# MATH 301

## INTRODUCTION TO PROOFS

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- Images and pre-images
- Image factorization
- Axiom of choice

# Relevant sections of the textbook

- Chapter 3
- Chapter 5

# Images of functions

A function $f\colon X \to Y$ induces a function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$f_*(U) = \{y \in Y \mid \exists x \in U\,(y = f(x))\}$$

for any subset $U$ of $X$. The subset $f_*(S)$ is called the image of $U$ under $f$.

Note that

$$\mathrm{id}_* = \mathrm{id}_{\mathcal{P}(X)}$$

## Proposition

*Show that a function $f\colon X \to Y$ is surjective if and only if $f_*(X) = Y$.*

We sometimes denote the set $f_*(X)$ by $\mathbf{Im}(f)$.

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions. We prove that

$$g_* \circ f_* = (g \circ f)_* \,.$$

Recall that in order to prove equality of functions we need to use function extensionality.
Suppose $T$ is a subset of $Z$. Then

$$\begin{aligned}
(g_* \circ f_*)\, U &= g_* \{y \in Y \mid \exists x \in U\, (y = f(x))\} \\
&= \big\{z \in Z \mid \exists y \in Y\, \exists x \in U\, (y = f(x) \wedge z = g(y)\big\} \\
&= \{z \in Z \mid \exists x \in U\, (z = g(f(x)))\} \\
&= (g \circ f)_* U
\end{aligned}$$

# Image factorization

## Proposition

*Every function $f \colon X \to Y$ factorizes as a surjection followed by an injection, i.e. there are surjection $p$ and injection $m$ such that $f = m \circ p$.*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \\
& \bullet \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

## Proof.

Define $p$ to be the assignment $p\colon X \to \mathbf{Im}(f)$ which takes $x$ to $f(x)$. This assignment is well-defined since $f$ is well-defined and that $f(x) \in \mathbf{Im}(f)$. Note that $p$ is surjective since for any $y \in \mathbf{Im}(f)$ there is some $x$ such that $f(x) = y$ by the definition of $\mathbf{Im}(f)$ and therefore there is some $x$ such that $p(x) = f(x) = y$.

Define $m$ to be the assignment $m\colon \mathbf{Im}(f) \to Y$ which takes $y$ to $y$. This assignment is well-defined since $\mathbf{Im}(f) \subseteq Y$. Note that $m$ is injective since $m(y) = m(y')$ implies $y = y'$ simply because $m(y) = y$ for all $y \in \mathbf{Im}(f)$.

Finally we have to show that $p$ and $m$ compose to $f$. To this end, note that for every $x \in X$

$$m(p(x)) = m(f(x)) = f(x)\,.$$

By function extensionality we have that $m \circ p = f$. □

# Graph surjects to image

## Exercise

1. *Show that the assignment which takes* $(x, f(x))$ *to* $f(x)$ *defines a function from* $\overline{\pi_2}\colon \mathbf{Gr}(f) \to \mathbf{Im}(f)$ *which is surjective.*
2. *Show that the following diagram of functions commute:*

$$\begin{tikzcd}
\mathbf{Gr}(f) \arrow[r, tail] \arrow[d, two heads, "\overline{\pi_2}"'] & X \times Y \arrow[d, "\pi_2"] \\
\mathbf{Im}(f) \arrow[r, tail] & Y
\end{tikzcd}$$

# Pre-images

A function $f\colon X \to Y$ induces a function

$$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^{-1}(S) = \{x \in X \mid f(x) \in S\}$$

for any subset $S$ of $Y$.

The subset $f^{-1}(S)$ is called the pre-image of $S$ under $f$.

Note that

$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}$$

# Injections and subsingletons

### Definition

*A set $U$ is said to be a* ***subsingleton*** *if it is a subset of the one-element set* $\mathbf{1}$.

### Proposition

*A function $f\colon X \to Y$ is injective if and only if for every $y \in Y$ the fibres $f^{-1}(y)$ are all subsingletons.*

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions. We prove that

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}\,.$$

Recall that in order to prove equality of functions we need to use function extensionality.
Suppose $T$ is a subset of $Z$. Then

$$\begin{aligned}
(f^{-1} \circ g^{-1})T &= f^{-1}\,\{y \in Y \mid g(y) \in T\} \\
&= \big\{x \in X \mid f(x) \in \{y \in Y \mid g(y) \in T\}\big\} \\
&= \{x \in X \mid g(f(x)) \in T\} \\
&= (g \circ f)^{-1}\,T
\end{aligned}$$

# Fibres

## Definition

*For a function $f : X \to Y$, and an element $y \in Y$, the subset*

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

*of $X$ is called the fibre of $f$ at $y$ and also the pre-image of $y$ under $f$. Although, technically incorrect, people write $f^{-1}(y)$ instead of $f^{-1}(\{y\})$.*

