Note

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1 Basic Construction

The Hamiltonian for t-J model is $H_{t\text{-}J} = H_t + H_J$ where

$$\begin{aligned} H_t &= -t \sum_{\langle ij \rangle, \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.), \\ H_J &= J \sum_{\langle ij \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right). \end{aligned} \tag{1}$$

Suppose the square lattice is formed with $N_x \cdot N_y = N$ sites and they have been numbered as $0, \cdots, N-1$ in a certain way, for instance, a *snake*. With consideration of one hole doped case, a generic basis can be defined in such a one-dimensional way

$$c_{0\sigma_0}^\dagger \cdots c_{h-1\sigma_{h-1}}^\dagger c_{h+1\sigma_{h+1}}^\dagger \cdots c_{N-1\sigma_{N-1}}^\dagger |0\rangle = (-)^h c_{h\sigma_h} |s\rangle \equiv |h; s\rangle, \tag{2}$$

where $|s\rangle \equiv c_{0\sigma_0}^\dagger \cdots c_{N-1\sigma_{N-1}}^\dagger |0\rangle$ is the half-filled spin background created by *ordered* fermionic operators. $|h; s\rangle$ thus can be represented as a bosonic configuration in computational program. Here our major task is to compute the vector multiplication required by the package ARPACKPP[Reuter *et al.*()Reuter, Gomes, and Sorensen]. H_J can be evaluated as same as the bosonic Heisenberg spin model as one diagonal block of the $H_{t\text{-}J}$ matrix in our representation. For H_t, we would like to compute the hole's hopping term from site h to site h' (electron's hopping frome h' to h)

$$\begin{aligned} &\sum_\sigma (c_{h\sigma}^\dagger c_{h'\sigma}) |h; s\rangle \\ &= c_{h\sigma_{h'}}^\dagger c_{h'\sigma_{h'}} (-)^h c_{h\sigma_h} |s\rangle = c_{h'\sigma_{h'}} (-)^{h+1} (c_{h\sigma_{h'}}^\dagger c_{h\sigma_h}) |s\rangle \\ &= (-)^{h-h'+1} (-)^{h'} c_{h'\sigma_{h'}} |s'\rangle, \end{aligned} \tag{3}$$

where spin summation $\sum_\sigma c_{h'\sigma}$ should match $\sigma_{h'}$ of which $c_{h'\sigma_{h'}}^\dagger$ in $|s\rangle$ otherwise leads to zero. Note that what $|s'\rangle$ differs from $|s\rangle$ is that the fermionic creation operator $c^\dagger_{h\sigma_h}$ in $|s\rangle$ is replaced by $c^\dagger_{h\sigma_{h'}}$ at site h. Its Hermatian conjugate part is similar. That is to say, in order to evaluate the non-zero matrix elements in terms of H_t which connects different bosonic Heisenberg sub-blocks of the total Hilbert space, despite considering the change of bosonic configuration in $|h'; s'\rangle$, an extra fermionic sign $(-)^{h-h'+1}$ should be taken in to consideration.

2 Translational operation

Suppose the square lattice is formed with $N_x \cdot N_y = N$ sites and they have been numbered as $k = 0, \cdots, N-1$ in a certain way, for instance, a *snake*. Here we choose the convention to label the number of fermionic opertors as $k = j \cdot N_x + i$, where integer pair (i, j) denotes the lattice coordinates with respect to x- and y-directions. With consideration of one hole doped case, a generic basis can be defined in such a one-dimensional way

$$c^\dagger_{0\sigma_0} \cdots c^\dagger_{h-1\sigma_{h-1}} c^\dagger_{h+1\sigma_{h+1}} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle = (-)^h c_{h\sigma_h} |s\rangle \equiv |h; s\rangle, \tag{4}$$

where $|s\rangle \equiv c^\dagger_{0\sigma_0} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle$ is the half-filled spin background created by *ordered* fermionic operators. In periodic boundary condition, translational operator can be defined as

$$\mathcal{T}_x c^\dagger_{k\sigma_k} \mathcal{T}_x^{-1} = c^\dagger_{k'\sigma_k}, \tag{5}$$

in which k and k' correspond to the coordinate (i, j) and $(i+1, j)$, respectively. Note that $i+1$ certainly takes the modulus of N_x. We are going to find what a state transformed under the operation of $\mathcal{T}_x$. Consider a generic basis operated by $\mathcal{T}_x$

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^h \mathcal{T}_x c_{h\sigma_h} \mathcal{T}_x^{-1} \mathcal{T}_x |s\rangle. \tag{6}$$

In the first place we compute

$$\begin{aligned} \mathcal{T}_x |s\rangle &= \mathcal{T}_x c^\dagger_{0\sigma_0} \cdots c^\dagger_{N-1\sigma_{N-1}} |0\rangle \\ &= \mathcal{T}_x c^\dagger_{0\sigma_0} \mathcal{T}_x^{-1} \cdots \mathcal{T}_x c^\dagger_{N-1\sigma_{N-1}} \mathcal{T}_x^{-1} \mathcal{T}_x |0\rangle \\ &= (-)^{(N_x-1)\cdot N_y} |s'\rangle, \end{aligned} \tag{7}$$

where the fermionic sign $(-)^{N_x-1}$ arises from the translational permutation of every horizontal row and there are N_y rows. $\mathcal{T}_x|0\rangle = |0\rangle$ is regarded as a basic assumption. $|s'\rangle$ is just the corresponding translated half-filled bosonic spin configuration. Then

$$\mathcal{T}_x (-)^h c_{h\sigma_h} |s\rangle = (-)^{h+(N_x-1)\cdot N_y} c_{h'\sigma_{h'}} |s'\rangle = \mathrm{sign} \cdot (-)^{h'} c_{h'\sigma_{h'}} |s'\rangle, \tag{8}$$

where an extra sign $= (-)^{h-h'+(N_x-1)\cdot N_y}$ must be multiplied in practical computer program. Note that $\mathcal{T}_x$ although does not change the spin of $c_{h\sigma_h}$, in the new half-filled configuration $|s'\rangle$, $\sigma_{h'}$ indeed corresponds to σ_h in $|s\rangle$.

The case for $\mathcal{T}_y$ is very similar to $\mathcal{T}_x$.

3 Ground state degeneracy

Numerical results turns out there is a $f = 6$ fold degeneracy for t-J model on 4×4 square lattice with both periodic conditions. $(H, \mathcal{T}_x, \mathcal{T}_y)$ can be a *complete set of commutating observables* of which eigenvalue quantum numbers can represent only one specific ground state.

- Firstly diagonalize the matrix $\langle \psi_i | \mathcal{T}_x | \psi_j \rangle, i, j = 0, \cdots, f-1$ and obtain the eigenvalues of $\mathcal{T}_x$ in the subspace with the same eigenvalue of H namely the subspace of degenerate ground states.
- Then you find there are still a $f' = 2$ fold degeneracy with eigenvalues of $\mathcal{T}_x$ and at this moment you should further diagonalize $\langle {\psi_i}' | \mathcal{T}_y | {\psi_j}' \rangle, i, j = 0, \cdots, f' - 1$ in the subspace with same eigenvalue of $\mathcal{T}_x$.

References

[Reuter *et al.*()Reuter, Gomes, and Sorensen] M. Reuter, F. M. Gomes, and D. Sorensen, "BSD arpack++ package," .