Subject Date No.

تحلیل

مرکب (1)

Subject Date No.

Complex Variables and Applications

" James Brown and Ruel churchill "

CH 1

Complex Numbers

→ sets of numbers :-

[1] Natural Numbers : $\mathbb{N} = \{1, 2, 3, \dots\}$

[2] Integer Numbers : $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

[3] Rational Numbers : $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \ , q \neq 0\}$

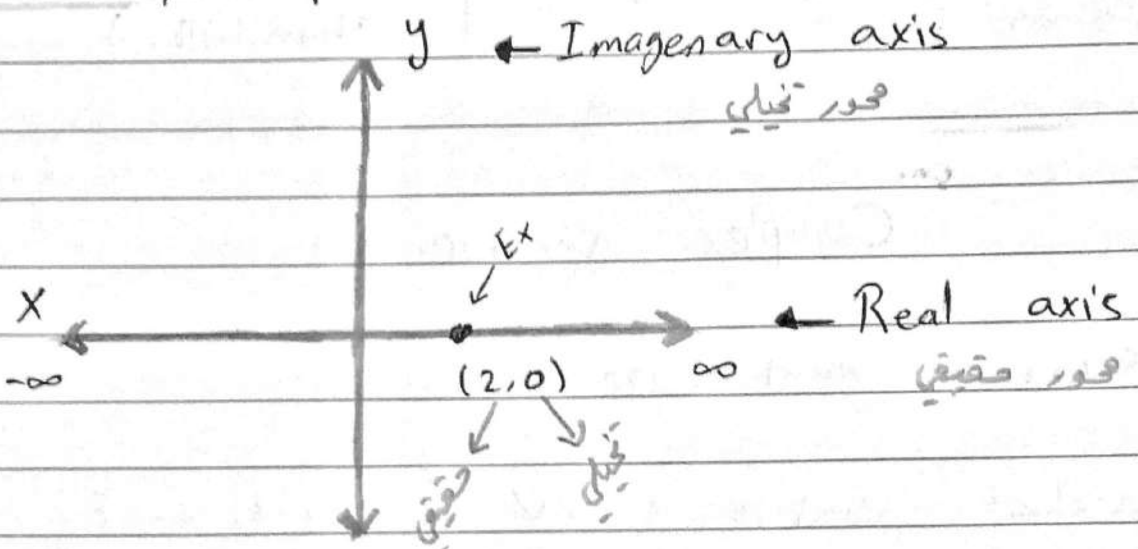
[4] Irrational Numbers : $\mathbb{I} = \{\sqrt{5}, \sqrt{7}, \pi, e, 21.34\dots\}$

[5] Real Numbers : $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

Note that :-

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

The complex plane ($\mathbb{C}$)



* The complex number can be defined as ordered pairs (x,y) of real numbers, so the points is the complex plaine are numbers.

* $(x,0) \equiv x$ on the real axis.

← نرسم هذه النقطة تبقى على المحور الحقيقي.

يقابل / يوافق

* Complex numbers of the form $(0, y)$ corrspond to points on the y-axis and are called pure imagenary numbers.

يعني يطابق ١٠٠٪

* We denote a complex numbers :-
(x,y) by Z, so that : $Z = (x,y)$

Z ← رمز مشهور يمثل (x,y)

* let $Z_1 = (x_1, y_1)$, $Z_2 = (x_2, y_2)$ be two complex numbers then :-

$\boxed{1}$ $Z_1 = Z_2$ iff $x_1 = x_2$ & $y_1 = y_2$

Ex find the value of x, y if $Z_1 = Z_2$ where $Z_1 = (3, 9)$, $Z_2 = (2x-4, 3y+5)$

$\rightarrow 2x - 4 = 3 \rightarrow 2x = 7 \rightarrow x = \frac{7}{2}$

$3y + 5 = 9 \rightarrow 3y = 4 \rightarrow y = \frac{4}{3}$

$\boxed{2}$ $Z_1 \pm Z_2 = (x_1, y_1) \pm (x_2, y_2)$
$= (x_1 \pm x_2 , y_1 \pm y_2)$

$\boxed{3}$ $\alpha Z_1 = \alpha (x_1, y_1) = (\alpha x_1 , \alpha y_1)$,
$\alpha \in \mathbb{R}$

$\boxed{4}$ $Z_1 \cdot Z_2 = (x_1, y_1) \cdot (x_2, y_2)$

$= (\underbrace{x_1 \cdot x_2 - y_1 \cdot y_2}, \underbrace{y_1 \cdot x_2 + y_2 \cdot x_1})$

الاول . الاول - الثاني . الثاني   القريب × القريب + البعيد × البعيد

<u>Ex</u> Let $Z_1 = (1, 3)$, $Z_2 = (2, 5)$ find :-

1) $Z_1 + Z_2 = (1, 3) + (2, 5) = (3, 8)$

2) $Z_1 - Z_2 = (1, 3) - (2, 5) = (-1, -2)$

3) $5 \cdot Z_1 = 5 \cdot (1, 3) = (5, 15)$

4) $Z_1 \cdot Z_2 = (1, 3) \cdot (2, 5)$

$= (2 - 15, 6 + 5) = (-13, 11)$

يجب أن يكون الناتج أزواج مرتبة

*** Remark :-**

النقطة $(0,1)$
تمثل $\sqrt{-1}$
$\rightarrow (0,1) = \sqrt{-1} = i$

$$[(0,1)]^2 = (0,1)\cdot(0,1)$$
$$= (0-1 \;,\; 0+0)$$
$$= (-1, 0)$$
$$= -1 \longrightarrow \text{Real part}$$ الجزء الحقيقي

← عند تمثيل النقطة $(-1,0)$ على محور الاعداد
يكون عدد حقيقي ...!؟ لذلك نستطيع أن
نكتبه منفرد على السطر دون زوج مرتب .

*** Remark :-**

1	2	3	4
$i = \sqrt{-1}$	$i^2 = -1$	$i^3 = -\sqrt{-1} = -i$	$i^4 = 1$

← إذا أعطى بالسؤال أس كبير نقسمه على 4 لأنه لدينا
٤ خيارات و حسب الأس الذي ينتج يكون ناتجه من هذه
الخيارات ، مثلاً :-

$$i^5 = \sqrt{-1}$$
$$i^6 = -1$$
$$i^7 = -\sqrt{-1}$$
$$i^8 = 1$$

وهكذا

Ex i^{203}

$\rightarrow$ $4 \overline{)203}$ quotient 50, 20, 003 ← الباقي

$\rightarrow i^{203} = i^{3} = -\sqrt{-1} = -i$

$\rightarrow i^{203} = -i$

Ex i^{57}

$\rightarrow$ $4 \overline{)57}$ quotient 14, 4, 17, 16, 1 ← الباقي

$\rightarrow i^{57} = i$

إذا كان الباقي :

④	③	②	①
$i^4 = 1$	$i^3 = -\sqrt{-1} = -i$	$i^2 = 1$	$i = \sqrt{-1}$
الباقي صفر			

((من الملاحظة السابقة))

* Any complex numbers can be write (x,y)

$$Z=(x,y) = (x,0)+(0,y)$$
$$= (x,0) + (0,1)\cdot(y,0)$$
$$= x + iy$$

$(0,y) \rightarrow (0,1)(y,0)$
$= (0-0, y+0)$
$= (0,y)$

من هنا جاءت → $Z = x + iy$

x = real part → $Re(Z) = x$

y = Imagenary part → $Im(Z) = y$

$i = \sqrt{-1}$

Ex find $Re(Z)$, $Im(Z)$

① $Z = 5-3i$ → $Re(Z) = 5$, $Im(Z) = -3$

② $Z = 4$ → $Re(Z) = 4$, $Im(Z) = 0$

③ $Z = 6i$ → $Re(Z) = 0$, $Im(Z) = 6$

Operation on complex numbers :

let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and
$z_3 = x_3 + iy_3$

$\boxed{1}$ Equality : خاصية المساواة

$z_1 = z_2$ iff $x_1 = x_2$ & $y_1 = y_2$

Ex if $z_1 = z_2$ find x, y
$z_1 = 2 + 3i$ $z_2 = x + (2y+5)i$?

→ $\boxed{x = 2}$

$2y + 5 = 3$
$2y = -2$
$\boxed{y = -1}$

$\boxed{2}$ Addition خاصية الجمع

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

Ex $z_1 = 2 + 3i$, $z_2 = 5 + 4i$, find $z_1 + z_2$?

$$z_1 + z_2 = (2+3i) + (5+4i)$$
$$= 7 + 7i$$

$\boxed{3}$ Subtraction خاصية الطرح

$z_1 = x_1 + iy_1 \quad , \quad z_2 = x_2 + iy_2 \,,$

$\rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

<u>Ex</u> $z_1 = 3 + 5i \quad , \quad z_2 = 2 + 7i$ find $z_1 - z_2$?

$$z_1 - z_2 = (3-2) + i(5-7)$$
$$= 1 + (-2)i$$
$$= 1 - 2i$$

$\boxed{4}$ Multiplication خاصية الضرب

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$
$$= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 y_2 + x_2 y_1)$$

(الاول × الاول - الثاني × الثاني) + i (القريب × القريب + البعيد × البعيد)

$(x_1 + iy_1)(x_2 + iy_2) \qquad (x_1 + iy_1)(x_2 + iy_2)$

*يجوز أيضاً طريقة توزيع الاقواس $(x_1 + iy_1)(x_2 + iy_2)$

<u>Ex</u> find $Re(Z_1 \cdot Z_2)$, $Im(Z_1 \cdot Z_2)$
where $Z_1 = 2+3i$, $Z_2 = 4+5i$

طريقة 1

$$Z_1 \cdot Z_2 = (2+3i)(4+5i)$$
$$= (8-15) + i(10+12)$$
$$= -7 + 22i$$

$Re(Z_1 \cdot Z_2) = -7$

$Im(Z_1 \cdot Z_2) = 22$

طريقة 2

$$Z_1 \cdot Z_2 = (2+3i)(4+5i)$$
$$= 8 + 10i + 12i - 15$$
$$= -7 + 22i$$

$\boxed{i \cdot i = -1}$

$Re(Z_1 \cdot Z_2) = -7$

$Im(Z_1 \cdot Z_2) = 22$

$\boxed{5}$ Division خاصية القسمة

$$\frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{(x_2 - iy_2)}{(x_2 - iy_2)}$$ ← ضرب المرافق للمقام

<u>Ex</u> $Z_1 = 2+3i$, $Z_2 = 4+5i$ find :
$Re\left(\frac{Z_1}{Z_2}\right)$, $Im\left(\frac{Z_1}{Z_2}\right)$

$$\frac{Z_1}{Z_2} = \frac{2+3i}{4+5i} \cdot \frac{(4-5i)}{(4-5i)} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}$$

$$= \frac{(8--15) + i(12+-10)}{(16--25) + i(\cancel{20+-20})}$$ ← لازم تروح

$$= \frac{23+2i}{41} = \frac{23}{41} + \frac{2i}{41}$$ → $Re\left(\frac{Z_1}{Z_2}\right) = \frac{23}{41}$

$Im\left(\frac{Z_1}{Z_2}\right) = \frac{2}{41}$

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Ex find $Re\left(\frac{3+4i}{2+7i}\right)$

$$\rightarrow \frac{(3+4i)}{(2+7i)} \cdot \frac{(2-7i)}{(2-7i)} = \frac{(3+4i)(2-7i)}{(2+7i)(2-7i)}$$

$$= \frac{(6+28)+i(8-21)}{(4+49)+i(-14+14)} = \frac{34}{53} - \frac{31i}{53}$$

$$\rightarrow Re\left(\frac{3+4i}{2+7i}\right) = \frac{34}{53}$$

Algebraic Properties :-

for any complex numbers :-

$Z_1 = x_1 + iy_1$, $Z_2 = x_2 + iy_2$ & $Z_3 = x_3 + iy_3$

we have :-

$\boxed{1}$ The commutative law الخاصية التبديلية

→ $Z_1 + Z_2 = Z_2 + Z_1$

$\boxed{2}$ The associative law الخاصية التجميعية

→ (i) $(Z_1 + Z_2) + Z_3 = Z_1 + (Z_2 + Z_3)$

→ (ii) $(Z_1 \cdot Z_2) \cdot Z_3 = Z_1 \cdot (Z_2 \cdot Z_3)$

$\boxed{3}$ The distributive law خاصية توزيع الضرب على الجمع

→ $Z_1 \cdot (Z_2 + Z_3) = (Z_1 \cdot Z_2) + (Z_1 \cdot Z_3)$

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$\boxed{4}$ The additive identity العنصر المحايد لعملية الجمع

is $0 = (0,0) = 0 + 0i$

$\rightarrow Z + 0 = Z$ * العنصر المحايد لا يؤثر على العملية

$\boxed{5}$ The additive inverse is $(-Z)$

معكوس جمعي

$\rightarrow Z^{-1} = -Z$

معكوس وليس أس

* معكوس الجمع ← السالب.