## Example

*Consider the function $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ which takes a real number to the greatest integer less than it. What are the fibres*

- $\lfloor - \rfloor^{-1}(0)$*?*
- $\lfloor - \rfloor^{-1}(\lfloor \pi \rfloor)$*?*

The operation of taking fibres of a function is itself a function. More specifically, given a function $f$, taking fibres of $f$ at different elements $y \in Y$ as a function is equal to the composite

$$Y \xrightarrow{\{-\}} \mathcal{P}(Y) \xrightarrow{f^{-1}} \mathcal{P}(X)\,,$$

that is for all $y \in Y$,

$$f^{-1}(y) = f^{-1}\{y\}$$

## Exercise

*Consider the family $\{f^{-1}(y) \mid y \in Y\}$. Show that all members of this family are mutually disjoint, and that their union is fact $X$.*

$$\coprod_{y \in Y} f^{-1}(y) \cong \bigcup_{y \in Y} f^{-1}(y) = X$$

As the last exercise suggests, we can associate to every function a family of sets given by fibres of that function at different elements of the codomain.

Interestingly, we also have the converse association: to a family $\{Y_x \mid x \in X\}$ we associate a function as follows: let the domain to be the disjoint union $\coprod_{x \in X} Y_x$ and let the codomain be $X$. The associated function $p \colon \{Y_x \mid x \in X\} \to X$ takes an element $\mathrm{in}(x) \in \coprod_{x \in X} Y_x$ to $x \in X$.

$$
\begin{tikzcd}
\begin{array}{c} Y \\ {\scriptstyle f}\big\downarrow \\ X \end{array} \arrow[r, shift left=2, "\mathbf{T} =_{\mathrm{def}} \text{taking fibres}"] & \{Y_x \mid x \in X\} \arrow[l, shift left=2, "\mathbf{U} =_{\mathrm{def}} \text{taking union}"]
\end{tikzcd}
$$

functions | families of sets

# Factorization of function via quotient

Recall from problem 5 of homework #4 that for each equivalence $\sim$ on a set $X$ we can construct a set $X/\sim$ whose elements are equivalence classes

$$[x]_\sim = \{y \in X \mid x \sim y\}$$

for all $x \in X$. Now collect all such equivalence classes into one set:

$$X/\sim =_{\text{def}} \{[x]_\sim \mid x \in X\}$$

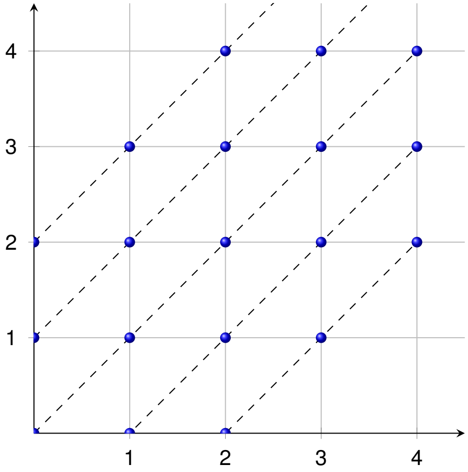
We call the set $X/\sim$ the quotient of $X$ by equivalence relation $\sim$.

# Example of quotient by an equivalence relation

Consider the relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ where

$$(m, n) \sim (m', n') \Leftrightarrow m + n' = n + m' \, .$$

For instance, the equivalence class $[(0, 0)]$ is the set $\{(0, 0), (1, 1), (2, 2), ...\}$.


4
3
2
1
1
2
3
4

We can define the operation of addition on $\mathbb{N} \times \mathbb{N}/\sim$ by an assignment $+_{\sim}: \mathbb{N} \times \mathbb{N}/\sim \times \mathbb{N} \times \mathbb{N}/\sim \longrightarrow \mathbb{N} \times \mathbb{N}/\sim$ which assigns to the pair $([(m, n)], [(m', n')])$ the class $[(m + m', n + n')]$.

### Exercise

*Show that the assignment $+_{\sim}$ is well-defined, i.e. it defines a function.*

### Exercise

*Show that the quotient $\mathbb{N} \times \mathbb{N}/\sim$ is isomorphic to the set $\mathbb{Z}$ of integers. Does your isomorphism preserve addition?*

### Exercise

*Define multiplication on the quotient $\mathbb{N} \times \mathbb{N}/\sim$. Does your isomorphism preserve addition?*

# Image factorization

## Proposition

*Suppose $f\colon A \to B$ is a function. We can factor $f$ into three functions*

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, two heads, "q"'] & & & B \\
& C \arrow[r, "g"'] & D \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

*that is $f = i \circ g \circ q$, where $q$ is a surjection, $g$ is a bijection, and $i$ is an injection.*

