Influence of hyperparameters on online aggregation with countable experts

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Aggregating forecasts from multiple experts is a valuable method to improve prediction accuracy. This paper examines the influence of hyperparameters on the accuracy of the aggregation algorithm for a countable number of experts. We implement a time series generator with specified properties and an aggregating forecasting model. We conduct a series of experiments with various hyperparameters of the algorithm (including initialization weights, mixing scheme, train window). The experiments confirm that these hyperparameters have a significant influence on the algorithm's performance.

Keywords: *online learning; aggregating algorithm; prediction with experts' advice; Fixed Share, Mixing Past Posteriors (MPP)*

1 Introduction

This work is inspired by the algorithm, developed in the article [4], considering online game of prediction with experts' advice. The data is presented as a time series, consisting of outcome pairs — «signal» and «response». In contrast to the classical statistical theory of sequential prediction, we make no assumptions about the nature of the data (which could be deterministic, stochastic, etc.). We use machine learning methods to build forecasters within a game-theoretic approach. The online learning master model considers a series of reference forecasters, referred to as experts, to build its opinion by combining their predictions.

The general prediction algorithm with expert advice follows this structure: Learning progresses in trials at discrete points in time. During each step , expert models, based on past observational data subsamples, provide their predictions. The master model then makes a decision using the chosen aggregating algorithm. At the end of the trial, the generator presents the true outcome, and both the master and expert models are scored using a loss function. The difference between the master's cumulative losses and the expert's cumulative losses is defined as regret. The traditional goal of the aggregating algorithm is to keep it as small as possible.

We use special assumptions about the data generation structure when building forecasting strategies. It is assumed that there are multiple generators, whose structure is unknown to the predictors. The time series is obtained by merging segments, each produced by one of the generators. These segments are called areas of stationarity, and can be studied using machine learning methods. Each corresponding local predictive model will be constructed based on data from the area of stationarity and can be then successfully applied in other areas of stationarity generated by the same generator.

In this formulation of the forecasting problem, the series of prediction steps is divided into segments that frame arbitrary sequences of expert strategies. The sequence of segments and its associated sequence of experts is called a partition. The modified goal of the aggregating algorithm is to perform well relative to the best partition. Accordingly, the new concept of algorithm regret is the difference between the algorithm's losses and the cumulative losses of the sequence of experts. This change allows for a more accurate modeling of real-life conditions, where the nature of responses may change over time and different experts may predict with varying degrees of success depending on the current trend.

The corresponding algorithm is called Fixed Share [5]. A further proposed generalization of it is the Mixing Past Posteriors (MPP) method [6]. The cumulative losses of the aggregating algorithm are related to convex combinations of expert losses. The concept of regret also changes. Now the algorithm's cumulative losses are compared to the cumulative losses of convex combinations of expert strategies.

A characteristic feature of the problem considered in [4] is the absence of a predefined set of competing expert strategies, as was the case in the works cited above. Instead, new expert strategies are constructed at each step of the online learning process. The master must aggregate the forecasts of all expert strategies constructed up to that time in real-time at each step. Algorithm GMPP, proposed in [4], is the foundation of our experiments.

2 Problem statement

Let $x_1, x_2, \dots$ be the sequence of signals belonging to the sample space $\mathcal{X}$. The masters goal is to predict the sequence of the corresponding responces $y_1, y_2, \dots$ belonging to the outcome space $\mathcal{Y}$. We assume that the master knows $a, b \in \mathbb{R}$ — the bounds of the responces sequence, s.t. $\forall i \quad a \leqslant y_i \leqslant b$.

We assume that there is an countable number of experts $i \in \mathcal{N}$, where $\mathcal{N}$ is the natural numbers set. The predictions of the master and experts belong to the decision space $\mathcal{D}$. Let $\lambda : \mathcal{D} \times \mathcal{Y} \to \mathbb{R}_+$ be the nonnegative loss function.

