Symmetry Brings Balance in Nature — Can We Prove It Mathematically? (Thomson Problem, Euler's Formula, Group Theory)

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1 Algebraic Proofs in 2D, 3D and beyond

1.1 In Two Dimension

- In **2D**, we place n vectors of equal length (say unit vectors) equally spaced around the origin on the unit circle.
- Each vector points at an angle $\theta_k = \frac{2\pi k}{n}$ from the positive x-axis.
- Their coordinates are $(\cos\theta_k, \sin\theta_k)$.
- So the total vector sum is:

$$\sum_{k=0}^{n-1}\left(\cos\left(\frac{2\pi k}{n}\right), \sin\left(\frac{2\pi k}{n}\right)\right) = \left(\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right), \sum_{k=0}^{n-1}\sin\left(\frac{2\pi k}{n}\right)\right)$$

- This is zero because:

$$\sum_{k=0}^{n-1} e^{2\pi ik/n} = 0 \quad \Rightarrow \quad \text{Re part: } \sum\cos\left(\frac{2\pi k}{n}\right) = 0, \quad \text{Im part: } \sum\sin\left(\frac{2\pi k}{n}\right) = 0$$

- And that sum vanishing is just the classic geometric sum of roots of unity (which was also proved in the video by Sumit Sah)
- As $n \to \infty$

$$\left(\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right), \sum_{k=0}^{n-1}\sin\left(\frac{2\pi k}{n}\right)\right)$$

$$= \left(n \times \sum_{k=0}^{n-1}\cos\left(2\pi\frac{k}{n}\right)\left(\frac{1}{n}\right), n \times \sum_{k=0}^{n-1}\sin\left(2\pi\frac{k}{n}\right)\left(\frac{1}{n}\right)\right)$$

Using Riemann Sum definition of integral , we know

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} f\left(a + (b-a)\frac{k}{n}\right)\frac{b-a}{n} = \sum_{k=1}^{n} f\left(a + (b-a)\frac{k}{n}\right)\frac{b-a}{n}$$

- Thus, the infinite sum translates to

$$\left(n \times \int_0^1 \cos(2\pi x)dx, n \times \int_0^1 \sin(2\pi x)dx\right)$$

$$= \left(\frac{n}{2\pi} \times \int_0^{2\pi} \cos(x)dx, \frac{n}{2\pi} \times \int_0^{2\pi} \sin(x)dx\right)$$

- This is zero because:

$$\int_0^{2\pi} e^{ix} = \left[-ie^{ix}\right]_0^{2\pi} = -i + i = 0$$

$$\Rightarrow \quad \text{Re part: } \int_0^{2\pi} \cos(x)dx = 0, \quad \text{Im part: } \int_0^{2\pi} \sin(x)dx = 0$$

1.2 In Three Dimension

- **The Goal:** We want to sum all unit vectors placed symmetrically on the surface of a sphere — using spherical coordinates — and show that their vector sum is zero:

$$\sum_{p=0}^{m}\sum_{k=0}^{n-1}\left(\sin\left(\frac{\pi p}{m}\right)\cos\left(\frac{2\pi k}{n}\right), \sin\left(\frac{\pi p}{m}\right)\sin\left(\frac{2\pi k}{n}\right), \cos\left(\frac{\pi p}{m}\right)\right) = (0,0,0)$$

- **Why this sum?**
  - We are essentially taking $m+1$ evenly spaced latitude slices (ϕ levels), with
  $$\phi_p = \frac{\pi p}{m}$$
  ranging from pole to pole (including both).
  - At each latitude ϕ_p, we place n vectors equally spaced around the azimuthal circle, meaning:
  $$\theta_k = \frac{2\pi k}{n} \quad \text{for } k = 0, 1, \ldots, n-1$$
  (we avoid having both $\theta = 0$ and $\theta = 2\pi$, as they refer to the same point).

- For each fixed p, we are placing a circle of vectors around a constant ϕ, and for each such vector, we compute:

$$(x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

Summing over all of them gives that double sum.