* الشيء ومعكوسه يعطي العنصر المحايد identity

$$
\begin{aligned}
Z + (-Z) &= (x + iy) + (-x - iy) \\
&= (x - x) + i(y - y) \\
&= 0 + 0i \\
&= (0,0)
\end{aligned}
$$

$\boxed{6}$ The multiplicative identity العنصر المحايد لعملية الضرب هو 1

is $1 = (1,0)$

$\rightarrow Z \cdot 1 = Z$

*بالجمع ← العنصر المحايد هو الصفر identity $= 0$

*بالضرب ← العنصر المحايد هو الواحد identity $= 1$

$\boxed{7}$ The multiplicati inverse

*معكوس Z في حالة الضرب ← $Z^{-1} = \frac{1}{Z}$

← معكوس وليس أس

ضرب المرافق

$$\rightarrow \frac{1}{Z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x-iy)^2}$$

$$\rightarrow \boxed{Z^{-1} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}}$$

Ex if $Z = 5 + 3i$ find multiplication inverse.

معكوس ضربي

$$\rightarrow Z^{-1} = \frac{1}{Z} = \frac{1}{(5+3i)} \cdot \frac{(5-3i)}{(5-3i)}$$

$$= \frac{5-3i}{(5+3i)(5-3i)} = \frac{5}{25+9} - \frac{3i}{25+9}$$

$$= \frac{5}{34} - i\,\frac{3}{34}$$

Ex show that : $Im(iZ) = Re(Z)$

$\rightarrow$ let $Z = x + iy$

$iZ = i(x+iy)$ $\leftarrow$ نضرب الطرفين بـ i

$iZ = \underbrace{-y}_{Re} + i\underbrace{x}_{Im}$

$Im(iZ) = x$ --- ①

$\rightarrow$ by ① & ②

$Im(iZ) = Re(Z)$

$\therefore Z = x + iy$

$Re(Z) = x$ --- ②

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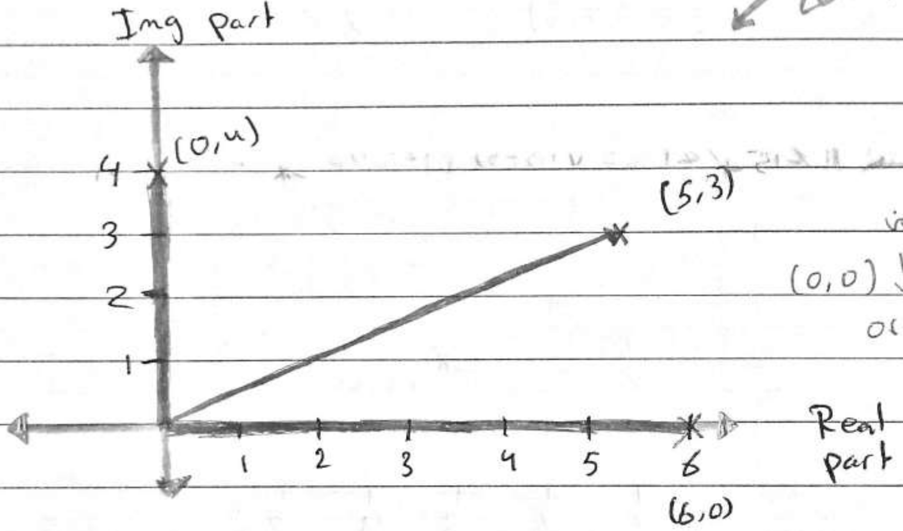
* Geometric interpretation *

$$\mathbb{C} = \{ Z : Z = x + iy \ , \ x,y \in \mathbb{R} \ , \ i = \sqrt{-1} \}$$

Ex
$Z_1 = 5 + 3i \longrightarrow (5,3)$
$Z_2 = 4i \longrightarrow (0,4)$
$Z_3 = 6 \longrightarrow (6,0)$

تمثيل بيانياً
complex



الرسم بيبدأ من
نقطة الاصل (0,0)
origin point

complex * ← يمثل خط مستقيم وليس نقطة.

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Ex $Z_1 = 3 + 2i$

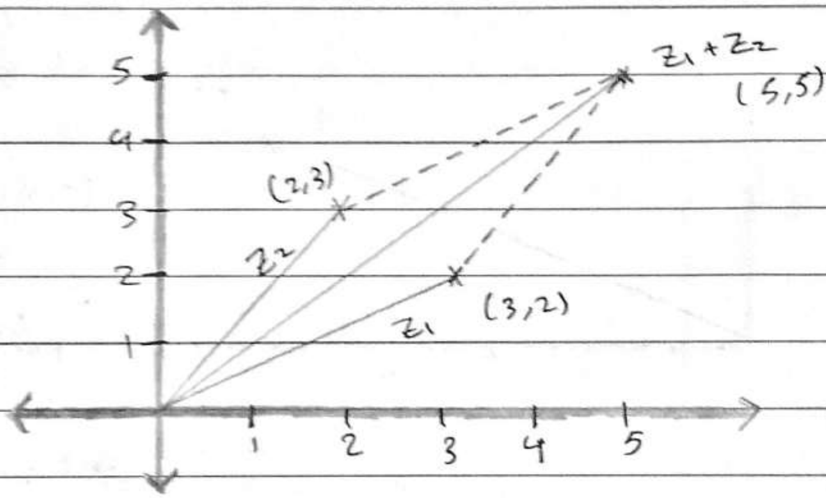
$Z_2 = 2 + 3i$

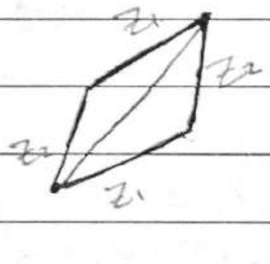
→ $Z_1 = (3, 2)$

→ $Z_2 = (2, 3)$

$Z_1 + Z_2 = (3 + 2i) + (2 + 3i) = 5 + 5i$

→ $Z_1 + Z_2 = (5, 5)$





* The mudulas , absolute or lenght of a complex numbers $Z = x + iy$ is defined by :-

$$|Z| = \sqrt{x^2+y^2} \quad , \quad |Z| \in \mathbb{R} \quad , \quad Z \in \mathbb{C}$$

* حساب طول متجه Z ← من $(0,0)$ لعند النقطة (x,y)
((المسافة بين نقطتين)) .

Ex find $|Z|$ if :-

① $Z_1 = 3 + 5i$

$$|Z_1| = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$$

② $Z_2 = 3i$

$$|Z_2| = \sqrt{0^2+3^2} = \sqrt{9} = 3$$

③ $Z_3 = 5$

$$|Z_3| = \sqrt{5^2+0^2} = \sqrt{25} = 5$$

$|-2| = 2$

⟸ المسافة من العنصر المحايد
إلى -2 هي
2

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* The distance between two points

$Z_1 = x_1 + iy_1$ is $|Z_1 - Z_2|$

$Z_2 = x_2 + iy_2$

$\rightarrow |Z_1 - Z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex find the distance between

$Z_1 = 3 + 2i$, $Z_2 = 5 + 4i$

$$\rightarrow |Z_1 - Z_2| = \sqrt{(3-5)^2 + (2-4)^2}$$
$$= \sqrt{(-2)^2 + (-2)^2}$$
$$= \sqrt{4+4} = \sqrt{8}$$

* Remark :

we can write $|Z_1| < |Z_2|$ but can't write $Z_1 < Z_2$

لا يجوز مقارنة بال complex لأنها عبارة عن أزواج مرتبة ولا يجوز المقارنة
بين الأزواج لكن يجوز المقارنة بطول المتجه لأنه عدد.

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$\underline{Ex}$ show that $|Z_1 - Z_2| = R$ is a circle in the complex plane with center Z_0 and radius R ?

← اثبت أنها دائرة مركزها Z_0 ونصف قطرها R.

→ $Z = x + iy$

$Z_0 = x_0 + iy_0$

$|Z - Z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2} = R$ ← (اخذ تربيع للطرفين)

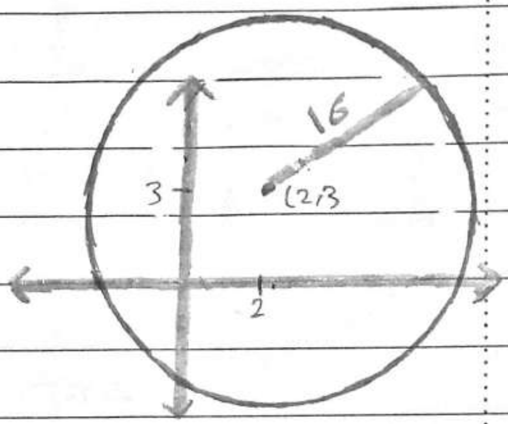
$(x-x_0)^2 + (y-y_0)^2 = R^2$

Circle $\begin{cases} \text{center } (x_0, y_0) \\ \text{radius } = R \end{cases}$

Ex Draw the following :

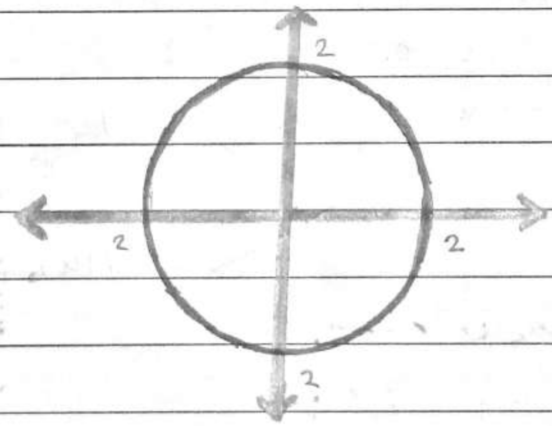
(i) $|Z - (2+3i)| = 16$

→ center $(x_0, y_0) = (2,3)$
radius $r = 16$



(ii) $|Z| = 2$

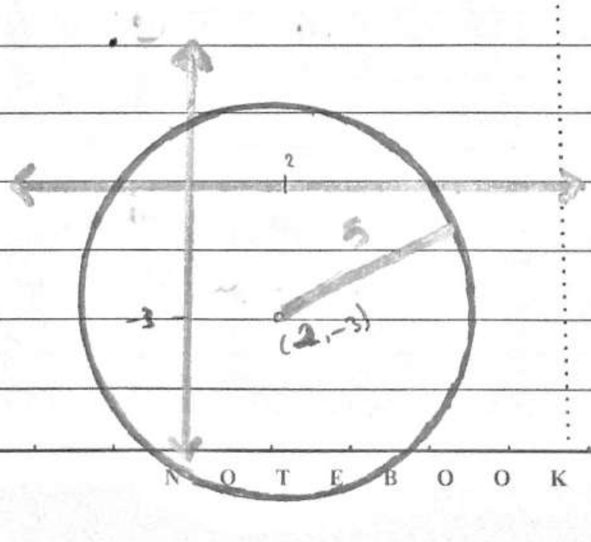
→ Center $(0,0)$
radius $r = 2$



(iii) $|Z - 2 + 3i| = 5$

→ $|Z - (2-3i)| = 5$

center $(2,-3)$
radius $r = 5$



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Remark :-

$$Re(Z) \leq |Re(z)| \leq |Z|$$ ← بطول أكبر شيء لأنه شيء + شيء

$$Im(Z) \leq |Im(Z)| \leq |Z|$$

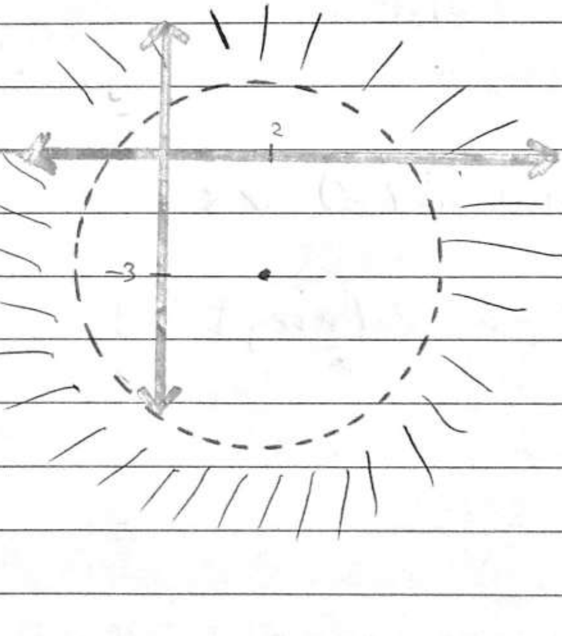
مثلاً (2) ← مطلقه أكبر وطوله اكبر لأنه شيء + شيء.

