TREE-WIDTH DRIVEN SDP FOR THE MAX-CUT PROBLEM

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ABSTRACT

This article discusses the well-studied Max-Cut problem in graph theory, which has found applications in various fields, particularly Machine Learning, Theoretical physics (the Ising model), and VLSI design. The original problem is NP-complete, and we call an effective solution such a polynomial algorithm that gives the answer closest to the true one. For a long time, the best accuracy achievable by polynomial algorithms was at least half of the optimal cut. It was only in 1995 that Goemans and Williamson introduced a polynomial algorithm using semidefinite programming and randomized rounding, guaranteeing an approximation of ≈ 0.878. This is the best possible approximation guarantee for the Max-Cut problem under the Unique games conjecture **?**. In this article, we propose a novel approach to solving the Max-Cut problem with improved accuracy, in the particular case where the graph exhibits a treewidth bounded by a pre-fixed value, providing a number of heuristics.

Keywords Max-Cut · LP · SDP · Treewidth · Derivative-free optimization

1 Introduction

In this article, we are working with the Max-Cut problem, where one is interested in finding the cut of the largest value according to a given graph. Max-Cut is an example of an NP-hard problem that we are interested in solving using a polynomial-time algorithm that produces an answer that is as close as possible to the optimal solution. For example, a simple probabilistic algorithm with an approximation factor of 0.5 involves dividing the vertices of the graph into two groups by flipping a coin. This approach, pioneered by Humans and Williamson, has proven to be a significant breakthrough, and it has been shown that this approximation factor is the best achievable in polynomial time under the assumption of the Unique Games Conjecture.

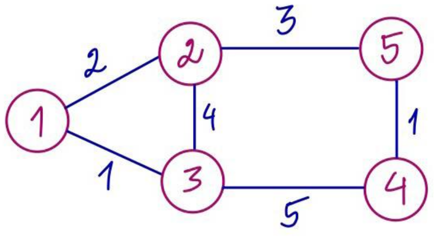
Our aim is to restrict ourselves to the specific case when the requirement that the treewidth of the graphs be bounded by a constant has been imposed in advance, and to develop a polynomial algorithm with the best approximation. Using Semi-Definite Programming and a bunch of heuristics we have developed a more accurate approximation technique for the Max-Cut problem.

2 Problem statement

Given a weighted, undirected graph $G = (V, E)$ i.e. each edge, (i, j), has a weight, $w_{ij} = w_{ji}$. The set of vertices is partitioned into two parts, S and $\bar{S} := V \setminus S$. Let us call the weight of this "cut" the sum of the weights of edges whose endpoints lie in different parts

$$W(S) := \sum_{(i,j)\ \in\ S\times\bar{S}} w_{ij}$$

The goal is to find the cut with maximal possible weight.



S	$W(S)$
$\varnothing$	0
$\{5\}$	$1+3=4$
$\{1,2,3\}$	$3+5=8$
$\{1,2,3,4\}$	$3+1=4$
$\{1,5,4\}$	$2+3+4+1+5=15$
$\{1,3,5\}$	$2+4+3+1+5=15$

Figure 1: Example

The maximum cut is 15. Actually, the graph is not bipartite, the total weight of all edges is 16, and there are no edges lighter than 1.

3 Theory

Let us rephrase the problem in the context of Integer Linear Programming (ILP) and reduce it to Semidefinite Programming (SDP)

For each vertex i in the graph, we define the indicator $x_i \in \{-1,+1\}$, characterizing the affiliation of $i \in S$ or $i \in \bar{S}$, respectively.
Similarly, for each edge (i,j), we define the indicator $y_{ij} = y_{ji} = x_i x_j \in \{-1,+1\}$ characterizing the belonging of the edge to the cut. Let x represent the vector $x = (x_1,\dots,x_n)$, where $n = |V|$.

Now the Max-Cut can be represented as

$$ILP = \max_{\substack{x_i \in \{-1,+1\} \\ y_{ij}=x_i x_j}} \frac{1}{4}\sum_{i\in V}\sum_{j\in V} w_{ij}(1-y_{ij}) = \max_{x_i^2=1} \frac{1}{4}\sum_{i\in V}\sum_{j\in V} w_{ij}(1-x_i x_j) =$$
$$= \max_{\substack{X=xx^T \\ X_{ii}=1}} \frac{1}{4}\sum_{i\in V}\sum_{j\in V} w_{ij}(1-X_{ij}) = SDP$$

Clearly, $X_{ij} = x_i x_j$, hence $x_i^2 = X_{ii} = 1 \implies x_i \in \{-1,+1\}$

In particular, the matrix X is

1. Symmetric with units on the diagonal
2. Positive semi-definite, indeed
   $\forall v \in \mathbb{R}^n: \quad v^T X v = v^T x x^T v = (x^T v)^T (x^T v) = (x^T v)^2$

