1 Introduction

Suppose, we have a parametric surface $\mathcal{M}$ in $\mathbb{R}^3$:

$$\mathcal{M} = (x(u,v), y(u,v), z(u,v)) = \mathbf{r}(u,v),$$

where $u, v \in U$. Then, we can derive the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$

2 Preliminaries

Define as following:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}, \quad \partial_1 = \partial_u, \quad \partial_2 = \partial_v$$

$$g_{11} = \mathbf{r}_u \cdot \mathbf{r}_u, \;\; g_{12} = g_{21} = \mathbf{r}_u \cdot \mathbf{r}_v, \;\; g_{22} = \mathbf{r}_v \cdot \mathbf{r}_v, \;\; |g| = g_{11} \cdot g_{22} - g_{12}^2$$

$$g^{11} = \frac{g_{22}}{|g|}, \quad g^{12} = g^{21} = -\frac{g_{12}}{|g|}, \quad g^{22} = \frac{g_{11}}{|g|}$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \quad \text{(sum over l)}$$

3 Laplace-Beltrami operator

$$\Delta_{\mathcal{M}} \phi = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \phi \right)$$

or

$$\Delta_{\mathcal{M}} \phi = g^{ij} \left(\partial_i \partial_j \phi - \Gamma^k_{ij} \partial_k \phi \right)$$

4 Examples

4.1 Sphere

Consider the unit sphere of radius R:

$$\mathbf{r}(\theta, \phi) = (R \sin\phi \cos\theta,\ R \sin\phi \sin\theta,\ R \cos\phi), \quad \theta \in [0, 2\pi),\ \phi \in [0, \pi].$$

The metric is

$$g_{\theta\theta} = R^2 \sin^2\phi, \quad g_{\phi\phi} = R^2, \quad g_{\theta\phi} = 0.$$

$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \cot\phi, \qquad \Gamma^{\phi}_{\theta\theta} = -\sin\phi\cos\phi.$$

Thus the Laplace–Beltrami operator is

$$\Delta_{\mathcal{M}}\phi = \frac{1}{R^2\sin^2\phi}\,\partial_{\theta\theta}\phi + \frac{1}{R^2\sin\phi}\,\partial_{\phi}(\sin\phi\,\partial_{\phi}\phi).$$

4.2 Cylinder

Consider the circular cylinder of radius R:

$$\mathbf{r}(\theta, z) = (R\cos\theta,\ R\sin\theta,\ z), \quad \theta \in [0, 2\pi),\ z \in \mathbb{R}.$$

The metric is

$$g_{\theta\theta} = R^2, \quad g_{zz} = 1, \quad g_{\theta z} = 0.$$

$$\Gamma^{k}_{ij} = 0.$$

Hence

$$\Delta_{\mathcal{M}}\phi = \frac{1}{R^2}\,\partial_{\theta\theta}\phi + \partial_{zz}\phi.$$

4.3 Torus

For a torus with major radius R and minor radius r:

$$\mathbf{r}(\theta, \phi) = ((R + r\cos\phi)\cos\theta,\ (R + r\cos\phi)\sin\theta,\ r\sin\phi), \quad \theta, \phi \in [0, 2\pi).$$

Metric coefficients:

$$g_{\theta\theta} = (R + r\cos\phi)^2, \quad g_{\phi\phi} = r^2, \quad g_{\theta\phi} = 0.$$

$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = -\frac{r\sin\phi}{R + r\cos\phi}, \qquad \Gamma^{\phi}_{\theta\theta} = \frac{(R + r\cos\phi)\sin\phi}{r}.$$

Therefore

$$\Delta_{\mathcal{M}}\phi = \frac{1}{(R + r\cos\phi)^2}\,\partial_{\theta\theta}\phi + \frac{1}{r^2}\,\partial_{\phi\phi}\phi - \frac{\sin\phi}{r(R + r\cos\phi)}\,\partial_{\phi}\phi.$$

4.4 Ellipsoid

Consider a triaxial ellipsoid:

$$\mathbf{r}(\theta, \phi) = (a\sin\phi\cos\theta,\ b\sin\phi\sin\theta,\ c\cos\phi), \quad \theta \in [0, 2\pi),\ \phi \in [0, \pi].$$

Metric coefficients:

$$g_{\theta\theta} = a^2\sin^2\phi\cos^2\theta + b^2\sin^2\phi\sin^2\theta,$$

$$g_{\phi\phi} = a^2\cos^2\phi\cos^2\theta + b^2\cos^2\phi\sin^2\theta + c^2\sin^2\phi,$$
$$g_{\theta\phi} = (a^2 - b^2)\sin\phi\cos\phi\sin\theta\cos\theta.$$
$$\Gamma^k_{ij} = \tfrac{1}{2}\,g^{k\ell}\big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\big).$$

Then the Laplace–Beltrami operator has the general form

$$\Delta_{\mathcal{M}}\phi = \frac{1}{\sqrt{|g|}}\left[\partial_\theta\Big(\sqrt{|g|}g^{\theta\theta}\partial_\theta\phi + \sqrt{|g|}g^{\theta\phi}\partial_\phi\phi\Big) + \partial_\phi\Big(\sqrt{|g|}g^{\phi\theta}\partial_\theta\phi + \sqrt{|g|}g^{\phi\phi}\partial_\phi\phi\Big)\right],$$

where $|g| = g_{\theta\theta}g_{\phi\phi} - g_{\theta\phi}^2$. In general this does not simplify further.

4.5 Paraboloid

Consider the paraboloid of revolution

$$\mathbf{r}(u,\theta) = (u\cos\theta,\ u\sin\theta,\ u^2), \quad u \geq 0,\ \theta \in [0, 2\pi).$$

Metric coefficients:

$$g_{uu} = 1 + 4u^2, \quad g_{\theta\theta} = u^2, \quad g_{u\theta} = 0.$$

$$\Gamma^u_{uu} = \frac{4u}{1+4u^2}, \qquad \Gamma^u_{\theta\theta} = -\frac{u}{1+4u^2}, \qquad \Gamma^\theta_{u\theta} = \Gamma^\theta_{\theta u} = \frac{1}{u}.$$

So

$$\Delta_{\mathcal{M}}\phi = \frac{1}{u\sqrt{1+4u^2}}\left[\partial_u\left(\frac{u}{\sqrt{1+4u^2}}\partial_u\phi\right) + \partial_\theta\left(\frac{\sqrt{1+4u^2}}{u}\partial_\theta\phi\right)\right].$$