Math 301: Introduction To Proofs

HW1 – Solutions

(10 pts)

P1. Soln.

First Proof. (Proof by $\boxed{\text{Contrapositive}}$)

Let P be the proposition "x is a rational number".

Let Q be the proposition "$x+y$ and $x-y$ are both rational."

We want to prove $\neg P \to \neg Q$.

Because $(Q \to P) \to \neg P \to \neg Q$ is a tautology, it is $\boxed{\text{sufficient}}$ to prove if $x+y$ and $x-y$ are both rational, then so is x.

But, observe that

$$\boxed{x = \frac{(x+y) + (x-y)}{2}}$$

which is a rational number. $\square$

Other possible proofs.

$\boxed{\text{Proof by contradiction,}}$

P2. Soln: (20 pts)

If we take $n = 1$ and $a = 4$, then

$$a^n - 1 = 4 - 1 = 3$$

is prime but n is not, and therefore, the proposition "If $a^n - 1$ is prime, then n is prime" is false.

We are looking for a proposition P, which states something about natural number a, and also makes the proposition

"If $a^n - 1$ is prime and P then n is prime."

true.

We claim that the proposition $P := (a=2)$ works.

Suppose $a^n - 1$ is a prime number. Notice that

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + 1). \quad (1)$$

We claim that $n \neq 1$, for otherwise $a-1$ would be prime. But, we also assumed $a = 2$. Hence 1 would be prime, which is absurd. Therefore, $n \neq 1$.

Next, we prove that n is a prime number.

Suppose $n = dk$ where d and k are natural numbers.

Since $n>1$ either $d>1$ or $k>1$.
without loss of generality,
assume $k>1$.

We have

$$2^n - 1 = 2^{dk} - 1 = (2^d)^k - 1 =$$

$$(2^d - 1)((2^d)^{k-1} + (2^d)^{k-2} + \cdots + 1)$$

Since $k>1$ we have

$$(2^d)^{k-1} + \cdots + 1 \geq 3.$$

Since $2^n - 1$ is prime,

we must have $2^d - 1 = 1$

Hence $d=1$.

We have showed that $n>1$ and in any factorization of n one of the factors necessarily equals 1. This means n is prime.

