Introduction to proofs HW3 Solution

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[2.6] (16 pts). Determine whether the following statements are true / false for all sets X, Y or true for some sets and false for others

(a) $P(X \cup Y) = P(X) \cup P(Y)$

It is true for some sets X, Y. e.g. when $X \subseteq Y$ or $X \supseteq Y$. (WLOG $X \subseteq Y$), then $P(X) \subseteq P(Y)$ and $X \cup Y = Y$

and thus $P(X \cup Y) = P(Y)$
$= P(Y) \cup P(X)$

not for all sets, X, Y.

e.g. $X = \{1, 2\}$ then $X \cup Y = \{1, 2, 3, 4\}$
$Y = \{3, 4\}$

$|P(X)| = |P(Y)| = 2^2 = 4$, and then $|P(X) \cup P(Y)| = |P(X)| + |P(Y)| - \underbrace{|P(X) \cap P(Y)|}_{\text{which is } \{\emptyset\}}$
$= 4 + 4 - 1 = 7$

but $|P(X \cup Y)| = 2^4 = 16 \neq 7$, which implies $P(X) \cup P(Y) \neq P(X \cup Y)$

□

(b) $P(X \cap Y) = P(X) \cap P(Y)$

This is true for any sets X, Y.

Proof: $A \in P(X \cap Y) \iff A \subseteq X \cap Y$
$\iff (A \subseteq X) \wedge (A \subseteq Y)$
$\iff (A \in P(X)) \wedge (A \in P(Y))$
$\iff A \in P(X) \cap P(Y)$

□

(c) $P(X \times Y) = P(X) \times P(Y)$

This is always false for any X, Y sets.

because $\emptyset \in P(X \times Y)$, but the elements in $P(X) \times P(Y)$ are "ordered pairs" of subsets of X and Y.

□

(d). $P(X \setminus Y) = P(X) \setminus P(Y)$.

This is always false for any sets X, Y

because $\emptyset \in P(X \setminus Y)$, but $\underbrace{\emptyset \notin P(X) \setminus P(Y)}_{\text{This is because } \emptyset \in P(X) \text{ and } \emptyset \in P(Y)}$

□

[2.14]. Determine if it is open. (16 pts)

(a). ϕ : Yes. because $(\phi)^c = \mathbb{R}$ is a closed set.

(c): $(0,1]$: No. because $1 \in (0,1]$, but any open neighborhood of 1, saying $(1-\varepsilon_1, 1+\varepsilon_2)$ does not lie in $(0,1]$.

(b): $(0,1)$: Yes. $\forall x \in (0,1)$, there exists $\varepsilon_x < \min\{1-x, x\}$, such that

$(x-\varepsilon_x, x+\varepsilon_x) \subseteq (0,1)$.

(d). $\mathbb{Z}$: No, because any open neighborhood of any integer, namely $(n-\varepsilon_1, n+\varepsilon_2)$

does not lie in $\mathbb{Z}$.

(e) $\mathbb{R}\setminus\mathbb{Z}$: ~~No. $\forall x \in \mathbb{R}\setminus\mathbb{Z}$, $\forall \varepsilon > 0$, any ε-neighbourhood of x, namely $(x-\varepsilon, x+\varepsilon)$~~.

~~contains at least one ration nu~~

Yes $\forall x \in \mathbb{R}\setminus\mathbb{Z}$, let $d_x = \min\{\lceil x\rceil - x,\ x - \lfloor x\rfloor\}$. then for any $\varepsilon < d_x$

$(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R}\setminus\mathbb{Z}$. because the closest integers to x, ($\lfloor x\rfloor$ and $\lceil x\rceil$)

are not contained in $(x-\varepsilon, x+\varepsilon)$.

(f) $\mathbb{Q}$: No. ~~because~~ $\forall r \in \mathbb{Q}$, $\forall \varepsilon > 0$, there exists infinitely many irrational numbers

in $(r-\varepsilon, r+\varepsilon)$.

□

[2.15] (8 pts) Prove that. $U \subseteq \mathbb{R}$ is open iff $\forall a \in U$, $\exists u, v \in \mathbb{R}$, s.t. $u < a < v$ and $(u,v) \subseteq U$

Proof: $\Rightarrow$) exactly follows from the definition.

