ex3.

the scope of x

the scope of y

(i) $(\exists x. \heartsuit(\text{you}, x)) \wedge (\exists y. \heartsuit(y, \text{you}))$

(ii) $\exists x. \heartsuit(\text{you}, x) \wedge (\exists x. \heartsuit(x, \text{you}))$

(iii) $\exists x. (\heartsuit(\text{you}, x) \wedge \heartsuit(x, \text{you}))$

the scope of x $\Longrightarrow$ you are happy

$(i) \equiv (ii) \not\equiv (iii)$

④ Free & bound variables.

bound to $\forall$

$\forall x.\ A(x)$

$\forall x.\ A(x) \lor B(x)$

$\forall x.\ (\mathrm{even}(x) \lor \mathrm{odd}(x))$

$\forall x.\ A(x) \land C(x, y)$

free

Ex. $P(x,y) :=$ x is a parent of y.

$C(x,y) :=$ x and y are a couple.

Consider the following statements.

(i) $\exists z\ P(x,z) \Rightarrow \exists y\ C(x,y)$

(ii) $\neg\left(C(x,y) \Rightarrow \exists z\ P(x,z) \wedge P(y,z)\right)$

In (i) $\begin{cases} \text{free vars: } x \\ \text{bounded vars: } y, z \end{cases}$

In (ii) $\begin{cases} \text{free vars: } x, y \\ \text{bounded vars: } z \end{cases}$

Ex. In the statement "for every integer n, there is a prime number p between n and $2n$"

n is a bound variable and
p is a free variable.

Formally,

$$\forall n \; \exists p \left(\text{Prime}(p) \wedge (n \leq p) \wedge (p \leq 2n) \right)$$

Ex. In the statement

$$a \geq 0 \Longrightarrow \exists b \; (a = b^2)$$

a is free and b is bounded.

The statement

$$\forall x.\ A(x)$$

is not at all about x.

e.g. $\forall x.\ \mathrm{even}(x) \vee \mathrm{odd}(x)$

says

every natural number is either even or odd;

no reference to x.

4½ Change of bound variables

(i)

$$\forall x \, (\text{even}(x) \vee \text{odd}(x)) \equiv$$

$$\forall y \, (\text{even}(y) \vee \text{odd}(y))$$

(ii) $\forall y \, (x \leq y) \equiv \forall z \, (x \leq z)$

⚠ z must be a fresh variable.

e.g.

$$\forall y \, (x \leq y) \not\equiv \forall x . (x \leq x)$$

(iii) $\underbrace{\forall y \, (x \leq y)}_{\text{statement about } x} \not\equiv \underbrace{\forall y \, (w \leq y)}_{\text{statement about } w}$

⚠ substitution (i.e. change) of a free variable by a fresh variable results in different formula.

$$P(x) \rightsquigarrow P(x)[w/x] \equiv P(w)$$

Binding priorities

Earlier in the course, we learned about the binding priorities of propositions:

(i) $\neg$ (ii) $\wedge, \vee$ (iii) $\Rightarrow$

Now, we add quantifier in between:

(i) $\neg$ (ii) $\forall, \exists$ (iii) $\wedge, \vee$, (iv) $\Rightarrow$

For instance the expression $\exists x.\ A(x) \wedge B(x)$ is parsed as $\left(\exists x. A(x)\right) \wedge B(x)$.

<u>Example</u>. Parse the following expressions by inserting brackets following the binding convention:

$$\exists x \ D(x) \Rightarrow \forall y \ D(y)$$

<u>Answer</u>:

$$\left(\exists x \ D(x)\right) \Rightarrow \left(\forall y \ D(y)\right) \qquad (1)$$

which is different from

$$\exists x \ \left(D(x) \Rightarrow \forall y \ D(y)\right) \qquad (2)$$

(1) is true if the universe of discourse is empty whereas (2) is false in that case

How to prove a universally quantified statement

Recall that in order to prove that $\sqrt{2}$ is not rational we used the lemma

$$\forall a \, \forall b \, (b \neq 0) \Rightarrow \neg (a^2 = 2b^2)$$

To prove the latter we start by letting

a, b to be arbitrary integers.