At each step t every expert $i \in \mathcal{N}$ provides his prediction $f_t^i = f_t^i(x_t) \in \mathcal{D}$. After obtaining them the master gives his prediction $p_t = p_t(x_t) \in \mathcal{D}$. Next, the generator reveals the true outcome y_t, and the losses are computed. Let $h_t = \lambda(p_t, y_t)$ be the master's loss , and $l_t^i = \lambda(f_t^i, y_t)$ be the loss of expert $i \leqslant t$. For $i > t$ (i.e. expert i is not yet initialized), $l_t^i = h_t$.

With designations $L_T^i = \sum_{t=1}^T l_t^i$, representing the cumulative loss of expert i during the first T steps, and $H_T = \sum_{t=1}^T h_t$, representing the master's cumulative loss during the first T steps, we can define the master's regret relative to the expert i as $R_T^i = H_T - L_T^i$.

The best partition is formed based on a hindsight analysis of the experiment results, assuming we know the segment boundaries. For each segment, the best partition predictions are set equal to the predictions of the best local expert (the one that has the lowest cumulative sum on that segment). Master's regret relative to the best partition is defined as $R_T = H_T - L_T$, where L_T is the cumulative loss of the best partition. We use this R_T as a metric in our experiments.

Algorithm GMPP

Initialize the weights w_1^i so that $\sum\limits_{i \in \mathcal{N}} w_1^i = 1$

for $t = 1, 2, \dots$ **do**

- Expert f^t initialization
- Signal x_t is received
- Experts' predictions $f_t^i = f_t^i(x_t), 1 \leqslant i \leqslant t$
- Computation of normalized weights of experts $1 \leqslant i \leqslant t$: $\widehat{w}_t^i = \dfrac{w_t^i}{\sum_{j=1}^t w_t^j}$
- Master's prediction evaluation $\gamma_t = Subst(\mathbf{f_t}, \widehat{\mathbf{w}}_\mathbf{t}) = \dfrac{a+b}{2} + \dfrac{1}{2\eta(b-a)} \cdot \ln \dfrac{\sum\limits_{i \in \mathcal{N}} \hat{w}_t^i e^{-\eta(b-f_i)^2}}{\sum\limits_{i \in \mathcal{N}} \hat{w}_t^i e^{-\eta(a-f_i)^2}}$,

  where $\widehat{\mathbf{w}}_\mathbf{t} = (\widehat{w}_t^1, \widehat{w}_t^2, \dots \widehat{w}_t^t)$, $\mathbf{f_t} = (f_t^1, f_t^2, \dots f_t^t)$
- True outcome y_t is revealed

- Computation of master's loss $h_t = \lambda(p_t, y_t)$ and experts' losses: $l_t^i = \begin{cases} \lambda(f_t^i, y_t), & \text{if } i \leqslant t \\ h_t, & \text{if } i > t \end{cases}$
- **Loss Update** weights modification

$$\widetilde{w}_t^i = \frac{w_t^i e^{\eta l_t^i}}{\sum_{j=1}^{t} w_t^j e^{-\eta l_t^i} + e^{-\eta h_t}(1 - \sum_{j=1}^{t} w_t^j)}$$

- **Mixing Update** weights modification

$$\widetilde{w}_{t+1}^i = \alpha_t \widetilde{w}_1^i + (1 - \alpha_t)\widetilde{w}_t^i$$

end for

3 Experiment

We implement the synthetic time series generator and the GMPP algorithm itself. We then conduct experiments to analyze the effects of varying hyperparameters.

3.1 Data generation

We define generators $G_1, G_2, \ldots, G_k$ by their weight vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k \in \mathbb{R}^d$. Here, d represents the dimensionality of the signals.

We generate signals as a series of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T \sim \mathcal{N}(0, I_d)$, indicating they are drawn from a zero-mean Gaussian distribution with an identity covariance matrix .