- **Why the vector sum vanishes?**
  - **X-component:**

$$\sum_{p=0}^{m}\sum_{k=0}^{n-1}\sin\left(\frac{\pi p}{m}\right)\cos\left(\frac{2\pi k}{n}\right)$$

Here, $\sin\left(\frac{\pi p}{m}\right)$ is constant for fixed p, so it factors out:

$$\sum_{p=0}^{m}\sin\left(\frac{\pi p}{m}\right)\left(\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right)\right)$$

The inner sum vanishes (as we showed in 2D via the roots of unity). So the entire x-component sum is zero.

  - **Y-component:** Identical reasoning:

$$\sum_{p=0}^{m}\sum_{k=0}^{n-1}\sin\left(\frac{\pi p}{m}\right)\sin\left(\frac{2\pi k}{n}\right) = 0$$

Again, since the sum over sine terms around a full circle vanishes.

  - **Z-component:** This part differs slightly. We compute:

$$\sum_{p=0}^{m}\sum_{k=0}^{n-1}\cos\left(\frac{\pi p}{m}\right)$$

Since $\cos\left(\frac{\pi p}{m}\right)$ is independent of k, this becomes:

$$\sum_{p=0}^{m}\left(n\cdot\cos\left(\frac{\pi p}{m}\right)\right) = n\cdot\sum_{p=0}^{m}\cos\left(\frac{\pi p}{m}\right)$$

Now this sum also vanishes by the same trick: write each cosine as $\Re(e^{i\phi})$, and we obtain a finite geometric sum over unit roots (or use cosine symmetry across $\pi/2$).

- **One subtle but important detail:** Why does p go from 0 to m, but k only from 0 to $n-1$?
  - $\theta = 0$ and $\theta = 2\pi$ refer to the same point on the azimuthal circle — so we don't double-count. Hence n distinct azimuths: $k = 0$ to $n-1$.
  - $\phi = 0$ and $\phi = \pi$ are distinct: they represent the north and south poles. So we do include both ends, hence $p = 0$ to m, which gives $m+1$ distinct latitudes.

Details of the Sum of $\sum_{p=0}^{m} \cos\left(\frac{\pi p}{m}\right)$

We use the identity:

$$\cos\left(\frac{\pi p}{m}\right) = \Re\left(e^{i\pi p/m}\right)$$

So the sum becomes:

$$\sum_{p=0}^{m} \cos\left(\frac{\pi p}{m}\right) = \Re\left(\sum_{p=0}^{m} e^{i\pi p/m}\right)$$

This is a geometric series with:

- First term: $a = 1$
- Common ratio: $r = e^{i\pi/m}$
- Number of terms: $m + 1$

Apply the geometric sum formula:

$$\sum_{p=0}^{m} e^{i\pi p/m} = \frac{1 - e^{i\pi(m+1)/m}}{1 - e^{i\pi/m}}$$

Simplify the numerator:

$$e^{i\pi(m+1)/m} = e^{i\pi(1+1/m)} = -e^{i\pi/m} \quad \Rightarrow \quad 1 - e^{i\pi(m+1)/m} = 1 + e^{i\pi/m}$$

So the full sum becomes:

$$\frac{1 + e^{i\pi/m}}{1 - e^{i\pi/m}}$$

Now multiply both numerator and denominator by $e^{-i\pi/(2m)}$ to symmetrize:

$$\frac{(1 + e^{i\pi/m})e^{-i\pi/(2m)}}{(1 - e^{i\pi/m})e^{-i\pi/(2m)}}$$

Simplify both:
Numerator:

$$(1+e^{i\pi/m})e^{-i\pi/(2m)} = e^{-i\pi/(2m)}+e^{i\pi/m}\cdot e^{-i\pi/(2m)} = e^{-i\pi/(2m)}+e^{i\pi/(2m)} = 2\cos\left(\frac{\pi}{2m}\right)$$

Denominator:

$$(1 - e^{i\pi/m})e^{-i\pi/(2m)} = e^{-i\pi/(2m)} - e^{i\pi/(2m)} = -2i\sin\left(\frac{\pi}{2m}\right)$$

Final result:

$$\frac{2\cos\left(\frac{\pi}{2m}\right)}{-2i\sin\left(\frac{\pi}{2m}\right)} = -i\cot\left(\frac{\pi}{2m}\right)$$

Thus the original complex geometric sum becomes:

$$\sum_{p=0}^{m} e^{i\pi p/m} = -i \cot\left(\frac{\pi}{2m}\right)$$

Taking real part:

$$\Re\left(-i \cot\left(\frac{\pi}{2m}\right)\right) = 0$$

since it is purely imaginary.

