MATH 301

INTRODUCTION TO PROOFS

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- Natural Deduction (cont'd)
- Using logic in math

In the last lecture

We learned about propositional logic and a way of encoding the inference rules of proposition called natural deduction.

In this lecture

We shall continue the trend of the last lecture:

- We shall introduce the inference rules of disjunction, contradiction and negation.
- We shall talk about the law of excluded middle and the law of double negation.

We shall also talk about the semantic aspects of (classical) propositional logic:

- We shall introduce the truth tables of propositions.
- We shall use the truth tables to talk about the meaning of propositions.

Overview

1. Natural deduction for disjunction and negation
2. Using logic in mathematical reasoning

Recall: New propositions from the old

- Recall that given propositions P and Q, we can make the following new propositions:

Proposition	Notation
P and Q	$P \wedge Q$
P or Q	$P \vee Q$
P implies Q	$P \Rightarrow Q$
P if and only if Q	$P \Leftrightarrow Q$
not P	$\neg P$

- Therefore, if $P : \mathbb{P}\mathrm{rop}$ and $Q : \mathbb{P}\mathrm{rop}$ then $P \wedge Q : \mathbb{P}\mathrm{rop}$, $P \vee Q : \mathbb{P}\mathrm{rop}$, $P \Rightarrow Q : \mathbb{P}\mathrm{rop}$, $P \Leftrightarrow Q : \mathbb{P}\mathrm{rop}$, $\neg P : \mathbb{P}\mathrm{rop}$, $\neg Q : \mathbb{P}\mathrm{rop}$, etc.

Few things to note

- Note that we use upper-case letters to denote propositions.
- $P \Rightarrow Q$: if P then Q, or P is sufficient for Q, or Q is necessary from P.
- $\neg P$: it is not the case that P.

Recall that

In the examples of natural deduction which we did in the last lecture, we learned to think

- what the formulas say,
- which rule of inference is invoked at each inference step, and
- which hypotheses are canceled at each stage.

If we look at any node of the tree, what has been established at that point is that the claim follows from all the hypotheses above it that haven't been canceled yet.

A natural deduction proof has the shape of a "tree" in which the nodes are decorated with propositions. The proposition occurring at the root of the tree is the conclusion, whereas the proposition at the leaves of the tree are its assumption.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[P \Rightarrow (Q \Rightarrow R)]}\,{}^{2} \quad \dfrac{\dfrac{}{[P \land Q]}\,{}^{1}}{P}}{Q \Rightarrow R} \quad \dfrac{\dfrac{}{[P \land Q]}\,{}^{1}}{Q}}{R}}{P \land Q \Rightarrow R}\,{}^{1}}{(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \land Q \Rightarrow R)}\,{}^{2}$$

The rules of inference for disjunction

The disjunction operator is the logical operator $\vee$, defined according to the following rules:

- If P is true, then $P \vee Q$ is true;
- If Q is true, then $P \vee Q$ is true;
- If $P \vee Q$ is true, and if R can be derived from P and from Q, then R is true.

$P \vee Q$ represents "P or Q".

The introduction rule

$$\frac{P}{P \vee Q}\,\vee\mathrm{I}_\ell \qquad \frac{Q}{P \vee Q}\,\vee\mathrm{I}_r$$

The elimination rule

$$\cfrac{P \vee Q \qquad \begin{array}{c} \overline{[P]}^{\,1} \\ \vdots \\ R \end{array} \qquad \begin{array}{c} \overline{[Q]}^{\,1} \\ \vdots \\ R \end{array}}{R}\;1\ \vee\mathrm{E}$$

Example.

We show that

$$((P \vee Q) \Rightarrow R) \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

is a tautology.

The rules of inference for negation

A **contradiction** is a proposition that is known or assumed to be false.

We will use the symbol $\perp$ to represent an arbitrary contradiction.

The expression $\neg P$ represents "not P" (or "P is false").

The elimination rule

$$\frac{\perp}{P}\ \perp\mathsf{E}$$

The rules of inference for negation

The **negation** operator is the logical operator $\neg$, defined according to the following rules:

- If a contradiction can be derived from the assumption that P is true, then $\neg P$ is true;
- If $\neg P$ and P are both true, then a contradiction may be derived.

The expression $\neg p$ represents "not P" (or "P is false").

The introduction rule

$$\dfrac{\dfrac{}{[P]}{}^{1} \quad \vdots \quad \bot}{\neg P}\,{}^{1}\ \neg\mathrm{I}$$

The elimination rule

$$\dfrac{\neg P \qquad P}{\bot}\ \neg\mathrm{E}$$

In order to prove a proposition P is false (that is, that $\neg P$ is true), it suffices to assume that P is true and derive a contradiction.

Example.

Construct a proof of

$$\big((P \lor Q) \land \neg Q\big) \Rightarrow P$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[\big((P \lor Q) \land \neg Q\big)]}\,{\scriptstyle 1}}{P \lor Q} \qquad \dfrac{\dfrac{}{[P]}\,{\scriptstyle 2}}{P} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[Q]}\,{\scriptstyle 2}}{Q} \qquad \dfrac{\dfrac{}{[\big((P \lor Q) \land \neg Q\big)]}\,{\scriptstyle 1}}{\neg Q}}{\perp}}{P}}{P}\,{\scriptstyle 2}}{\big((P \lor Q) \land \neg Q\big) \Rightarrow P}\,{\scriptstyle 1}$$

Example.

Show that

$$P \Rightarrow \neg\neg P$$

is a tautology.

Truth Tables

A truth table is a table showing the truth value of a propositional logic formula as a function of its inputs.

Useful for

- Formally defining what a connective “means”.
- Deciphering what a complex propositional formula means.
- Finding out which formulas are not tautology.

Overview

1. Natural deduction for disjunction and negation
2. Using logic in mathematical reasoning

Example.

Show that 0 *is the only real solution to the equation*

$$x + \sqrt{x} = 0\,.$$

$$\begin{aligned}
& x + \sqrt{x} = 0 & \\
\Rightarrow\ & x = -\sqrt{x} & \text{rearranging} \\
\Rightarrow\ & x^2 = x & \text{squaring} \\
\Rightarrow\ & x(x-1) = 0 & \text{rearranging} \\
\Rightarrow\ & x = 0 \text{ or } x = 1 &
\end{aligned}$$

Now certainly 0 is a solution to the equation, since $0 + \sqrt{0} = 0 + 0 = 0$. However, 1 is *not* a solution, since $1 + \sqrt{1} = 1 + 1 = 2$.

Hence it is actually the case that, given a real number x, we have

$$x + \sqrt{x} = 0 \quad \Leftrightarrow \quad x = 0$$

Checking the converse here was vital to our success in solving the equation!

Note that *the formal expression of our reasoning* is of the form

$$\big((P \vee Q) \wedge \neg Q\big) \Rightarrow P\,.$$

Example

Proposition.

Let $n \in \mathbb{Z}$. Then n^2 leaves a remainder of 0 or 1 when divided by 3.

The proof is constructed using the argument by cases which is exactly the elimination rule of disjunction (from Definition 1.1.12).

$$\dfrac{p_1 \vee p_2 \vee p_3 \qquad \begin{array}{c}[p_1]\\ \vdots \\ q\end{array} \qquad \begin{array}{c}[p_2]\\ \vdots \\ q\end{array} \qquad \begin{array}{c}[p_3]\\ \vdots \\ q\end{array}}{q}\ (\vee\text{E})$$

Determine what p_1, p_2, p_3 and q are, and construct the proof.

Thanks for your attention!

Time for your questions!