This series of vectors is then randomly divided into a sequence of segments. We denote these segments as $S_1, S_2, \ldots, S_m$. Each segment S_i is a sequence of vectors itself, containing elements from $\boldsymbol{x}_{s_{i-1}+1}$ to $\boldsymbol{x}_{s_i}$ (where s_i is the index of the last element in segment i, $s_0 = 0$). Importantly, each segment S_i is assigned to a specific generator $G_{g(i)}$ according to a random function g. This function determines which generator is responsible for creating responses for that segment.

Ultimately, we obtain the series of responses $y_1, y_2, \ldots, y_T$ by taking the dot product of each signal vector with the weight vector of the corresponding generator and adding the noise: $y_i = \langle \boldsymbol{w}_{g(i)}, \boldsymbol{x}_i \rangle + \varepsilon_i$, where $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$

In our experiment, we fix the following data characteristics:

- Time series length $T = 2000$
- For adequate algorithm behaviour in the beggining of the algorithm, we add few segments (one from each generator) in front of the main time series. This part is not taken into account in regret calculation
- Signals dimention $d = 10$
- Generator weight vectors are drawn from a uniform distribution $U_{[-10,10]^d}$
- Responses bounds $(a, b) = (-40, 40)$
- Segments sizes are drawn from a uniform distribution $U_{[50,300]}$
- Number of generators $k = 5$
- Function g randomly alternates generators for adjacent segments
- White noise variance $\sigma^2 = 1$

Also, to ensure robust performance from the outset, we insert a short priming sequence at the beginning of the time series. This sequence includes one segment from each generator. Importantly, the data from this priming sequence is excluded from the regret calculation.

3.2 Experts

Each expert f_t is initialized as a linear regression model. This model is trained on a fixed window of l past observations. The training process yields a weight vector, which is calculated as $\theta_t = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$, where $\boldsymbol{X}^T = (\boldsymbol{x}_{t-1}, \boldsymbol{x}_{t-2}, \dots \boldsymbol{x}_{t-l}),\ \boldsymbol{y} = (y_{t-1}, y_{t-2}, \dots y_{t-l})$.

Given a new signal vector $\boldsymbol{x}$, the expert's prediction is obtained by calculating the dot product between the weight vector and the signal vector: $f_t(\boldsymbol{x}) = \langle \boldsymbol{\theta}_t, \boldsymbol{x} \rangle$

3.3 Hyperparameters

We investigate the impact of various hyperparameters on the GMPP algorithm's performance, focusing on initialization weights, mixing scheme, mixing update coefficients, and expert window size.

1. Initialization Weights (w_1^t):
   - Default: $w_1^t = \frac{1}{(t+1)\ln^2(t+1)}$
   - Experimental:
     - $w_1^t = \frac{1}{e^t}$, exploring exponential decay as a common alternative.
     - $w_1^t = \frac{1}{(t+4)\ln(t+4)\ln^2\ln(t+4)}$, representing a slower decaying function.
     - $w_1^t = \frac{1}{t^\alpha}$, with $\alpha \in (1, 2]$, to analyze the impact of faster decay rates.
     - $w_1^t = \frac{1}{t^\alpha}$, with $\alpha \in (0, 1]$, to analyze the behavior with diverging weights.
2. Mixing Update Coefficients (α_t):
   - Default: $\alpha_t = \frac{1}{t+1}$
   - Experimental:
     - $\alpha_t = \frac{1}{(t+1)^\beta}$, with $\beta \in (0, 2]$, to analyze different decay speeds.
     - $\alpha_t = \frac{1}{t+c}$, with varying c, to analyze the influence of a constant shift.
     - $\alpha_t = \frac{1}{c}$, with $100 \leqslant c \leqslant 10000$
     - $\alpha_t = \frac{1}{e^{t/3}}$, exponential decay, rapidly reducing the influence of initial weights.
3. Mixing Update Scheme:
   While GMPP in [3] utilizes a specific scheme, we explore various Mixing Fixed-Share Update schemes as presented in [6]:

$$\widetilde{w}_{t+1}^i = \sum_{q=1}^{t} \beta_t(q)\widetilde{w}_q^i$$

   - Start Vector Share (default in GMPP) — emphasizes the initial and the most recent weight vectors:

$$\beta_t(q) = \begin{cases} \alpha_t, & q = 1 \\ 0, & 1 < q < t \\ 1 - \alpha_t, & q = t \end{cases}$$