$\Leftarrow$): If $\forall a \in U$, $\exists u, v \in \mathbb{R}$, s.t. $u < a < v$. $(u,v) \subseteq U$.

then define $d = \min\{a-u, v-a\}$. and thus $(a-d, a+d) \subseteq (u,v) \subseteq U$

which meets the definition of open set

□

[2.16] (20 pts) (a) $n \geq 1$. $U_1 \ldots U_n$ open $\subseteq \mathbb{R}$. Prove that $U_1 \cap \cdots \cap U_n$ is open.

Proof: Pick any $a \in U_1 \cap \cdots \cap U_n$, We know $a \in U_i$ for each $i \in \{1,2,\ldots,n\}$.

so. $\exists\, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$. s.t. $(a-\varepsilon_i, a+\varepsilon_i) \subseteq U_i$ for each i

Now take $\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, so $(a-\varepsilon_0, a+\varepsilon_0) \subseteq U_i \;\; \forall i$

$\Rightarrow (a-\varepsilon_0, a+\varepsilon_0) \subseteq U_1 \cap \cdots \cap U_n$

$\Rightarrow U_1 \cap \cdots \cap U_n$ is open

$\square$

(b). $(0, 1+\frac{1}{n})$ is open $\forall n \geq 1$, but $\bigcap_{n=1}^{\infty} (0, 1+\frac{1}{n})$ is not.

Proof: Any open interval is open.

Next we show $(0,1] = \bigcap_{n=1}^{\infty} (0, 1+\frac{1}{n})$.

$\subseteq$: $\forall x \in (0,1]$ we have $0 < x \leq 1$. so, $0 < x \leq 1+\frac{1}{n}$ for any n

$\Rightarrow x \in (0, 1+\frac{1}{n}) \;\; \forall n \Rightarrow x \in \bigcap_{n=1}^{\infty} (0, 1+\frac{1}{n})$

$\supseteq$: $\forall x \in \bigcap_{n=1}^{\infty} (0, 1+\frac{1}{n})$, we have $0 < x < 1+\frac{1}{n} \;\; \forall n$.

$\Rightarrow 0 < x \leq 1$ ($x > 1$ is false; If $x > 1$. then $\exists\, \varepsilon_0 > 0$. $x \geq 1+\varepsilon_0$. Picking $n_0 \geq [\frac{1}{\varepsilon_0}]+1$ yields $x > 1+\frac{1}{n_0}$. Contradicts)

$(0,1]$ is not open. (cf. 2.14(c)).

$\square$

Problem 1: (8 pts) $\{A_{ij} \mid i \in I, j \in J\}$ is a family of sets

(1) $\bigcup_{i \in I} \bigcup_{j \in J} A_{ij} = \bigcup_{j \in J} \bigcup_{i \in I} A_{ij}$.

$x \in \bigcup_i \bigcup_j A_{ij} \Leftrightarrow \exists\, i_0 \in I,\; x \in \bigcup_j A_{i_0 j} \Leftrightarrow \exists\, i_0 \in I\; \exists\, j_0 \in J.\; x \in A_{i_0, j_0}$.

$\Leftrightarrow \exists\, j_0 \in J.\; x \in \bigcup_{i \in I} A_{i, j_0}$

$\Leftrightarrow x \in \bigcup_{j \in J} \bigcup_{i \in I} A_{ij}$

(2) $\bigcap_{i\in I}\bigcap_{j\in J} A_{i,j} = \bigcap_{j\in J}\bigcap_{i\in I} A_{i,j}$

$$x\in \bigcap_{i\in I}\bigcap_{j\in J} A_{i,j} \iff \forall i_0\in I.\ x\in\bigcap_{j\in J} A_{i_0,j}$$
$$\iff \forall i_0\in I,\ \forall j_0\in J,\ x\in A_{i_0,j_0}.$$
$$\iff \forall j_0\in J\ \ x\in\bigcap_{i\in I} A_{i,j_0}$$
$$\iff \cancel{\forall}\ x\in\bigcap_{j\in J}\bigcap_{i\in I} A_{i,j}.$$

Problem 2 (16 pts). Let X, Y be classical sets. Prove that $X\setminus(X\setminus Y) = X\cap Y$.
Can we drop the condition of being classical about either X or Y and have the same conclusion. If so, which one?

