MATH 301

INTRODUCTION TO PROOFS

Sina Hazratpour
Johns Hopkins University
Fall 2021

- Propositions and Proofs
- Propositional Logic

Overview

In this lecture

In this lecture, we shall introduce propositional logic and we shall use it to analyze the forms of mathematical reasoning.

Symbolic logic

- A map of Amsterdam is an an idealized model of Amsterdam. It depicts a caricature 2-dim image of buildings, Amstel river, various canals, roads, bicycle lanes, etc. We can consult a map to help us find the best route from one place to another.

Symbolic logic

- A map of Amsterdam is an an idealized model of Amsterdam. It depicts a caricature 2-dim image of buildings, Amstel river, various canals, roads, bicycle lanes, etc. We can consult a map to help us find the best route from one place to another.
- In a similar way, symbolic logic is an idealized model of mathematical language and proof.

Symbolic logic

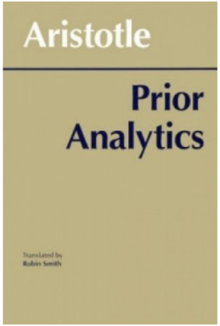
- A map of Amsterdam is an an idealized model of Amsterdam. It depicts a caricature 2-dim image of buildings, Amstel river, various canals, roads, bicycle lanes, etc. We can consult a map to help us find the best route from one place to another.
- In a similar way, symbolic logic is an idealized model of mathematical language and proof.
- Curiously, mathematicians did not really study the proofs that they were constructing until the 20th century. Once they did, they discovered that logic itself was a deep topic with many implications for the rest of mathematics.

Symbolic logic begins with Aristotle

- A formal study of patterns of reasoning, known today as 'syllogism', was first done by Aristotle in his book '*Prior Analytics*' (circa 350 BCE).

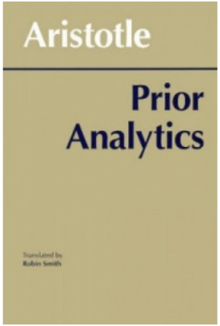
Symbolic logic begins with Aristotle

- A formal study of patterns of reasoning, known today as 'syllogism', was first done by Aristotle in his book '*Prior Analytics*' (circa 350 BCE).



Symbolic logic begins with Aristotle

- A formal study of patterns of reasoning, known today as 'syllogism', was first done by Aristotle in his book '*Prior Analytics*' (circa 350 BCE).
- The crucial observation of Aristotle: the correctness of inferences of statements has nothing to do with the content, truth or falsity of the individual statements, but, rather, the general patterns/rules of reasoning.



Symbolic logic begins with Aristotle

- A formal study of patterns of reasoning, known today as ‘syllogism’, was first done by Aristotle in his book ‘*Prior Analytics*’ (circa 350 BCE).
- The crucial observation of Aristotle: the correctness of inferences of statements has nothing to do with the content, truth or falsity of the individual statements, but, rather, the general patterns/rules of reasoning.
- He showed us that we can classify valid patterns of inference by their logical form.

Examples of syllogism

Every man is an animal.
Every animal is mortal.
Therefore every man is mortal.

Every A is B.
Every B is C.
Therefore every A is C.

$$\frac{BaA \quad CaB}{CaA}$$

Examples of syllogism

Every man is an animal. Every animal is mortal. Therefore every man is mortal.	Every A is B. Every B is C. Therefore every A is C.	$\begin{array}{c} BaA \\ CaB \\ \hline CaA \end{array}$
Every human is mortal. No cyborg is mortal. Therefore no cyborg is a human.	Every N is M. No X is M. Therefore no X is N.	$\begin{array}{c} MaN \\ MeX \\ \hline NeX \end{array}$

Historical context

> *First, to say about what and of what this is an investigation: it is about demonstration and of demonstrative science. Then, to define what is a premise, what is a term, and what a syllogism, and which kind of syllogism is perfect and which imperfect. [...] A premise, then, is a sentence that affirms or denies something of something, and this is either universal or particular or indeterminate. By 'universal' I mean belonging to all or to none of something; by 'particular' , belonging to some, or not to some, or not to all ; by 'indeterminate', belonging without universality or particularity, as in 'of contraries there is a single science' or 'pleasure is not a good'.*
>
> *(Aristotle, Prior Analytics Book I, translated by Gisela Striker)*

- An argument is a sequence of claims.