Conclusion:

$$\sum_{p=0}^{m} \cos\left(\frac{\pi p}{m}\right) = \Re\left(\sum_{p=0}^{m} e^{i\pi p/m}\right) = \Re\left(-i \cot\left(\frac{\pi}{2m}\right)\right) = 0$$

- As $n, m \to \infty$, our original vector sum translates to

$$\frac{nm}{2\pi^2}\left(\int_0^{\pi}\int_0^{2\pi} \sin(x_2)\sin(x_1)dx_1dx_2, \int_0^{\pi}\int_0^{2\pi} \sin(x_2)\cos(x_1)dx_1dx_2, \int_0^{\pi}\int_0^{2\pi} \cos(x_2)dx_1dx_2\right)$$

- This goes to zero because

$$\int_0^{2\pi} \sin(x_1)dx_1 = 0, \quad \int_0^{2\pi} \cos(x_1)dx_1 = 0, \quad \int_0^{\pi} \cos(x_2)dx_2 = 0$$

1.3 In Four Dimensions

We aim to prove that the sum of unit vectors, symmetrically placed on the 3-sphere (the unit sphere in $\mathbb{R}^4$), adds up to the zero vector:

$$\sum_{p_1=0}^{m_1}\sum_{p_2=0}^{m_2}\sum_{k=0}^{n-1} \vec{v}_{p_1,p_2,k} = \vec{0}$$

Each vector is represented in hyperspherical coordinates as:

$$\vec{v}_{p_1,p_2,k} = (\sin\phi_1 \sin\phi_2 \cos\theta,\ \sin\phi_1 \sin\phi_2 \sin\theta,\ \sin\phi_1 \cos\phi_2,\ \cos\phi_1)$$

where

$$\phi_1 = \frac{\pi p_1}{m_1}, \quad \phi_2 = \frac{\pi p_2}{m_2}, \quad \theta = \frac{2\pi k}{n}$$

and

$$\phi_1 \in [0,\pi], \phi_2 \in [0,\pi], \theta \in [0, 2\pi)$$

Step-by-step Breakdown

Structure of the Sum. We distribute vectors evenly using a grid in spherical coordinates:

- $p_1 = 0, \ldots, m_1$ gives $m_1 + 1$ samples for $\phi_1 \in [0, \pi]$,
- $p_2 = 0, \ldots, m_2$ gives $m_2 + 1$ samples for $\phi_2 \in [0, \pi]$,
- $k = 0, \ldots, n-1$ gives n azimuthal directions, avoiding double-counting 0 and 2π.

Thus the total sum is:

$$\sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} \sum_{k=0}^{n-1} (\sin\left(\frac{\pi p_1}{m_1}\right) \sin\left(\frac{\pi p_2}{m_2}\right) \cos\left(\frac{2\pi k}{n}\right),\ \sin\left(\frac{\pi p_1}{m_1}\right) \sin\left(\frac{\pi p_2}{m_2}\right) \sin\left(\frac{2\pi k}{n}\right),$$

$$\sin\left(\frac{\pi p_1}{m_1}\right) \cos\left(\frac{\pi p_2}{m_2}\right),\ \cos\left(\frac{\pi p_1}{m_1}\right))$$

We analyze each component individually.

