

Singular Value Decomposition (SVD)

Matrix Algebra 4 Statistical Learning

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Introduction

Two Special Decompositions

Last time we talked about the Eigen Value Decomposition (EVD).

In these slides, we'll talk about a closely related decomposition of EVD: the so-called **Singular Value Decomposition (SVD)**

Recap

Matrix decompositions, also known as matrix factorizations

$$\mathbf{M} = \mathbf{AB} \quad \text{or} \quad \mathbf{M} = \mathbf{ABC}$$

are a means of expressing a matrix as a product of usually two or three **simpler** matrices.

Types of matrices

Two types of matrices

We said that in data analysis we typically concentrate on two types of matrices:

- ▶ general **rectangular** matrices used to represent data tables.
- ▶ **positive semi-definite** matrices used to represent covariance matrices, correlation matrices, and any matrix that results from a crossproduct.

SVD

Singular Value Decomposition

- ▶ One of the most important decompositions in matrix algebra
- ▶ Can be applied to **any** rectangular matrix
- ▶ ANY: rectangular or square, singular or nonsingular.

Singular Value Decomposition

Let \mathbf{M} be an $n \times p$ matrix.

For convenience, let's also assume that \mathbf{M} is a "tall" matrix with $n > p$, although this is not essential.

Likewise, let's assume for the moment that the rank of \mathbf{M} is $r \leq p < n$

Singular Value Decomposition

We can represent \mathbf{M} as a triple product given by:

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

where

- ▶ \mathbf{U} is a $n \times r$ matrix which is **orthonormal** by columns:
 $\mathbf{U}^T\mathbf{U} = \mathbf{I}_r$
- ▶ \mathbf{D} is a $r \times r$ **diagonal** matrix consisting of r positive diagonal entries.
- ▶ \mathbf{V} is a $p \times r$ matrix which is **orthonormal** by columns:
 $\mathbf{V}^T\mathbf{V} = \mathbf{I}_r$

Singular Value Decomposition

The representation of \mathbf{M} as the triple product \mathbf{UDV}^T is called its *singular value decomposition*, occasionally referred to as “basic structure”.

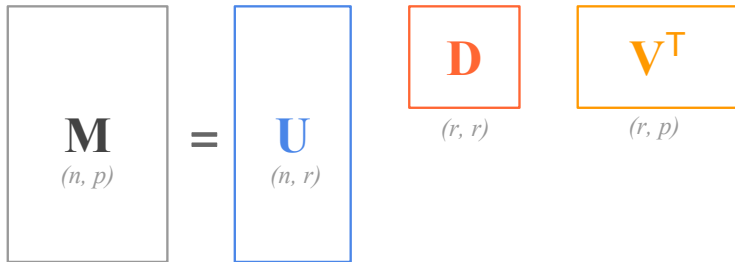
- ▶ \mathbf{U} is the matrix of **left singular vectors** of \mathbf{M}
- ▶ \mathbf{D} is the matrix of **singular values** of \mathbf{M}
- ▶ \mathbf{V} is the matrix of **right singular vectors** of \mathbf{M}

SVD

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}$$

$$\mathbf{M} = \begin{bmatrix} u_{11} & \cdots & u_{1r} \\ u_{21} & \cdots & u_{2r} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nr} \end{bmatrix} \begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_r \end{bmatrix} \begin{bmatrix} v_{11} & \cdots & v_{p1} \\ \vdots & \ddots & \vdots \\ v_{1r} & \cdots & v_{pr} \end{bmatrix}$$

SVD Diagram



When \mathbf{M} is of rank $r < p$

SVD Diagram

The diagram illustrates the Singular Value Decomposition (SVD) of a matrix \mathbf{M} . It consists of four rectangular boxes arranged horizontally, separated by an equals sign. The first box on the left contains the matrix \mathbf{M} with dimensions (n, p) below it, enclosed in a black border. The second box contains the matrix \mathbf{U} with dimensions (n, p) below it, enclosed in a blue border. The third box contains the matrix \mathbf{D} with dimensions (p, p) below it, enclosed in a red border. The fourth box contains the matrix \mathbf{V}^T with dimensions (p, p) below it, enclosed in an orange border.

$$\mathbf{M}_{(n, p)} = \mathbf{U}_{(n, p)} \mathbf{D}_{(p, p)} \mathbf{V}^T_{(p, p)}$$

When \mathbf{M} is of full rank p

About SVD

Singular Value Decomposition

We can think of the SVD structure as *the basic structure of a matrix*. What do we mean by “basic”? Well, this has to do with what each of the matrices $\mathbf{U}\mathbf{D}\mathbf{V}^T$ represent.

