TREE-WIDTH DRIVEN SDP FOR MAX CUT PROBLEM

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ABSTRACT

This paper addresses the well-known Max Cut problem, which has various applications both in machine learning and theoretical physics. The Max Cut problem is computationally NP-hard over general graphs. This paper presents a novel empirical approach aimed at enhancing the quality of Max-Cut approximations within polynomial time bounds. While the problem is tractable for graphs with small tree-width, its solution over general graphs often relies on Semi-Definite Programming or Lovász-Schrijver relaxations. We achieve the described improvement of approximation quality by combining relaxation approaches, the tree-width ideas and various heuristics described in the paper.

Keywords SDP · Treewidth · Max Cut · Lovász-Schrijver relaxations

1 Introduction

In this paper, we will discuss a non-asymptotic improvement of the solution to the MAX CUT problem – the problem of finding the maximum cut in undirected graphs. It involves partitioning the vertices of a graph into two sets such that the number of edges between the two sets (the cut) is maximized. This problem has applications in many spheres, including machine learning, theoretical physics, and theoretical computer science. It serves as a basis for developing approximation algorithms and heuristic methods for solving other optimization problems. Currently, for graphs in general, the best solutions proposed by X. Goemans and David P. Williamson find a cut that contains at least approximately 88% of total weight of the edges in the optimal cut [1]. There are families of graphs for which this bound is asymptotically optimal unless Unique games conjecture is false [16].

In this article, we focus on a non-asymptotic improvement of the solution in polynomial time on arbitrary graphs. The known solution [1] utilizes Semi-Definite Programming problems, and here, we present reasoning that allows solving them with greater accuracy by combining optimization ideas, tree-width approach, and heuristics. We decide on the efficiency of provided algorithm by comparing it with well-known ones using the different datasets[10 - 13].

2 Problem statement

We focus on weighted undirected graphs, where each edge (i, j) is assigned a weight w_{ij}. As the graph is undirected,$w_{ij} = w_{ji}$. Such graphs are represented as G = (V, E), where V denotes the set of vertices and E is the symmetric matrix with w_{ij} indicating the weight of edge (i, j). Later we will refer to weighted undirected graphs simply as graphs.

Given a fixed graph G = (V, E) with the sum of weights denoted by W, a cut in the graph is defined as a subset $S \subseteq V$. The complement of S is denoted by $T = V \setminus S$. Notably, a cut partitions the vertices into two sets: S and T. Additionally, the edges are divided into three categories: those entirely within S, those entirely within T, and those split by the cut, where one vertex lies in S and the other in T. Let's define W(S) to be the weight of the cut:

$$W(S) = \sum_{i \in S} \sum_{j \notin S} w_{ij}$$

Our goal is to find in polynomial time cut $S_{found} \subseteq V$, such that the value $W(S_{found})$ is as big as possible

$$W(S_{found}) \to \max_{S_{found}}$$

We decide on the efficiency of provided algorithm by comparing it with well-known ones using the different datasets [10 - 13].

3 Theory

We find the approximate solution using SDP solution. First of all, we show, how SDP problem is connected to Max Cut.

Let matrix L be L = $\begin{bmatrix} (\sum_{i=1}^n w_{1i}) & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & (\sum_{i=1}^n w_{2i}) & -w_{23} & \dots & -w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & (\sum_{i=1}^n w_{ni}) \end{bmatrix}$

Later we call such matrix the Laplacian of the graph.
Then we state that $W(S) = \frac{1}{4} x \top Lx$, where $x_i = 1$ if $x \in S$ and $x_i = -1$ if $x \in T$. It is easy to see: the coefficient before each variable w_{ij} 1 in both sides of equality equals to 1 if i and j are from parts of partition and equals to 0 otherwise.

It means that our task is equivalent to finding

$$OPT = \max_{x_i^2 = 1} x^T Lx$$

since

$$\max_{S \subseteq V} W(S) = \frac{1}{4} \max_{x_i^2=1} x^T Lx$$

Let's look at the dual problem for OPT. We will also call SDP a problem relaxed by forgetting about the rank condition.

$$OPT = \max_{x_i^2=1} x^T Lx = \max_{\substack{x_i^2=1 \\ X = x^T x}} LX = \max_{\substack{X \succeq 0 \\ diag(X) = 1_n \\ rank(X)=1}} \operatorname{Tr}(LX) \leq \max_{\substack{X \succeq 0 \\ diag(X)=1_n}} \operatorname{Tr}(LX)$$

Let's find the dual for OPT problem.

$$Dual = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i (1 - x_i^2) - \sum_{i,j}^{n} x_i x_j L_{ij} = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2$$

if $-L - Diag(\lambda) \not\succeq 0$ then

$$\min_{x} \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2 = -\infty$$

since we can multiply the vector, which proves that $-L - Diag(\lambda) \not\succeq 0$, by constant and get the arbitrary small value.