- An argument is a sequence of claims.
- Some claims are called premises and some claims are called conclusions.

- An argument is a sequence of claims.
- Some claims are called premises and some claims are called conclusions.
- A conclusion follows from some or no premises.

- An argument is a sequence of claims.
- Some claims are called premises and some claims are called conclusions.
- A conclusion follows from some or no premises.
- Expressions like 'so', 'consequently', 'hence' and 'therefore' are used to indicate that the claim that follows is the conclusion of the argument.

- An argument is a sequence of claims.
- Some claims are called premises and some claims are called conclusions.
- A conclusion follows from some or no premises.
- Expressions like ‘so’, ‘consequently’, ‘hence’ and ‘therefore’ are used to indicate that the claim that follows is the conclusion of the argument.
- Expressions like ‘because’, ‘since’, and ‘after all’ are used to indicate that the claims that follow are the premises of the argument.

Overview

Propositions

...

New propositions from the old

- We can make the following new propositions from propositions P and Q.

Proposition	Notation
P and Q	$P \wedge Q$
P or Q	$P \vee Q$
P implies Q	$P \Rightarrow Q$
P if and only if Q	$P \Leftrightarrow Q$
not P	$\neg P$

New propositions from the old

- We can make the following new propositions from propositions P and Q.

Proposition	Notation
P and Q	$P \wedge Q$
P or Q	$P \vee Q$
P implies Q	$P \Rightarrow Q$
P if and only if Q	$P \Leftrightarrow Q$
not P	$\neg P$

- Therefore, if $P : \mathbb{P}\mathrm{rop}$ and $Q : \mathbb{P}\mathrm{rop}$ then $P \wedge Q : \mathbb{P}\mathrm{rop}$, $P \vee Q : \mathbb{P}\mathrm{rop}$, $P \Rightarrow Q : \mathbb{P}\mathrm{rop}$, $P \Leftrightarrow Q : \mathbb{P}\mathrm{rop}$, $\neg P : \mathbb{P}\mathrm{rop}$, $\neg Q : \mathbb{P}\mathrm{rop}$, etc.

Propositional logic: natural deduction style

- The propositional logic tells us precisely which inferences about propositions are valid and why.

Propositional logic: natural deduction style

- The propositional logic tells us precisely which inferences about propositions are valid and why.
- An inference is valid if it can be justified by fundamental rules of reasoning that reflect the meaning of the logical terms involved.

Propositional logic: natural deduction style

- The propositional logic tells us precisely which inferences about propositions are valid and why.
- An inference is valid if it can be justified by fundamental rules of reasoning that reflect the meaning of the logical terms involved.
- In natural deduction, every proof is a proof from hypotheses. In other words, in any proof, there is a finite collection of hypotheses $P_1, P_2, \ldots, P_n$ and a conclusion Q, and the proof shows that how Q follows from $P_1, P_2, \ldots, P_n$.

Overview

The rules of inference for implication

The implication operator is the logical operator $\Rightarrow$, defined according to the following rules:

The rules of inference for implication

The implication operator is the logical operator $\Rightarrow$, defined according to the following rules:

- If Q can be derived from the assumption that P is true, then $P \Rightarrow Q$ is true;

The introduction rule

$$\frac{\begin{array}{c}[P] \\ \vdots \\ Q\end{array}}{P \Rightarrow Q}\ {\Rightarrow}\mathrm{I}$$

The rules of inference for implication

The implication operator is the logical operator $\Rightarrow$, defined according to the following rules:

- If Q can be derived from the assumption that P is true, then $P \Rightarrow Q$ is true;
- If $P \Rightarrow Q$ is true and P is true, then Q is true.

