Minimum Spanning Tree
-
DPHPC

Th. Cambier R. Dang-Nhu Th. Dardinier C. Trassoudaine

ETH Zürich

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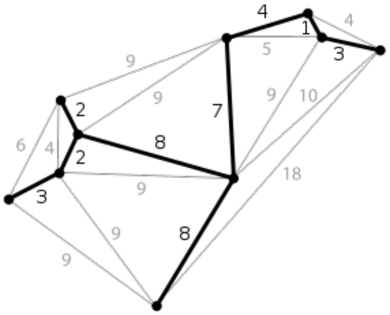
Problem definition - reminder

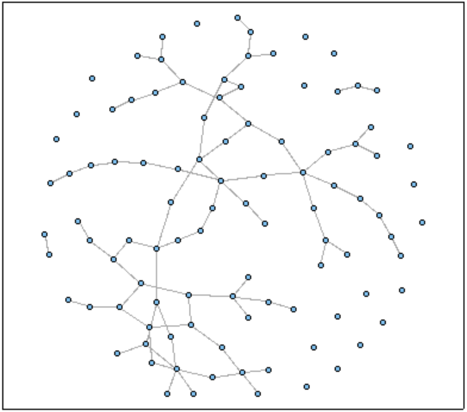
The MST problem

A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted (un)directed graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.



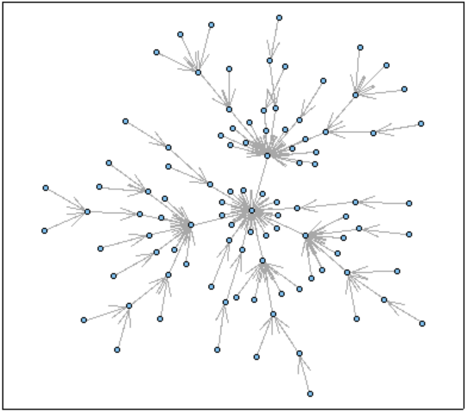
Input sets: $G(n, p)$

$G(100, 0.02)$



Input sets: $PA(n, m)$

$PA(100, 1)$



Input sets: 9^{th} DIMACS challenge dataset

USA Roads

Name	Description	# nodes	# arcs	l
USA	Full USA	23,947,347	58,333,344	-
CTR	Central USA	14,081,816	34,292,496	[
W	Western USA	6,262,104	15,248,146	[
E	Eastern USA	3,598,623	8,778,114	[
LKS	Great Lakes	2,758,119	6,885,658	[
CAL	California and Nevada	1,890,815	4,657,742	[
NE	Northeast USA	1,524,453	3,897,636	[
NW	Northwest USA	1,207,945	2,840,208	[
FLA	Florida	1,070,376	2,712,798	[
COL	Colorado	435,666	1,057,066	[
BAY	San Francisco Bay Area	321,270	800,172	[
NY	New York City	264,346	733,846	[

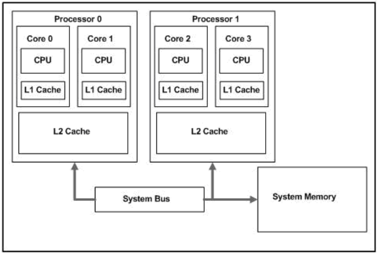
Setup

Software

- OMP
- Intel Threading Building Blocks (TBB)
- Parallel Streaming Transformation Loader Service (PSTL)

EULER

- 1 node limitation (OMP)
- 2 sockets filled with 18 cores - up to 3.7Ghz
- Inter-sockets bus speed: 10.4 GT/s



Algorithms and parallel implementations

Sollin

1. For each connected component, find adjacent edge with minimum weight
2. Add edge to mst (each edge add at most twice)
3. Merge connected components

Kruskal

```
A = ∅
for all v ∈ G.V do
    MAKE-SET(v)
end for
Sort (asc.) (weight(u, v))_{(u,v)∈G.E}
for all (u, v) in G.E ordered by weight do
    if FIND-SET(u) ≠ FIND-SET(v) then
        A = A ∪ (u, v)
        UNION(u, v)
    end if
end for
return A
```

Boost implementations

- Boost-Kruskal used as a reference

Parallel sorting on Kruskal

m is the number of edges, p is the number of cores, $T(m)$ is the runtime with m edges.

- Sequential: $O\left(m\ln(m) + m\right)$
- Parallel sort (**Intel TBB**): $O\left(\frac{m\ln(m)}{p} + m\right)$

Filter Kruskal

E: set of edges, $m = |E|$, p is the number of cores, $T(m)$ is the runtime with m edges.

1. If $m \leq \mathit{threshold}$ then solve with Kruskal
2. Find pivot for edges (weight)
3. Partition E into $E_{\leq}, E_{>}$
4. $A_{\leq} = \mathit{filterKruskal}(E_{\leq})$
5. $E_{>} = \mathit{filter}(E_{>})$
6. $A_{>} = \mathit{filterKruskal}(E_{>})$
7. Return $\mathit{merge}(A_{\leq}, A_{>})$

Filter Kruskal

E: set of edges, $m = |E|$, p is the number of cores, $T(m)$ is the runtime with m edges.