And if $-L - Diag(\lambda) \succeq 0$, then

$$\min_{x} \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2 = \min_{x} \sum_{i=1}^{n} \lambda_i$$

This means that if we denote $\xi_i := -\lambda_i$, Dual problem can be rewritten this way:

$$Dual = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i (1 - x_i^2) - \sum_{i,j}^{n} x_i x_j L_{ij} = \max_{\substack{\lambda: \\ -L - Diag(\lambda) \succeq 0}} \sum_{i=1}^{n} \lambda_i = \max_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} -\xi_i = \min_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} \xi_i$$

Later we will try to approximate Dual value using tree-width ideas, but first of all let's prove the following Lemma.

Lemma 3.1.

$$Dual = \min_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} \xi_i = \min_{L_T \succeq L} \max_{x: x_i^2 = 1} x^T L_T x = TreeRel$$

where L_T can be represented as $L_T = L_{tree} + Diagonal$, where L_{tree} corresponds to Laplacian of a tree graph and Diag is a diagonal matrix with non-negative values.

Proof. It is well-known, that tree is a bipartite graph and hence the MaxCut value equals to the total sum of edges in the graph

$$\max_{x: x_i^2 = 1} x^T L_{tree} x = 4 \sum_i \sum_{j>i} w_{ij}$$

Then it is easy to conclude, that if $L_T = L_{tree} + Diagonal$, then

$$\max_{x: x_i^2 = 1} x^T L_T x = 4 \sum_i \sum_{j>i} w_{ij} + \mathrm{Tr}(Diagonal)$$

Now we show inequalities between $Dual = \min_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} \xi_i$ and $TreeRel = \min_{L_T \succeq L} 4 \sum_i \sum_{j>i} w_{ij} + \mathrm{Tr}(Diagonal)$. $Dual \geqslant TreeRel$ is obvious since for each matrix $Diag(\xi) \succeq L$ it is possible to take $L_T = Diag(\xi)$ (which means that we simply take the tree with all the weights equal to 0.

Finally, $Dual \leqslant TreeRel$. In order to show that for each $L_T = L_{tree} + Diagonal \succeq L$ we can construct a vector ξ, such that $Diag(\xi) \succeq L$ and $4 \sum_i \sum_{j>i} w_{ij} + \mathrm{Tr}(Diagonal) = \sum_{i=1}^{n} \xi_i$. Let's take $\xi_i = Diagonal_{ii} + 2 \sum_{j=1}^{n} w_{ij}$. Then indeed

$$\sum_{i=1}^{n} \xi_i = 4 \sum_i \sum_{j>i} w_{ij} + \mathrm{Tr}(Diagonal)$$

Finally, we notice that $Diag(\xi) - L_T$ is SDP and hence $Diag(\xi) \succeq L_T \succeq L$ which completes the proof.

Let t_{ij} be the weight of an edge between vertexes i and j in the tree.

$$Diag(\xi) - L_T = \begin{bmatrix} (\sum_{i=1}^{n} t_{1i}) & t_{12} & t_{13} & \dots & t_{1n} \\ t_{21} & (\sum_{i=1}^{n} t_{2i}) & t_{23} & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & t_{n3} & \dots & (\sum_{i=1}^{n} t_{ni}) \end{bmatrix}$$

This matrix is symmetric and diagonally dominant with positive diagonal entries. It is known that such matrix is PSD. Or in this case it is obvious since

$$x^T (Diag(\xi) - L_T) x = \sum_{i=1}^{n} \sum_{j=i+1}^{n} t_{ij} (x_i + x_j)^2 \ \geqslant \ 0$$

■

Later we will refer to the Dual as SDP.

k-diagonal hierarchy Let's define k-diagonal hierarchy this way:

$$D_k = \min_{\substack{T: T = T^\top \succeq A \\ T \in (2k+1)-diag}} \max x^\top T x$$

Then we can see, that

$$OPT = D_n \ \leqslant \ \dots \ \leqslant \ D_1 = SDP$$

since inequalities are obvious, the first equality is true due to T = A being the optimal matrix for OPT and the second equality is true due to Lemma 3.1.