The introduction rule

$$\dfrac{\begin{matrix}[P] \\ \vdots \\ Q\end{matrix}}{P \Rightarrow Q} \Rightarrow\mathrm{I}$$

The elimination rule

$$\dfrac{P \Rightarrow Q \qquad P}{Q} \Rightarrow\mathrm{E}$$

The rules of inference for implication

The implication operator is the logical operator $\Rightarrow$, defined according to the following rules:

- If Q can be derived from the assumption that P is true, then $P \Rightarrow Q$ is true;
- If $P \Rightarrow Q$ is true and P is true, then Q is true.

$P \Rightarrow Q$ represents the expression "if P, then Q".

The introduction rule

$$\frac{\begin{array}{c}[P] \\ \vdots \\ Q\end{array}}{P \Rightarrow Q} \Rightarrow\mathrm{I}$$

The elimination rule

$$\frac{P \Rightarrow Q \qquad P}{Q} \Rightarrow\mathrm{E}$$

The rules of inference for conjunction

The conjunction operator is the logical operator $\wedge$, defined according to the following rules:

The rules of inference for conjunction

The conjunction operator is the logical operator $\wedge$, defined according to the following rules:

- If P is true and Q is true, then $P \wedge Q$ is true;

The introduction rule

$$\frac{P \qquad Q}{P \wedge Q}\ {\scriptstyle \wedge\mathrm{I}}$$

The rules of inference for conjunction

The conjunction operator is the logical operator $\wedge$, defined according to the following rules:

- If P is true and Q is true, then $P \wedge Q$ is true;
- If $P \wedge Q$ is true, then P is true;

The introduction rule

$$\frac{P \qquad Q}{P \wedge Q}\ \wedge\mathrm{I}$$

The elimination rule

$$\frac{P \wedge Q}{P}\ \wedge\mathsf{E}_{\ell} \qquad\qquad \frac{P \wedge Q}{Q}\ \wedge\mathsf{E}_{\mathsf{r}}$$

The rules of inference for conjunction

The conjunction operator is the logical operator $\wedge$, defined according to the following rules:

- If P is true and Q is true, then $P \wedge Q$ is true;
- If $P \wedge Q$ is true, then P is true;
- If $P \wedge Q$ is true, then Q is true.

$P \wedge Q$ represents "P and Q".

The introduction rule

$$\frac{P \qquad Q}{P \wedge Q}\ \wedge\mathrm{I}$$

The elimination rule

$$\frac{P \wedge Q}{P}\ \wedge\mathrm{E}_\ell \qquad\qquad \frac{P \wedge Q}{Q}\ \wedge\mathrm{E}_\mathrm{r}$$

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$.

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$. Suppose we know the propositions

$$(P_1 \Rightarrow P_2)\,,\ (P_2 \Rightarrow P_3)\,,\ \ldots\,,\ (P_{n-1} \Rightarrow P_n)$$

to be true.

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$. Suppose we know the propositions

$$(P_1 \Rightarrow P_2)\,,\,(P_2 \Rightarrow P_3)\,,\;\ldots\;,\,(P_{n-1} \Rightarrow P_n)$$

to be true. Then $P_1 \Rightarrow P_n$ is true.

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$. Suppose we know the propositions

$$(P_1 \Rightarrow P_2)\,,\ (P_2 \Rightarrow P_3)\,,\ \ldots\ ,\ (P_{n-1} \Rightarrow P_n)$$

to be true. Then $P_1 \Rightarrow P_n$ is true.

For instance if $n = 3$ we have

$$\dfrac{\dfrac{\dfrac{}{[P_1]}\,{}^{1} \qquad P_1 \Rightarrow P_2}{P_2} \qquad P_2 \Rightarrow P_3}{}$$

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$. Suppose we know the propositions

$$(P_1 \Rightarrow P_2)\,,\ (P_2 \Rightarrow P_3)\,,\ \ldots\,,\ (P_{n-1} \Rightarrow P_n)$$

to be true. Then $P_1 \Rightarrow P_n$ is true.