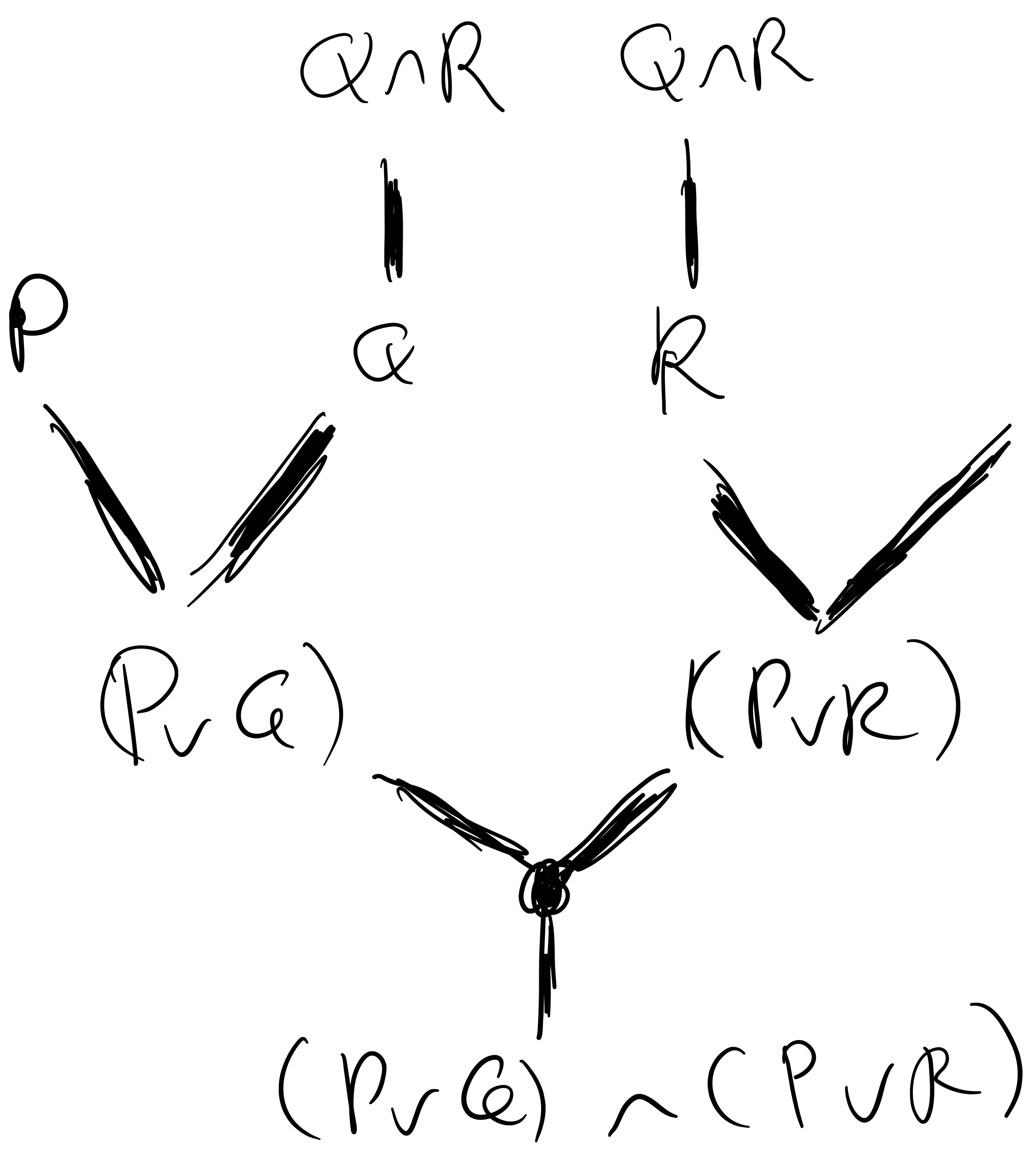
P3. Soln (20 pts)

(1) (10 pts)

$$\dfrac{\dfrac{}{P\lor(Q\land R)}(1) \qquad \dfrac{\dfrac{\dfrac{}{P}2}{P\lor Q}(\lor_l I) \quad \dfrac{\dfrac{}{P}2}{P\lor R}(\lor_l I)}{(P\lor Q)\land(P\lor R)}(\land I) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Q\land R}3}{Q}(\land_l E)}{P\lor Q}(\lor_r I) \quad \dfrac{\dfrac{\dfrac{}{Q\land R}3}{R}(\land_r E)}{P\lor R}(\lor_r I)}{(P\lor Q)\land(P\lor R)}}{(P\lor Q)\land(P\lor R)}(2,3)$$

$$\dfrac{}{P\lor(Q\land R)\Rightarrow(P\lor Q)\land(P\lor R)}(1)$$

(2) (10 pts)



P4. John: (30 pts)

First Part (15 pts)

$\overline{}$ (3)

$\neg(P \wedge \neg Q)$

$(P \wedge \neg Q) \to \bot$

$\dfrac{\neg\neg Q}{Q}$ (DN)

$P \to \neg\neg Q \qquad \neg\neg Q \to Q$

$P \to Q$

(1)

$(P \to Q) \to P$

$\neg P \to \neg(P \to Q)$

$\overline{}$ (2)

$\neg P$

$(\Rightarrow E)$

$\neg(P \to Q)$

$(\Rightarrow E)$

$\bot$

(3)

$\neg\neg(P \wedge \neg Q)$

(DN)

$P \wedge \neg Q$

$(\wedge E)$

$\overline{}$ (2)

$\neg P \qquad P$

$(\Rightarrow E)$

$\bot$

(2)

$\neg\neg P$

(DN)

P

(1)

$((P \to Q) \to P) \to P$

Note: A proof using the law of excluded middle (EM) — which says $\forall P{:}\mathrm{Prop}.\,(P \vee \neg P)$ — is also acceptable since

$$EM \Longleftrightarrow DN \quad \square$$

Second Part (15 pts)

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P$	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	0	1

P5. Soln (20 pts)

First, we define the following propositions.