Here is how the proof of irrationality of $\sqrt{2}$ goes:

Let a and b be arbitrary integers.
Suppose $b \neq 0$, and
suppose $a^2 = 2b^2$

$\vdots$

Contradiction. $\square$

Here's the last proof presented in natural deduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \neq 0}\,1 \qquad\qquad \dfrac{}{a^2 = 2b^2}\,2}{\vdots \qquad\qquad\qquad \vdots}}{\perp}}{\neg (a^2 = 2b^2)}\,2}{b \neq 0 \Rightarrow \neg (a^2 = 2b^2)}\,1}{\dfrac{\forall b\ (b \neq 0 \Rightarrow \neg (a^2 = 2b^2))}{\forall a\ \forall b\ (b \neq 0 \Rightarrow \neg (a^2 = 2b^2))}}
$$

intro rule for $\forall$

$$\frac{A(x)}{\forall x . A(x)} \ (\forall \text{ intro})$$

⚠ x should not be free in any hypothesis which has not been cancelled!

e.g.

$$\frac{\begin{matrix} \overline{P(x)}^{\,1} \\ \vdots \\ Q(x) \end{matrix}}{\forall x . Q(x)}$$

x free in $P(x)$

Not allowed!

Elim rule for $\forall$

$$\frac{\forall x. A(x)}{A(t)}\ (\forall\text{elim})$$

t : an arbitrary term (?)

$$\frac{\forall x \exists y\ y > x}{\exists y.\ y > y+1}$$ $(x = y+1)$

truth is not preserved!
what did we do wrong?

Example. We construct a natural deduction proof of

$$\forall x\, A(x) \Rightarrow \forall x\, B(x) \Rightarrow \forall y\,(A(y) \wedge B(y))$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\, A(x)}{\scriptstyle 1}}{A(y)}{\scriptstyle(\forall E)} \qquad \dfrac{\dfrac{}{\forall x\, B(x)}{\scriptstyle 2}}{B(y)}{\scriptstyle(\forall E)}}{A(y) \wedge B(y)}{\scriptstyle(\wedge I)}}{\forall y\,(A(y) \wedge B(y))}}{\forall x\, B(x) \rightarrow \forall y\,(A(y) \wedge B(y))}{\scriptstyle 2}}{\forall x\, A(x) \rightarrow \forall x\, B(x) \rightarrow \forall y\,(A(y) \wedge B(y))}{\scriptstyle 1}
$$

Ex. In a town there is a barber that shaves all and only the men who do not shave themselves. Show that this is a contradiction.

Define $S(x,y) =$ x shaves y.

$$\dfrac{\dfrac{}{S(b,b)\vee\neg S(b,b)}(LEM) \qquad \dfrac{\overline{S(b,b)}^{\,1} \qquad \dfrac{\overline{S(b,b)}^{\,1} \quad \dfrac{\forall x.(S(b,x)\Leftrightarrow\neg S(x,x))}{S(b,b)\Leftrightarrow\neg S(b,b)}}{\neg S(b,b)}}{\bot} \qquad \dfrac{\overline{\neg S(b,b)}^{\,2} \qquad \dfrac{\overline{\neg S(b,b)}^{\,1} \quad \dfrac{\forall x.(S(b,x)\Leftrightarrow\neg S(x,x))}{S(b,b)\Leftrightarrow\neg S(b,b)}}{S(b,b)}}{\bot}}{\bot}\,1$$

Ex. Suppose E and O are predicates with one variable ranging over natural numbers. Suppose also that

(i) $\forall n.\ (\neg E(n) \Rightarrow O(n))$

Prove that $\forall n.\ O(n) \vee E(n)$

$$
\dfrac{
\dfrac{
E(n) \vee \neg E(n)
\qquad
\dfrac{\dfrac{}{E(n)}\,1}{O(n) \vee E(n)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\forall n.(\neg E(n) \to O(n))}{\neg E(n) \to O(n)} \qquad \dfrac{}{\neg E(n)}\,1}{O(n)}}{O(n) \vee E(n)}}{}
}{O(n) \vee E(n)}
}{\forall n.\ O(n) \vee E(n)}
$$

Intro rule for $\exists$

$$\frac{A(t)}{\exists x\, A(x)}$$

Not quite correct!