k-diagonal algorithm Now we are ready to describe the k-diagonal algorithm. First, we note that for fixed k if T is (2 k + 1)-diagonal matrix

$$\max_{x_i^2=1} x^\top T x$$

can be solved in O(n) time by dynamic programming. It is possible to calculate a two-dimensional array dp, where for $1 \leqslant i \leqslant n, 0 \leqslant mask \leqslant 2^{k-1} - 1$

$$dp[i][mask] = \max_{\substack{x_i^2=1 \\ x_{i-k+1},\ldots x_{i-1}, x_i = mask}} x^\top T_i x,$$

where 1)T_i, is $n \times n$ matrix which is made out of matrix T by setting all the values outside top left $k \times k$ matrix to zeros. 2) It is known, how this and the previous k - 1 vertexes are distributed between parts of cut: ones in $mask$ correspond to the vertexes from one part of the cut, and mimus ones in $mask$ correspond to the vertexes from another one. $dp[i][b_1, ..., b_n]$ can be easily recalculated by $dp[i][-1, b_1, ..., b_{n-1}]$, $dp[i][1, b_1, ..., b_{n-1}]$ and $L[i-k][i], L[i-k+1][i], ..., L[i][i]$

It means, that we can find the optimal value of H_k using gradient-free methods of optimization and solving $\max_{x_i^2=1} x^\top T x$ with oracle, which uses described dynamic programming and works for O(n) time.

Finally, we can restore the final cut corresponding as the cut we get from this optimization.

4 Computational experiment

4.1 Data

We consider well known BiqMac dataset for testing our solution and comparing it with others. There are different types of graphs in this dataset:

1. g05-n.i For each dimension unweighted graphs with edge probability 0.5. n=60,80,100.
2. pm1s-n.i For each dimension weighted graphs with edge weights chosen uniformly from 0,1 and density 0.1. n=80,100
3. pm1d-n.i For each dimension weighted graphs with edge weights chosen uniformly from 0,1 and density 0.99. n=80,100
4. pwd-100.i For each density graphs with integer edge weights chosen from [0,10] and density d=0.1,0.5,0.9, n=100.

4.2 Plan of experiment

1. Implement standard MaxCut solution

2. Implement new MaxCut solution

3. Compare the average ratio of cut value divided by total maximum cut with SDP solution and new solutions using different sets of graphs

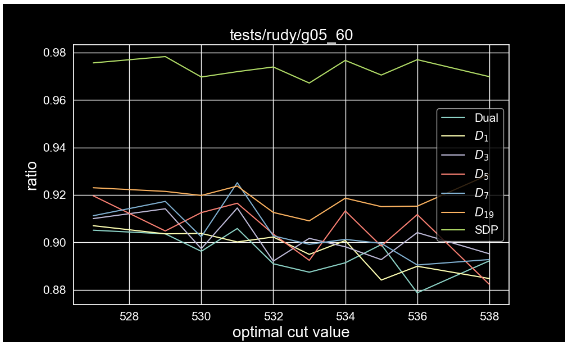
We refer to the solution described in [7] and implemented in [3] as basic solution or SDP, while the Dual solution refers to one solving problem which we call Dual. We refer to the k-diagonal optimization solutions as D_k.

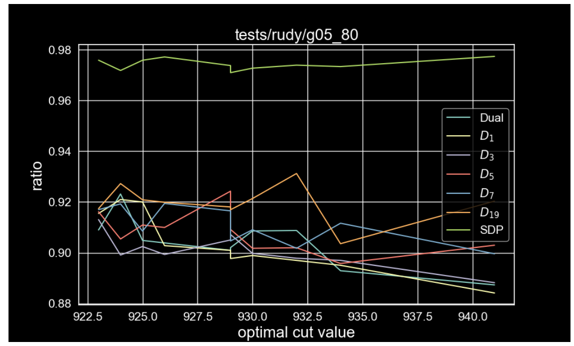
4.3 Comparison

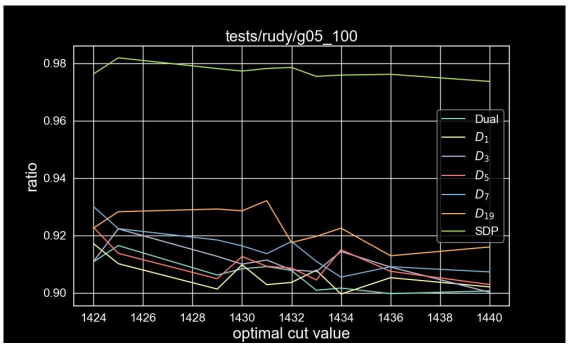
Unfortunately, as we can observe, our solution does not show significant increase in accuracy on many different types of graphs. But it is more effective on the graphs, which laplacians have a small number of diagonals (it shows optimal results for such graphs).