1. If $m \leq threshold$ then solve with Kruskal
2. Find pivot for edges (weight): $O(1)$
   - Concurrently sample 256 elements from the list (**OpenMP**)
   - Sort the list and return the median (**TBB**)
3. Partition E into $E_{\leq}, E_{>}$
4. $A_{\leq} = filterKruskal(E_{\leq})$
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3. Partition E into $E_{\leq}, E_{>}$: $O\left(\frac{m}{p}\right)$
    - Partition function (**Intel Parallel STL**)
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5. $E_{>} = \mathit{filter}(E_{>})$: $O\left(\frac{m}{p}\right)$
6. $A_{>} = \mathit{filterKruskal}(E_{>})$: $O\left(T\left(\frac{m}{2}\right)\right)$
   - In practice way less than $\frac{m}{2}$
7. Return $\mathit{merge}(A_{\leq}, A_{>})$

Filter Kruskal

E: set of edges, $m = |E|$, p is the number of cores, $T(m)$ is the runtime with m edges.

1. If $m \leq$ *threshold* then solve with Kruskal
2. Find pivot for edges (weight): $O(1)$
3. Partition E into $E_{\leq}, E_{>}$: $O\left(\frac{m}{p}\right)$
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7. Return $\mathit{merge}(A_{\leq}, A_{>})$: $O(1)$

Filter Kruskal: Complexity analysis

- Worst case: $T(m) \leq O\left(\frac{m}{p}\right) + 2T\left(\frac{m}{2}\right)$
- Best case: $T(m) \leq O\left(\frac{m}{p}\right) + T\left(\frac{m}{2}\right)$

Filter Kruskal: Complexity analysis

- Worst case: $T(m) \leq O\left(\frac{m}{p}\right) + 2T\left(\frac{m}{2}\right)$
  $T(m) = O\left(\frac{m\ln(m)}{p} + m\right)$
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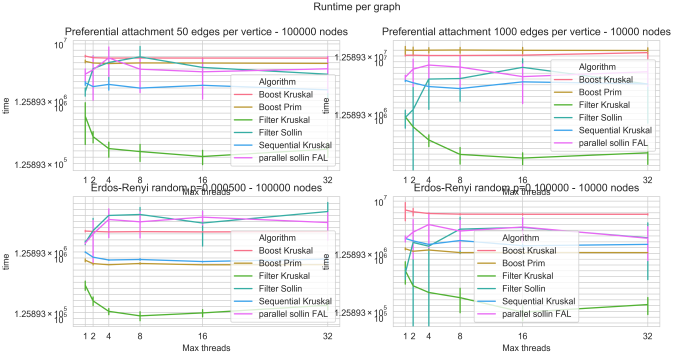
- $O(m) \leq T(m) \leq O\left(\frac{m\ln(m)}{p} + m\right)$

Kruskal vs Sollin

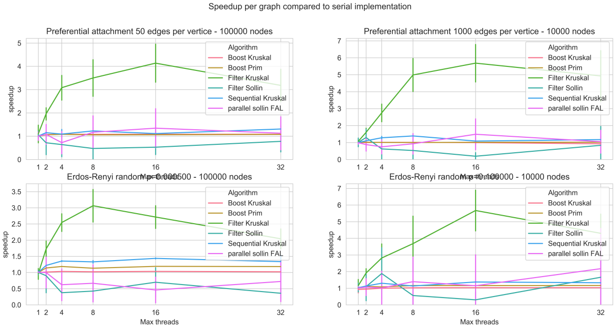
- Same complexity, but Sollin computing steps are more fine-grained
- Sollin is entirely paralelizable
- Filter Kruskal can be adapted to Filter Sollin

Results overview

Runtimes


Runtime per graph
Preferential attachment 50 edges per vertice - 100000 nodes
Preferential attachment 1000 edges per vertice - 10000 nodes
Erdos-Renyi random p=0.000500 - 100000 nodes
Erdos-Renyi random p=0.100000 - 10000 nodes
time
Max threads
1 2 4 8 16 32
Algorithm
Boost Kruskal
Boost Prim
Filter Kruskal
Filter Sollin
Sequential Kruskal
parallel sollin FAL

Speedups


Speedup per graph compared to serial implementation
Preferential attachment 50 edges per vertice - 100000 nodes
Preferential attachment 1000 edges per vertice - 10000 nodes
Erdos-Renyi random p=0.000500 - 100000 nodes
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speedup
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Amdahl's law

S_p is the speedup with p cores, f is the part of the program that is sequential.

$$S_p = \frac{1}{\frac{1-f}{p} + f}$$

$$f = \frac{\frac{p}{S_p} - 1}{p - 1}$$

Amdahl's law: Kruskal

Graph: Erdos-Renyi (100,000 nodes, $p = 0.0005$)

Cores	Median speed-up	Standard deviation	f
1	1	0.0129805395	-
2	1.1881513396	0.0400305172	0.6832872491
4	1.3340592415	0.0130798641	0.6661225318
8	1.3134515984	0.01048841	0.7272602975
16	1.4399618642	0.0088220116	0.6740936918
32	1.3877640643	0.0442081933	0.7115701488

Amdahl's law: Filter Kruskal

Graph: Preferential attachment (10,000 nodes, 1,000 edges per vertex)

Cores	Median speed-up	Standard deviation	f
1	1	0.174182691	-
2	1.6268639022	0.2472258016	0.2293591353
4	2.669168898	0.5213238241	0.1661979381
8	5.3584384444	0.9757941767	0.0704246098
16	5.7447285937	1.108109986	0.1190108026
32	5.6322979489	1.497976882	0.1510166972