Finally, let us consider the following problem

$$SDP^* = \max \frac{1}{4}\sum_{i\in V}\sum_{j\in V} w_{ij}(1-X_{ij}), \text{ where } X = \begin{pmatrix} 1 & x_{12} & \dots & x_{1n} \\ x_{12} & 1 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & 1 \end{pmatrix} \succcurlyeq 0$$

The difference between the SDP and SDP^* is as follows

	SDP	SDP^*
X	$X = xx^T,\ x \in \mathbb{R}^n$	$X = LL^T,\ x \in \mathbb{R}^{n\times m}$
X_{ij}	$X_{ij} = x_i x_j \in \{-1,+1\}$	X_{ij} arbitrary element

Such formulation is a relaxation of the Max-Cut problem (matrix X still meets properties 1 and 2) and it can be easily solved with cvxpy:

```python
import numpy as np
import cvxpy as cp

n = 5
W = np.array([[0, 2, 1, 0, 0],
              [2, 0, 4, 0, 3],
              [1, 4, 0, 5, 0],
              [0, 0, 5, 0, 1],
              [0, 3, 0, 1, 0]])
X = cp.Variable((n, n), symmetric=True)
constraints = [X >> 0]
constraints += [X[i][i] == 1 for i in range(n)]
objective = cp.Maximize(0.25 * cp.sum(cp.multiply(W, (1 - X))))
prob = cp.Problem(objective, constraints)
prob.solve()
print("The optimal value is ", prob.value)
print("The solution is", X.value)
```

The optimal value is 15.000002187529262

$$\text{A solution is } X = \begin{pmatrix} 1 & -1.00000047 & 1.0000005 & -1.00000052 & 1.00000057 \\ -1.00000047 & 1 & -1.00000027 & 1.00000038 & -1.00000037 \\ 1.0000005 & -1.00000027 & 1 & -1.00000026 & 1.0000004 \\ -1.00000052 & 1.00000038 & -1.00000026 & 1 & -1.00000042 \\ 1.00000057 & -1.00000037 & 1.0000004 & -1.00000042 & 1 \end{pmatrix}$$

Indeed, the matrix $X = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$ corresponds to the maximum cut $S = \{1, 3, 5\}$ of the value $W(S) = 15$.

TODO: write about random rounding and 0.878 algorithm

$$OPT = \max_{x_i^2=1} x^T Lx = 4 \cdot \; MaxCut, \text{ where } L \text{ is the Laplacian of the graph}$$

$$Dual = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i(1 - x_i^2) - \sum_{i,j}^{n} x_i x_j L_{ij} = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2 = \min_{Diag(\xi) \succcurlyeq L} \sum_{i=1}^{n} \xi_i$$

Here is cvxpy solver for dual problem:

```python
import numpy as np
import cvxpy as cp

n = 5
L = np.array([[ 3, -2, -1,  0,  0],
              [-2,  9, -4,  0, -3],
              [-1, -4,  9, -5,  0],
              [ 0,  0, -5,  6, -1],
              [ 0, -3,  0, -1,  4]])
X = cp.Variable((n, n), diag=True)
constraints = [X >> L]
objective = cp.Minimize(cp.trace(X))
prob = cp.Problem(objective, constraints)
prob.solve()
print("The optimal value is", 0.25 * prob.value)
```

The optimal value is 14.749999991267886

Lemma 1.

$$Dual = \min_{Diag(\xi) \succcurlyeq L} \sum_{i=1}^{n} \xi_i = \min_{Diag(\xi) \succcurlyeq L} \max_{x_i^2=1} x^T Diag(\xi) x = \min_{L_T \succcurlyeq L} \max_{x_i^2=1} x^T L_T x = TreeRel$$

where L_T can be represented as $L_T = L_{tree} + Diag$, where L_{tree} corresponds to Laplacian of a tree graph and $Diag$ is a diagonal matrix with non-negative values.

$$H_k = \min_{\substack{T\colon T=T^\top \succcurlyeq L \\ tw(T) \leqslant k}} \max_{x_i^2=1} x^\top T x, \qquad OPT = H_k \leqslant \ldots \leqslant H_1$$

where optimization is taken over all graph with tree-width less than k. That is internal problem can be solved by dynamic programming.

Instead of insisting on $treewidth \leqslant k$ matrix T can be restricted for having less or equal than k diagonals.

$$D_k = \min_{\substack{T\colon T=T^\top \succcurlyeq L \\ T\ is \leqslant k-diagonal}} \max_{x_i^2=1} x^\top T x \qquad \text{then} \quad H_k \leqslant D_k$$

Optimization for this problem can be performed using Derivative-free methods. We will work with Hill climbing algorithm.

Possible implementation:

```
def hill_climbing(f, x0):
    x = x0  # initial solution
    while True:
        neighbors = generate_neighbors(x)  # generate neighbors of x
        # find the neighbor with the highest function value
        best_neighbor = max(neighbors, key=f)
        if f(best_neighbor) <= f(x):  # if the best neighbor is not better than x,
 stop
            return x
        x = best_neighbor  # otherwise, continue with the best neighbor
```

References

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