MATH 301

INTRODUCTION TO PROOFS

Sina Hazratpour
Johns Hopkins University
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- integers
- rational numbers
- real numbers

Relevant sections of the textbook

- Section B.2. (incomplete!)

Quotients by relations

Recall from problem 5 of homework #4 that for each a binary relation R on a set X we can construct a set X/R whose elements are R-classes

$$[x]_R = \{y \in X \mid R(x, y)\}$$

for all $x \in X$.

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We call the set X/R the quotient of X by the relation R.

Example

Consider the set of natural numbers with the usual ordering $\leqslant : \mathbb{N} \to \mathbb{N} \to \mathbf{2}$ *defined for* $m \in \mathbb{N}$ *recursively by*

$m \leqslant 0$ *if and only if* $m = 0$, *and*

$m \leqslant \mathrm{succ}(n)$ *if and only if* $m = \mathrm{succ}(n)$ *or* $m \leqslant n$.

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We have classes $[n]$ *forming a chain in the subset relation ordering:* $[0] \supset [1] \supset [2] \supset \ldots$. *Note that* $0 \leqslant 1$ *but* $[0] \neq [1]$.

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- What is a class $[a]$ for a vertex a?
- Show that if $R(a, b) \wedge R(b, a)$ then it is not necessarily true that $[a] = [b]$.

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- What is a class $[a]$ for a vertex a?
- Show that if $R(a, b) \wedge R(b, a)$ then $[a] = [b]$.

Proposition

Prove that the function $q\colon X \to X/R$ assigning to each x in X the class $[x]$ in X/R is a surjection.

The universal mapping property of quotient construction

Proposition

Let R be a symmetric and transitive relation on a set X. For any set Y, precomposing with q yields a bijection

$$(X/R \to Y) \cong \{f\colon X \to Y \mid \forall x, y \in X,\ R(x, y) \Rightarrow f(x) = f(y)\}$$

Recall that a relation R on a set A is called an equivalence if it satisfies the following conditions:

- **reflexivity**: $\forall a \in A,\ R(a,a)$,
- **symmetry**: $\forall a,b \in A,\ R(a,b) \to R(b,a)$, and
- **transitivity**: $\forall a,b,c \in A,\ R(a,b) \Rightarrow R(b,c) \Rightarrow R(a,c)$.

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We usually denote an equivalence relation by the symbol $\sim$ (instead of R).

Quotients by equivalence relations

For each equivalence $\sim$ on a set X we can construct a set $X/\sim$ whose elements are equivalence classes

$$[x]_\sim = \{y \in X \mid x \sim y\}$$

for all $x \in X$.

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We call the set $X/\sim$ the quotient of X by equivalence relation $\sim$.

For an equivalence relation $\sim$, the surjection $q\colon X \to X/\sim$ has an extra nice property:

$$q(x) = q(y) \Leftrightarrow R(x, y)$$

Example of quotient by an equivalence relation

Consider the relation $\sim$ on $\mathbb{N} \times \mathbb{N}$. where

$$(m, n) \sim (m', n') \Leftrightarrow m + n' = n + m' \, .$$

Prove that this relation is an equivalence.

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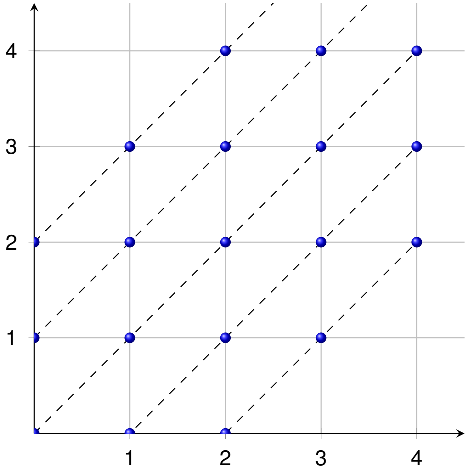
The equivalence class $[(0, 0)]$ is the set $\{(0, 0), (1, 1), (2, 2), ...\}$. What is the equivalence class $[(0, 1)]$?



4
3
2
1
1
2
3
4

Integers

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- In this case, there are *canonical representatives* of the equivalence classes: those of the form $(n, 0)$ or $(0, n)$.

Addition on integers

We can define the operation of addition on $\mathbb{Z}$ by an assignment $+_\sim : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ which assigns to the pair $([(m, n)], [(m', n')])$ the class $[(m + m', n + n')]$.

Exercise

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- *Show that for all integers a we have $0+a=a=a+0$.*
- *Show that for all integers a,b,c we have $(a+b)+c=a+(b+c)$.*
- *Show that for all integers a,b we have $a+b=b+a$.*

We can define the operation of multiplication on $\mathbb{Z}$ by an assignment $\cdot_\sim : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ which assigns to the pair $([(m, n)], [(m', n')])$ the class $[(m \cdot m', n \cdot n')]$.

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- *Show that for all integers a, b we have $a + b = b + a$.*

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Exercise

Show that the relation above is indeed an equivalence relation.

Image factorization

Proposition

Suppose $f\colon A \to B$ is a function. We can factor f into a surjection followed by a bijection followed by an injection.

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, twoheadrightarrow, "q"'] & & & B \\
& C \arrow[r, "g"'] & D \arrow[ur, rightarrowtail, "i"'] &
\end{tikzcd}$$

that is there are functions q, g, i such that $f = i \circ g \circ q$, where q is a surjection, g is a bijection, and i is an injection.

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Our factorization diagram becomes

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In fact, $g \circ q = p \colon X \to \mathbf{Im}(f)$.

Definition

For a function $f\colon X \to X$, define

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$$X/\sim_f \;\cong\; \mathrm{Fix}(f) \cong \mathbf{Im}(f)$$

Exercise

Construct an idempotent $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ *such that* $\mathrm{Fix}(f)$ *is in bijection with the set of integers.*

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Lemma

Suppose $P : \mathbb{Z} \to \mathbb{P}\mathrm{rop}$ *is a predicate over integers, and*

- $P(0)$ *holds,*
- $\forall n : \mathbb{N},\ P(n) \Rightarrow P(\mathrm{succ}(n))$*, and*
- $\forall n : \mathbb{N},\ P(-n) \to P(-\mathrm{succ}(n))$*.*

Then we have $\forall z : \mathbb{Z},\ P(z)$*.*

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We construct the *field* of rationals $\mathbb{Q}$ along the same lines as well, namely as the quotient

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where

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Here too we have a canonical choice of representatives, namely fractions in lowest terms.

The arithmetic of rational numbers

We write down the arithmetical operations on $\mathbb{Q}$ so that we can compute with fractions.

The order on rational numbers

We equip $\mathbb{Q}$ with a total order.

Real numbers

Questions

Thanks for your attention!