$$\frac{\forall y\, (y=y)}{\exists x\, \forall y\, (y=x)}$$

A wrong inference

Note: when we substitute x for t in $\frac{A(t)}{\exists x\, A(x)}$

Elim rule for $\exists$

$$\dfrac{\exists x\, A(x) \qquad \begin{array}{c}\overline{A(y)}^{\,1} \\ \vdots \\ B\end{array}}{B}$$

Ex. Let's prove

$$\exists x\,(A(x) \vee B(x)) \Rightarrow \exists x\, A(x) \vee \exists x\, B(x)$$

$$\dfrac{\dfrac{\overline{\exists x\,(A(x) \vee B(x))}^{\,1} \qquad \dfrac{\overline{A(y) \vee B(y)}^{\,2} \qquad \dfrac{\dfrac{\overline{A(y)}^{\,3}}{\exists x\, A(x)}}{\exists x A(x) \vee \exists x B(x)} \qquad \dfrac{\dfrac{\overline{B(y)}^{\,3}}{\exists x\, B(x)}}{\exists x A(x) \vee \exists x B(x)}}{\exists x\, A(x) \vee \exists x\, B(x)}\,3}{\exists x\, A(x) \vee \exists x\, B(x)}\,2}{\exists x\,(A(x) \vee B(x)) \Rightarrow \exists x\, A(x) \vee \exists x\, B(x)}\,1$$

Recall

$$\dfrac{\exists x\, P(x) \qquad \begin{matrix}\overline{P(y)}^{\,1} \\ \vdots \\ Q\end{matrix}}{Q}$$

What if the domain of quantification is empty?!

$P(x)$: predicate with variable x ranging over $X = \emptyset$.

$\exists x. P(x)$ is $\bot$ (false)

$\forall x. P(x)$ is $\top$ (true)

But, why?

$$\dfrac{\exists x. A(x) \qquad \begin{array}{c} \overline{A(y)}^{1} \\ \vdots \\ B \end{array}}{B}$$

this holds trivially for any B.

Therefore, $\dfrac{\exists x. A(x)}{B}$

similar to $\dfrac{\bot}{B}$

Q. Can we derive

$$\frac{\forall x.\ A(x)}{\exists x.\ A(x)}$$

$$\cfrac{\cfrac{\forall x.\ A(x)}{A(y)}}{\exists x.\ A(x)}$$

Aristotle thought we can.

But what if the domain/universe of discourse of A is empty? We get non-sense!

$$\frac{\top}{\bot}$$

Most likely, Aristotle must have excluded empty universes of discourse.

Many Sorted Logic

So far, in each example we have considered our variables range over the same universe of discourse.

$\forall x \exists y \; x \leq y$ where x, y are integers.

$\exists x \forall y \; \heartsuit(y, x)$ where x, y are humans.

...

Consider the following example from Euclidean geometry.

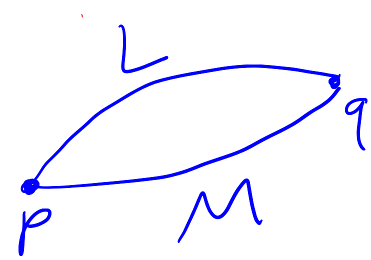
Suppose we want to say that for any two distinct points on the Euclidean plane there is a unique line which passes through them.

$$\forall p \, \forall q \, \forall L \, \forall M$$

$$\big(Point(p) \wedge Point(q) \wedge Line(L) \wedge Line(M) \wedge On(p,L) \wedge On(q,L) \wedge On(p,M) \wedge On(q,M) \big)$$

$$\Longrightarrow (p \neq q) \Longrightarrow L = M$$

This is very Tedious!



Instead, we introduce **Sorts**.

In the previous example,

Instead of writing

Point (p), Point(q), . . .   and

Line (L), Line(M), . . .   we

simply introduce two Sorts

Point and Line.

When we write $p : \text{Point}$

we mean $\text{Point}(p)$

When we write $L : \text{Line}$

we mean $\text{Line}(L)$.