**Proof.**

We have to construct the sets $C$, $D$ and a surjection $q$, a bijection $g$ and an injection $i$. We define an equivalence relation $\sim$ on $A$ by

$$x \sim y \Leftrightarrow f(x) = f(y)\,.$$

Now we define $C$ to be $A/\sim$, and $D$ to be the image $f_*(A)$ of $A$ under $f$. We also define $q$ to be the obvious quotient map and $i$ to be the obvious inclusion. Clearly, $q$ is surjective and $i$ is injective. We define $g$ to be the assignment which takes an equivalence class $[x]$ to the element $f(x) \in B$. Note that $g$ is well-defined, since if $[x] = [y]$ then $x \sim y$ and therefore, by the definition of $\sim$, we have $f(x) = f(y)$. We now show that $g$ is a bijection. $g$ is injective since for every $x, y \in A$, if $g([x]) = g([y])$ then $f(x) = f(y)$ and therefore, $[x] = [y]$. Also, $g$ is surjective: given $b$ in $f_*(A)$ there is some $a \in A$ such that $b = f(a) = g([a])$. $\square$

Our factorization diagram becomes

$$
\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, two heads, "q"'] & & & B \\
& A/\sim \arrow[r, "g"'] & f_*(A) \arrow[ur, tail, "i"'] &
\end{tikzcd}
$$

In fact, $g \circ q = p\colon X \to \mathbf{Im}(f)$.

# The set of functions

Suppose $X$ and $Y$ are sets. We can define a new set consisting of all the functions from $X$ to $Y$. We denote this set by $Y^X$. Explicitly,

$$Y^X = \{f\colon X \to Y\} \cong \{R \subset X \times Y \mid R \text{ is a functional relation}\}$$

## Exercise

*Suppose $X$ is a finite set with $m$ elements and Suppose $Y$ is a finite set with $n$ elements. Then the set $Y^X$ has $n^m$ elements.*

# The set of functions behaves like exponentials

## Proposition

*Suppose $X, Y, Z$ are sets. We have*

- $X^{\emptyset} \cong 1$
- $\emptyset^{X} \cong 1$ *if and only if* $X = \emptyset$. *In particular* $\emptyset^{\emptyset} \cong 1$.
- $(X^{Y})^{Z} \cong X^{Y \times Z}$.
- $X^{Y+Z} \cong X^{Y} \times X^{Z}$

Let $\Omega$ be a set with two elements, for instance $\{\top, \bot\}$. We show that

$$\Omega^X \cong \mathcal{P}(X)$$

that is the power set of $X$ is isomorphic to the set of functions from $X$ to $\Omega$. To this end we construct two functions $f$ and $g$ and prove that they are inverse of each other. We have functions $\lambda(\varphi : \Omega^X).\{x \in X \mid \varphi(x) = \top\} : \Omega^X \to \mathcal{P}(X)$, and $\lambda(S : \mathcal{P}(X)).\chi_S : \mathcal{P}(X) \to \Omega^X$ where, we recall, that $\chi_S$ is the characteristic function of $S \subseteq X$.

# Dependent product of sets

Let $\{X_i \mid i \in I\}$ be a family of sets.
Define the set $\prod_{i \in I} X_i$ to be

$$\{h \colon I \to \bigcup_{i \in I} X_i \mid \forall i \, (h(i) \in X_i)\}$$

Note that if $I$ is a finite set, say $I = \{1, 2, \cdots, n\}$ then

$$\prod_{i \in I} X_i \cong X_1 \times X_2 \times \cdots \times X_n$$

In case where $I$ is a finite set, if each $X_i$ is inhabited then the cartesian product $\prod_{i \in I} X_i$ is also inhabited. But we cannot prove this for a general $I$.

# Axiom of choice

Axiom of Choice (AC) asserts that the set $\prod_{i \in I} X_i$ is inhabited for *any* indexing set $I$ and any family $(X_i \mid i \in I)$ of *inhabited* sets.

### Warning

*The axiom of choice is highly non-constructive: if a proof of a result that does not use the axiom of choice is available, it usually provides more information than a proof of the same result that does use the axiom of choice.*

# Logical incarnation of Axiom of Choice

## Proposition

*The axiom of choice is equivalent to the statement that for any sets $X$ and $Y$ and any formula $p(x, y)$ with free variables $x \in X$ and $y \in Y$, the sentence*

$$\forall x \in X \, \exists y \in Y \, p(x, y) \Rightarrow \exists (f : X \to Y) \, \forall x \in X, \; p(x, f(x)) \tag{1}$$

