Neural Networks Loss Landscape Convergence in Hessian Low-Dimensional Space

Tem Nikitin Nikita Kiselev Andrey Grabovoy Vladislav Meshkov

Moscow Institute of Physics and Technology

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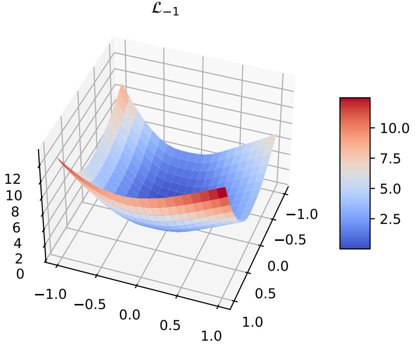
Goals

- Study how neural network loss landscape changes with dataset size
- Define and measure $\Delta_k = \mathbb{E}\big(\mathcal{L}_{k+1}(\mathbf{w}) - \mathcal{L}_k(\mathbf{w})\big)^2$
- Develop Hessian-based low-dimensional projection method
- Derive theoretical bound for Δ_k via top-d eigenvalues
- Validate threshold k^* beyond which further data yield negligible change

One-slide



- Hessian projection onto top-d eigenvectors
- Monte Carlo estimate of $\Delta_k^{emp} = \mathbb{E}\Big(\mathcal{L}_{k+1} - \mathcal{L}_k\Big)^2$.
- Analytical bound: $\Delta_k^{th} \geq \Delta_k^{emp}$
- Detect dataset threshold k^*

Main message: Loss landscape stabilizes after sufficient sample size.

Literature

- Wu *et al.* (2017): loss landscapes vs dataset size [2]
- Sagun *et al.* (2018): Hessian low effective rank [7]
- Li *et al.* (2018): visualizing loss surfaces [11]
- Ghorbani *et al.* (2019): eigenvalue density analysis [8]
- Bousquet & Elisseeff (2002): stability and generalization bounds [12]

Problem Statement: Hypothesis and Model

Hypothesis

Beyond some k^*, adding new samples changes the local loss landscape by less than a tolerance Δ_{tol}, i.e. $\forall\, k \geq k^* :\ \Delta_k < \Delta_{tol}$.

Model

- MLP with ReLU activations for K-class classification
- Empirical loss: $\mathcal{L}_k(\mathbf{w}) = \frac{1}{k}\sum_{i=1}^{k} \ell_i(\mathbf{w})$
- Hessian: $\mathbf{H}_k(\mathbf{w}) = \nabla^2_{\mathbf{w}}\mathcal{L}_k(\mathbf{w})$

Problem Statement: Quality Criteria

- Convergence rate: $\Delta_k = O(1/k)$
- Theoretical bound via top-d eigenvalues upper-bounds empirical Δ_k
- Plateau in eigenvalue differences $\lambda_i^{k+1} - \lambda_i^k$ indicates threshold

Problem Solution: Theoretical Analysis

- Project parameters: $\mathbf{w} = \mathbf{w}^* + \mathbf{P}\boldsymbol{\theta}$, $\mathbf{P} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$
- Taylor approx: $\mathcal{L}_k(\mathbf{w}^* + \mathbf{P}\boldsymbol{\theta}) \approx \mathcal{L}_k(\mathbf{w}^*) + \frac{1}{2}\boldsymbol{\theta}^\top \boldsymbol{\Lambda}_k \boldsymbol{\theta}$

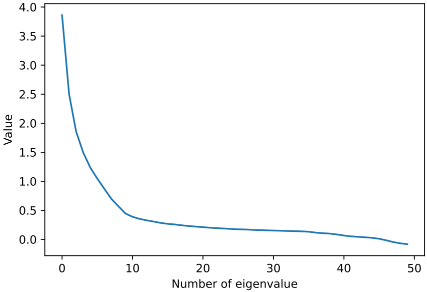


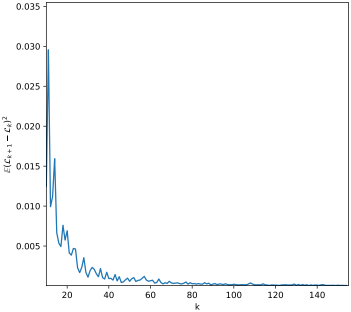
Figure 1: Eigenvalue decay (p.4)

- Bound:

$$\Delta_k \approx \frac{\sigma^4}{4}\left(2\sum_{i=1}^{d}(\lambda_{k+1}^i - \lambda_k^i)^2 + \left(\sum_{i=1}^{d}(\lambda_{k+1}^i - \lambda_k^i)\right)^2\right).$$

Goals of Computational Experiment

- Datasets: MNIST, Fashion-MNIST (60k train, 10k test)
- MLP: 2 hidden layers, $\sim 10^5$ parameters
- Subspace dimension $d = 10$, Monte Carlo samples $K = 64$, $\sigma = 1$
- Compare empirical Δ_k vs theoretical bound across k



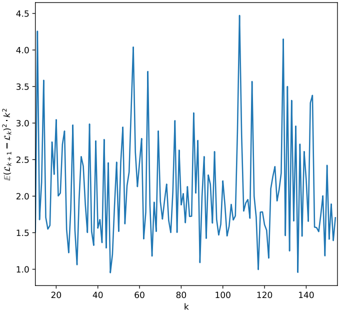


Figure 8: Monte Carlo Δ_k vs k and $\Delta_k \cdot k^2$ (p.11)

Error Analysis

- Empirical Δ_k consistently below theoretical estimate
- Gap due to neglected eigen-modes beyond top-d
- Monte Carlo variance decreases with sample size K

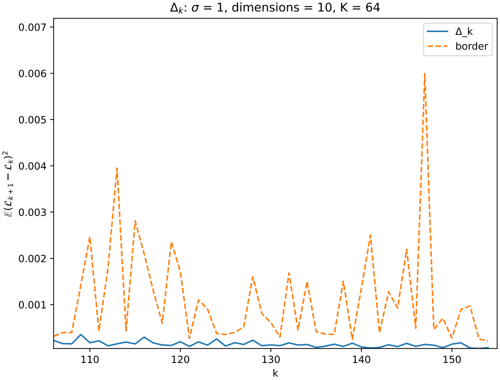


Figure 9: Theoretical (dashed) vs empirical (solid) Δ_k (p.12)

Results and Conclusions

- Identified threshold k (MNIST), (Fashion-MNIST)
- Loss landscape stabilizes: additional data negligible beyond k^*
- Hessian-based bound provides reliable upper-bound for Δ_k
- Practical guideline for dataset sizing and early stopping