To determine the required boundary conditions for this result to hold, let us consider the functions *f(x)* = J_y*(xx)* and *g(x)* = J_y*(β x)*, which, as will be proved below, respectively satisfy the equations

$$x^2f'' + xf' + (\alpha^2x^2 - v^2)f=0, \tag{18.85}$$

$$x^2g'' + xg' + (\beta^2x^2 - v^2)g=0. \tag{18.86}$$

Show that f(x) = $J_y(\alpha x)$ *satisfies (18.85).*

If *f(x)* = $J_y(\alpha x)$ and we write $w=\alpha x$, then

$$\frac{df}{dx} = \alpha\frac{dJ_y(w)}{dw} \quad and \quad \frac{d^2f}{dx^2} = \alpha^2\frac{d^2J_y(w)}{dw^2}$$

When these expressions are substituted into(18.85), its LHS becomes

$$x^2\alpha^2\frac{d^2J_y(w)}{dw^2} + x\alpha\frac{dJ_y(w)}{dw} + (\alpha^2x^2 - v^2)J_y(w)$$

$$=w^2\frac{d^2J_y(w)}{dw^2} + w\frac{dJ_y(w)}{dw} + (w^2 - v^2)J_y(w)$$

But from Bessel's equation itself, this final expression is equal to zero, thus verifying that *f(x)* does satisfy (18.85)

Now multiplying(18.85) by *f(x)* and (18.86) by *g(x)* and subtracting them gives

$$\frac{d}{dx}[x(fg'-gf')]=(\alpha^2-\beta^2)xfg, \tag{18.87}$$

where we have used the fact that

$$\frac{d}{dx}[x(fg'-gf')]= x(fg''-gf'')+x(fg'-gf').$$

By integrating(18.87) over any given range x = a to x=b, we obtain

$$\int_a^b xf(x)g(x)dx = \frac{1}{\alpha^2-\beta^2}\Big[xf(x)g'(x) - xg(x)f'(x)\Big]_a^b,$$

which, on setting *f(x)*=$J_y(\alpha x)$ and *g(x)* = J_y*(β x)*, becomes

$$\int_a^b xJ_y(\alpha x)J_y(\beta x)dx = \frac{1}{\alpha^2-\beta^2}\Big[\beta xJ_y(\alpha x)J_y'(\beta x) - \alpha xJ_y(\beta x)J_y'(\alpha x)\Big]_a^b. \tag{18.88}$$

If $\alpha \neq \beta$, and the interval [a,b] is such that the expression on the RHS of (18.88) equals zero, then we obtain the orthogonality condition(18.84). This happens, for example, if $J_y(\alpha\beta)$ and $J_y(\beta x)$ vanish at x = a and x= b, or if $J_y(\alpha\beta)$ and $J_y(\beta x)$ vanish at x = a and x = b, or for many more general conditions. It should be noted that the boundary term is automatically zero at the point x = 0, as one might expect from the fact that the Sturm- Loiville form of Bessel's equation has *p(x)=x*.

If $\alpha = \beta$, the RHS of (18.88) takes the indeterminant form 0/0. This may be