Note

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1 Translational operation

Suppose the square lattice is formed with $N_x \cdot N_y = N$ sites and they have been numbered as $k = 0, \cdots, N-1$ in a certain way, for instance, a *snake*. Here we choose the convention to label the number of fermionic opertors as $k = j \cdot N_x + i$, where integer pair (i, j) denotes the lattice coordinates with respect to x- and y-directions. With consideration of one hole doped case, a generic basis can be defined in such a one-dimensional way

$$c^\dagger_{0\sigma_0} \cdots c^\dagger_{h-1\sigma_{h-1}} c^\dagger_{h+1\sigma_{h+1}} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle = (-)^h c_{h\sigma_h} |s\rangle \equiv |h; s\rangle, \tag{1}$$

where $|s\rangle \equiv c^\dagger_{0\sigma_0} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle$ is the half-filled spin background created by *ordered* fermionic operators. In periodic boundary condition, translational operator can be defined as

$$\mathcal{T}_x c^\dagger_{k\sigma_k} \mathcal{T}_x^{-1} = c^\dagger_{k'\sigma_k}, \tag{2}$$

in which k and k' correspond to the coordinate (i, j) and $(i+1, j)$, respectively. Note that $i+1$ certainly takes the modulus of N_x. We are going to find what a state transformed under the operation of $\mathcal{T}_x$. Consider a generic basis operated by $\mathcal{T}_x$

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^h \mathcal{T}_x c_{h\sigma_h} \mathcal{T}_x^{-1} \mathcal{T}_x |s\rangle. \tag{3}$$

In the first place we compute

$$\begin{aligned}
\mathcal{T}_x |s\rangle &= \mathcal{T}_x c^\dagger_{0\sigma_0} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle \\
&= \mathcal{T}_x c^\dagger_{0\sigma_0} \mathcal{T}_x^{-1} \cdots \mathcal{T}_x c^\dagger_{N-1\sigma_{N-1}} \mathcal{T}_x^{-1} \mathcal{T}_x |0\rangle \\
&= (-)^{(N_x-1)\cdot N_y} |s'\rangle,
\end{aligned} \tag{4}$$

where the fermionic sign $(-)^{N_x-1}$ arises from the translational permutation of every horizontal row and there are N_y rows. $\mathcal{T}_x|0\rangle = |0\rangle$ is regarded as a basic assumption. $|s'\rangle$ is just the corresponding translated half-filled bosonic spin configuration. Then

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^{h+(N_x-1)\cdot N_y} c_{h'\sigma_{h'}} |s'\rangle = \text{sign} \cdot (-)^{h'} c_{h'\sigma_{h'}} |s'\rangle, \tag{5}$$

where an extra sign $= (-)^{h-h'+(N_x-1)\cdot N_y}$ must be multiplied in practical computer program. Note that $\mathcal{T}_x$ although does not change the spin of $c_{h\sigma_h}$, in the new half-filled configuration $|s'\rangle$, $\sigma_{h'}$ indeed corresponds to σ_h in $|s\rangle$.

The case for $\mathcal{T}_y$ is very similar to $\mathcal{T}_x$.