Note

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1 Translational operation

Suppose the square lattice is formed with $N_x \cdot N_y = N$ sites and they have been numbered as $k = 0, \cdots, N-1$ in a certain way, for instance, a *snake*. Here we choose the convention to label the number of fermionic opertors as $k = j \cdot N_x + i$, where integer pair (i, j) denotes the lattice coordinates with respect to x- and y-directions. With consideration of one hole doped case, a generic basis can be defined in such a one-dimensional way

$$c_{0\sigma_0}^\dagger \cdots c_{h-1\sigma_{h-1}}^\dagger c_{h+1\sigma_{h+1}}^\dagger \cdots c_{N-1\sigma_{N-1}}^\dagger |0\rangle = (-)^h c_{h\sigma_h} |s\rangle \equiv |h; s\rangle, \tag{1}$$

where $|s\rangle \equiv c_{0\sigma_0}^\dagger \cdots c_{N-1\sigma_{N-1}}^\dagger |0\rangle$ is the half-filled spin background created by *ordered* fermionic operators. In periodic boundary condition, translational operator can be defined as

$$\mathcal{T}_x c_{k\sigma_k}^\dagger \mathcal{T}_x^{-1} = c_{k'\sigma_k}^\dagger, \tag{2}$$

in which k and k' correspond to the coordinate (i, j) and $(i+1, j)$, respectively. Note that $i+1$ certainly takes the modulus of N_x. We are going to find what a state transformed under the operation of $\mathcal{T}_x$. Consider a generic basis operated by $\mathcal{T}_x$

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^h \mathcal{T}_x c_{h\sigma_h} \mathcal{T}_x^{-1} \mathcal{T}_x |s\rangle. \tag{3}$$

In the first place we compute

$$\begin{aligned} \mathcal{T}_x |s\rangle &= \mathcal{T}_x c_{0\sigma_0}^\dagger \cdots c_{N-1\sigma_{N-1}}^\dagger |0\rangle \\ &= \mathcal{T}_x c_{0\sigma_0}^\dagger \mathcal{T}_x^{-1} \cdots \mathcal{T}_x c_{N-1\sigma_{N-1}}^\dagger \mathcal{T}_x^{-1} \mathcal{T}_x |0\rangle \\ &= (-)^{(N_x-1)\cdot N_y} |s'\rangle, \end{aligned} \tag{4}$$

where the fermionic sign $(-)^{N_x-1}$ arises from the translational permutation of every horizontal row and there are N_y rows. $\mathcal{T}_x|0\rangle = |0\rangle$ is regarded as a basic assumption. $|s'\rangle$ is just the corresponding translated half-filled bosonic spin configuration. Then

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^{h+(N_x-1)\cdot N_y} c_{h'\sigma_{h'}} |s'\rangle = \text{sign} \cdot (-)^{h'} c_{h'\sigma_{h'}} |s'\rangle, \tag{5}$$

where an extra sign $= (-)^{h-h'+(N_x-1)\cdot N_y}$ must be multiplied in practical computer program. Note that $\mathcal{T}_x$ although does not change the spin of $c_{h\sigma_h}$, in the new half-filled configuration $|s'\rangle$, $\sigma_{h'}$ indeed corresponds to σ_h in $|s\rangle$.

The case for $\mathcal{T}_y$ is very similar to $\mathcal{T}_x$.

2 Ground state degeneracy

Numerical results turns out there is a $f = 6$ fold degeneracy for t-J model on 4×4 square lattice with both periodic conditions. $(H, \mathcal{T}_x, \mathcal{T}_y)$ can be a *complete set of commutating observables* of which eigenvalue quantum numbers can represent only one specific ground state.

- Firstly diagonalize the matrix $\langle\psi_i|\mathcal{T}_x|\psi_j\rangle, i, j = 0, \cdots, f-1$ and obtain the eigenvalues of $\mathcal{T}_x$ in the subspace with the same eigenvalue of H namely the subspace of degenerate ground states.

- Then you find there are still a $f' = 2$ fold degeneracy with eigenvalues of $\mathcal{T}_x$ and at this moment you should further diagonalize $\langle{\psi_i}'|\mathcal{T}_y|{\psi_j}'\rangle, i, j = 0, \cdots, f'-1$ in the subspace with same eigenvalue of $\mathcal{T}_x$.