For instance if $n = 3$ we have

$$\dfrac{\dfrac{\dfrac{}{[P_1]}\,1 \qquad P_1 \Rightarrow P_2}{P_2} \qquad P_2 \Rightarrow P_3}{P_3}$$

Example: chain of implications

Suppose $P_1, \ldots, P_n : \mathbb{P}\mathrm{rop}$. Suppose we know the propositions

$$(P_1 \Rightarrow P_2)\,,\ (P_2 \Rightarrow P_3)\,,\ \ldots\ ,\ (P_{n-1} \Rightarrow P_n)$$

to be true. Then $P_1 \Rightarrow P_n$ is true.

For instance if $n = 3$ we have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[P_1]}{}^{1} \qquad P_1 \Rightarrow P_2}{P_2} \qquad P_2 \Rightarrow P_3}{P_3}}{P_1 \Rightarrow P_3}{}^{1}$$

A lazy (but wrong) way of writing

$$(P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_3) \wedge \ldots \wedge (P_{n-1} \Rightarrow P_n)\,. \tag{1}$$

is

$$P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow \ldots \Rightarrow P_n\,.$$

However since the expression (1) is tedious as it repeats the propositions $P_2, \ldots, P_{n-1}$, we allow for the short hand notation

$$\begin{aligned} & P_1 \\ \Rightarrow\ & P_2 \\ \Rightarrow\ & \ldots \\ \Rightarrow\ & P_n \end{aligned}$$

to denote that the chain of implications (1) leads to the conclusion $P_1 \Rightarrow P_n$.

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{}{[P \wedge Q]}\,1}{Q}$$

1

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\dfrac{\qquad\qquad\qquad}{} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{}}{} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{Q}}{}$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\overline{} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{P}}{} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{Q}}{}$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}\,{}^{2} \qquad \dfrac{\dfrac{}{[P \wedge Q]}\,{}^{1}}{P}}{} \qquad\qquad \dfrac{\dfrac{}{[P \wedge Q]}\,{}^{1}}{Q}}{}$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\overline{[P \Rightarrow (Q \Rightarrow R)]}^{\,2} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{P}}{Q \Rightarrow R} \qquad \dfrac{\overline{[P \wedge Q]}^{\,1}}{Q}}{}$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}\;2 \qquad \dfrac{\dfrac{}{[P \wedge Q]}\;1}{P}}{Q \Rightarrow R} \qquad \dfrac{\dfrac{}{[P \wedge Q]}\;1}{Q}}{R}$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}\;{}^{2} \qquad \dfrac{\dfrac{}{[P \wedge Q]}\;{}^{1}}{P}}{Q \Rightarrow R} \qquad \dfrac{\dfrac{}{[P \wedge Q]}\;{}^{1}}{Q}}{R}}{P \wedge Q \Rightarrow R}\;{}^{1}
$$

Example.

We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}\,{}^{2} \quad \dfrac{\dfrac{}{[P \wedge Q]}\,{}^{1}}{P}}{Q \Rightarrow R} \quad \dfrac{\dfrac{}{[P \wedge Q]}\,{}^{1}}{Q}}{R}}{P \wedge Q \Rightarrow R}\,{}^{1}}{(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)}\,{\color{red}{}^{2}}$$

A natural deduction proof has the shape of a “tree” in which the nodes are decorated with propositions. The proposition occurring at the root of the tree is the conclusion, whereas the proposition at the leaves of the tree are its assumption.