Ex (i) $|Z - 2 + 3i| > 5$

$\rightarrow |Z - (2 - 3i)| > 5$

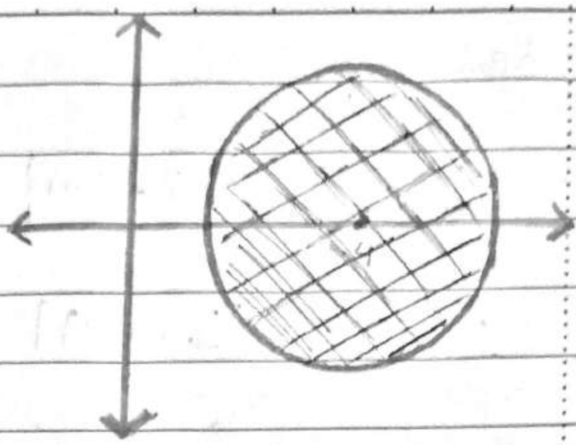
→ center $(2, -3)$
radius $r = 5$

رسمت بخطوط متقطعة ← بسبب عدم وجود المساواة في السؤال



منطقة الحل خارج الدائرة (متباينة)

(ii) $|Z-4| \leq 2$



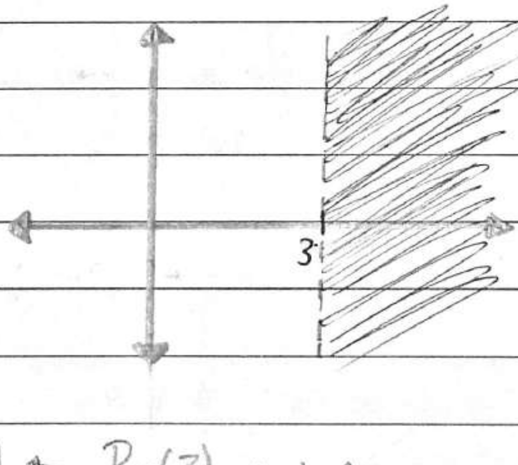
$\rightarrow$ Center $(4,0)$

$r = 2$

← إذا أعطانا إشارة أصغر تكون منطقة الحل داخل الدائرة
وإذا أعطانا إشارة أكبر تكون منطقة الحل خارج الدائرة.

← إذا أعطانا المساواة في إشارة المتباينة يكون خط الدائرة متصل
وإذا لم يكن هناك مساواة في إشارة المتباينة يكون خط الدائرة
غير متصل (متقطع) بسبب عدم وجود مساواة.

(iii) $Re(Z) > 3$



$\rightarrow Re(x+iy) > 3$

$x > 3$

← يريد $Re(Z)$ ← الجزء الحقيقي لـ Z وهو X
لو طلب $Im(Z)$ ← الجزء التخيلي لـ Z هو y

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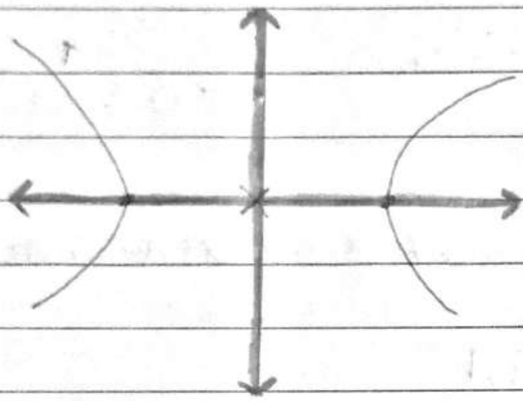
(4) $Re(Z^2) = 1$

$Z^2 = (x+iy)(x+iy)$
$= (x^2-y^2)+i(2xy)$

$\to Re(Z^2) = x^2-y^2 = 1$ ← معادلة قطع زائد

$x^2+y^2=1$
← معادلة دائرة مركزها (0،0)
ونصف قطرها 1

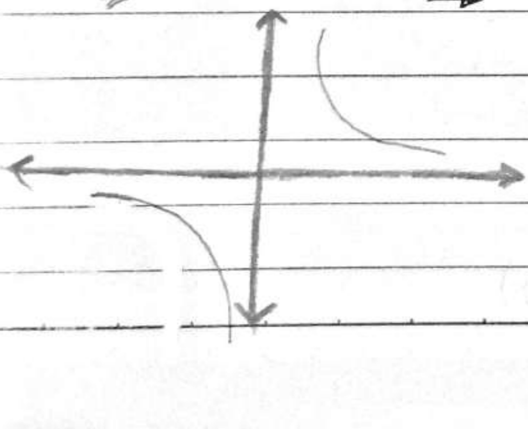


→ hyperbola قطع زائد

$x^2-y^2=1$
← معادلة قطع زائد
مركزها (0،0)

نفس السؤال السابق لكن Im ← (5) $Im(Z^2) = 1$

$\to 2xy = 1 \to y = \frac{1}{2x}$ ← رسمتها تقرب لكن لا تقطع



* الموجب ← ربع الاول والثالث.
* السالب ← ربع الثاني والرابع

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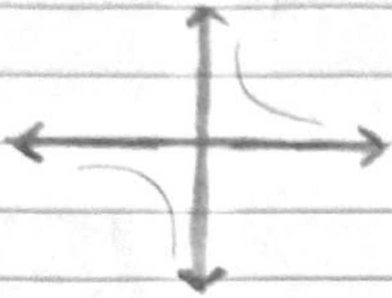
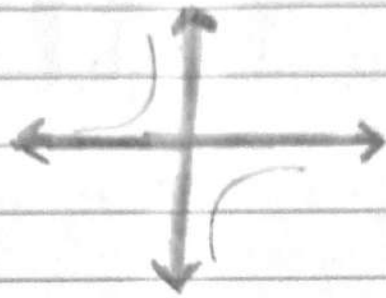
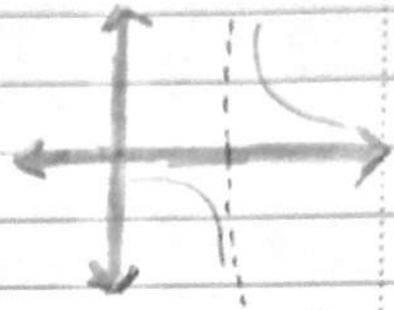
Remark :

$y = \frac{1}{x}$ $y = \frac{-1}{x}$ $y = \frac{1}{x-2}$







ازاحة الرسمة

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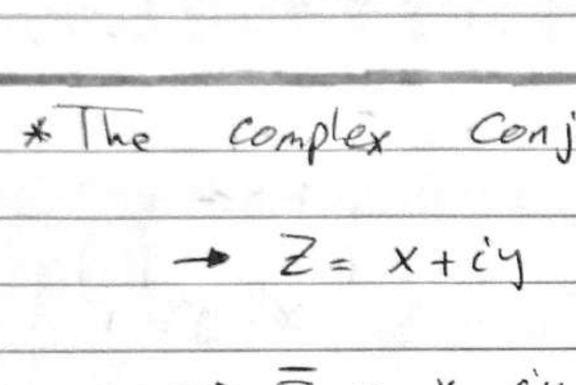
* The complex conjugate of complex numbers Z

→ $Z = x + iy$

→ $\bar{Z} = x - iy$ → conjugate of Z

يعني Im تتغير اشارته
إذا كان موجب يصبح سالب
وإذا كان سالب يصبح موجب



Ex find the conjugate :-

(i) $Z = 3 + 5i \longrightarrow \bar{Z} = 3 - 5i$

(ii) $Z = 4 \longrightarrow \bar{Z} = 4$

(iii) $Z = 5i \longrightarrow \bar{Z} = -5i$

* Properties :-

$\boxed{1}$ $Z = \bar{Z}$ iff $Z = X \leftarrow$ عندما لا يوجد Im
أي لا يوجد iy

$\boxed{2}$ $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

proof :

$$\overline{Z_1 + Z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$$
$$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$$
$$= (x_1 + x_2) - i(y_1 + y_2)$$
$$= (x_1 + x_2) - iy_1 - iy_2$$
$$= (x_1 - iy_1) + (x_2 - iy_2)$$
$$= \overline{Z_1} + \overline{Z_2}$$

$\boxed{3}$ $\overline{Z_1 \cdot Z_2} = \overline{Z_1} \cdot \overline{Z_2}$

$\boxed{4}$ $\overline{\left(\frac{Z_1}{Z_2}\right)} = \left(\frac{\overline{Z_1}}{\overline{Z_2}}\right)$

$\boxed{5}$ $\overline{\overline{Z}} = Z$

proof :-

let $Z = x + iy \rightarrow \overline{Z} = x - iy$

$\rightarrow \overline{\overline{Z}} = x + iy = Z$

$\boxed{6}$ $Re(Z) = \dfrac{Z + \overline{Z}}{2}$

$\boxed{7}$ $Im(Z) = \dfrac{Z - \overline{Z}}{2i}$

من أجل اثباتهما نأخذ
الطرف الثاني من أجل الوصول
للطرف الأول (الأيسر)

proof :

$$\frac{Z + \overline{Z}}{2} = \frac{(x + \cancel{iy}) + (x - \cancel{iy})}{2} = \frac{2x}{2}$$

$$= x = Re(Z)$$

$\boxed{8}$ $Z \cdot \overline{Z} = |Z|^2$

Subject Date No.

Ex find $Re\left(\overline{\frac{2+5i}{3i}}\right)$

ضرب بالمرافق

$$\rightarrow \left(\overline{\frac{2+5i}{3i}}\right) = \frac{\overline{(2+5i)}}{\overline{3i}} = \frac{2-5i}{-3i} \cdot \frac{3i}{3i}$$

$$= \frac{(2-5i)\cdot 3i}{9} = \frac{6i+15}{9} = \frac{15}{9} + \frac{6i}{9}$$

$$\rightarrow Re\left(\overline{\frac{2+5i}{3i}}\right) = \frac{15}{9}$$

* Properties of modulus :- خصائص القيمة المطلقة

$\boxed{1}$ $|z| = |\bar{z}|$ $\bar{z}$ طول = z طول

proof :-

$$z = x + iy \rightarrow \bar{z} = x - iy$$

$$|z| = \sqrt{x^2+y^2} \quad \text{--- } ①$$

$$|\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} \quad \text{--- } ②$$

$$\rightarrow ① = ② \quad \therefore \quad |z| = |\bar{z}|$$

Subject Date No.

$\boxed{2}$ $|Z_1 \cdot Z_2| = |Z_1| \cdot |Z_2|$ الطول يوزع في حالة الضرب والقسمة

proof :

$$|Z|^2 = Z \cdot \bar{Z}$$

$$|Z_1 \cdot Z_2|^2 = (Z_1 \cdot Z_2) \cdot \overline{(Z_1 \cdot Z_2)}$$
$$= (Z_1 \cdot Z_2) \cdot (\bar{Z}_1 \cdot \bar{Z}_2)$$
$$= (Z_1 \cdot \bar{Z}_1) \cdot (Z_2 \cdot \bar{Z}_2)$$
$$= |Z_1|^2 \cdot |Z_2|^2$$

$\rightarrow |Z_1 \cdot Z_2|^2 = |Z_1|^2 \cdot |Z_2|^2$

$|Z_1 \cdot Z_2|^2 = (|Z_1| \cdot |Z_2|)^2$ ← أخذ جذر للطرفين

$\rightarrow |Z_1 \cdot Z_2| = |Z_1| \cdot |Z_2|$

Subject — Date — No.