   - Uniform Past Share — assigns equal weight to all past weight vectors :

$$\beta_t(q) = \begin{cases} \alpha_t \frac{1}{t}, & 1 \leqslant q < t \\ 1 - \alpha_t, & q = t \end{cases}$$

– Decaying Past Share — assigns decreasing weights to past weight vectors:

$$\beta_t(q) = \begin{cases} \alpha_t \frac{1}{(t-q)^\gamma} \frac{1}{Z_t}, & 1 \leqslant q < t \\ 1 - \alpha_t, & q = t \end{cases}$$

,with $Z_t = \sum_{q=1}^{t-1} \frac{1}{(t-q)^\gamma}, \gamma > 0.$

We also propose a new mixing scheme:

– Increasing Past Share — assigns increasing weights to past weight vectors:

$$\beta_t(q) = \begin{cases} \alpha_t (t-q)^\gamma \frac{1}{Z_t}, & 1 \leqslant q < t \\ 1 - \alpha_t, & q = t \end{cases}$$

,with $Z_t = \sum_{q=1}^{t-1} (t-q)^\gamma, \gamma > 0.$

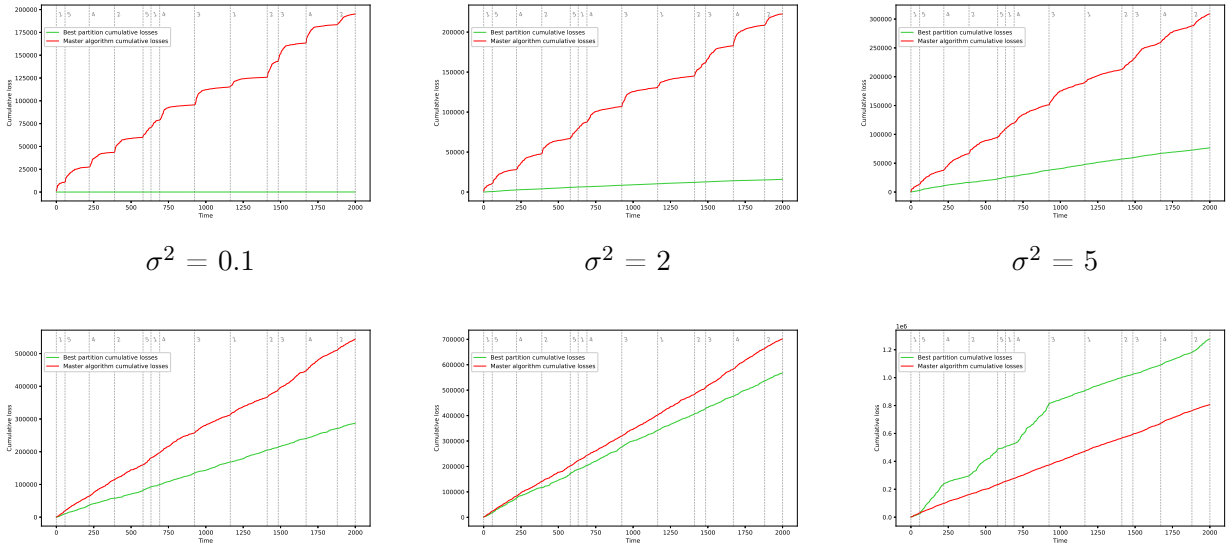
4. Window Size (l): We vary train window size $l \in \{5, 10, 20, 50, 100\}$ to check algorithm performance in situations when experts lack information or when they often train on pieces made by different generators.