First Coordinate (x_1):

$$\sum_{p_1} \sum_{p_2} \sum_{k} \sin\left(\frac{\pi p_1}{m_1}\right) \sin\left(\frac{\pi p_2}{m_2}\right) \cos\left(\frac{2\pi k}{n}\right)$$

Factor out the terms independent of k:

$$\sum_{p_1} \sum_{p_2} \sin\left(\frac{\pi p_1}{m_1}\right) \sin\left(\frac{\pi p_2}{m_2}\right) \left(\sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right)\right)$$

The inner sum vanishes:

$$\sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) = 0$$

Hence, the x_1-component vanishes.

Second Coordinate (x_2):

$$\sum_{p_1} \sum_{p_2} \sum_{k} \sin\left(\frac{\pi p_1}{m_1}\right) \sin\left(\frac{\pi p_2}{m_2}\right) \sin\left(\frac{2\pi k}{n}\right)$$

Since

$$\sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = 0$$

So the x_2-component also vanishes.

Third Coordinate (x_3)**:**

$$\sum_{p_1=0}^{m_1}\sum_{p_2=0}^{m_2}\sin\left(\frac{\pi p_1}{m_1}\right)\cos\left(\frac{\pi p_2}{m_2}\right)\sum_{k=0}^{n-1}1 = n\sum_{p_1}\sum_{p_2}\sin\left(\frac{\pi p_1}{m_1}\right)\cos\left(\frac{\pi p_2}{m_2}\right)$$

Since

$$\sum_{p_2}\cos\left(\frac{\pi p_2}{m_2}\right) = 0$$

Thus, the x_3-component also vanishes.

Fourth Coordinate (x_4)**:**

$$\sum_{p_1=0}^{m_1}\cos\left(\frac{\pi p_1}{m_1}\right)\sum_{p_2=0}^{m_2}\sum_{k=0}^{n-1}1 = n(m_2+1)\sum_{p_1}\cos\left(\frac{\pi p_1}{m_1}\right)$$

Since

$$\sum_{p_1=0}^{m_1}\cos\left(\frac{\pi p_1}{m_1}\right) = 0$$

Hence, the x_4-component vanishes.

Final Conclusion

All four components vanish independently:

$$\sum_{p_1=0}^{m_1}\sum_{p_2=0}^{m_2}\sum_{k=0}^{n-1}\vec{v}_{p_1,p_2,k} = (0,0,0,0)$$

Therefore, the total vector sum of all symmetrically placed unit vectors on the 3-sphere is zero.

- As $n, m_1, m_2 \to \infty$, our original vector sum translates to

$$\frac{nm_1m_2}{2\pi^3}\begin{pmatrix} \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\sin(x_3)\sin(x_2)\sin(x_1)\,dx_1dx_2dx_3, \\ \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\sin(x_3)\sin(x_2)\cos(x_1)\,dx_1dx_2dx_3, \\ \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\sin(x_3)\cos(x_2)\,dx_1dx_2dx_3, \\ \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\cos(x_3)\,dx_1dx_2dx_3 \end{pmatrix}$$

- This goes to zero because

$$\int_0^{2\pi}\sin(x_1)dx_1 = 0, \quad \int_0^{2\pi}\cos(x_1)dx_1 = 0, \quad \int_0^{\pi}\cos(x_2)dx_2 = 0, \quad \int_0^{\pi}\cos(x_3)dx_3 = 0$$

1.4 In N dimensions

We want to show that the sum of all unit vectors symmetrically placed on the unit $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$, using discretized hyperspherical coordinates, adds up to the zero vector:

$$\sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} \cdots \sum_{p_{n-2}=0}^{m_{n-2}} \sum_{k=0}^{n_\theta - 1} \vec{v}_{p_1,\ldots,p_{n-2},k} = \vec{0}$$

Hyperspherical Coordinates in $\mathbb{R}^n$

For a point on the unit $(n-1)$-sphere:

$$\begin{aligned}
x_1 &= \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-2} \cos\theta \\
x_2 &= \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-2} \sin\theta \\
x_3 &= \sin\phi_1 \sin\phi_2 \cdots \cos\phi_{n-2} \\
x_4 &= \sin\phi_1 \sin\phi_2 \cos\phi_{n-3} \\
&\vdots \\
x_{n-1} &= \sin\phi_1 \cos\phi_2 \\
x_n &= \cos\phi_1
\end{aligned}$$

