MORE EXAMPLES OF NATURAL DEDUCTION

SINA HAZRATPOUR

In the following problems, $P, Q, R, \ldots$ are propositional variables.

Exercise 1. Give a natural deduction proof of the following propositional formulas:
 (1) $P \vee Q \Leftrightarrow Q \vee P$
 (2) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$
 (3) $P \Rightarrow \neg\neg P$
 (4) $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$
 (5) $\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$
 (6) $(P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow \neg(Q \Rightarrow \neg R))$
 (7) $(P \vee (Q \wedge P)) \Rightarrow P$

Exercise 2. Two propositional formulas, A and B, are said to be *logically equivalent* if $A \Leftrightarrow B$ is provable. Show that the following propositional formulas are logically equivalent.
 (1) $P \vee Q$ and $Q \vee P$
 (2) $(P \vee Q) \vee R$ and $P \vee (Q \vee R)$
 (3) $\neg P \wedge \neg Q$ and $\neg(P \vee Q)$

Exercise 3. Use a truth table for each of the following propositional formulas to verify the fact that they are tautologies.
 (1) $(((P \Rightarrow Q) \Rightarrow Q) \Rightarrow Q) \Rightarrow P \Rightarrow Q$
 (2) $(P \Rightarrow (Q \wedge R)) \Rightarrow (P \Rightarrow Q) \wedge (P \Rightarrow R)$
 (3) $\neg(((\neg P \Leftrightarrow P) \Leftrightarrow P) \Leftrightarrow P)$

Exercise 4. Use appropriate truth tables to show that the following propositional formulas are not tautologies by observing that not all the rows of the truth tables consist solely of 1's (We used 1 to denote the truth value "`true`").
 • $(P \Rightarrow Q) \Rightarrow Q \Rightarrow P$
 • $(P \Rightarrow Q) \Rightarrow \neg P \Rightarrow \neg Q$