As a metric of quality we use R_T — master's regret relative to the best partition. We run experiments 4 times for each configuration of hyperparameters, and take the mean of the earned regrets.

4 Results

We analyze the impact of various hyperparameters on the GMPP algorithm's performance, focusing on initialization weights, mixing schemes, mixing update coefficients, and expert window size. Our primary performance metric is the mean regret relative to the best partition, calculated from four independent runs for each hyperparameter configuration.

4.1 Influence of Noise



$\sigma^2 = 0.1$ $\sigma^2 = 2$ $\sigma^2 = 5$

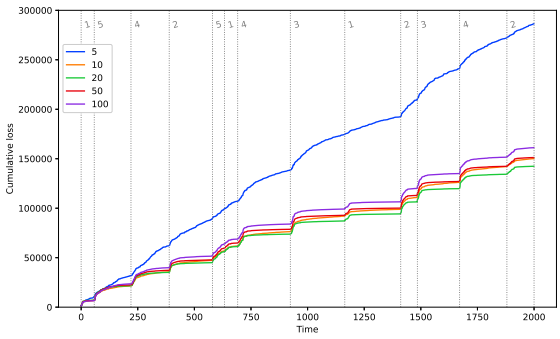
$\sigma^2 = 10$ $\sigma^2 = 15$ $\sigma^2 = 20$

Figure 1 Total losses with different noise variance σ^2 (alpha function is defaut, weight function is $1/x^{1.01}$, window size = 10)

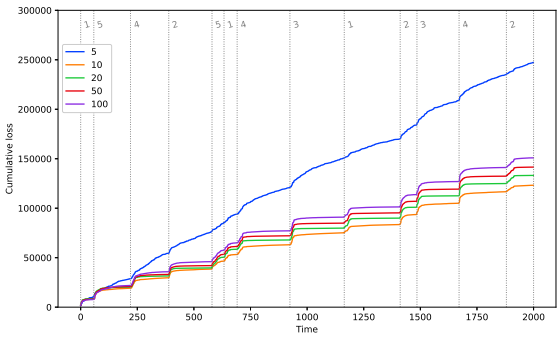
Table 1 Regret with different Mixing Update schemes and Noise (alpha function is defaut, weight function is $1/x^{1.01}$, window size = 10)

	Mixing scheme			
Noise variance	decaying past	increasing past	start	uniform past
0.10	131348.25	114564.74	131578.01	115927.25
1	123066.72	110438.09	132268.30	110569.83
2	109753.81	105398.29	136043.26	103136.06
5	76132.46	92343.75	144554.62	83630.45
6	17155.14	89032.63	146382.86	25178.60
7	-18273.00	29417.82	144827.02	-9208.74
8	-83796.21	-13307.43	147510.57	-55905.83
10	-649917.41	-120166.34	90089.56	-377184.32
12	-828973.51	-1123130.94	-420588.94	-1354731.42
15	-1376777.17	-1508510.73	-1205600.59	-1862352.43
20	-8506756.45	-9004436.41	-8784588.25	-9007978.86
50	-12031878.44	-14242897.85	-4704221.50	-14879230.26