Where:

$$\theta \in [0, 2\pi), \quad \phi_1, \ldots, \phi_{n-2} \in [0, \pi]$$

Grid Sampling

We define:

$$\theta_k = \frac{2\pi k}{n_\theta}, \quad \text{for } k = 0, 1, \ldots, n_\theta - 1$$

$$\phi_{j,p_j} = \frac{\pi p_j}{m_j}, \quad \text{for } p_j = 0, \ldots, m_j$$

The vector at multi-index $(p_1, \ldots, p_{n-2}, k)$ is then:

$$\vec{v}_{p_1,\ldots,p_{n-2},k} = \begin{pmatrix} \sin\phi_1 \cdots \sin\phi_{n-2} \cos\theta \\ \sin\phi_1 \cdots \sin\phi_{n-2} \sin\theta \\ \sin\phi_1 \cdots \cos\phi_{n-2} \\ \sin\phi_1 \cos\phi_{n-3} \\ \vdots \\ \cos\phi_1 \end{pmatrix} \in \mathbb{R}^n$$

Componentwise Cancellation

Let us prove that each coordinate sum vanishes.

Coordinates 1 and 2

These involve $\cos\theta$ and $\sin\theta$, both multiplied by the same product $\sin\phi_1 \cdots \sin\phi_{n-2}$:

$$x_1 = A \cdot \cos\theta$$
$$x_2 = A \cdot \sin\theta$$

So we write:

$$\sum_{k=0}^{n_\theta - 1} \cos\left(\frac{2\pi k}{n_\theta}\right) = 0, \quad \sum_{k=0}^{n_\theta - 1} \sin\left(\frac{2\pi k}{n_\theta}\right) = 0$$

Thus for any fixed $p_1, \dots, p_{n-2}$, the total sum over k gives zero in both x_1 and x_2 components:

$$\sum \vec{v}^{(1)}_{p_1,\dots,p_{n-2},k} = 0, \quad \sum \vec{v}^{(2)}_{p_1,\dots,p_{n-2},k} = 0$$

Coordinates 3 to n

These components no longer depend on θ, so:

$$\sum_{k=0}^{n_\theta - 1} 1 = n_\theta$$

We just get:

$$n_\theta \cdot \sum_{p_1} \sum_{p_2} \cdots \sum_{p_{n-2}} (\text{component expression})$$

Each of these remaining components is a product of several $\sin(\phi_j)$ terms and one $\cos(\phi_k)$ for some $k \in \{1, \dots, n-2\}$. Just like before:

$$\sum_{p_j=0}^{m_j} \cos\left(\frac{\pi p_j}{m_j}\right) = 0$$

by our earlier geometric series proof.

Hence for each x_j with $j \geq 3$, the total sum vanishes.

Conclusion

Each coordinate of the total vector sum independently vanishes, so:

$$\sum_{p_1=0}^{m_1} \cdots \sum_{p_{n-2}=0}^{m_{n-2}} \sum_{k=0}^{n_\theta - 1} \vec{v}_{p_1,\dots,p_{n-2},k} = \vec{0} \in \mathbb{R}^n$$

Hence, the sum of all symmetrically arranged unit vectors on the $(n-1)$-sphere is zero.

- The argument for the integral translation of the vector sum is same.

2 Group Theory proof in 2D

$$\text{Let } G = \{e^{2\pi ik/n} : k = 0, ..., n-1\} \tag{1}$$

These are invertible and satisfy the group property

$$e^{2\pi ik/n}e^{2\pi i\ell/n} = e^{2\pi i((k+\ell) \text{ mod n})/n} \in G \tag{2}$$

So G is a finite group.