$\boxed{3}$ Triangle inqualite متباينة مثلثية

$$\rightarrow |Z_1 + Z_2| \leqslant |Z_1| + |Z_2|$$

نأخذ الطرف الأيسر ونربعه

"الابتداء بالطرف الأيسر"

$$|Z_1 + Z_2|^2 = (Z_1 + Z_2) \cdot \overline{(Z_1 + Z_2)}$$

$$= (Z_1 + Z_2) \cdot (\overline{Z}_1 + \overline{Z}_2)$$

$$= Z_1 \cdot \overline{Z}_1 + Z_1 \cdot \overline{Z}_2 + Z_2 \cdot \overline{Z}_1 + Z_2 \cdot \overline{Z}_2$$

$$= |Z_1|^2 + Z_1 \cdot \overline{Z}_2 + Z_2 \cdot \overline{Z}_1 + |Z_2|^2$$

نفس المعنى

$$= |Z_1|^2 + \underbrace{Z_1 \cdot \overline{Z}_2 + \overline{Z_1 \cdot \overline{Z}_2}}_{2\,Re(Z_1 \cdot \overline{Z}_2)} + |Z_2|^2$$

من المعلومة

$$Re(Z) = \frac{Z + \overline{Z}}{2}$$

$$\Rightarrow Z + \overline{Z} = 2\,Re(Z)$$

ملاحظة * $Re(Z) \leqslant |Re(Z)| \leqslant |Z|$

حولنا الـ = الى المتباينة ≤ من أجل استخدام هذه الملاحظة

$$\rightarrow \leqslant |Z_1|^2 + 2|Z_1 \overline{Z}_2| + |Z_2|^2$$

$$\leqslant (|Z_1| + |Z_2|)^2$$

$$\rightarrow |Z_1 + Z_2|^2 \leqslant (|Z_1| + |Z_2|)^2 \leftarrow$$ أخذ جذر للطرفين

$$|Z_1 + Z_2| \leqslant |Z_1| + |Z_2| \quad \#$$

Subject Date No.

$\boxed{4}$ $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

proof :-

يعمل أخذ $-z_2$ مع z_1 وكل

$$|z_1| = |z_1 + z_2 - z_2|$$ باضافة وطرح z_2

$$= |z_1 + z_2 + (-z_2)|$$

من الاثبات السابق $\leq |z_1 + z_2| + |(-z_2)|$ (سالب مع مطلق يختفي)

$$\leq |z_1 + z_2| + \cancel{|z_2|}$$

$$-|z_2| \qquad\qquad - \cancel{|z_2|}$$

$$\rightarrow |z_1| - |z_2| \leq |z_1 + z_2|$$

$\rightarrow$ let $z_2 = -z_2$ (نقلب إشارة z_2)

$$|z_1| - |-z_2| \leq |z_1 - z_2|$$

$$|z_1| - |z_2| \leq |z_1 - z_2| \quad \ldots (1)$$

$$|z_2| = |z_2 + z_1 - z_1|$$

$$= |z_2 + z_1| + |(-z_1)|$$

$$\leq |z_1 + z_2| + |z_1|$$

$$-|z_1| \qquad\qquad - |z_1|$$

مثل متغير محصور

مثل $|x| \leq 5$

$-5 \leq x \leq 5$

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

let $z_2 = -z_2$

$$\rightarrow |z_2| - |z_1| \leq |z_1 - z_2|$$

$$-(|z_1| - |z_2|) \leq |z_1 - z_2|$$

تكملة

Subject Date No.

تكملة $\rightarrow -(|Z_1| - |Z_2|) \leq |Z_1 - Z_2|$

$$|Z_1| - |Z_2| \geq -|Z_1 - Z_2| \quad \dots (2)$$

by (1) & (2) $\rightarrow -|Z_1 - Z_2| \leq |Z_1| - |Z_2| \leq |Z_1 - Z_2|$

قاعدة القيمة المطلقة

$$\rightarrow \big||Z_1| - |Z_2|\big| \leq |Z_1 - Z_2|$$

Ex if $|Z| = 2$ then $\left|\dfrac{1}{Z^4 - 4Z^2 + 3}\right| \leq \dfrac{1}{3}$

$$Z^4 - 4Z^2 + 3 = (Z^2 - 3)(Z^2 - 1)$$

(1) $\rightarrow |Z^2 - 3| \geq \big||Z|^2 - |3|\big|$

$\geq \big||2|^2 - |3|\big|$

$\geq |4 - 3| = 1 \rightarrow |Z^2 - 3| \geq 1$

(2) $\rightarrow |Z^2 - 1| \geq \big||Z|^2 - |1|^2\big|$

$\geq \big||2|^2 - |1|^2\big|$

$\geq |4 - 1| = 3 \rightarrow |Z^2 - 1| \geq 3$

by (1) & (2) $\rightarrow |Z^2 - 3| \cdot |Z^2 - 1| \geq 1 \cdot 3 \rightarrow$ (قلب البسط والمقام)

$\rightarrow \dfrac{1}{|Z^2 - 3| \cdot |Z^2 - 1|} \leq \dfrac{1}{3} \rightarrow \left|\dfrac{1}{|Z^2 - 3| \cdot |Z^2 - 1|}\right| \leq \dfrac{1}{3}$

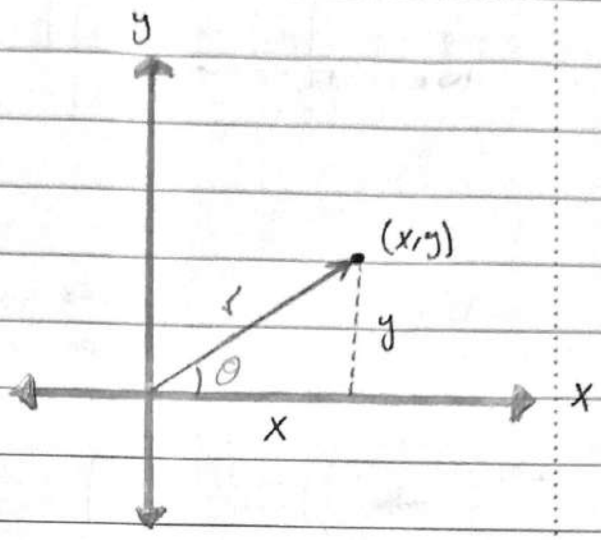
عند قلب البسط مع المقام تقلب اشارة المتباينة

NOTEBOOK

Subject

Date No.

Polar coordinate :-



Cartesian form :-

$Z = x + iy \rightarrow (x, y)$

التحويل من (x,y)
إلى (r,θ)

$\cos\theta = \frac{x}{r} \rightarrow \boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r} \rightarrow \boxed{y = r\sin\theta}$

$\boxed{r = \sqrt{x^2 + y^2}} \leftarrow$ قانون فيثاغورس

$\tan\theta = \frac{y}{x} \rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

Cartesian form $\longrightarrow$ polar form

$Z = x + iy$ $\qquad$ $Z = r\cos\theta + ir\sin\theta$

$\rightarrow Z = r(\cos\theta + i\sin\theta)$

Remark

* $\theta = \arg(Z)$ $\leftarrow$ arg small letter

$\hookrightarrow \theta + 2k\pi$

زاوية

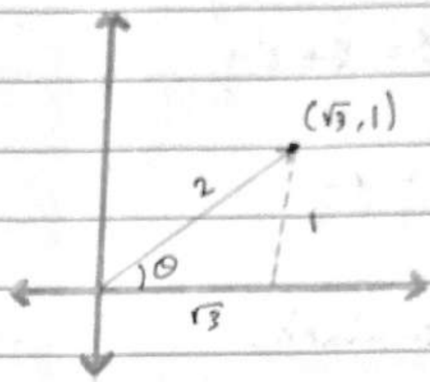
* $Arg(Z)$: argument in $(-\pi, \pi]$ $\leftarrow$ Arg capital letter

$\rightarrow \arg(Z) = Arg(Z) + 2k\pi$

عدد اللفات $k = 0, 1, 2, \ldots$

Ex write Z in the polar form :-

① $Z = \sqrt{3} + i$



$\rightarrow (x,y) = (\sqrt{3}, 1)$

$\rightarrow r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\rightarrow \sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}$

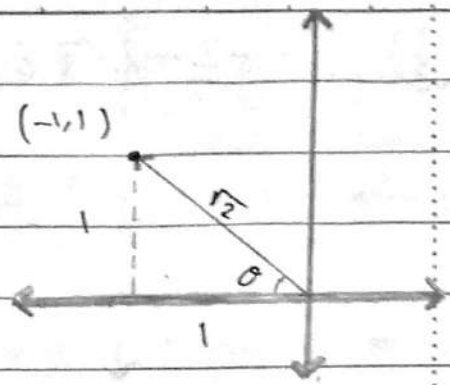
$\arg(Z) = \frac{\pi}{6} + 2K\pi$

$\therefore$ porlar form : $Z = r\left[\cos\theta + i\sin\theta\right]$

$\rightarrow Z = 2\left(\cos\left(\frac{\pi}{6} + 2K\pi\right) + i\sin\left(\frac{\pi}{6} + 2K\pi\right)\right)$

$K = 0, 1, 2, \ldots$

② $Z = -1 + i$



$\rightarrow (x, y) = (-1, 1)$

$\rightarrow r = \sqrt{x^2 + y^2} = \sqrt{2}$

$\rightarrow \sin\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4} \rightarrow$ يقع في الربع الثاني

وهذه الزاوية هي المتممة لـ 180

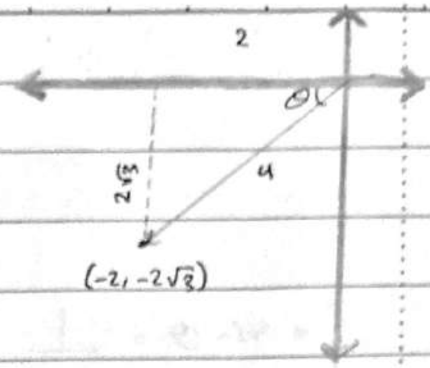
$\rightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\rightarrow \arg(Z) = \frac{3\pi}{4} + 2K\pi$

$\therefore$ polar form :- $Z = r(\cos\theta + i\sin\theta)$

$\rightarrow Z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2K\pi\right) + i\sin\left(\frac{3\pi}{4} + 2K\pi\right)\right)$

$K = 0, 1, 2, \ldots$



③ $Z = -2 - 2\sqrt{3}\, i$

$\rightarrow (x, y) = (-2, -2\sqrt{3})$

$\rightarrow r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

$\rightarrow \sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{3}$ (بالربع الثالث)

$\rightarrow \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$\rightarrow \arg(Z) = \frac{4\pi}{3} + 2K\pi$

$\therefore$ polar form : $Z = r(\cos\theta + i\sin\theta)$

$\rightarrow Z = 4\left(\cos\left(\frac{4\pi}{3} + 2K\pi\right) + i\sin\left(\frac{4\pi}{3} + 2K\pi\right)\right)$

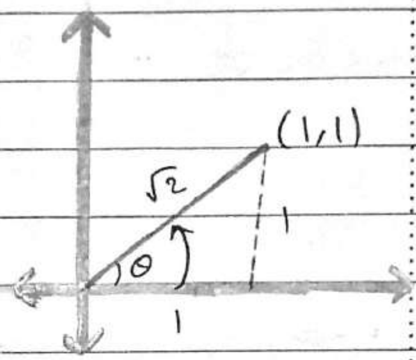
$K = 0, 1, 2, \ldots$

Subject Date No.