A natural deduction proof has the shape of a “tree” in which the nodes are decorated with propositions. The proposition occurring at the root of the tree is the conclusion, whereas the proposition at the leaves of the tree are its assumption.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}{}^{2} \quad \dfrac{\dfrac{}{[P \wedge Q]}{}^{1}}{P}}{Q \Rightarrow R} \quad \dfrac{\dfrac{}{[P \wedge Q]}{}^{1}}{Q}}{R}}{P \wedge Q \Rightarrow R}{}^{1}}{(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)}{}^{2}$$

The rules of inference for disjunction

The disjunction operator is the logical operator $\vee$, defined according to the following rules:

The rules of inference for disjunction

The disjunction operator is the logical operator $\vee$, defined according to the following rules:

- If P is true, then $P \vee Q$ is true;

The introduction rule

$$\frac{P}{P \vee Q}\ \vee\mathrm{I}_\ell \qquad \frac{Q}{P \vee Q}\ \vee\mathrm{I}_\mathrm{r}$$

The rules of inference for disjunction

The disjunction operator is the logical operator $\vee$, defined according to the following rules:

- If P is true, then $P \vee Q$ is true;
- If Q is true, then $P \vee Q$ is true;

The introduction rule

$$\frac{P}{P \vee Q}\ \vee\mathrm{I}_\ell \qquad \frac{Q}{P \vee Q}\ \vee\mathrm{I}_r$$

The elimination rule

$$\frac{P \vee Q \qquad \begin{array}{c}\overline{[P]}^{\,1} \\ \vdots \\ R\end{array} \qquad \begin{array}{c}\overline{[Q]}^{\,1} \\ \vdots \\ R\end{array}}{R}\ 1\ \vee\mathrm{E}$$

The rules of inference for disjunction

The disjunction operator is the logical operator $\vee$, defined according to the following rules:

- If P is true, then $P \vee Q$ is true;
- If Q is true, then $P \vee Q$ is true;
- If $P \vee Q$ is true, and if R can be derived from P and from Q, then R is true.

$P \vee Q$ represents "P or Q".

The introduction rule

$$\frac{P}{P \vee Q}\ \vee\mathrm{I}_\ell \qquad \frac{Q}{P \vee Q}\ \vee\mathrm{I}_\mathrm{r}$$

The elimination rule

$$\frac{P \vee Q \qquad \begin{array}{c} \overline{[P]}^{\,1} \\ \vdots \\ R \end{array} \qquad \begin{array}{c} \overline{[Q]}^{\,1} \\ \vdots \\ R \end{array}}{R}\ 1\ \vee\mathrm{E}$$

Example.

We show that

$$((P \vee Q) \Rightarrow R) \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

is a tautology.

The rules of inference for negation

A **contradiction** is a proposition that is known or assumed to be false.

The rules of inference for negation

A **contradiction** is a proposition that is known or assumed to be false.

We will use the symbol $\perp$ to represent an arbitrary contradiction.

The rules of inference for negation

A **contradiction** is a proposition that is known or assumed to be false.

We will use the symbol $\perp$ to represent an arbitrary contradiction.

The expression $\neg P$ represents "not P" (or "P is false").

The elimination rule

$$\frac{\perp}{P}\;\perp\mathsf{E}$$

The rules of inference for negation

The **negation** operator is the logical operator $\neg$, defined according to the following rules:

- If a contradiction can be derived from the assumption that P is true, then $\neg P$ is true;
- If $\neg P$ and P are both true, then a contradiction may be derived.

The expression $\neg p$ represents "not P" (or "P is false").

The introduction rule

$$\dfrac{\dfrac{}{[P]}^{1} \\ \vdots \\ \bot}{\neg P}\, 1 \ \neg\mathrm{I}$$

The elimination rule

$$\dfrac{\neg P \qquad P}{\bot}\,\neg\mathrm{E}$$

In order to prove a proposition P is false (that is, that $\neg P$ is true), it suffices to assume that P is true and derive a contradiction.

Example.

Show that

$$\big((P \vee Q) \wedge \neg Q\big) \Rightarrow P\,.$$

Example.