- ▶ \mathbf{U} is the orthonormalized matrix which is the most basic component. It's like the skeleton of the matrix.
- ▶ \mathbf{D} is referred to as the *spectrum* and it is a scale component.
- ▶ \mathbf{V} is an orientation component, also referred to as the *rotation* matrix.

About SVD

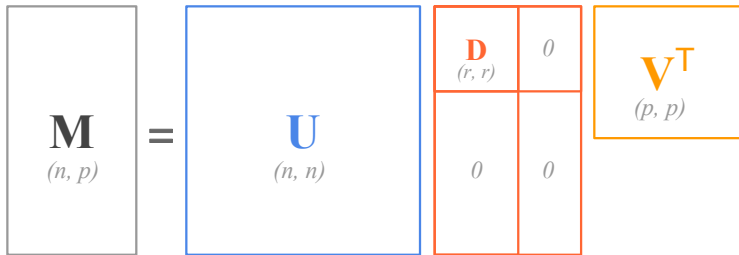
Note that:

- ▶ \mathbf{U} cannot be orthogonal ($\mathbf{U}\mathbf{U}^T = \mathbf{I}_n$) unless $r = p$
- ▶ \mathbf{V} cannot be orthogonal ($\mathbf{V}\mathbf{V}^T = \mathbf{I}_p$) unless $r = p = m$
- ▶ All elements of \mathbf{D} can be taken to be positive, ordered from large to small (with ties allowed)

Full SVD

If we let \mathbf{D} be the first r rows and r columns embedded in a (larger) $n \times p$ *rectangular* matrix, with $n - r$ rows and $p - r$ columns of zeros elsewhere, both \mathbf{U} and \mathbf{V}^T could be made fully orthogonal and, in this sense, properly constitute rotation matrices of order $n \times n$ and $p \times p$, respectively. This generalization can be called the **full SVD** of a matrix.

“Full” SVD Diagram



When \mathbf{M} is of rank $r \leq p$

About SVD

The preceding generalization is a significant one. It tells us that ANY matrix with real-valued entries can be represented as the product of:

- ▶ a rotation (possibly followed by a reflection), followed by
- ▶ a stretch, followed by
- ▶ a rotation

SVD

- ▶ \mathbf{U} is unitary, and its columns form a basis for the space spanned by the columns of \mathbf{M} .

$$\mathbf{U}^T \mathbf{U} = \mathbf{I}_p$$

- ▶ \mathbf{V} is unitary, and its columns form a basis for the space spanned by the rows of \mathbf{M} .

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}_p$$

- ▶ \mathbf{D} has non-negative real numbers on the diagonal (assuming \mathbf{M} is real).

SVD in R

svd() in R

svd() function

R provides the function `svd()` to perform a singular value decomposition of a given matrix

svd() output

A list with the following components

- `d` a vector containing the singular values
- `u` a matrix whose columns contain the left singular vectors
- `v` a matrix whose columns contain the right singular vectors

SVD example in R

```
# X matrix
set.seed(22)
X = matrix(rnorm(20), 5, 4)

# singular value decomposition
SVD = svd(X)

# elements returned by svd()
names(SVD)

## [1] "d" "u" "v"

# vector of singular values
(d = SVD$d)

## [1] 3.9516353 2.0223602 1.4748193 0.4324292
```