" الفرق بين arg و Arg "

Ex find $arg(Z)$, $Arg(Z)$

① $Z = 1+i$



→ $(x,y) = (1,1)$ ← بالربع الاول

→ $r = \sqrt{1^2+1^2} = \sqrt{2}$

→ $\sin\theta = \frac{1}{\sqrt{2}}$ → $\theta = \frac{\pi}{4}$

→ $arg(Z) = \frac{\pi}{4} + 2K\pi$, $K = 0, 1, 2, \ldots$

→ $Arg(Z) = \frac{\pi}{4}$ ⟶ بالربع الاول والثاني تبقى نفس القيمة

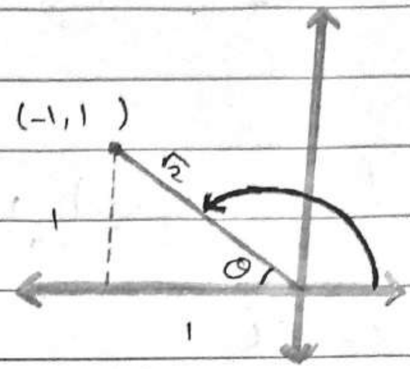
↓

محصورة بين

$(-\pi, \pi]$

② $Z = -1 + i$

→ $(x, y) = (-1, 1)$ ← بالربع الثاني



→ $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

→ $\sin \theta = \frac{1}{\sqrt{2}}$ → $\theta = \frac{\pi}{4}$ ← لكن بالربع الثاني

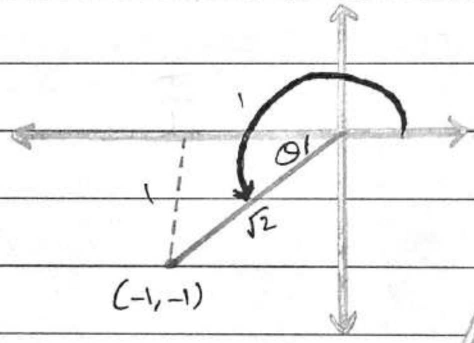
→ $\theta = \pi - \frac{\pi}{4}$ → $\theta = \frac{3\pi}{4}$

→ $\arg(Z) = \frac{3\pi}{4} + 2K\pi \quad , K = 0, 1, 2, \ldots$

→ $Arg(Z) = \frac{3\pi}{4}$

Arg ← نمشي عكسي في حالة الربع الثالث أو الرابع ونضرب الزاوية بسالب

③ $Z = -1 - i$



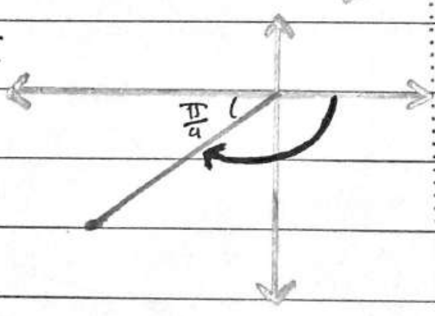
→ $(x, y) = (-1, -1)$ ← بالربع الثالث

→ $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

→ $\sin \theta = \frac{1}{\sqrt{2}}$ → $\theta = \frac{\pi}{4}$ ← لكن بالربع الثالث

$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

→ $\arg(Z) = \frac{5\pi}{4} + 2K\pi \quad , K = 0, 1, 2, \ldots$



→ $Arg(Z) = -\left(\pi - \frac{\pi}{4}\right)$

$= -\frac{3\pi}{4}$

④ $Z = 1 - i$

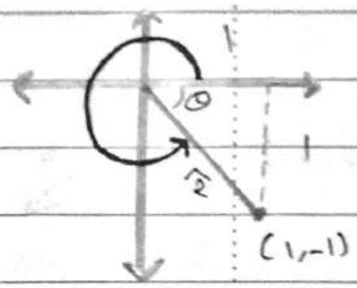
إذا كانت الزاوية
سالبة
يعني أننا مشينا
بشكل عكسي

$\rightarrow (x, y) = (1, -1)$

$\rightarrow r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

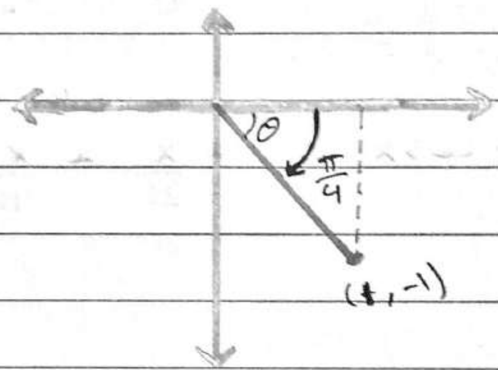
$\rightarrow \sin\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = 2\pi - \frac{\pi}{4} \rightarrow \theta = \frac{7\pi}{4}$

$\theta = \frac{\pi}{4}$



$\rightarrow \arg(Z) = \frac{7\pi}{4} + 2k\pi$

$\rightarrow \mathrm{Arg}(Z) = -\frac{\pi}{4}$



Subject Date No.

Euler's Formula :-

الصيغة العامة لقاعدة أويلر → $$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

proof :- $$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

احفظ

مرة موجب ومرة سالب

شكل $\sin x$ (الاعداد الفردية) ← $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

شكل $\cos x$ (الاعداد الزوجية) ← $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

مكان كل x نكتب $i\theta$ → $$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} + \frac{(-i\theta^3)}{3!} + \frac{\theta^4}{4!} + \dots$$

هنا نأخذ الحدود الفردية لحال و الزوجية التي بها i لحال

$$= \underbrace{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right]}_{\text{شكل } \cos} + i\underbrace{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right]}_{\text{شكل } \sin}$$

$$\rightarrow e^{i\theta} = \cos\theta + i\sin\theta$$

Subject Date No.

Ex find :-

① $e^{\pi i}$ $\rightarrow$ $= \cos\pi + i\sin\pi$

$= -1 + i \cdot 0$

$= -1$

② $e^{\frac{\pi}{3} i}$ $\rightarrow$ $= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$

③ $e^{-\frac{\pi}{2} i}$ $\rightarrow$ $= \cos(\frac{-\pi}{2}) + i\sin(\frac{-\pi}{2})$

$= \cos(\frac{\pi}{2}) + i \cdot -\sin(\frac{\pi}{2})$

$= 0 - i \cdot 1$

$= -i$

$\boxed{*}$ $\cos(-\theta) = \cos\theta$

$\boxed{*}$ $\sin(-\theta) = -\sin\theta$

* Note :-

The complex number Z has the following form :-

$\boxed{1}$ $Z = x + iy = (x, y)$ $\rightarrow$ cartesian form

$\boxed{2}$ $Z = r(\cos\theta + i\sin\theta)$ $\rightarrow$ polar form

$\boxed{3}$ $Z = re^{i\theta}$ $\rightarrow$ exponatial form

$re^{i\theta} = r(\cos\theta + i\sin\theta)$

ضرب الطرفين بـ r

N O T E B O O K

Ex let $Z = \frac{1}{1-i}$ write Z in :-

1) cartesian form.
2) polar form
3) exponantial form

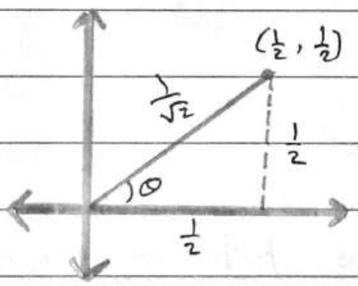
ضرب بالمرافق

$\rightarrow$ ① $Z = \frac{1}{1-i} \cdot \frac{(1+i)}{(1+i)} = \frac{1+i}{1^2+1^2} = \underbrace{\frac{1}{2}}_{x} + \underbrace{\frac{i}{2}}_{y}$

$\rightarrow Z = \frac{1}{2} + \frac{i}{2}$

$\rightarrow (x,y) = (\frac{1}{2}, \frac{1}{2})$

$\rightarrow$ ② $r = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$



$\rightarrow \tan \theta = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \rightarrow \theta = \frac{\pi}{4}$

$\rightarrow Z = r(\cos\theta + i\sin\theta) = \frac{1}{\sqrt{2}}\left(\cos(\frac{\pi}{4} + 2K\pi) + i\sin(\frac{\pi}{4} + 2K\pi)\right)$

$\rightarrow$ ③ $Z = re^{i\theta} = \frac{1}{\sqrt{2}} e^{i(\frac{\pi}{4} + 2K\pi)}$

تطبيق مباشر من معلومات الفرع السابق

*** Remark :-**

$$|Z| = |re^{i\theta}| = |r|$$

Job ↗

proof :-

$$Z = re^{i\theta}$$

$$= \underbrace{r\cos\theta}_{x} + \underbrace{ir\sin\theta}_{y}$$

$$\rightarrow |Z| = \sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{r^2}$$

$$= |r|$$

$\cos^2\theta + \sin^2\theta = 1$

Ex find $|Z|$:-

① $Z = \underbrace{2}_{r} e^{i\frac{\pi}{5}} \longrightarrow |Z| = |2| = 2$

② $Z = e^{i(\frac{\pi}{3}+2k\pi)} \longrightarrow |Z| = |1| = 1$

③ $Z = -2e^{i(\frac{\pi}{3})} \longrightarrow |Z| = |-2| = 2$

Ex prove that :-

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

proof :-

$$e^{i\theta_1} \cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)$$

$(i \cdot i = -1)$

$$= \cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

متطابقات →

$$= [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] + i[\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2]$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= e^{i(\theta_1+\theta_2)}$$

*Remark :- متطابقات مهمة

→ $\cos(a+b) = \cos a\cos b - \sin a\sin b$

→ $\sin(a+b) = \sin a\cos b + \cos a\sin b$

Ex show that $\arg(Z_1 \cdot Z_2) = \arg(Z_1) + \arg(Z_2)$

proof :-

$$Z_1 = r_1 e^{i\theta_1} \qquad \arg(Z_1) = \theta_1$$

$$Z_2 = r_2 e^{i\theta_2} \qquad \arg(Z_2) = \theta_2$$

$$\rightarrow Z_1 \cdot Z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$

$$= r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$$ $\leftarrow$ من الاثبات السابق

$$\rightarrow \arg(Z_1 \cdot Z_2) = \theta_1 + \theta_2$$

$$= \arg(Z_1) + \arg(Z_2)$$ ⋇

(**H.W**) show that $\arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$

proof

$$\frac{Z_1}{Z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$$

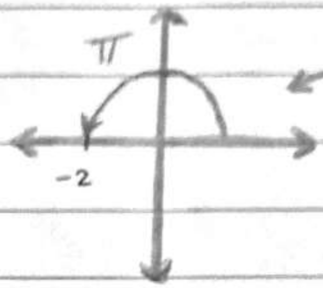
$$\rightarrow \arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$$ ⋇

Ex show that $Arg(Z_1 \cdot Z_2) \neq Arg(Z_1) + Arg(Z_2)$

→ proof   for example

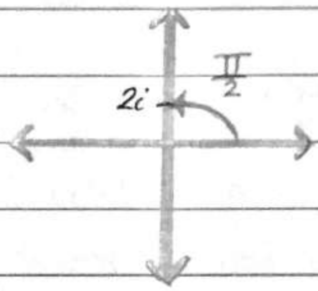
$Z_1 = -2$

$Z_2 = 2i$

→ 

$Z_1 = -2$

$Arg(Z_1) = \pi$

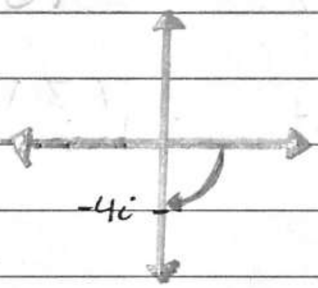
→ 

$Z_2 = 2i$

$Arg(Z_2) = \frac{\pi}{2}$

$\therefore Z_1 \cdot Z_2 = -2 \cdot 2i$

$= \underbrace{-4i}_{\text{Im part}}$



$Arg(Z_1 \cdot Z_2) = \frac{-\pi}{2}$

→ $Arg(Z_1 \cdot Z_2) \neq Arg(Z_1) + Arg(Z_2)$

$\frac{-\pi}{2} \neq \pi + \frac{\pi}{2}$ → $\frac{-\pi}{2} \neq \frac{3\pi}{2}$

Subject Date No.

