0,0)\right] + \int_0^1 \frac{dy}{y}\left[f(0,y) - f(0,0)\right] - \ln(\varepsilon)\, f(0,0)\,, \tag{A.6}$$

that I rewrite as:

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \int_0^1 dx \int_0^1 dy \left\{ \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y) \right\} f(x,y)\,. \tag{A.7}$$

Comparing the equation above with Eq. (A.1), one deduces that:

$$\delta(xy - \varepsilon) \xrightarrow[\varepsilon\to 0]{} \frac{\delta(y)}{[x]_+} + \frac{\delta(x)}{[y]_+} - \ln(\varepsilon)\,\delta(x)\delta(y)\,. \tag{A.8}$$

Finally, substituting:

$$x \to \frac{1-x}{x}\,, \quad y \to \frac{1-z}{z}\,, \quad \text{and} \quad \varepsilon \to \frac{q_T^2}{Q^2}\,, \tag{A.9}$$

it is easy to recover Eq. (5.8). In particular, one needs to use the fact that:

$$\delta\left(\frac{1-x}{x}\right) = \delta(1-x)\,. \tag{A.10}$$

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