MATH 301: INTRODUCTION TO PROOFS
HOMEWORK 4

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Problem 1. Decide whether the following are functions, giving reasons for your answers:

(1) The assignment $g\colon \mathbb{N} \to \mathbb{N}$ given by $n \mapsto$ "the $(n+1)$ prime number".
(2) The assignment $e\colon \mathbb{P}\mathrm{rop} \times \mathbb{P}\mathrm{rop} \to \mathbb{P}\mathrm{rop}$ which assigns to a pair of propositions P and Q a proposition R such that $P \wedge R \Rightarrow Q$.
(3) The assignment $f\colon \mathbb{R} \to \mathbb{R}$ which assigns to a real number x a real number y such that $y^2 = x$.
(4) The assignment $f'\colon \mathbb{C} \to \mathbb{C}$ which assigns to a complex number z a complex number w such that $w^2 = z$.
(5) The assignment $h\colon \mathbb{Q} \to \mathbb{Q}$ given by $h(y) = 1/y$.
(6) The assignment $j\colon \mathbb{Q} \to \mathbb{Q}$ given by $h(y) = 1/y$.
(7) The assignment $k\colon \mathbb{N} \to \mathbb{N}$ given by $m \mapsto d$ where d is a divisor of m.

Problem 2. Let $f\colon X \to Y$ be a function. Prove or find a counterexample for the following assertions

(1) $U \subset f^{-1}(f(U))$ for all $U \subset X$
(2) $f^{-1}(f(U)) \subset U$ for all $U \subset X$
(3) $V \subset f(f^{-1}(V))$ for all $V \subset Y$
(4) $f(f^{-1}(V)) \subset V$ for all $V \subset Y$

Problem 3.

(1) As in the lecture, write $\mathbf{1} = \{0\}$ for the set with one element. Let A be a set. Construct an isomorphism i between A and the $A^{\mathbf{1}}$.
(2) As in the lecture, write $\mathbf{2} = \{0, 1\}$ for the set with two elements. Construct an isomorphism j between the cartesian product $A \times A$ and the $A^{\mathbf{2}}$.
(3) Show the "diagonal" function $A \to A \times A$ is an injection.
(4) Use (3) to prove that the function $A^{\mathbf{1}} \to A^{\mathbf{2}}$ induced by the unique function $\mathbf{2} \to \mathbf{1}$ is an injection.

Problem 4. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions.

(1) Show that if f and g are surjective, then $g \circ f$ is surjective.
(2) Show that if $g \circ f$ is surjective, then g is surjective.

Problem 5. Let n be a natural number and define an equivalence relation $\sim_n$ on $\mathbb{Z}$ by

$$a \sim_n b \ \text{ if } \ a - b \text{ is divisible by } n.$$

(1) Show that $\sim_n$ is an equivalence relation.

(2) For $a \in \mathbb{Z}$, define the set

$$[a]_{\sim_n} = \{b \in \mathbb{Z} \mid a \sim_n b\},$$

and let C_n be the set

$$\{[a]_{\sim_n} \mid a \in \mathbb{Z}\}.$$

Show that C_n is a finite set.

(3) Show that the assignment

$$[a]_{\sim_n} + [b]_{\sim_n} = [a+b]_{\sim_n}$$

defines a function

$$+\colon C_n \times C_n \to C_n.$$

In other words, show that the equivalence class $[a+b]$ only depends on the equivalence classes $[a], [b]$ and not on the representatives a, b.

(4) Similarly, show that the assignment

$$[a]_{\sim_n} * [b]_{\sim_n} = [a \cdot b]_{\sim_n},$$

where $a \cdot b$ is just the usual multiplication of integers, defines a function

$$*\colon C_n \times C_n \to C_n.$$

(5) Fill in the multiplication table for C_4.

$*$	[0]	[1]	[2]	[3]
[0]				
[1]				
[2]				
[3]				

(6) Prove that for any integer m, the remainder of m^2 when divided by 4 is not 3. State your proof strategy.