De Moive's formula :-

$$(\cos\theta + i\sin\theta)^n = \cos \underline{n\theta} + i \sin \underline{n\theta}$$

proof :-

$$(e^{i\theta})^n = e^{n\theta i} = \cos n\theta + i \sin n\theta$$

Euler's formula

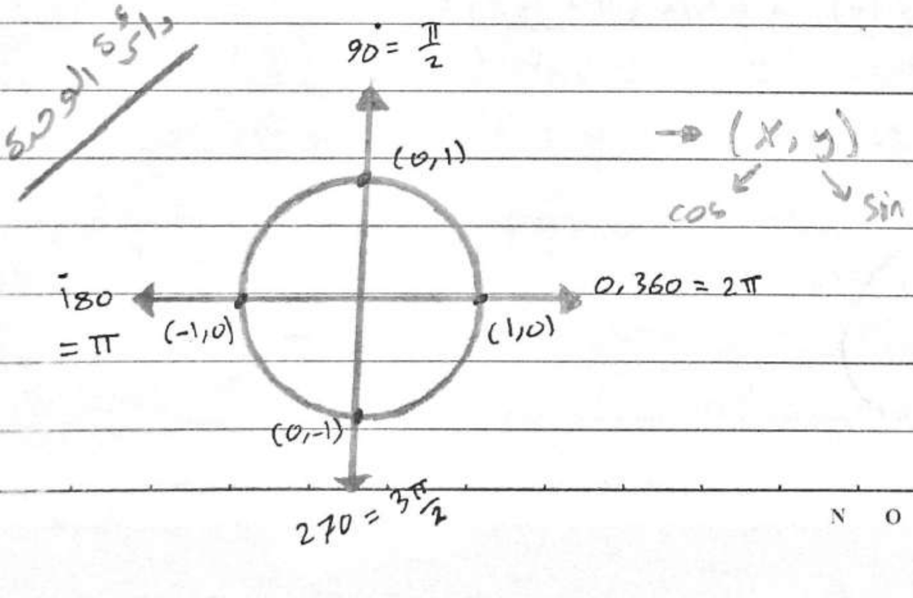
مرفوع الكل للأس n

$$\rightarrow (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

Ex find :-

① $(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})^3$

$$\rightarrow (\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = 0 + i(-1) = -i$$

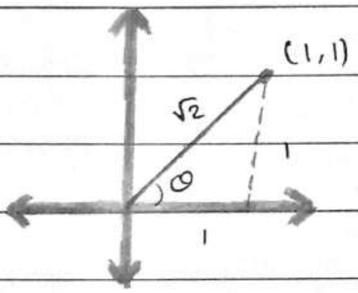


Subject Date No.

② $(1+i)^4$ $\longrightarrow$ عندما يعطينا سؤال مثل هذا نحولها إلى صيغة cos, sin أي polar coordinate

$\rightarrow (x,y) = (1,1)$

$\rightarrow r = \sqrt{x^2+y^2} = \sqrt{2}$



$\rightarrow \sin\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4}$

$\rightarrow Z = 1+i$

$= r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right)$

$(Z)^4 = \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right)\right)^4$

$= 4\left(\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right)^4$ ← De Moivre's formula

$= 4\left(\cos(\pi + \underbrace{8k\pi}_{\text{عدد اللفات}}) + i\sin(\pi + \underbrace{8k\pi}_{\text{عدد اللفات}})\right)$

$= 4\cdot(-1 + i(0))$

$= -4$

Ex Use De Moiver's formula to show that :-

(i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(ii) $\sin 2\theta = 2\sin\theta\cos\theta$

proof :-

$(\cos\theta + i\sin\theta)^2 \rightarrow$ حل بطريقتين

$\rightarrow$ De Moiver's $\rightarrow \cos 2\theta + i\sin 2\theta$ --- ①

$\rightarrow (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$ --- ②

خاصية الضرب

① = ② $\rightarrow$ تساوي المعادلتين

$\cos 2\theta + i\sin 2\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$

$\therefore Re(z) = Re(z)$

$\rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta$ ✓

$\therefore Im(z) = Im(z)$

$\rightarrow \sin 2\theta = 2\sin\theta\cos\theta$ ✓

Ex Use De moivre's formula to show that.

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \rightarrow$ H.W

(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

proof

(i) $\rightarrow \sin 3\theta = 3\sin\theta - 4\sin^3\theta \leftarrow$ H.W

$$\sin 3\theta = \sin(2\theta + \theta)$$
$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \leftarrow$$ كل المتطابقة
$$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + (1-\sin^2\theta - \sin^2\theta)\sin\theta$$
$$= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

(ii) $\rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos 3\theta = \cos(2\theta + \theta)$$
$$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \leftarrow$$ كل المتطابقة
$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cdot\cos\theta)\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

Root of complex numbers :-

$Z^n - Z_0 = 0$ or $Z^n = Z_0$ or $Z = (Z_0)^{\frac{1}{n}}$

← هذه ٣ صيغ نفس المعنى.

حفظ

$$\boxed{Z_k = r_0^{\frac{1}{n}} \left[\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i \sin\left(\frac{\theta_0 + 2k\pi}{n}\right) \right]}$$

عدد الجذور → $k = 0, 1, 2, \ldots, n-1$

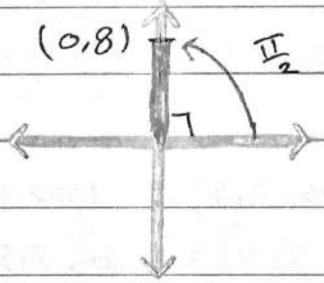
Ex solve $Z^3 - 8i = 0$ ⟵ لازم نوجد قيمة n و K و θ_0 و r_0

نكتب المسألة بدلالة Z → $Z = (8i)^{\frac{1}{3}}$ → $n = 3$

$k = 0, 1, 2$

→ $Z_0 = 8i$ → $(0, 8)$

→ $r_0 = \sqrt{0^2 + 8^2} = 8$



$\theta_0 = \frac{\pi}{2}$

$$\rightarrow Z_k = (8)^{\frac{1}{3}} \left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) \right]$$

تكملة ⟹

تكملة

1^{st} root $Z_0 = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

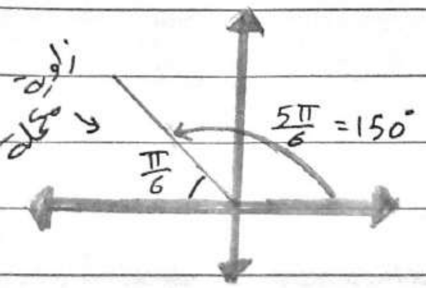
$k=0$

$$= 2\left[\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right] = \sqrt{3} + i$$

2^{nd} root $Z_1 = 2\left[\cos\left(\frac{\frac{\pi}{2}+2\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi}{3}\right)\right]$

$k=1$

$$= 2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$



$$= 2\left[-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right] = -\sqrt{3} + i$$

3^{ed} root $Z_2 = 2\left[\cos\left(\frac{\frac{\pi}{2}+4\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+4\pi}{3}\right)\right]$

$k=2$

$$= 2\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right)$$

$$= 2\left(0 + i(-1)\right)$$

$$= -2i$$

الربع الثالث $= 270^\circ$

$\frac{9\pi}{6} = \frac{3\pi}{2}$

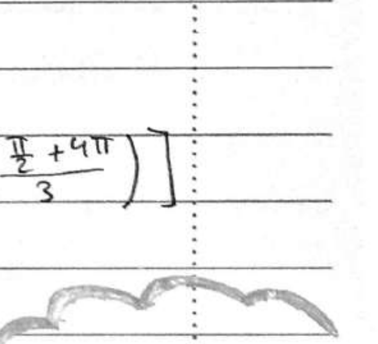
التأكد من الحل → $Z_0 + Z_1 + Z_2 = 0$

* عند جمع الحدود لازم يكون مجموعهم يساوي صفر.

$(\sqrt{3} + i) + (-\sqrt{3} + i) + (-2i) = 0$

تمثيل الـ root بيانيا. →

يجب أن يكون الطول r متساوي →



Subject Date No.

(**H.W**) find all value of $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{\frac{1}{4}}$

(i) Algebraically جبرياً (حل عادي)

(ii) Geometrically هندسياً (تمثيل بياني)

Subject Date No.

(H.W) find all values of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{\frac{3}{4}}$

Ex find all values of $z^4 - 1 = 0$

(H.W)

CH2 Analytic functions

Def: A function f denoted an a set of complex number S is a rule that assigns to each Z in S a complex number ω.

→ The number ω is called the value of f at Z and is denoted by $(\omega = f(Z))$

→ The set S is called the Domaine of f

$$* \; W = f(Z) = f(x+iy)$$
$$= u + iv$$

نفسها ← مثل الاقترانات

ص = ق(س)

→ $y = f(x)$

كان نفس المبدأ

$$= \underline{u(x,y)} + i\,\underline{V(x,y)}$$

→ $Re(\underline{f(z)}) = u(x,y)$

→ $Im(\underline{f(z)}) = V(x,y)$

Subject Date No.

Ex Decribe the Dom for each the following functions :-

① $f(z) = \dfrac{5z^2+3}{z}$

اصفار المقام

$\rightarrow Dom(f(z)) = \mathbb{C} - \{0\}$

المجال للاقتران مثل
مجال الاقترانات العادية
لكن هنا المجال بـ $\mathbb{C}$ وليس $\mathbb{R}$

② $f(z) = 4z^3 + 2z^2 + 5$ ← كثير حدود

$\rightarrow Dom(f(z)) = \mathbb{C}$

تحلل بالـcomplex
لكن لا تحلل بـ $\mathbb{R}$

③ $f(z) = \dfrac{3}{z^4-1}$

$$z^2+1 = z^2 - (\sqrt{-1})^2$$
$$= z^2 - i^2$$
$$= (z-i)(z+i)$$

$$\rightarrow z^4 - 1 = (z^2-1)(z^2+1) = 0$$
$$= (z-1)(z+1)(z^2+1) = 0$$
$$(z-1)(z+1)(z-i)(z+i)$$

$\rightarrow z = 1, -1, i, -i$

$\therefore Dom(f(z)) = \mathbb{C} - \{1, -1, i, -i\}$

* There are two kinds of complex functions :-

$\boxed{1}$ Single - value function .

Ex ① $f(Z) = Z^2+1$

$\hookrightarrow f(i) = i^2+1$

② $f(Z) = Arg(Z+3)$

$\boxed{2}$ Multiple-Value function

Ex $f(Z) = arg(Z)$

Subject Date No.

Ex write the function $f(Z)$ in the form
$f(Z) = u + iv$:-

قبل التعويض (x,y)
بعد التعويض u,v

① $f(Z) = Z^2$

$Z = x + iy \rightarrow f(x+iy) = (x+iy)^2$

$= (x+iy)(x+iy)$

$= \underbrace{(x^2 - y^2)}_{u} + i\underbrace{(2xy)}_{v}$

② $f(Z) = |Z|^2$

الطول

$Z = x + iy \rightarrow f(x+iy) = |(x+iy)|^2$

$= \left|\sqrt{x^2+y^2}\right|^2 = \underbrace{x^2 + y^2}_{u} + \underbrace{0}_{v}i$

③ $f(Z) = iZ$

$\rightarrow f(x+iy) = i(x+iy)$

$= \underbrace{-y}_{u} + i\underbrace{x}_{v}$

Subject Date No.

Remark :-

$Z = x + iy \longrightarrow Z = u(x,y) + i\,v(x,y)$

number صيغة   function صيغة

Ex write $f(Z) = Z^2$ in polar form.

polar بصيغة → $Z = re^{i\theta}$

$$f(Z) = f(re^{i\theta}) = (re^{i\theta})^2$$
$$= r^2 e^{2i\theta}$$
$$= r^2(\cos 2\theta + i\sin 2\theta)$$
$$= \underbrace{r^2\cos 2\theta}_{u} + i\,\underbrace{r^2\sin 2\theta}_{v}$$

* limits :-

Ex find :-

① $\lim_{Z \to 2} (Z^2 + 5Z + 3)$

مثل العادي → $= 4 + 10 + 3 = 17$
تعويض مباشر

② $\lim\limits_{Z \to 1+i} (3Z+2)$

$= 3(1+i)+2 = 3+3i+2 = 5+3i$

③ $\lim\limits_{Z \to 2+i} (4x+5y+3)$

عبارة عن (2, 1)

نعوضها مكان x

نعوضها مكان y

$= 4 \times 2 + 5 \times 1 + 3$

$= 8 + 5 + 3 = 16$

④ $\lim\limits_{Z \to i} \dfrac{Z^4 - 1}{Z - i}$ ناتجها بالتعويض المباشر $\to \frac{0}{0}$

بالcomplex حلل

$= \lim\limits_{Z \to i} \dfrac{(Z^2-1)(Z^2+1)}{(Z-i)} = \lim\limits_{Z \to i} \dfrac{(Z^2-1)(Z+i)\cancel{(Z-i)}}{\cancel{(Z-i)}}$

$= (i^2-1)(i+i)$

$= (-1-1)(2i) = -2 \cdot 2i = -4i$

⑤ $\lim\limits_{Z \to \infty} \dfrac{5Z^2 + 3Z + 2}{4Z - 6Z^2 + 1}$

اعلى اس في البسط / اعلى اس في المقام

$= \lim\limits_{Z \to \infty} \dfrac{5Z^2}{-6Z^2} = \dfrac{-5}{6}$

Subject Date No.