For $v \in \mathbb{C}$ define

$$\overline{v} = \frac{1}{|G|}\sum_{g\in G} gv \tag{3}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi ik/n}v \tag{4}$$

Now multiply both sides by $e^{2\pi i\ell/n}$

$$e^{2\pi i\ell/n}\overline{v} = \frac{1}{n}e^{2\pi i\ell/n}\sum_{k=0}^{n-1} e^{2\pi ik/n}v \tag{5}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i(k+\ell)/n}v \tag{6}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i/n}v \tag{7}$$

$$= \overline{v} \tag{8}$$

This means either $\overline{v} = 0$ or, dividing by $\overline{v}$, we get $e^{2\pi i\ell/n} = 1$ for all ℓ, which is a contradiction, i.e., subtract to get

$$0 = (e^{2\pi i\ell/n} - 1)\overline{v} \tag{9}$$

For $\ell \neq 0$ (which is possible iff $n > 1$), $e^{2\pi i\ell/n} - 1 \neq 0$. Therefore dividing by it gives $\overline{v} = 0$.

3 A 2D Proof Without involving Euler's Formula (Purely Based on Linear Algebra and Group Theory):

Mathematically, actions like rotations and reflections can be represented using transformation matrices. For example, if we wish to rotate a vector $\vec{v} \in \mathbb{R}^2$ by an angle θ, we use the following rotation matrix:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For rotation by $\frac{2k\pi}{n}$ radians:

$$R\left(\frac{2k\pi}{n}\right) = \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) & -\sin\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) & \cos\left(\frac{2k\pi}{n}\right) \end{bmatrix}$$

We can construct a group G, where $G = \{R_k\}_{k=0}^{n-1} = \{R_0, R_1, \cdots, R_{n-1}\}$, and

$$R_k = R\left(\frac{2\pi k}{n}\right)$$

Now, consider the unit vector $\vec{v}_k$. It can be represented in matrix form as:

$$\vec{v}_k = \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) \end{bmatrix}$$

We know vectors add up to result in a single vector. Let $\vec{\mathbb{V}}$ be the sum of $\vec{v}_k$ as k goes from 0 to $n-1$, i.e.,

$$\vec{\mathbb{V}} = \sum_{k=0}^{n-1} \vec{v}_k = \sum_{k=0}^{n-1} \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) \end{bmatrix}$$

Now, we perform a rotation action, say $R_1 \in G$, multiplying by $R\left(\frac{2\pi}{n}\right)$ on both sides:

$$R_1\vec{\mathbb{V}} = \sum_{k=0}^{n-1} R\left(\frac{2\pi}{n}\right) \vec{v}_k$$

Which becomes:

$$= \sum_{k=0}^{n-1} \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{2k\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2k\pi}{n}\right) + \cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2k\pi}{n}\right) \end{bmatrix}$$

Using the compound angle formula:

$$= \sum_{k=0}^{n-1} \begin{bmatrix} \cos\left(\frac{2(k+1)\pi}{n}\right) \\ \sin\left(\frac{2(k+1)\pi}{n}\right) \end{bmatrix}$$

Each vector changed individually (rotated by $\frac{2\pi}{n}$), but the set of vectors just got rearranged. The sum remained unchanged.

If we apply some other $R_i \in G$, the result becomes:

$$\vec{v}_{k+i \mod n} = \begin{bmatrix} \cos\left(\frac{2(k+i)\pi}{n}\right) \\ \sin\left(\frac{2(k+i)\pi}{n}\right) \end{bmatrix}$$

Again, the k-th vector becomes the $(k+i)$-th, modulo n. The full set is just permuted.

Since rotation doesn't change the vector sum, and the group acts transitively, the sum must be invariant under every rotation. i.e,

$$R_i \vec{\mathbb{V}} = \vec{\mathbb{V}}$$

Mathematically, this is possible when R_i is I, which is not possible for $n > 1$, which means $\vec{\mathbb{V}} = 0$. Thus, the only vector in $\mathbb{R}^2$ unchanged by all non-trivial rotations is the zero vector. So:

$$\vec{\mathbb{V}} = \vec{0}$$

The symmetry of the setup forces the result, which completes our proof.

So, without any complex numbers or Euler's formula, we showed purely using linear algebra and symmetry that the sum of evenly spaced vectors around a circle is zero.