$\forall p \forall q \ \forall L \ \forall M$

$(Point(p) \wedge Point(q) \wedge Line(L) \wedge Line(M)$

$\wedge \ On(p, L) \wedge On(q, L) \wedge On(p, M) \wedge On(q, M))$

$\Rightarrow (p \neq q) \Rightarrow L = M$

$\forall p, q : Point \quad \forall L, M : Line$

$On(p, L) \wedge On(q, L) \wedge On(p, M), On(q, M)$

$\Rightarrow (p \neq q) \Rightarrow L = M$

Using <u>equality</u> in logic we can express statements like

(i) there are at least two elements x for which $A(x)$ is true.

$$\exists x,y \; (\neg(x=y) \wedge A(x) \wedge A(y))$$

(ii) there are at most one element x for which $A(x)$ is true.

$$\forall x,y \; (A(x) \wedge A(y) \Rightarrow x=y)$$

(iii) there are at most n elements x for which $A(x)$ is true.

$$\forall x_1,\dots,x_{n+1} \; (A(x_1) \wedge \dots \wedge A(x_{n+1}) \Rightarrow$$
$$(x_1=x_2) \vee \dots \vee (x_1=x_{n+1})$$
$$\vee (x_2=x_3) \vee \dots \vee (x_2=x_{n+1})$$
$$\vee \dots \quad (x_n=x_{n+1}))$$

Special notation for unique existence

We denote the statement that there is a unique x such that $A(x)$ is true by $\exists! x\, A(x)$.

Note that

$$\exists! x\ A(x) \equiv$$

$$\left(\exists x\, A(x)\right) \wedge \left(\forall x \forall y \left(A(x) \wedge A(y) \Rightarrow x = y\right)\right)$$

$$\equiv$$

$$\exists x \left(A(x) \wedge \forall y \left(A(y) \Rightarrow x = y \right) \right)$$

Counterexamples

Given a formula of the form

$$\forall x.\ A(x)$$

a counterexample is a term t

such that $\neg A(t)$.

Example. Find counterexamples to the statements below

(i) Every prime integer is odd.

(ii) Every integer has a prime factor.

(iii) Every perfect number is even.

$6 = 1 + 2 + 3$

A counterexample is a proof of $\neg \forall x.\, A(x)$ because

$$\neg \forall x.\, A(x) \iff \exists x.\, \neg A(x)$$

is a tautology, using Excluded Middle.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg \forall x.\, A(x)}^{\,1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg \exists x.\, \neg A(x)}^{\,2} \qquad \dfrac{\overline{\neg A(x)}^{\,3}}{\exists x.\, \neg A(x)}}{\bot}\,3}{\neg\neg A(x)}}{A(x)}}{\forall x.\, A(x)}}{\bot}\,2}{\neg\neg \exists x.\, \neg A(x)}\,\mathrm{LEM}}{\exists x.\, \neg A(x)}}{\neg \forall x.\, A(x) \Rightarrow \exists x.\, \neg A(x)}$$

Conversely,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.\, \neg A(x)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg A(y)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\forall x.\, A(x)$}
\UnaryInfC{$A(y)$}
\BinaryInfC{$\bot$}
\RightLabel{2}
\UnaryInfC{$\neg \forall x.\, A(x)$}
\RightLabel{3}
\BinaryInfC{$\neg \forall x.\, A(x)$}
\RightLabel{1}
\UnaryInfC{$\exists x.\, \neg A(x) \Rightarrow \neg \forall x.\, A(x)$}
\end{prooftree}
$$

<u>Exercise</u>. Prove the dual equivalence yourself.

Suppose variable x is not free in B.

We prove the equivalence

$$\exists x\,(A(x) \Rightarrow B) \iff (\forall x\, A(x) \Rightarrow B)$$

(1) First, let's prove $\quad \exists x\,(A(x) \Rightarrow B) \Rightarrow (\forall x\, A(x)) \Rightarrow B$

$$\dfrac{\dfrac{\dfrac{}{\exists x\,(A(x)\Rightarrow B)}{}^{1} \qquad \dfrac{\dfrac{\dfrac{}{A(y)\Rightarrow B}{}^{3} \qquad \dfrac{\dfrac{}{\forall x\, A(x)}{}^{2}}{A(y)}}{B}{}^{2}}{\forall x\, A(x) \Rightarrow B}}{\forall x\, A(x) \Rightarrow B}{}^{3}}{\exists x\,(A(x)\Rightarrow B) \Rightarrow (\forall x\, A(x)) \Rightarrow B}{}^{1}$$

<u>Cor.</u> (Drinker's paradox)

$$\exists x\,(D(x) \Rightarrow \forall x\, D(x)) \Leftarrow (\forall x\, D(x) \Rightarrow \forall x\, D(x))$$