⑥ $$\lim_{Z\to\infty} \frac{4Z^2+3Z+1}{5Z^3-2}$$

$$= \lim_{Z\to\infty} \frac{4Z^2}{5Z^3} = \lim_{Z\to\infty} \frac{4}{5Z} = \frac{4}{\infty} = 0$$

$\frac{\text{عدد}}{\infty} = 0$

$\frac{\infty}{\text{عدد}} = \pm\infty$ حسب اشارة العدد

⑦ $$\lim_{Z\to 0} \frac{Z^2}{|Z|^2}$$

$$= \lim_{\substack{Z\to 0\\(0,0)}} \frac{Z^2}{x^2+y^2} = \frac{0}{0}$$

هنا نأخذ مرتين مرة $(x=0)$

ومرة $(y=0)$

مثل النهاية مرة يمين ومرة

يسار

$\Rightarrow$ (i) $(x=0)$

$$\rightarrow \lim_{(x,y)\to(0,0)} \frac{(-y^2+0)}{y^2} = -1$$

$\Rightarrow$ (ii) $(y=0)$

$$\rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$$

$\Rightarrow$ d.n.e

يجب أن يكون الطرفين

نفس الإجابة حتى

يكون للسؤال حل

عدا ذلك d.n.e

Subject Date No.

Continuity :- الاتصال

<u>Def</u> :- A function is continuous at a point z_0 if all three the following conditions are :-

① $\lim_{z \to z_0} f(z)$ exists

② $f(z_0)$ exists

③ $\lim_{z \to z_0} f(z) = f(z_0)$

نفس شروط النهاية بالوضع العادي :-

- النهاية موجودة
- الصورة موجودة
- النهاية = الصورة

<u>Ex</u> ① $f(z) = z^2 + 4z + 7 \rightarrow$ poly.

$\rightarrow$ continuous

② $f(z) = \dfrac{z^2 + 7z - 1}{z - 3}$

$\rightarrow$ cont $\mathbb{C} - \{3\}$ ← متصلة عدا اصفار المقام

③ $f(z) = 5yi + \frac{2}{x}$

$\rightarrow$ cont $\mathbb{C} - \forall x = 0$

Derivative :

التعريف العام للمشتقة ① → $f'(Z) = \lim\limits_{Z \to Z_0} \dfrac{f(Z) - f(Z_0)}{Z - Z_0}$ → نها ع←ن (د(ع) - د(ن)) / (ع - ن)

التعريف العام للمشتقة ② → $f'(Z) = \lim\limits_{h \to 0} \dfrac{f(Z+h) - f(Z)}{h}$ → نها هـ←٠ (د(ن+هـ) - د(ن)) / هـ

Ex if $f(Z) = Z^2$ find $f'(Z)$ by defined of derivative → باستخدام التعريف العام للمشتقة

$$f'(Z) = \lim_{Z \to Z_0} \frac{f(z) - f(Z_0)}{Z - Z_0}$$

$$= \lim_{Z \to Z_0} \frac{Z^2 - Z_0^2}{Z - Z_0} = \lim_{Z \to Z_0} \frac{\cancel{(Z - Z_0)}(Z + Z_0)}{\cancel{(Z - Z_0)}}$$

$$= Z_0 + Z_0 = 2Z_0$$

Differentiation rules :

① $f(z) = c \longrightarrow f'(z) = 0 \quad , c \in \mathbb{C}$

ثابت

② $f(z) = z^n \longrightarrow f'(z) = n z^{n-1}$

③ $(f \pm g)'(z) = f'(z) \pm g'(z)$

④ $(f \cdot g)'(z) = f(z) \cdot g'(z) + g(z) \cdot f'(z)$

← الاول × مشتقة الثاني + الثاني × مشتقة الاول

⑤ $\left(\frac{f}{g}\right)'(z) = \frac{g(z) \cdot f'(z) - f(z) \times g'(x)}{(g(z))^2}$

$\leftarrow \dfrac{\text{المقام × مشتقة البسط − البسط × مشتقة المقام}}{\text{(المقام)}^2}$

Subject Date No.

Ex Find $f'(z)$:-

① $f(z) = z^2 + 5z + 3$

$\hookrightarrow f'(z) = 2z + 5$

② $f(z) = (2z+3)^4$

$\hookrightarrow f'(z) = 4(2z+3)^3 \cdot 2$

③ $f(z) = \dfrac{z^2+3}{2z-1}$

$\hookrightarrow f'(z) = \dfrac{(2z-1)(2z) - (z^2+3)\cdot 2}{(2z-1)^2}$

Cauch - Riemann equations :-

Thm suppose that $f(z) = u(x,y) + iv(x,y)$
if $f'(z_0)$ exists then

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= v_x \end{aligned}}$$

أي
معادلة تحقق شرط
Cauch-Riemann
له
مشتقة على الأكيد

Thm if u_x, v_x, u_y and v_y are continuous and

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

then $f'(z)$ exists and:-

$$\boxed{f'(z) = u_x + iv_x}$$

Ex Show that $f(Z) = Z^2$ then $f'(Z) = 2Z$ (by cauch-Riemann)

$$f(Z) = Z^2$$
$$= (x+iy)^2$$
$$= \underbrace{(x^2-y^2)}_{u} + \underbrace{2xyi}_{v}$$

نشتقها مرتين
اشتقاق جزئي $u = x^2 - y^2$

مرة بالنسبة لـ x → $u_x = 2x$

مرة بالنسبة لـ y → $u_y = -2y$

نشتقها مرتين
اشتقاق جزئي $V = 2xy$

مرة بالنسبة لـ x → $V_x = 2y$

مرة بالنسبة لـ y → $V_y = 2x$

$$\rightarrow u_x = 2x = V_y$$
$$u_y = 2y = -V_x$$

$\therefore f'(Z)$ exiest

$$\therefore f'(Z) = u_x + iV_x$$
$$= 2x + i2y$$
$$= 2(x+iy)$$
$$= 2Z$$

Subject Date No.

Ex show that if $f(z) = Re(z)$
then $f'(z)$ d.n.e

$f(z) = Re(z)$
$= Re(x+iy)$
$= \underbrace{x}_{u} + \underbrace{0i}_{v}$

$u = x$
$\to u_x = 1$
$\to u_y = 0$

$v = 0$
$\to v_x = 0$
$\to v_y = 0$

$\to u_x \neq v_y$ $\therefore f'(z)$ d.n.e
$1 \neq 0$

Subject Date No.

Ex show that if $f(z) = e^z$, $f'(z) = e^z$

$$f(z) = e^z$$
$$= e^{x+iy} = e^x \cdot e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$= \underbrace{e^x \cos y}_{u} + \underbrace{e^x \sin y}_{v}\, i$$

$u = e^x \cos y$

$\to u_x = e^x \cos y$

$\to u_y = -e^x \sin y$

$v = e^x \sin y$

$\to v_x = e^x \sin y$

$\to v_y = e^x \cos y$

$\therefore u_y = -e^x \sin y = -v_x$

$u_x = e^x \cos y = v_y$

$\therefore f'(z)$ exesist $\to$ $f'(z) = u_x + i v_x$

$$= e^x \cos y + i e^x \sin y$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x \cdot e^{iy}$$
$$= e^{x+iy} = e^z \quad \#$$

Ex $f(z) = |z|^2$

$f(z) = |x+iy|^2 = \underbrace{x^2+y^2}_{u} + \underbrace{0i}_{v}$

$u = x^2+y^2$

$\rightarrow u_x = 2x$

$\rightarrow u_y = 2y$

$v = 0$

$\rightarrow v_x = 0$

$\rightarrow v_y = 0$

$\rightarrow u_x = 2x = v_y$ iff $x=0$ ← لها اشتقاق

$\rightarrow u_y = 2y = v_x$ iff $y=0$ فقط عند النقطة 0

عدا ذلك لا يوجد

لها اشتقاق

$\therefore f'(0)$ exesist

Subject Date No.

Cauch - Riemann equation in Polar coordinate :-

Thm let $f(Z) = u_{(r,\theta)} + i\, v_{(r,\theta)}$, then if :-

1. u_r , u_θ , v_r and v_θ are continuous

2. $$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r\, v_r \end{aligned}}$$

* يجب أن تتحقق الشرطين من أجل أن تكون $f(z)$ exist

$\rightarrow$ then $f'(Z)$ exists and $f'(Z) = e^{-i\theta}(u_r + i v_r)$

Subject Date No.

Ex $f(Z) = \frac{1}{Z}$ find $f'(Z)$ by C-R in polar coordinate.

بالسؤال حدد طريقة الحل

$$\Rightarrow f(Z) = \frac{1}{Z} = \frac{1}{r \cdot e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

ملاحظة

$$= \frac{1}{r}\left(\cos(-\theta) + i \sin(-\theta)\right)$$

$*\cos(-\theta) = \cos\theta$

$*\sin(-\theta) = -\sin\theta$

$$= \frac{1}{r}\left(\cos\theta - i \sin\theta\right) = \underbrace{\frac{1}{r}\cos\theta}_{u} - i \underbrace{\frac{1}{r}\sin\theta}_{v}$$

نشتق مرتين اشتقاق جزئي $\Rightarrow$

$$u = \frac{1}{r}\cos\theta \qquad v = \frac{-1}{r}\sin\theta$$

مرة بالنسبة لـ r $\quad u_r = \frac{-\cos\theta}{r^2} \qquad v_r = \frac{1}{r^2}\sin\theta$

مرة بالنسبة لـ θ $\quad u_\theta = \frac{-\sin\theta}{r} \qquad v_\theta = \frac{-\cos\theta}{r}$

حسب القاعدة

$$\rightarrow u_r = \frac{-\cos\theta}{r^2} \overset{?}{=} \frac{1}{r} v_\theta = \frac{1}{r} \cdot \frac{-1}{r}\cos\theta$$

$$= \frac{-\cos\theta}{r^2} \quad \checkmark$$

حسب القاعدة

$$\rightarrow u_\theta = \frac{-\sin\theta}{r} \overset{?}{=} -r\, v_r = -r \cdot \frac{\sin\theta}{r^2}$$

$$= -\frac{\sin\theta}{r} \quad \checkmark$$

$\therefore f'(Z)$ exist

تكملة

تكملة الحل

$$\therefore f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$\Rightarrow f'(z) = e^{-i\theta}\left(\frac{-1}{r^2}\cos\theta + i\,\frac{1}{r^2}\sin\theta\right)$$

$$= e^{-i\theta}\cdot\frac{-1}{r^2}\left(\cos\theta - i\sin\theta\right)$$ نرجع رجوع بالخطوات

$$= e^{-i\theta}\cdot\frac{-1}{r^2}\left(\cos(-\theta) + i\sin(-\theta)\right)$$

$$= e^{-i\theta}\cdot\frac{-1}{r^2}\cdot e^{-i\theta} = \frac{-1}{r^2\, e^{2i\theta}}$$

$$= \frac{-1}{(r\cdot e^{i\theta})^2} = \frac{-1}{z^2}$$ ※

Ex $f(z) = \sqrt{z}$ find $f'(z)$ by C-R in polar coordinate

$$\rightarrow f(z) = \sqrt{z} = \sqrt{r e^{i\theta}} = (r e^{i\theta})^{\frac{1}{2}}$$

$$= r^{\frac{1}{2}}\cdot e^{\frac{1}{2}i\theta} = r^{\frac{1}{2}}\left(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta\right)$$

$$= \underbrace{r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta}_{u} + i\,\underbrace{r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta}_{v}.$$