A = "Alice is a knight"

B = "Bob is a knight"

E = "Eve is a knight"

From the fact that Alice says that Bob is a knight we get

(i) $(A \Rightarrow B) \wedge (\neg A \Rightarrow \neg B)$

From the fact that Bob says that Alice is a knight but Eve is a knave we get

(ii) $(B \Rightarrow A \wedge \neg E) \wedge (\neg B \Rightarrow \neg(A \wedge \neg E))$

From the fact that Eve says that Alice is a knight and Bob is a knight we get

(iii) $(E \Rightarrow A \wedge B) \wedge (\neg E \Rightarrow \neg(A \wedge B))$

(i) has the same truth value as

$A \Leftrightarrow B$

(ii) has the same truth value as

$B \Leftrightarrow (A \wedge \neg E)$

(iii) has the same truth value as

$E \Leftrightarrow A \wedge B$

In a truth table involving proposition A, B, E we must look for a row where the value of $(A \Leftrightarrow B) \wedge (B \Leftrightarrow A \wedge \neg E) \wedge (E \Leftrightarrow A \wedge B)$ is T.

A	B	E	$\neg E$	$A \Leftrightarrow B$	$B \Leftrightarrow (A \wedge \neg E)$	$E \Leftrightarrow A \wedge B$	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0	0	0					T

Therefore, Alice, Bob and Eve are all knaves.

P6. Soln (5 pts)

First soln. (Direct Proof)

Take x to be $\sqrt{10}$.

Take y to be $2\log_{10}^{3}$.

Include the proof of irrationality of $\sqrt{10}$ which we did in the class.

We show that y is irrational which is equivalent to show $y/2$ is irrational.

Suppose there are integers p, q where q is non-zero and $\log_{10}^{3} = \frac{p}{q}$. It follows that

$$3 = 10^{\log_{10}^{3}} = 10^{\frac{p}{q}}.$$

Hence

$$3^{q} = 10^{p}.$$

Since $q \geq 1$, the LHS is an integer divisible by 3

Whereas the RHS is not. Therefore $\log_{10}^{3}$ cannot be a rational number.

However,

$$x^y = \sqrt{10}^{\,2\log_{10}^{3}} = \left((\sqrt{10})^2\right)^{\log_{10}^{3}} = 10^{\log_{10}^{3}} = 3$$

which is a rational number.

Second Proof. (Indirect)

In this proof, we use the law of excluded middle (EM)

(EM) $\boxed{\text{Either } \sqrt{2}^{\sqrt{2}} \text{ is rational or it is not.}}$

If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done:

Take x to be $\sqrt{2}$ and take y to be $\sqrt{2}$.

Both x and y are irrational because

[... include proof of irrationality of $\sqrt{2}$ ...]

If $\sqrt{2}^{\sqrt{2}}$ is irrational, however, then take

$$x = \sqrt{2}^{\sqrt{2}} \quad \text{and}$$

$$y = \sqrt{2}. \quad \text{we have}$$

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\,2} = 2$$

which is rational. $\square$

A Bird's eye view of our proof

$$\dfrac{\dfrac{}{P \vee \neg P}\,(\text{LEM}) \qquad \begin{array}{c} \overline{P}^{\,1} \\ \vdots \\ R \end{array} \qquad \begin{array}{c} \overline{\neg P}^{\,1} \\ \vdots \\ R \end{array}}{R}\,(1)$$

$$R = \exists x \exists y \left(x \notin \mathbb{Q} \wedge y \notin \mathbb{Q} \wedge x^y \in \mathbb{Q} \right)$$

where $x \notin \mathbb{Q} := \neg \exists m \exists n \left(n \neq 0 \wedge x = \frac{m}{n} \right)$

Some common mistakes which cost points

① Using variables not introduced before (1 pt)

② Not stating your proof strategy (1 pt)

e.g. proof by cases, proof by contradiction, contrapositi

③ Not being clear about what your assumption and goals are (2 pts)

④