SVD example in R (con't)

```
# matrix of left singular vectors
```

```
(U = SVD$u)
```

```
##           [,1]      [,2]      [,3]      [,4]
## [1,] -0.4251177 -0.53913435 -0.7232572  0.00979433
## [2,]  0.5268694 -0.76862769  0.2860048  0.05610045
## [3,]  0.5752546  0.04999546 -0.4421464  0.13107213
## [4,]  0.2215220  0.05272644 -0.1702161 -0.95123359
## [5,] -0.4021114 -0.33655016  0.4130778 -0.27337073
```

```
# matrix of right singular vectors
```

```
(V = SVD$v)
```

```
##           [,1]      [,2]      [,3]      [,4]
## [1,]  0.5708354 -0.7406782  0.33862988  0.1042716
## [2,] -0.2741800 -0.5295008 -0.76797328  0.2338189
## [3,]  0.2772481  0.3206239 -0.04462207  0.9046229
## [4,]  0.7225689  0.2611992 -0.54180782 -0.3407543
```

SVD example in R (con't)

```
# U orthonormal (U'U = I)
```

```
t(U) %*% U
```

```
##           [,1]           [,2]           [,3]           [,4]
## [1,] 1.000000e+00  1.387779e-16  2.775558e-17  0.000000e+00
## [2,] 1.387779e-16  1.000000e+00 -2.775558e-17 -8.326673e-17
## [3,] 2.775558e-17 -2.775558e-17  1.000000e+00  5.551115e-17
## [4,] 0.000000e+00 -8.326673e-17  5.551115e-17  1.000000e+00
```

```
# V orthonormal (V'V = I)
```

```
t(V) %*% V
```

```
##           [,1]           [,2]           [,3]           [,4]
## [1,] 1.000000e+00 -1.110223e-16 -5.551115e-17  1.110223e-16
## [2,] -1.110223e-16  1.000000e+00  8.326673e-17  1.942890e-16
## [3,] -5.551115e-17  8.326673e-17  1.000000e+00 -8.326673e-17
## [4,] 1.110223e-16  1.942890e-16 -8.326673e-17  1.000000e+00
```

SVD example in R (con't)

```
# X equals U D V'  
U %*% diag(d) %*% t(v)
```

```
##           [,1]      [,2]      [,3]      [,4]  
## [1,] -0.5121391  1.85809239 -0.76390728 -0.9221536  
## [2,]  2.4851837 -0.06602641  0.08196190  0.8615624  
## [3,]  1.0078262 -0.16276495  0.74302828  2.0029422  
## [4,]  0.2928146 -0.19986068 -0.08402219  0.9365510  
## [5,] -0.2089594  0.30056173 -0.79289452 -1.6157349
```

```
# compare to X  
X
```

```
##           [,1]      [,2]      [,3]      [,4]  
## [1,] -0.5121391  1.85809239 -0.76390728 -0.9221536  
## [2,]  2.4851837 -0.06602641  0.08196190  0.8615624  
## [3,]  1.0078262 -0.16276495  0.74302828  2.0029422  
## [4,]  0.2928146 -0.19986068 -0.08402219  0.9365510  
## [5,] -0.2089594  0.30056173 -0.79289452 -1.6157349
```


SVD and Cross-products

Data Matrix

Data

The analyzed data can be expressed in matrix format \mathbf{X} :

$$\underset{n \times p}{\mathbf{X}} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}$$

- ▶ n objects in the rows
- ▶ p variables in the columns

Relation of SVD and Cross-Product Matrices

The cross-product matrix of columns of \mathbf{X} can be expressed as:

$$\mathbf{X}^T \mathbf{X} = \mathbf{V} \mathbf{D}^2 \mathbf{V}^T$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of columns can be expressed as:

$$\begin{aligned}\mathbf{X}^T\mathbf{X} &= (\mathbf{U}\mathbf{D}\mathbf{V}^T)^T(\mathbf{U}\mathbf{D}\mathbf{V}^T) \\ &= (\mathbf{V}\mathbf{D}\mathbf{U}^T)(\mathbf{U}\mathbf{D}\mathbf{V}^T) \\ &= \mathbf{V}\mathbf{D}(\mathbf{U}^T\mathbf{U})\mathbf{D}\mathbf{V}^T \\ &= \mathbf{V}\mathbf{D}^2\mathbf{V}^T\end{aligned}$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of rows of \mathbf{X} can be expressed as:

$$\mathbf{X}\mathbf{X}^T = \mathbf{U}\mathbf{D}^2\mathbf{U}^T$$

Relation of SVD and Cross-Product Matrices

The cross-product matrix of rows can be expressed as:

$$\begin{aligned}\mathbf{XX}^T &= (\mathbf{UDV}^T)(\mathbf{UDV}^T)^T \\ &= (\mathbf{UDV}^T)(\mathbf{VDU}^T) \\ &= \mathbf{UD}(\mathbf{V}^T\mathbf{V})\mathbf{DU}^T \\ &= \mathbf{UD}^2\mathbf{U}^T\end{aligned}$$