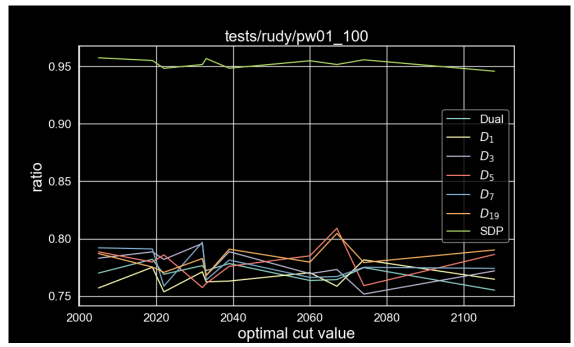
Test	$\mathbf{Dual}$	$\mathbf{D_1}$	$\mathbf{D_3}$	$\mathbf{D_5}$	$\mathbf{D_7}$	$\mathbf{D_{19}}$	$\mathbf{SDP}$
g05-60	0.8951	0.8971	0.902	0.9055	0.9042	0.9187	0.9731
g05-80	0.9042	0.9033	0.9009	0.9079	0.9107	0.9197	0.9743
g05-100	0.9062	0.906	0.9107	0.9102	0.9152	0.923	0.9772
pw01-100	0.7706	0.7658	0.7772	0.7788	0.7767	0.7828	0.9525
pw05-100	0.8919	0.89	0.8947	0.8969	0.8964	0.9036	0.9736
pw09-100	0.9359	0.9378	0.9396	0.943	0.9441	0.9541	0.9839

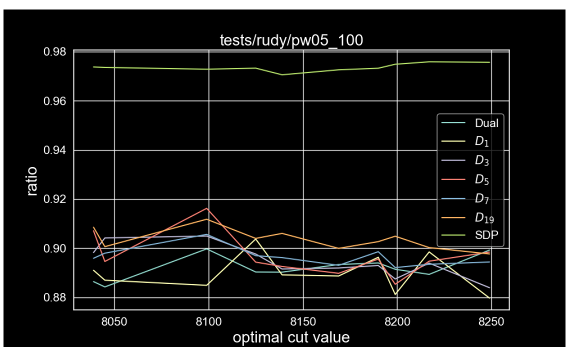
Table 1: Results comparision

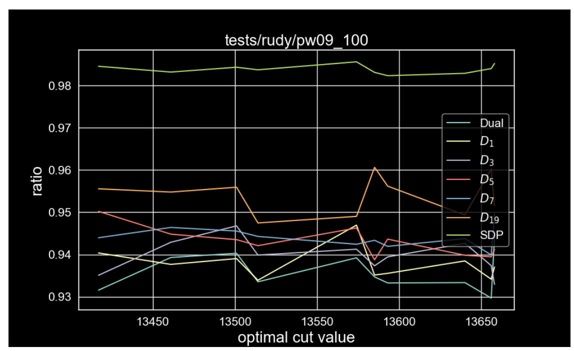






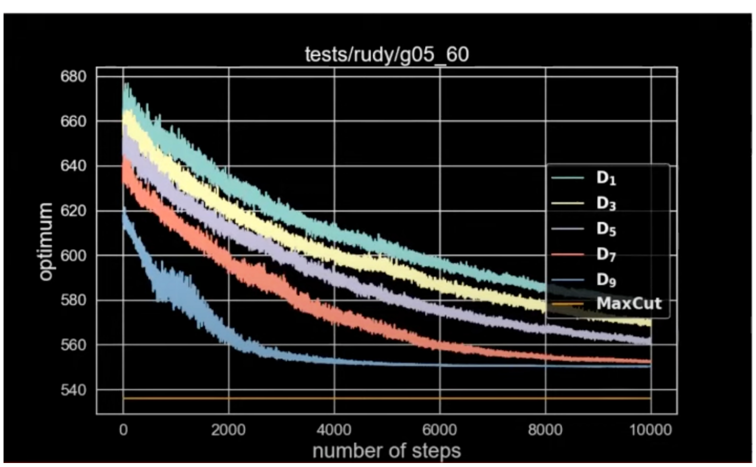






4.4 Error analysis

We show, how the quality of approximation for D_k depends on the number of steps made by gradient-free optimizers.



5 Conclusion

This paper presents a new approach to finding maximum cut in the graph, using idea of approximating maxcut by tractable maxcuts of graphs, which laplacian may be presented as k-diagonal matrices. The accuracy of approximation of new approach compared with the well-known Goemans-Williamson solution.

6 Future plans

In future we want to implement the similar oracul for graphs with bounded tree-width, using the tree-width hierarchy.

$$H_k = \min_{\substack{T:T=T^\top \succeq A \\ tw(T) \leqslant k}} \max x^\top Tx, \qquad OPT = H_k \leqslant \ldots \leqslant H_1 = Dual$$

And we will try to explore different heuristics, for example:

Having T, optimal approximation with $(T) = 1$ (tree), incrementally add edges of the graph with the largest weight to T keeping $(T) \leqslant k$.

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