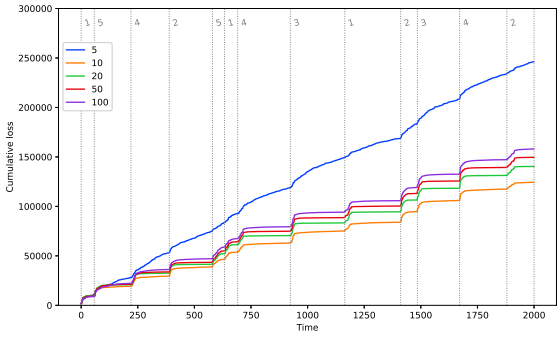
4.2 Influence of Training Window Size



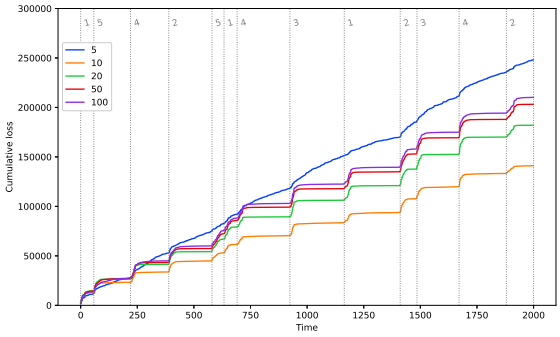
Start Vector Share



Increasing Past Share



Uniform Past Share



Decaying Past Share

Figure 2 Total losses with different sizes of train window (alpha function is defaut, weight function is $1/x^{1.01}$, $\sigma^2 = 1$)

Table 2 Regret with different Mixing Update schemes and Train Windows (alpha function is defaut, weight function is $1/x^{1.01}$, $\sigma^2 = 1$)

	Mixing scheme			
Train Window	decaying past	increasing past	start	uniform past
5	140895.88	155163.20	186550.57	150308.76
10	123066.72	110438.09	132268.30	110569.83
20	163303.00	121289.12	125763.68	127764.22
50	187007.96	132165.07	136397.74	139560.71
100	195888.54	143925.39	148888.73	150253.75

4.3 Impact of Initialization Weights

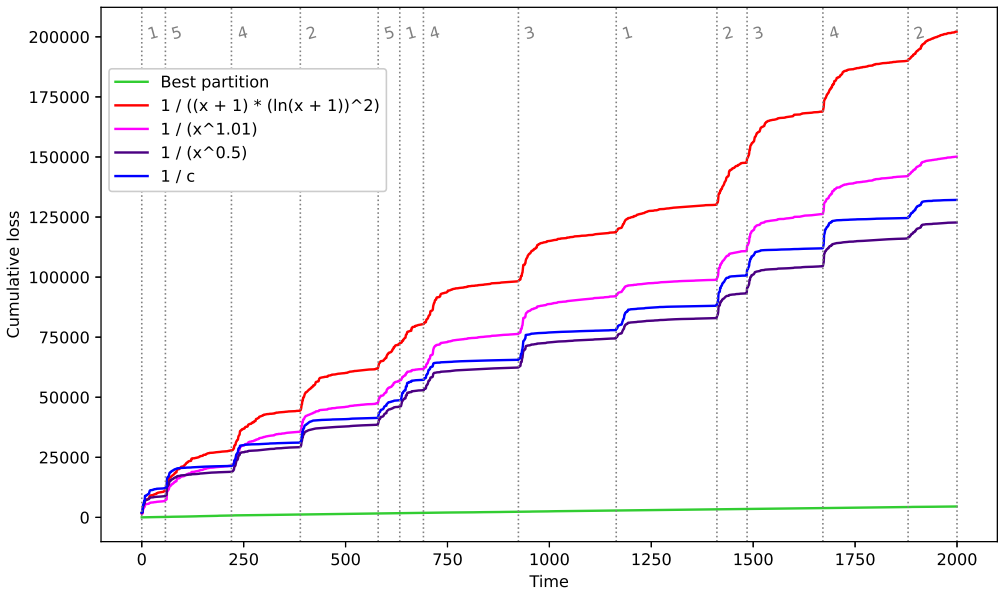


Figure 3 Total loss for different weight functions (alpha function is defaut, $\sigma^2 = 1$, window size = 10)

Table 3 Regret with different weight functions and mixing types (alpha function is defaut, $\sigma^2 = 1$, window size = 10)