*holds.*

**Proof**. Assume axiom of choice. Let $X$ and $Y$ be arbitrary sets and $p(x, y)$ any formula with free variables $x \in X$ and $y \in Y$. For each $x \in X$, define $Y_x = \{y \in Y \mid p(x, y)\}$. Note that $Y_x$ is inhabited for each $x \in X$ by the assumption $\forall x \in X, \exists y \in Y, p(x, y)$. By the axiom of choice there exists a function $h\colon X \to \bigcup_{x \in X} Y_x$ such that $h(x) \in Y_x$ for all $x \in X$. We compose the function $h$ with the inclusion $\cup_{x \in X} Y_x \rightarrowtail Y$, which we get from the fact that $Y_x \subseteq Y$ for each $x \in X$, to obtain a function $f\colon X \to Y$. But then $p(x, f(x)) = p(x, h(x))$ is true for each $x \in X$ by definition of the sets $Y_x$.

Conversely, suppose that we have a family $(X_i \mid i \in I)$ of inhabited sets. Consider the cartesian product $\prod_{i \in I} X_i$. We want to show that this product is inhabited. Define

$$p(i, x) =_{\text{def}} (x \in X_i)$$

Now, we apply the sentence (1) to the sets $I$, $\bigcup_{i \in I} X_i$ and the formula $p(i, x)$ just defined: we find a function $f \colon I \to \bigcup_{i \in I} X_i$ such that $p(i, f(i))$ for all $i \in I$. But, by definition of $p(i, x)$, we conclude that $f(i) \in X_i$ for all $i \in I$. Hence, $f$ is a member of $\prod_{i \in I} X_i$. $\square$

# Axiom of Choice and surjections

Given a function $p \colon Y \to X$, consider the associated family $\{Y_x \mid x \in X\}$ of sets obtained by taking fibres of $p$ at different elements of $x$.

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**Lemma**

*A maps $p \colon Y \to X$ is surjective if and only if the fibres $Y_x$ are inhabited for all $x \in X$.*

## Axiom of Choice and surjections

Given a function $p\colon Y \to X$, consider the associated family $\{Y_x \mid x \in X\}$ of sets obtained by taking fibres of $p$ at different elements of $x$.

**Lemma**

*A maps $p\colon Y \to X$ is surjective if and only if the fibres $Y_x$ are inhabited for all $x \in X$.*

**Lemma**

*An element of $\prod_{x\in X} Y_x$ is the same thing as a section of $p\colon Y \to X$.*

# Axiom of Choice and surjections

## Proposition

*Axiom of choice is equivalent to the statement that every surjection has a section.*

## Proof.

Assume AC. Let $p\colon Y \to X$ be a surjection. Therefore all the fibres $Y_x$ are inhabited. By AC, the product $\prod_{x \in X} Y_x$ is inhabited. Hence, by the last lemma above, $p$ has a section. $\square$

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Assume AC. Let $p\colon Y \to X$ be a surjection. Therefore all the fibres $Y_x$ are inhabited. By AC, the product $\prod_{x \in X} Y_x$ is inhabited. Hence, by the last lemma above, $p$ has a section. Conversely, suppose that every surjection has a section. Let $\{Y_x \mid x \in X\}$ be family of sets where the set $Y_x$ is inhabited for every $x \in X$. Consider the associated function $\sqcup_{x \in X} Y_x \to X$. Note that this map is surjective by our assumption and the first lemma above. Hence, it has a section which is the same thing as an element of $\prod_{x \in X} Y_x$. Therefore AC holds. $\square$

**Theorem (Diaconescu, Goodman-Myhill)**

*The axiom of choice implies the law of excluded middle.*

# Cantors' theorem: $A < P(A)$

## Lemma

*If a function $\sigma : A \to B^A$ is surjective then every function $f : B \to B$ has a fixed point.*

## Proof.

Because $\sigma$ is a surjection, there is $a \in A$ such that $\sigma(a) = \lambda x : A\,.\, f(\sigma(x)(x))$, but then $\sigma(a)(a) = f(\sigma(a)(a))$. $\square$

## Corollary

*There is no surjection $A \to P(A)$.*

Let's associate to each *finite set* $X$ a number $\texttt{card}(X)$, called the "cardinality" of $X$, which measures how many (distinct) elements the set $X$ has. We then have

- $\texttt{card}(X + Y) = \texttt{card}(X) + \texttt{card}(Y)$ and
- $\texttt{card}(X \times Y) = \texttt{card}(X) \times \texttt{card}(Y)$.

More generally, for any finite set $I$ and a family of finite sets $\{X_i \mid i \in I\}$, we have

- $\texttt{card}(\bigsqcup_{i \in I} X_i) = \sum_{i \in I} \texttt{card}(X_i)$ and
- $\texttt{card}(\prod_{i \in I} X_i) = \prod_{i \in I} \texttt{card}(X_i)$

# Questions

Thanks for your attention!