Minimum Spanning Tree
-
DPHPC

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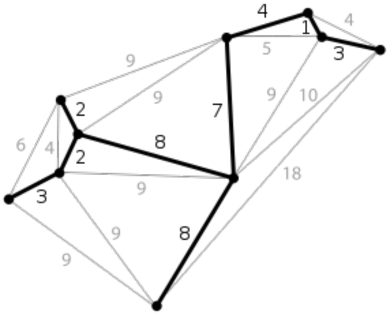
Problem definition - reminder

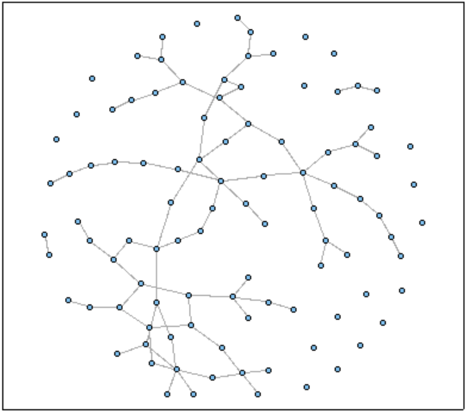
The MST problem

A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted (un)directed graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.



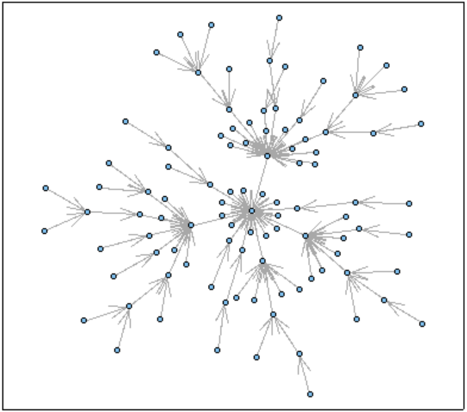
Input sets: $G(n, p)$

$$G(100, 0.02)$$



Input sets: $PA(n)$

$PA(100)$



Input sets: 9^{th} DIMACS challenge dataset

USA Roads

Name	Description	# nodes	# arcs	L
USA	Full USA	23,947,347	58,333,344	-
CTR	Central USA	14,081,816	34,292,496	[
W	Western USA	6,262,104	15,248,146	[
E	Eastern USA	3,598,623	8,778,114	[
LKS	Great Lakes	2,758,119	6,885,658	[
CAL	California and Nevada	1,890,815	4,657,742	[
NE	Northeast USA	1,524,453	3,897,636	[
NW	Northwest USA	1,207,945	2,840,208	[
FLA	Florida	1,070,376	2,712,798	[
COL	Colorado	435,666	1,057,066	[
BAY	San Francisco Bay Area	321,270	800,172	[
NY	New York City	264,346	733,846	[

Algorithms and parallel implementations

Sollin

1: F = set(one-vertex trees)
2: **while** $| F | > 1$ **do**
3:     TODO
4: **end while**

Kruskal

```
A = ∅
for all v ∈ G.V do
    MAKE-SET(v)
end for
Sort (asc.) (weight(u, v))_{(u,v)∈G.E}
for all (u, v) in G.E ordered by weight do
    if FIND-SET(u) ≠ FIND-SET(v) then
        A = A ∪ (u, v)
        UNION(u, v)
    end if
end for
return A
```

Boost implementations

- Boost-Kruskal used as a reference

Parallel sorting on Kruskal

Filter Kruskal

E: set of edges, $m = |E|$, p is the number of cores, $T(m)$ is the runtime with m edges.

1. If $m \leq \textit{threshold}$ then solve with Kruskal
2. Find pivot for edges (weight)
3. Partition E into $E_{\leq}, E_{>}$
4. $A_{\leq} = \textit{filterKruskal}(E_{\leq})$
5. $E_{>} = \textit{filter}(E_{>})$
6. $A_{>} = \textit{filterKruskal}(E_{>})$
7. Return $\textit{merge}(A_{\leq}, A_{>})$

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1. If $m \leq threshold$ then solve with Kruskal
2. Find pivot for edges (weight): $O(1)$
   - Concurrently sample 256 elements from the list (**OpenMP**)
   - Sort the list and return the median (**TBB**)
3. Partition E into $E_{\leq}, E_{>}$
4. $A_{\leq} = filterKruskal(E_{\leq})$
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3. Partition E into $E_{\leq}, E_{>}$: $O\left(\frac{m}{p}\right)$
   - Partition function (**Intel Parallel STL**)
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2. Find pivot for edges (weight): $O(1)$
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4. $A_{\leq} = filterKruskal(E_{\leq})$: $T\left(\frac{m}{2}\right)$
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6. $A_{>} = filterKruskal(E_{>})$: $O\left(T\left(\frac{m}{2}\right)\right)$
   - In practice way less than $\frac{m}{2}$
7. Return $merge(A_{\leq}, A_{>})$

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7. Return $\mathit{merge}(A_{\leq}, A_{>})$: $O(1)$

Filter Kruskal: Complexity analysis

- Worst case: $T(m) \leq O\left(\frac{m}{p}\right) + 2T\left(\frac{m}{2}\right)$
- Best case: $T(m) \leq O\left(\frac{m}{p}\right) + T\left(\frac{m}{2}\right)$

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  $T(m) = O\left(\frac{m}{p} + m\right) = O(m)$
- $O(m) \leq T(m) \leq O\left(\frac{m\ln(m)}{p} + m\right)$

Amdahl's law

S_p is the speedup with p cores, f is the part of the program that is sequential.

$$S_p = \frac{1}{\frac{1-f}{p} + f}$$

$$f = \frac{\frac{p}{S_p} - 1}{p - 1}$$

Amdahl's law: Kruskal

Graph: Erdos-Renyi (100,000 nodes, $p = 0.0005$)

Cores	Median speed-up	Standard deviation	f
1	1	0.0129805395	-
2	1.1881513396	0.0400305172	0.6832872491
4	1.3340592415	0.0130798641	0.6661225318
8	1.3134515984	0.01048841	0.7272602975
16	1.4399618642	0.0088220116	0.6740936918
32	1.3877640643	0.0442081933	0.7115701488

Amdahl's law: Filter Kruskal

Graph: Preferential attachment (10,000 nodes, 1,000 edges per vertex)

Cores	Median speed-up	Standard deviation	f
1	1	0.174182691	-
2	1.6268639022	0.2472258016	0.2293591353
4	2.669168898	0.5213238241	0.1661979381
8	5.3584384444	0.9757941767	0.0704246098
16	5.7447285937	1.108109986	0.1190108026
32	5.6322979489	1.497976882	0.1510166972

Filter Sollin

Results overview

EULER Cluster

Scalability

Speedups