Sign Operator for (L_0, L_1)-Smooth Optimization with Heavy-Tailed Noise

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Course: My first scientific paper (Strijov's practice)/Group 206

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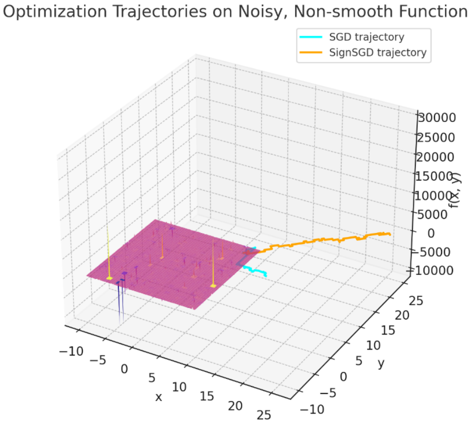
Goal of Research

Objectives

- Define (L_0, L_1)-smoothness.
- Develop sign-based methods (Sign-SGD, minibatch-SignSGD, momentum-SignSGD) for heavy-tailed (HT) noise.
- Establish theoretical convergence bounds under (L_0, L_1)-smoothness and HT noise.
- Validate results through computational experiments.

Idea



Convergence rates improve significantly with Sign-methods.

$\|\nabla^2 f(x)\|_2 \leq L_0 + L_1\|\nabla f(x)\|$

$\mathbb{E}_\xi[|\nabla f(x,\xi)_i - \nabla f(x)_i|^\kappa] \leq \sigma_i^\kappa,\ \kappa \in (1,2]$

Subjects: Sign-based methods, (L_0, L_1)-smoothness, high-probability convergence, heavy-tailed noise.

Literature

Title	Year	Authors	Paper
Sign Operator for Coping with Heavy-Tailed Noise	2025	Kornilov et al.	arXiv
signSGD: Compressed Optimisation for Non-Convex Problems	2018	J. Bernstein et al.	PMLR
Methods for Convex (L0,L1)-Smooth Optimization	2024	Gorbunov et al.	arXiv
Robustness to Unbounded Smoothness of Generalized SignSGD	2022	M. Crawshaw et al.	NeurIPS

Hypothesis and Model

Hypothesis

Sign-based optimization methods outperform traditional gradient-based methods in (L_0, L_1)-smooth problems with heavy-tailed noise, achieving faster convergence and robustness.

Model

Consider a function $f : \mathbb{R}^d \to \mathbb{R}$ that is (L_0, L_1)-smooth:

$$\|\nabla f(x) - \nabla f(y)\| \leq (L_0 + L_1 \|\nabla f(u)\|) \, \|x - y\|,$$

with gradient estimates $\nabla f(x, \xi)$ under HT noise:

- $\mathbb{E}_\xi[\nabla f(x, \xi)] = \nabla f(x)$,
- $\mathbb{E}_\xi[|\nabla f(x, \xi)_i - \nabla f(x)_i|^\kappa] \leq \sigma_i^\kappa$, $\kappa \in (1, 2]$.

Objective: Minimize $f(x)$ in sparse, noisy, and communication-constrained settings.

Solution: Theoretical Part

Theorem (HP complexity for minibatch-L0L1-SignSGD)

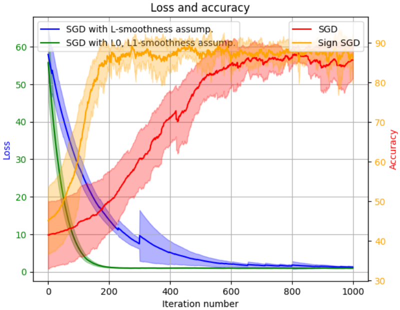
Consider lower-bounded $(L0, L1)$-smooth function f and HT gradient estimates. Then Alg. minibatch-SignSGD requires the sample complexity N to achieve $\frac{1}{T}\sum_{k=1}^{T}\|\nabla f(x^k)\|_1 \le \varepsilon$ with probability at least $1-\delta$ for:

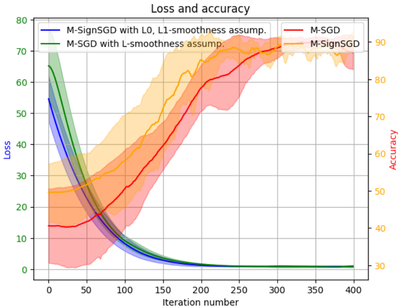
Optimal tuning. *In case $\varepsilon \ge \frac{8L_0}{L_1\sqrt{d}}$, we use stepsize* $\gamma = \frac{1}{48L_1 d \log\frac{1}{\delta}\sqrt{d}} \Rightarrow 80L_0 d\gamma \log(1\delta) \le \varepsilon/2$ *and batchsize* $B_k \equiv \max\left\{1, \left(\frac{16\|\vec{\sigma}\|_1}{\varepsilon}\right)^{\frac{\kappa}{\kappa-1}}\right\}$. $T = O\left(\frac{\Delta_1 L_1 \log\frac{1}{\delta} d^{\frac{3}{2}}}{\varepsilon}\right)$. *The total number of oracle calls is:*

$$\varepsilon \ge \frac{8L_0}{L_1\sqrt{d}} \Rightarrow N = O\left(\frac{\Delta_1 L_1 \log(1\delta) d^{\frac{3}{2}}}{\varepsilon}\left[1+\left(\frac{\|\vec{\sigma}\|_1}{\varepsilon}\right)^{\frac{\kappa}{\kappa-1}}\right]\right),$$

$$\varepsilon < \frac{8L_0}{L_1\sqrt{d}} \Rightarrow N = O\left(\frac{\Delta_1 L_0 \log(1\delta) d}{\varepsilon^2}\left[1+\left(\frac{\|\vec{\sigma}\|_1}{\varepsilon}\right)^{\frac{\kappa}{\kappa-1}}\right]\right).$$

Computational Experiment: Goals and Statistics





Convergence and accuracy rates improve significantly with Sign-methods.

Error Analysis

Error comparison

Method	Mean Loss	Mean Accuracy (%)	Loss Variance
M-SignSGD	3.636586	82.868848	73.560639
M-SGD	7.729881	73.468356	209.468179
SignSGD	6.717110	79.121894	155.107155
SGD	16.446888	62.961501	234.206647

Table: comparison of convergence of several methods under the assumptions

Results and Conclusions

Results

- Sign-based methods outperform SGD in convergence under (L_0, L_1)-smoothness and HT noise.
- minibatch-SignSGD reduces sample complexity for $\kappa < 2$.
- Momentum-SignSGD and minibatch-SignSGD convergence are bounded and proved.

Conclusions

- (L_0, L_1)-smoothness enables better rates under (L_0, L_1) and HT-noise assumptions.
- Sign-based methods are noise-robust and communication-efficient.