Math 301: Introduction To Proofs

HW1 – Solutions

(10 pts)

P1. Soln.

First Proof. (Proof by $\boxed{\text{Contrapositive}}$)

Let P be the proposition

"x is a rational number".

Let Q be the proposition

"$x+y$ and $x-y$ are both rational."

We want to prove $\neg P \to \neg Q$.

Because $(Q \to P) \to \neg P \to \neg Q$

is a tautology, it is $\boxed{\text{sufficient}}$

to prove if $x+y$ and $x-y$ are both rational, then

so is x.

But, observe that

$$\boxed{x = \frac{(x+y)+(x-y)}{2}}$$

which is a rational number. $\square$

Other possible proofs.

$\boxed{\text{Proof by contradiction,}}$

P2. Soln: (20 pts)

If we take $n = 1$ and $a = 4$, then

$$a^n - 1 = 4 - 1 = 3$$

is prime but n is not, and therefore, the proposition

"If $a^n - 1$ is prime, then n is prime"

is false.

We are looking for a proposition P, which states something about natural number a, and also makes the proposition

"If $a^n - 1$ is prime and P then n is prime."

true.

We claim that the proposition $P := (a=2)$ works.

Suppose $a^n - 1$ is a prime number. Notice that

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + 1). \qquad (1)$$

We claim that $n \neq 1$, for otherwise $a-1$ would be prime. But, we also assumed $a = 2$: Hence 1 would be prime, which is absurd. Therefore, $n \neq 1$.

Next, we prove that n is a prime number.

Suppose $n = dk$ where d and k are natural numbers.

Since $n > 1$ either $d > 1$ or $k > 1$.
without loss of generality,
assume $k > 1$.

We have

$$2^n - 1 = 2^{dk} - 1 = (2^d)^k - 1 =$$

$$(2^d - 1)((2^d)^{k-1} + (2^d)^{k-2} + \cdots + 1)$$

Since $k > 1$ we have

$$(2^d)^{k-1} + \cdots + 1 \geq 3.$$

Since $2^n - 1$ is prime,

we must have $2^d - 1 = 1$

Hence $d = 1$.

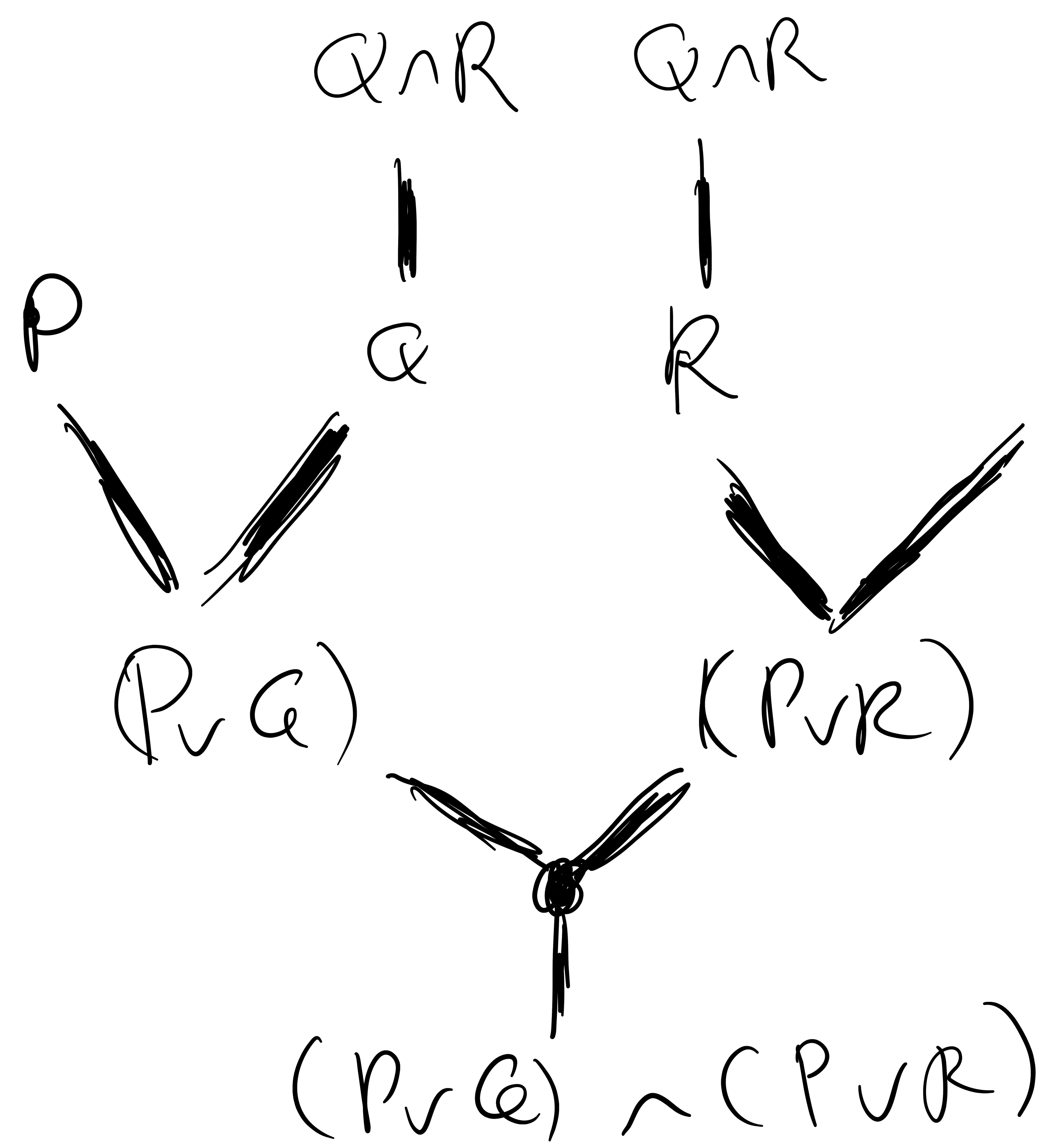
We have showed that $n > 1$ and in any factorization of n one of the factors necessarily equals 1. This means n is prime.

