MATH 301

INTRODUCTION TO PROOFS

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- Sets and logic
- Operations on sets
- Power sets

Relevant sections of the textbook

Chapter 2

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The simple characterisation of sets as collections leads to logical inconsistencies, such as the infamous *Russell's paradox*.
These logical paradoxes can be overcome by restricting ourselves to working inside a universe $\mathcal{U}$, which we consider to be a set which is so big that it contains all of the mathematical objects that we want to talk about.

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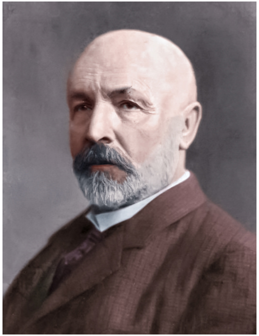
- Set theory is a foundation for mathematics. This means
  1. All abstract mathematical concepts can be expressed in the language of set theory.
  2. All concrete mathematical objects can be encoded as sets.

“ By a set we mean any collection M of determinate, distinct objects (called the elements of M) of our intuition or thought into a whole.”
(Georg Cantor, 1985)



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- The collection of everything in the universe of discourse is called the universal set, denoted by $\mathcal{U}$.

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- Note that the predicate P can have many variables.

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4. $\{x \mid c(x) = \mathtt{yellow} \wedge s(x) = \mathtt{triangle}\} = \varnothing = \{\}$

Instead of

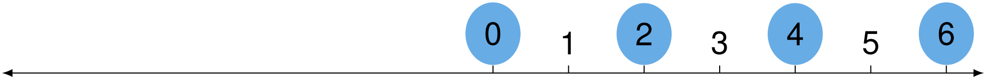
$$\cancel{E = \{2, 4, 6, \ldots\}}$$

we use

$$E = \{n \in \mathbb{N} \mid n \text{ is even}\}\,.$$

More formally, this set is written as

$$\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\}\,.$$



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- $\{a \in \mathbb{R} \mid a \text{ is equal to } 1, 2, 3, \text{ or } \pi\}$

An alternative to set-builder notation

An alternate form of set-builder notation uses an expression involving one or more variables to the left of the vertical bar, and the range of the variable(s) to the right. The elements of the set are then the values of the expression as the variable(s) vary:

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Example

The expression $\{2n \mid n \in \mathbb{N}\}$ *denotes the set of even numbers. It is shorthand for* $\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\}$.

Example

We can use a mix of the two notations:

$$\{p^2 + 1 \mid p \text{ is prime}\}.$$

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- For distinct objects a and b, we have $\{x \in \mathcal{U} \mid (x = a) \vee (x = b)\}$ is the set $\{a, b\}$.

Inhabited vs non-empty

Definition

A set X is inhabited if it has at least one element. Formally, a set X is inhabited if the sentence

$$\exists x \in X$$

– or equivalently the sentence $\exists x \, (x \in X)$ – is true.

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\text{Complement} \quad & A^{c} = \{x \mid \neg\,(x \in A)\ \} \\
\text{Relative complement} \quad & X \setminus Y = \{x \in X \mid x \notin Y\} =_{\text{def}} \{x \mid (x \in X) \wedge \neg(x \in Y)\}
\end{aligned}$$

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- $\forall x\ (x \in A \setminus B \leftrightarrow x \in A \wedge \neg x \in B)$

Equality of sets

1. Are the sets

$$\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\} \quad \text{and} \quad \{n \in \mathbb{Q} \mid \exists k \in \mathbb{N},\ n = 2k\}$$

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2. How about 'the set of prime numbers less than 2' and 'the set of even prime numbers greater than 2'?

3. How about

$$\{x \in \mathbb{Q} \mid x^2 < 2\} \quad \text{and} \quad \{x \in \mathbb{Q} \mid x^2 \leqslant 2\}\,?$$

Extensional equality of sets

Definition

Two sets A and B are equal precisely when they have the same elements.

The formal sentence expressing $A = B$ is

$$\forall x \, (x \in A \Leftrightarrow x \in B) \, .$$

Therefore, using the extensional definition of equality of sets, the answers to the questions (1)-(3) of the previous slide are all positive.