	Mixing Type			
Weight Function	Decaying Past	Increasing Past	Start	Uniform Past
$1/((x+1)\ln^2(x+1))$	122513.26	122383.91	175594.64	117175.93
$1/c$	128114.78	117241.85	113998.26	119703.68
$1/(x^{0.1})$	127435.12	115101.52	111276.08	117830.57
$1/(x^{0.3})$	126769.51	112900.36	108946.40	115819.07
$1/(x^{0.5})$	126191.99	111221.38	108630.68	114153.63
$1/(x^{0.7})$	125352.96	109775.49	112027.20	112378.87
$1/(x^{0.9})$	123969.39	109408.73	122832.21	110783.66
$1/(x^{1.01})$	123066.72	110438.09	132268.30	110569.83
$1/(x^{1.1})$	123963.12	111050.44	135253.09	111366.88
$1/(x^{1.2})$	124846.17	111873.03	137976.99	112298.14
$1/(x^2)$	133051.62	124473.63	148080.01	123449.14
$1/((x+4)\ln(x+4)\ln^2(\ln(x+4)))$	121229.77	119413.95	169065.24	114823.76

4.4 Impact of Mixing Update Coefficients

Table 4 Regret with different Alpha and Weight Functions in Start Vector Share Update scheme ($\sigma^2 = 1$, window size = 10)

	Weight function			
Alpha function	$\frac{1}{(x+1)\ln^2(x+1)}$	$\frac{1}{t^{1.01}}$	$\frac{1}{t^{0.5}}$	$\frac{1}{c}$
$1/(x+1)$	175594.64	132268.30	108630.68	113998.26
$1/(x+1)^{0.5}$	245494.10	207099.32	164079.31	141554.07
$1/(x+1)^{1.5}$	130339.21	125699.71	130185.66	131329.66
$1/(x+1)^2$	136029.09	134929.54	133876.06	132709.43
$1/e^{x/3}$	136413.87	135409.06	134012.15	132760.88
$1/(x+10)$	175411.17	132208.09	108638.55	114051.47
$1/(x+100)$	173760.98	131623.56	108732.56	114568.90
$1/(x+1000)$	162129.20	127165.47	110389.16	118512.52
$1/100$	220008.81	163117.40	129035.87	117521.60
$1/500$	198094.83	141385.79	111706.58	108969.41
$1/1000$	184324.73	135612.25	109449.15	111709.95
$1/5000$	147561.46	121905.50	115147.43	123741.66
$1/10000$	136414.42	119357.81	120634.74	127510.93
$1/50000$	130783.65	125141.06	129963.25	131542.49

4.5 Gamma parameter in Increasing Past Mixing Update

Table 5 Regret with different γ in Increasing Past Share Mixing Update scheme (alpha function is defaut, $\sigma^2 = 1$, window size = 10)

	Weight function			
γ	$\frac{1}{(x+1)\ln^2(x+1)}$	$\frac{1}{t^{1.01}}$	$\frac{1}{t^{0.5}}$	$\frac{1}{c}$
0.5	119783.90	112279.88	118155.78	110151.26
1	122383.91	111221.38	117241.85	110438.09
2	126865.67	110089.54	116216.44	111563.64
4	133447.69	109160.56	115299.68	113922.30