Relation of SVD and Cross-Product Matrices

One of the interesting things about SVD is that \mathbf{U} and \mathbf{V} are matrices whose columns are eigenvectors of product moment matrices that are *derived* from \mathbf{X} . Specifically,

- ▶ \mathbf{U} is the matrix of eigenvectors of (symmetric) $\mathbf{X}\mathbf{X}^T$ of order $n \times n$
- ▶ \mathbf{V} is the matrix of eigenvectors of (symmetric) $\mathbf{X}^T\mathbf{X}$ of order $p \times p$

Of additional interest is the fact that \mathbf{D} is a diagonal matrix whose main diagonal entries are the square roots of Λ , the *common* matrix of eigenvalues of $\mathbf{X}\mathbf{X}^T$ and $\mathbf{X}^T\mathbf{X}$.

Relation between EVD and SVD

The EVD of the cross-product matrix of columns (or minor product moment) $\mathbf{X}^T\mathbf{X}$ can be expressed as:

$$\mathbf{X}^T\mathbf{X} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

in terms of the SVD factorization of \mathbf{X} :

$$\mathbf{X}^T\mathbf{X} = \mathbf{V}\mathbf{D}^2\mathbf{V}^T$$

Relation between EVD and SVD

The EVD of the cross-product matrix of rows (or major product moment) \mathbf{XX}^T can be expressed as:

$$\mathbf{XX}^T = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$$

in terms of the SVD factorization of \mathbf{X} :

$$\mathbf{XX}^T = \mathbf{U}\mathbf{D}^2\mathbf{U}^T$$

Rank Reduction

SVD Rank-Reduction Theorem

In terms of the diagonal elements l_1, l_2, \dots, l_r of \mathbf{D} , the columns $\mathbf{u}_1, \dots, \mathbf{u}_r$ of \mathbf{U} , and the columns $\mathbf{v}_1, \dots, \mathbf{v}_r$ of \mathbf{V} , the basic structure of \mathbf{X} may be written as

$$\mathbf{X} = l_1 \mathbf{u}_1 \mathbf{v}_1^T + l_2 \mathbf{u}_2 \mathbf{v}_2^T + \dots + l_p \mathbf{u}_r \mathbf{v}_r^T$$

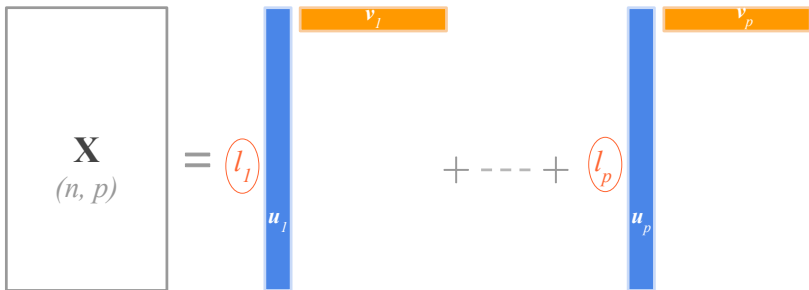
which shows that the matrix \mathbf{X} of rank p is a linear combination of r matrices of rank 1.

SVD Rank-Reduction Theorem

A very interesting and alternative way to represent the SVD is with the following formula:

$$\mathbf{X} = \sum_{k=1}^r l_k \mathbf{u}_k \mathbf{v}_k^T$$

SVD Diagram



SVD as sum of rank one matrices (assuming $r = p$)

SVD Rank-Reduction Theorem

SVD alternative formula:

$$\mathbf{X} = \sum_{k=1}^r l_k \mathbf{u}_k \mathbf{v}_k^T$$

- ▶ This expresses the SVD as a sum of r rank-1 matrices.
- ▶ This result is formalized in what is known as the **SVD theorem** described by Carl Eckart and Gale Young in 1936, and it is often referred to as the *Eckart-Young theorem*.
- ▶ This theorem applies to practically any arbitrary rectangular matrix.