As an exercise we prove the distributvity of intersection ($\cap$) over union ($\cup$) of sets.

Theorem

Let A, B, and C denote sets of elements of some domain. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

Let x be arbitrary, and suppose x is in $A \cap (B \cup C)$. Then x is in A, and either x is in B or x is in C. In the first case, x is in A and x is in B, and hence x is in $A \cap B$. In the second case, x is in A and C, and hence x is in $A \cap C$. Therefore, x is in $(A \cap B) \cup (A \cap C)$.

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First, suppose x is in $A \cap B$. Then x is in both A and B. Since x is in B, it is also in $B \cup C$, and so x is in $A \cap (B \cup C)$.

The second case is similar: suppose x is in $A \cap C$. Then x is in both A and C, and so also in $B \cup C$. Hence, in this case also, x is in $A \cap (B \cup C)$, as required. $\square$

You should be able to see elements of natural deduction implicitly in the proof above. Explicitly, we need to construct a natural deduction proof of the sentence

$$\forall x \, (x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C)) \, .$$

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\begin{prooftree}
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\AxiomC{}
\RightLabel{1}
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\end{prooftree}
$$

Subsets

Definition

If A and B are sets, A is said to be a subset of B, written $A \subseteq B$, if every element of A is an element of B.

Formally, $A \subseteq B$ is expressed by the sentence

$$\forall x \, (x \in A \Rightarrow x \in B)$$

Exercise

Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Subsets (II)

Let's prove few facts about the subset relationship:

Exercise

1. *Prove that for all sets A we have $A \subseteq A$.*
2. *Prove that for all sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subset C$.*
3. *Prove that for all sets A we have $\emptyset \subseteq A$.*
4. *Prove that for all sets A, B, if $A \cup B = B$ then $A \subseteq B$.*
5. *Prove that for all sets A, B, if $A \cap B = A$ then $A \subseteq B$.*

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5. *Prove that for all sets A, B, if $A \cap B = A$ then $A \subseteq B$.*

Remark

It is true that $\varnothing \subseteq \varnothing$*, but false that* $\varnothing \in \varnothing$*. Indeed,*

- $\varnothing \subseteq \varnothing$ *means* $\forall x \in \varnothing, x \in \varnothing$*; but propositions of the form* $\forall x \in \varnothing, p(x)$ *are always true.*
- *The empty set has no elements; if* $\varnothing \in \varnothing$ *were true, it would mean that* $\varnothing$ *had an element (that element being* $\varnothing$*). So it must be the case that* $\varnothing \notin \varnothing$*.*

$$A \cup A^c = \mathcal{U}$$

$$A \cup A = A$$

$$A \cup \emptyset = A$$

$$A \cup \mathcal{U} = \mathcal{U}$$

$$A \cup B = B \cup A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cup B)^c \subset A^c \cap B^c$$

$$A \cap A^c = \emptyset$$

$$A \cap A = A$$

$$A \cap \emptyset = \emptyset$$

$$A \cap \mathcal{U} = A$$

$$A \cap B = B \cap A$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cap B)^c \supseteq A^c \cup B^c$$

$$A \cup A^c = \mathcal{U} \qquad A \cap A^c = \emptyset$$

$$A \cup A = A \qquad A \cap A = A$$

$$A \cup \emptyset = A \qquad A \cap \emptyset = \emptyset$$

$$A \cup \mathcal{U} = \mathcal{U} \qquad A \cap \mathcal{U} = A$$

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cup B)^c \subset A^c \cap B^c \qquad (A \cap B)^c \supseteq A^c \cup B^c$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cap (A \cup B) = A$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cup (A \cap B) = A$$

Classical sets

Definition

We call a set A classical if $A \subseteq A^{cc}$.

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Exercise

Show that if A is a classical set then $A^{cc} = A$.