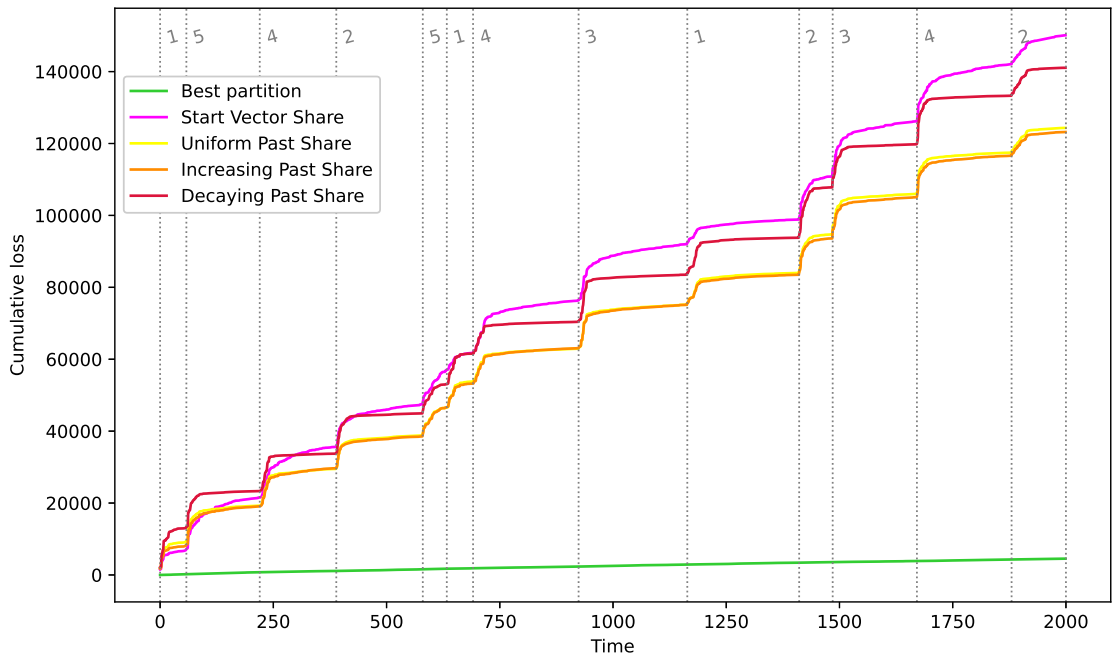


Figure 4 Total loss with different mixing schemes
(alpha function is defaut, weight function is $1/x^{1.01}$, $\sigma^2 = 1$, window size $= 10$)

Table 6 Mean values with different Mixing Update schemes and Weight and Alpha functions ($\sigma^2 = 1$, window size = 10)

Mixing scheme	start			increasing past			uniform past			decaying past		
α_i \ w_1^i	$\frac{1}{x^{1.1}}$	$\frac{1}{x^{0.5}}$	$\frac{1}{c}$	$\frac{1}{x^{1.1}}$	$\frac{1}{x^{0.5}}$	$\frac{1}{c}$	$\frac{1}{x^{1.1}}$	$\frac{1}{x^{0.5}}$	$\frac{1}{c}$	$\frac{1}{x^{1.1}}$	$\frac{1}{x^{0.5}}$	$\frac{1}{c}$
$1/(t+1)^{1.5}$	125699	130185	131329	131002	131849	131691	132418	132462	131940	134772	133653	132550
$1/(t+1)^2$	134929	133876	132709	135243	133935	132721	135301	133958	132731	135387	133999	132753
$1/\ln(t+1)$	418163	340332	309671	400131	337777	312108	390906	333528	309569	283210	252792	236401
$1/e^{t/3}$	135409	134012	132760	135409	134012	132760	135409	134012	132760	135409	134012	132760
$1/(t+10)$	132208	**108638**	114051	110259	111266	117298	110521	114212	119758	123137	126238	128145
$1/(t+100)$	131623	**108732**	114568	**109743**	111718	117835	110604	114760	120269	123716	126653	128428
$1/(t+1000)$	127165	110389	118512	110754	115361	121521	113447	118679	123629	127161	129057	129996
$1/100$	163117	129035	117521	129873	125704	120981	126579	125443	122193	116458	119146	121009
$1/500$	141385	111706	**108969**	112133	110618	112210	110553	111701	114376	115819	120327	124078
$1/1000$	135612	**109449**	111709	**109844**	110794	115281	**109804**	113087	117809	120757	124499	127207
$1/5000$	121905	115147	123741	115333	121026	125928	119177	123913	127377	130666	131268	131337
$1/10000$	119357	120634	127510	121016	125808	128881	124693	127919	129766	132825	132555	132022
$1/50000$	125141	129963	131542	130965	131894	131885	132477	132530	132100	134850	133704	132608

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