Show that

$$\big((P \vee Q) \wedge \neg Q\big) \Rightarrow P\,.$$

$$
\dfrac{\dfrac{\dfrac{\overline{[((P \vee Q) \wedge \neg Q)]}^{\,1}}{P \vee Q} \qquad \dfrac{\overline{[P]}^{\,2}}{P} \qquad \dfrac{\dfrac{\dfrac{\overline{[Q]}^{\,2}}{Q} \qquad \dfrac{\overline{[((P \vee Q) \wedge \neg Q)]}^{\,1}}{\neg Q}}{\bot}}{P}}{P}\;{}^{2}}{\big((P \vee Q) \wedge \neg Q\big) \Rightarrow P}\;{}^{1}
$$

Example.

Show that 0 *is the only real solution to the equation*

$$x + \sqrt{x} = 0\,.$$

$$\begin{aligned} & x + \sqrt{x} = 0 & \\ \Rightarrow\ & x = -\sqrt{x} & \text{rearranging} \\ \Rightarrow\ & x^2 = x & \text{squaring} \\ \Rightarrow\ & x(x-1) = 0 & \text{rearranging} \\ \Rightarrow\ & x = 0 \text{ or } x = 1 & \end{aligned}$$

Example.

Show that 0 *is the only real solution to the equation*

$$x + \sqrt{x} = 0\,.$$

$$\begin{aligned}
& x + \sqrt{x} = 0 & \\
\Rightarrow\ & x = -\sqrt{x} & \text{rearranging} \\
\Rightarrow\ & x^2 = x & \text{squaring} \\
\Rightarrow\ & x(x-1) = 0 & \text{rearranging} \\
\Rightarrow\ & x = 0 \text{ or } x = 1 &
\end{aligned}$$

Now certainly 0 is a solution to the equation, since $0 + \sqrt{0} = 0 + 0 = 0$.

Example.

Show that 0 *is the only real solution to the equation*

$$x + \sqrt{x} = 0\,.$$

$$\begin{aligned} & x + \sqrt{x} = 0 & \\ \Rightarrow\ & x = -\sqrt{x} & \text{rearranging} \\ \Rightarrow\ & x^2 = x & \text{squaring} \\ \Rightarrow\ & x(x-1) = 0 & \text{rearranging} \\ \Rightarrow\ & x = 0 \text{ or } x = 1 & \end{aligned}$$

Now certainly 0 is a solution to the equation, since $0 + \sqrt{0} = 0 + 0 = 0$. However, 1 is *not* a solution, since $1 + \sqrt{1} = 1 + 1 = 2$.

Hence it is actually the case that, given a real number x, we have

$$x + \sqrt{x} = 0 \quad \Leftrightarrow \quad x = 0$$

Checking the converse here was vital to our success in solving the equation!

Example

Proposition.

Let $n \in \mathbb{Z}$. Then n^2 leaves a remainder of 0 or 1 when divided by 3.

We use the elimination rule of disjunction (from Definition 1.1.12).

$$\dfrac{p_1 \vee p_2 \vee p_3 \qquad \begin{array}{c}[p_1]\\ \rightsquigarrow \\ q\end{array} \qquad \begin{array}{c}[p_2]\\ \rightsquigarrow \\ q\end{array} \qquad \begin{array}{c}[p_3]\\ \rightsquigarrow \\ q\end{array}}{q}\ (\vee\text{E})$$

Determine what p_1, p_2, p_3 and q are.

Proposition.

Consider the polynomial $p(x) = x^2 + ax + b$ whose coefficients a, b are real numbers and whose discriminant $\Delta = a^2 - 4b$ is non-zero. If $p(x)$ has two distinct roots, then their difference is either a real number or a purely imaginary number. Furthermore, if $\Delta \geqslant 0$, then the difference of the roots is a real number.

First, let us find the logical form of the proposition above.

$$\Big(\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C},$$

$$(\alpha^2 + a\alpha + b = 0) \wedge (\beta^2 + a\beta + b = 0) \wedge (\alpha \neq \beta) \wedge (\alpha - \beta = x + yi)\Big)$$

$$\Rightarrow \big(x = 0 \vee y = 0\big)$$