A digression: numbers from sets

We can define "fake" natural numbers by way of sets: $\underline{0} = \emptyset$

$\underline{1} = \{\underline{0}\} = \{\emptyset\} = \{\{\}\}$

$\underline{2} = \{\underline{0}, \underline{1}\} = \{\emptyset, \{\emptyset\}\} = \{\{\}, \{\{\}\}\}$

$\vdots$

$\underline{n} = \{\underline{0}, \underline{1}, \cdots, \underline{n-1}\}$

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Are any of these sets satisfactory definitions of natural numbers?

Indexed Families of Sets

If I is a set, we will sometimes wish to consider a family $(A_i)_{i \in I}$ of sets indexed by elements of I. For example, we might be interested in a sequence

$$A_0, A_1, A_2, \ldots$$

of sets indexed by the natural numbers.

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Example

- *For each natural number n, we can define the set A_n to be the set of people alive today that are of age n.*
- *For every positive real number r we can define B_r to be the interval $[-r, r]$. Then $(B_r)_{r \in \mathbb{R}}$ is a family of sets indexed by the real numbers.*
- *For every natural number n we can define $C_n = \{k \in \mathbb{N} \mid k \text{ is a divisor of } n\}$ as the set of divisors of n.*

Union and intersection of indexed families

Given a family $(A_i)_{i \in I}$ of sets indexed by I, we can form its union:

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

We can also form the intersection of a family of sets:

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}$$

So an element x is in $\bigcup_{i \in I} A_i$ if and only if x is in A_i for *some* i in I,

and

x is in $\bigcap_{i \in I} A_i$ if and only if x is in A_i for every i in I.

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These operations are represented in symbolic logic by the existential and the universal quantifiers. We have:

$$\forall x \; (x \in \bigcup_{i \in I} A_i \leftrightarrow \exists i \in I \; (x \in A_i))$$

$$\forall x \; (x \in \bigcap_{i \in I} A_i \leftrightarrow \forall i \in I \; (x \in A_i))$$

Suppose that I contains just two elements, say $I = \{0, 1\}$.
Let $(A_i)_{i \in I}$ be a family of sets indexed by I.
Because I has two elements, this family consists of just the two sets A_0 and A_1.
Then the union and intersection of this family are just the union and intersection of the two sets.

$$\bigcup_{i \in I} A_i = A_0 \cup A_1.$$
$$\bigcap_{i \in I} A_i = A_0 \cap A_1.$$

This means that the union and intersection of two sets are just a special case of the union and intersection of a family of sets.

Exercise

Prove the following identities of sets:

1. $$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$
2. $$A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$$
3. $$\bigcup_{i \in I} \bigcup_{j \in J} A_{i,j} = \bigcup_{j \in J} \bigcup_{i \in I} A_{i,j}$$
4. $$\bigcap_{i \in I} \bigcap_{j \in J} A_{i,j} = \bigcap_{j \in J} \bigcap_{i \in I} A_{i,j}$$

The power set

Let X be a set. The power set of X, written $\mathcal{P}(X)$ is the set of all subsets of X.

Formally,

$$\mathcal{P}(X) =_{\text{def}} \{S \mid S \subseteq X\}$$

Therefore,

$$S \subseteq X \quad \Leftrightarrow \quad S \in \mathcal{P}(X)$$

Note that the power set of every set is inhabited since for a set X we have $\varnothing \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

Cartesian product of sets

With the tools we have developed we can define the cartesian product $A \times B$ of sets A and B to be the set containing exactly ordered pairs

$$(a, b) =_{\text{def}} \{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

where $a \in A$ and $b \in B$.
In other words,

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

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Notice that if $a = b$, the set (a, b) has only one element:

$$(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}.$$

The following theorem shows that the definition of cartesian product of sets is reasonable.

Theorem

$(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$.

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We leave the proof to the reader as an exercise.

Exercise

1. *Prove that that* $A \subseteq B$ *if and only if* $P(A) \subseteq \mathcal{P}(B)$.
2. *Prove that* $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. *Find an expression for* $(A \cup B) \times (C \cup D)$ *consisting of unions of cartesian products, and prove that your expression is correct.*

The End

THANKS FOR YOUR ATTENTION!