SVD Rank-Reduction Theorem

What if you take $m < r$ terms?

$$\hat{\mathbf{X}} = \sum_{k=1}^m l_k \mathbf{u}_k \mathbf{v}_k^T$$

How would $\hat{\mathbf{X}}$ compare to \mathbf{X} ?

SVD Rank-Reduction Theorem

The SVD theorem of Eckart and Young is related to the important problem of approximating a matrix.

The basic result says that if \mathbf{X} is an $n \times p$ rectangular matrix, then the best r -dimensional approximation $\hat{\mathbf{X}}$ to \mathbf{X} is obtained by minimizing:

$$\min \quad \|\mathbf{X} - \hat{\mathbf{X}}\|^2$$

SVD Rank-Reduction Theorem

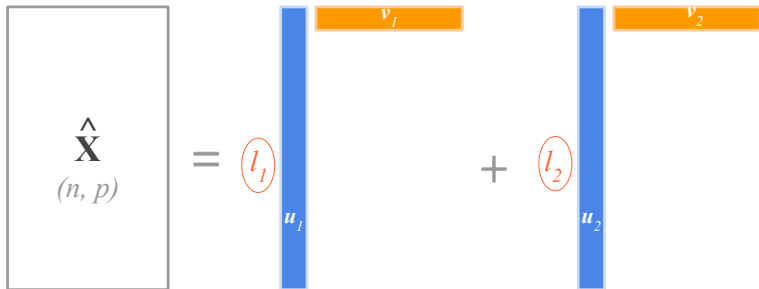
The minimization problem:

$$\min \quad \|\mathbf{X} - \hat{\mathbf{X}}\|^2$$

is a special type of approximation: a least squares approximation.

The solution is obtained by taking the first m elements of matrices \mathbf{U} , \mathbf{D} , \mathbf{V} so that $\hat{\mathbf{X}} = \mathbf{U}_m \mathbf{D}_m \mathbf{V}_m^T$

SVD rank-two approximation



The diagram illustrates the SVD rank-two approximation of a matrix $\hat{\mathbf{X}}$. On the left, a large rectangle contains the matrix $\hat{\mathbf{X}}$ with dimensions (n, p) below it. This is followed by an equals sign. To the right of the equals sign are two terms added together. The first term consists of a red circle containing l_1 next to a blue vertical rectangle labeled u_1 at its base, which is then multiplied by an orange horizontal rectangle labeled v_1 at its top. The second term is similar, with a red circle containing l_2 next to a blue vertical rectangle labeled u_2 at its base, multiplied by an orange horizontal rectangle labeled v_2 at its top.

$$\hat{\mathbf{X}}_{(n, p)} = l_1 u_1 v_1 + l_2 u_2 v_2$$

SVD as sum of two rank one matrices

SVD Rank-Reduction Theorem

The best 2-rank approximation $\hat{\mathbf{X}}$ of \mathbf{X} is given by:

$$\hat{\mathbf{X}} = l_1 \mathbf{u}_1 \mathbf{v}_1^T + l_2 \mathbf{u}_2 \mathbf{v}_2^T$$

We can say that the “information” contained in $n \times p$ values is compressed into $n \times 2$ values.

References

- ▶ **Multivariate Analysis** by Maurice Tatsuoka (1988). *Chapter 5: More Matrix Algebra*. Macmillan Publishing.
- ▶ **Mathematical Tools for Applied Multivariate Analysis** by J.D. Carroll, P.E. Green, and A. Chaturvedi (1997). *Chapter 5: Decomposition of Matrix Transformations: Eigenstructures and Quadratic Forms*. Academic Press.
- ▶ **Principal Component Analysis** by Ian Jolliffe (2002). *Chapter 3, Section 3.5: The Singular Value Decomposition*. Springer.
- ▶ **Hands-on Matrix Algebra using R** by Hrishikesh Vinod (2011). *Chapter 12: Kronecker Products and Singular Value Decomposition*. World Scientific.