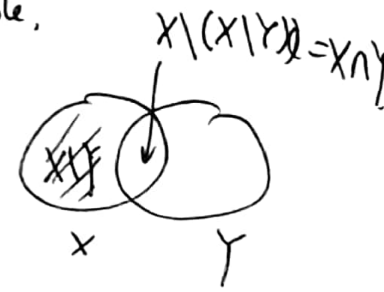
Proof: $\subseteq$: $\cancel{X\setminus(X}$ $x\in X\setminus(X\setminus Y) \Rightarrow x\in X$ and $x\notin X\setminus Y = X\cap Y^c$.

~~i.e. x∈X and ¬(x∈X and x~~

so $x\in (X\cap Y^c)^c = X^c\cup Y \Rightarrow x\in X^c$ or $x\in Y$.

But $x\in X$, which implies ~~th~~ the case "$x\in X^c$" is impossible,

so $\left.\begin{array}{l} x\in Y \\ x\in X\end{array}\right\} \Rightarrow x\in X\cap Y$.



$\supseteq$: If $x\in X\cap Y$, then $x\in X$ and $x\in Y$

$x\in Y \Rightarrow x\in X^c\cup Y \Rightarrow x\in (X^c\cap Y^c)^c = (X\setminus Y)^c$

which together with $x\in X$ ~~⇒ x ∈ X~~ yields $x\in X\setminus(X\setminus Y)$.

. Y must be classical, because we use the fact that

$Y = (Y^c)^c$ (and thus $Y\subseteq (Y^c)^c$)

□

Problem 3 (16pts) For classical sets $A, X, Y, \{X_i \mid i \in I\}$. Prove de Morgan's law:

(Assume all the sets here are not empty (without loss of generality)).

(1) $A \setminus (X \cup Y) = (A \setminus X) \cap (A \setminus Y)$

~~(2)~~

Proof: $x \in A \setminus (X \cup Y) \Leftrightarrow x \in A$ and $x \notin X \cup Y$

$\Leftrightarrow (x \in A) \wedge \neg((x \in X) \vee (x \in Y))$

$\Leftrightarrow (x \in A) \wedge (x \notin X) \wedge (x \notin Y)$

$\Leftrightarrow ((x \in A) \wedge (x \notin X)) \wedge ((x \in A) \wedge (x \notin Y)) \Leftrightarrow x \in (A \setminus X) \cap (A \setminus Y)$

$\square$

(2) $A \setminus (X \cap Y) = (A \setminus X) \cup (A \setminus Y)$

Proof: $x \in A \setminus (X \cap Y) \Leftrightarrow (x \in A) \wedge x \notin (X \cap Y)$

$\Leftrightarrow (x \in A) \wedge ((x \notin X) \vee (x \notin Y))$

$\Leftrightarrow ((x \in A) \wedge (x \notin X)) \vee ((x \in A) \wedge (x \notin Y))$

$\Leftrightarrow (x \in (A \setminus X)) \vee (x \in (A \setminus Y))$

$\Leftrightarrow x \in (A \setminus X) \cup (A \setminus Y)$

(3) $A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i)$

Proof: $x \in A \setminus \bigcup_{i \in I} X_i \Leftrightarrow x \in A,\ x \notin \bigcup_{i \in I} X_i$.

$\Leftrightarrow x \in A$, and $\forall i \in I.\ x \notin X_i$

$\Leftrightarrow \forall i \in I.\ x \in A$ and $x \notin X_i$

$\Leftrightarrow \forall i \in I.\ x \in A \setminus X_i \Leftrightarrow x \in \bigcap_{i \in I} A \setminus X_i$.

(4) $A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} A \setminus X_i$

Proof: $x \in A \setminus \bigcap_{i \in I} X_i \Leftrightarrow x \in A.\ x \notin \bigcap_{i \in I} X_i$

$\Leftrightarrow x \in A$. and $\exists i \in I.\ x \notin X_i$

$\Leftrightarrow \exists i \in I.\ x \in A \setminus X_i \Leftrightarrow x \in \bigcup_{i \in I} A \setminus X_i$