P3. Soln (20 pts.)

(1) (10 pts)

$$\dfrac{\dfrac{\dfrac{}{P\vee(Q\wedge R)}\,(1) \qquad \dfrac{\dfrac{\dfrac{}{P}\,2}{P\vee Q}\,(\vee_l I) \quad \dfrac{\dfrac{}{P}\,2}{P\vee R}\,(\vee_l I)}{(P\vee Q)\wedge(P\vee R)}\,(\wedge I) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Q\wedge R}\,3}{Q}\,(\wedge_l E)}{P\vee Q}\,(\vee_r I) \quad \dfrac{\dfrac{\dfrac{}{Q\wedge R}\,3}{R}\,(\wedge_r E)}{P\vee R}\,(\vee_r I)}{(P\vee Q)\wedge(P\vee R)}}{(P\vee Q)\wedge(P\vee R)}\,(2,3)}{P\vee(Q\wedge R)\Rightarrow(P\vee Q)\wedge(P\vee R)}\,(1)$$

(2) (10 pts)



P4. John: (30 pts)

First Part (15 pts)

$\overline{\qquad}$ (3)

$\neg(P \wedge \neg Q)$

$(P \wedge \neg Q) \to \bot$ $\qquad$ $\dfrac{\neg\neg Q}{Q}$ (DN) $\qquad\qquad$ (1)

$(P \to Q) \to P$ $\qquad\qquad$ (2)

$P \to \neg\neg Q$ $\qquad$ $\neg\neg Q \to Q$ $\qquad$ $\neg P \to \neg(P \to Q)$ $\qquad$ $\neg P$ $\qquad$ $(\Rightarrow E)$

$P \to Q$ $\qquad\qquad$ $\neg(P \to Q)$ $\qquad$ $(\Rightarrow E)$

$\bot$

(3)

$\neg\neg(P \wedge \neg Q)$

(DN)

$P \wedge \neg Q$

$(\wedge E)$

$\overline{\qquad}$ (2)

$\neg P$ $\qquad\qquad$ P

$(\Rightarrow E)$

$\bot$

(2)

$\neg\neg P$

(DN)

P

(1)

$((P \to Q) \to P) \to P$

Note: A proof using the law of excluded middle (EM) which says $\forall P : Prop. (P \vee \neg P)$ is also acceptable

Since

$$EM \iff DN \quad . \square$$

Second Part (15 pts)

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P$	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	0	1

P5. Soln (20pts)

First, we define the following propositions.

A = " Alice is a knight "

B = " Bob is a knight "

E = " Eve is a knight "

From the fact that Alice says that Bob is a knight we get

(i) $(A \Rightarrow B) \wedge (\neg A \Rightarrow \neg B)$

From the fact that Bob says that Alice is a knight but Eve is a knave we get

(ii) $(B \Rightarrow A \wedge \neg E) \wedge (\neg B \Rightarrow \neg (A \wedge \neg E))$

From the fact that Eve says that Alice is a knight and Bob is a knight we get

(iii) $(E \Rightarrow A \wedge B) \wedge (\neg E \Rightarrow \neg (A \wedge B))$

(i) has the same truth value as

$$A \Leftrightarrow B$$

(ii) has the same truth value as

$$B \Leftrightarrow (A \wedge \neg E)$$

(iii) has the same truth value as

$$E \Leftrightarrow A \wedge B$$

In a truth table involving proposition A, B, E we must look for a row where the value of $(A \Leftrightarrow B) \wedge (B \Leftrightarrow A \wedge \neg E) \wedge (E \Leftrightarrow A \wedge B)$ is T.

A	B	E	$\neg E$	$A \Leftrightarrow B$	$B \Leftrightarrow (A \wedge \neg E)$	$E \Leftrightarrow A \wedge B$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0					T

Therefore, Alice, Bob and Eve are all knaves.

P6. Soln (5 pts)

First soln. (Direct Proof)

Take x to be 10.

Take y to be $\log_{10} 3$.

We show that y is irrational.

Suppose there are integers p, q where q is non-zero and

$y = \log_{10} 3 = \frac{p}{q}$. It follows that

$3 = 10^{\log_{10} 3} = 10^{\frac{p}{q}}$. Hence

$3^q = 10^p$. But, the

LHS is an integer divisible by 3

while the RHS is not. Therefore $y = \log_{10}^{3}$ cannot be a rational number.

However,

$x^y = 10^{\log_{10}^{3}} = 3$ is a rational number.

Second Proof. (Indirect)

In this proof, we use the law of excluded middle (EM)

(EM)

$\boxed{\text{Either } \sqrt{2}^{\sqrt{2}} \text{ is rational or it is not.}}$

If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done:

Take x to be $\sqrt{2}$ and take y to be $\sqrt{2}$.
Both x and y are irrational because

[... include proof of irrationality of $\sqrt{2}$ ...]

If $\sqrt{2}^{\sqrt{2}}$ is irrational, however, then take

$x = \sqrt{2}^{\sqrt{2}}$ and

$y = \sqrt{2}$. we have

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{2} = 2$$

which is rational. □

Some common mistakes which cost points

1. Using variables not introduced before (1 pt)

2. Not stating your proof strategy (1pt)
e.g. proof by cases, proof by contradiction, contrapositn

3. Not being clear about what your assumption and goals are (2pts)

4.