تكملة →

$$U = r^{\frac{1}{2}} \cos\tfrac{1}{2}\theta$$
$$\to U_r = \tfrac{1}{2} r^{-\frac{1}{2}} \cos\tfrac{1}{2}\theta$$
$$\to U_\theta = \tfrac{-1}{2} r^{\frac{1}{2}} \sin\tfrac{1}{2}\theta$$

$$V = r^{\frac{1}{2}} \sin\tfrac{1}{2}\theta$$
$$\to V_r = \tfrac{1}{2} r^{-\frac{1}{2}} \sin\tfrac{1}{2}\theta$$
$$\to V_\theta = \tfrac{1}{2} r^{\frac{1}{2}} \cos\tfrac{1}{2}\theta$$

$U_r = \frac{1}{2} r^{-\frac{1}{2}} \cos\frac{1}{2}\theta \overset{?}{=} \frac{1}{r} V_\theta$ (حسب القاعدة)

$$= \frac{1}{r} \cdot \frac{1}{2} r^{\frac{1}{2}} \cos\tfrac{1}{2}\theta$$

$$= \frac{1}{2} r^{-\frac{1}{2}} \cos\tfrac{1}{2}\theta \checkmark$$

$U_\theta = \frac{-1}{2} r^{\frac{1}{2}} \sin\frac{1}{2}\theta \overset{?}{=} -r \cdot V_r$ (حسب القاعدة)

$$= -r \cdot \frac{1}{2} \cdot r^{-\frac{1}{2}} \sin\tfrac{1}{2}\theta$$

$$= \frac{-1}{2} r^{\frac{1}{2}} \sin\tfrac{1}{2}\theta \checkmark$$

$\therefore f'(Z)$ exist

$$\therefore f'(Z) = e^{-i\theta}(U_r + iV_r)$$
$$= e^{-i\theta}\left(\tfrac{1}{2} r^{-\frac{1}{2}} \cos\tfrac{1}{2}\theta + i\,\tfrac{1}{2} r^{-\frac{1}{2}} \sin\tfrac{1}{2}\theta\right)$$
$$= e^{-i\theta} \cdot \tfrac{1}{2} r^{-\frac{1}{2}} \left(\cos\tfrac{1}{2}\theta + i \sin\tfrac{1}{2}\theta\right)$$
$$= e^{-i\theta} \cdot \tfrac{1}{2} r^{-\frac{1}{2}} \cdot e^{\frac{1}{2}i\theta}$$
$$= \tfrac{1}{2} r^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}i\theta} = \frac{1}{2(r^{\frac{1}{2}} \cdot e^{\frac{1}{2}i\theta})} = \frac{1}{2(r \cdot e^{i\theta})^{\frac{1}{2}}}$$
$$= \frac{1}{2\sqrt{Z}}$$

Analytic function :-

- A function f is analytic at Z_0 iff $f'(Z)$ exists and $f'(Z)$ exists $\forall Z \in nbd(Z_0)$
  ↳ neighborhood

* لازم يكون $f(Z)$ ← متصل
← وقابل للاشتقاق (مشتقة يمين = مشتقة يسار)
↳ n.b.d

if $f'(Z_0)$ d.n.e but $f'(Z)$ at some point in every nbd exists then Z_0 is called a <u>singular point</u>.
← معناها النقاط التي تكون المشتقة عندها غير موجودة.

- A function f is <u>entire</u> iff $f(Z)$ is analytic $\forall Z \in \mathbb{C}$
← عندما تكون جميع النقاط
أي لا يوجد singular point
مثل كثير الحدود.

Ex ① $f(z) = \frac{1}{z-1}$

→ singular point : $z-1=0$ → $\boxed{z=1}$

$\therefore f(z)$ - Analytic : $\forall z \in \mathbb{C} - \{1\}$

② $f(z) = z^2 + 3z + 1$

→ $\nexists$ singular point
→ entire
→ Analytic $\forall z \in \mathbb{C}$

جميعها نفس المعنى

③ $f(z) = \frac{1}{z^4 - 1}$

→ singular point $z^4 - 1 = 0$

$(z^2-1)(z^2+1) = 0$

$(z-1)(z+1)(z-i)(z+i) = 0$

→ $z = 1, -1, i, -i$

$\therefore f(z)$ analytic $\forall z \in \mathbb{C} - \{1, -1, i, -i\}$

Subject 83 Date No.

Harmonic functions

A real-valued function $H(x,y)$ is said to be harmonic iff it has continuous partial derivative of first and second order and

$$H_{xx} + H_{yy} = 0 \quad ((\text{laplace equations}))$$

المشتقة الثانية بالنسبة لـ x ← H_{xx}

المشتقة الثانية بالنسبة لـ y ← H_{yy}

<u>Ex</u> show that $H(x,y) = e^{-y} \sin x$ is harmonic.

→ $H(x,y) = e^{-y} \sin x$

المشتقة الاولى بالنسبة لـ x ← $H_x = e^{-y} \cos x$	المشتقة الاولى بالنسبة لـ y ← $H_y = -e^{-y} \sin x$
المشتقة الثانية بالنسبة لـ x ← $H_{xx} = -e^{-y} \sin x$	المشتقة الثانية بالنسبة لـ y ← $H_{yy} = e^{-y} \sin x$

$\Rightarrow H_{xx} + H_{yy} \overset{?}{=} 0$

$-e^{-y} \sin x + e^{-y} \sin x = 0$ ✓

$\therefore H(x,y)$ is harmonic

Subject 84 Date No.

<u>Thm</u> if $f(Z) = u_{(x,y)} + i\,V_{(x,y)}$ is analytic then u and V are harmonic.

ملاحظة :- بالكومبلكس إذا تحقق شرط C-R فهي ← analytic

$$\boxed{\begin{array}{l} u_x = V_y \\ u_y = -V_x \end{array}}$$

← النظرية :- إذا كان الاقتران $f(Z)$ قابل للاشتقاق فإن u و V ← harmonic

---

<u>Ex</u> let $f(Z) = \sin x \cosh y + i \cos x \sinh y$ show that :- u and V are harmonic.

معنا طريقتين للحل أما C-R أو harmonic

الطريقة الاولى

$u = \sin x \cosh y$

→ $u_x = \cos x \cosh y$

→ $u_y = \sin x \sinh y$

$V = \cos x \sinh y$

→ $V_x = -\sin x \sinh y$

→ $V_y = \cos x \cosh y$

→ $u_x = \cos x \cosh y = V_y$ ✓

$u_y = \sin x \sinh y = -V_x$ ✓

$\therefore$ by C-R $f(Z)$ is analytic.

$\therefore$ u and V harmonic function.

تكملة

الطريقة الثانية

$u = \sin x \cosh y$

$\to u_x = \cos x \cosh y$

$\to u_{xx} = -\sin x \cosh y$

$u = \sin x \cosh y$

$\to u_y = \sin x \sinh y$

$\to u_{yy} = \sin x \cosh y$

$\Rightarrow u_{xx} + u_{yy} \stackrel{?}{=} 0$

$-\sin x \cosh y + \sin x \cosh y = 0$ ✓

$V = \cos x \sinh y$

$\to V_x = -\sin x \sinh y$

$\to V_{xx} = -\cos x \sinh y$

$V = \cos x \sinh y$

$\to V_y = \cos x \cosh y$

$\to V_{yy} = \cos x \sinh y$

$\Rightarrow V_{xx} + V_{yy} \stackrel{?}{=} 0$

$-\cos x \sinh y + \cos x \sinh y = 0$ ✓

$\to u, V \Rightarrow$ harmonic functions.

Subject Date No.

* If $f(z) = u(x,y) + iV(x,y)$ is analytic then we say that V is a harmonic conjugate of u.

لما نعوض V + نعوض بال analytic

لتحقق شرط C-R

Ex let $u(x,y) = 2x - x^3 + 3xy^2$:-

① show that u is harmonic function.

② find a harmonic conjugate of u.

① → $u = 2x - x^3 + 3xy^2$

→ $u_x = 2 - 3x^2 + 3y^2$

→ $u_{xx} = -6x$

$u = 2x - x^3 + 3xy^2$

→ $u_y = 6xy$

→ $u_{yy} = 6x$

$u_{xx} + u_{yy} \overset{?}{=} 0 \rightarrow -6x + 6x = 0$ ✓

$\therefore u$ is harmonic function.

تكملة →

② $\rightarrow$ $U_x = V_y = 2 - 3x^2 + 3y^2$ (حسب القاعدة)

← لكن هنا نحن نريد V ولا نريد Vy لذلك نكامل بالنسبة لـ y.

$$\rightarrow \int V_y \, dy = \int (2 - 3x^2 + 3y^2) \, dy$$

$$V = 2y - 3x^2 y + y^3 + \phi(x)$$

ثابت بالنسبة لـ y

ثم نشتقها بالنسبة لـ x

$$V_x = -6xy + \phi'(x)$$

$\rightarrow -U_y = V_x = -6xy + \phi'(x)$ (حسب القاعدة)

$$\cancel{-6xy} = \cancel{-6xy} + \phi'(x)$$

$$\rightarrow \phi'(x) = 0 \longrightarrow \phi(x) = c$$

ما هو الشيء الذي مشتقه صفر = الثابت

$$\therefore V = 2y - 3x^2 y + y^3 + c$$

Subject Date No.

(H.W) $u(x,y) = 2x - 2xy$

① Show that u is harmonic function.
② find a harmonic conjugate of u.

① → $u = 2x - 2xy$
$\hookrightarrow u_x = 2 - 2y$
$\hookrightarrow u_{xx} = 0$

$u = 2x - 2xy$
$\hookrightarrow u_y = -2x$
$\hookrightarrow u_{yy} = 0$

$\Rightarrow u_{xx} + u_{yy} \overset{?}{=} 0$
$0 + 0 = 0$ ✓

② $u = 2x - 2xy$ حسب القاعدة
$\hookrightarrow u_x = 2 - 2y \quad = v_y$
$\hookrightarrow u_y = -2x \quad = -v_x$

$u_x = v_y \Rightarrow \int v_y = \int 2 - 2y \Rightarrow v = 2y - y^2 + \phi(x)$

$\Rightarrow v_x = \phi'(x) = -u_y$
$\phi'(x) = 2x \longrightarrow \phi(x) = x^2$

$\rightarrow v = 2y - y^2 + x^2$

(H.W) $u(x,y) = \sinh x \sin y$

① show that u is harmonic function.
② find a harmonic conjugate of u.

① $u = \sinh x \sin y$
$\rightarrow u_x = \cosh x \sin y$
$\rightarrow u_{xx} = \sinh x \sin y$

$u = \sinh x \sin y$
$\rightarrow u_y = \sinh x \cos y$
$\rightarrow u_{yy} = -\sinh x \sin y$

$\rightarrow u_{xx} + u_{yy} \overset{?}{=} 0 \Rightarrow \sinh x \sin y - \sinh x \sin y = 0$

② $u = \sinh x \sin y$ بالعكس
$\rightarrow u_x = \cosh x \sin y \qquad = v_y$
$\rightarrow u_y = \sinh x \cos y \qquad = -v_x$

$u_x = \int v_y = \int \cosh x \sin y$

$\rightarrow v = -\cosh x \cos y + \phi(x)$
$\rightarrow v_x = -\sinh x \cos y + \phi'(x) \qquad = -u_y$

$\cancel{-\sinh x \cos y} + \phi'(x) = \cancel{-\sinh x \cos y}$

$\rightarrow \phi'(x) = 0 \quad \rightarrow \phi(x) = c$

$\therefore v = -\cosh x \cos y + c.$

Subject　　　　Date　　　　No.

(H.W) $u(x,y) = \frac{y}{x^2+y^2}$

① Show that u is harmonic function.
② Find a harmonic conjugate of u.

① $u = \frac{y}{x^2+y^2}$

$\rightarrow u_x = \frac{-y \cdot 2x}{(x^2+y^2)^2}$ $\Rightarrow u_{xx} = \frac{(x^2+y^2)^2 \cdot (-2y) + 2yx \cdot 4x(x^2+y^2)}{(x^2+y^2)^4}$

اجابة نهائية ل u_{xx} $\Rightarrow u_{xx} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

$\rightarrow u_y =$

CH3 Elementary Function

⇒ The exponential function :- الاقتران الأسي/ الموضوعي

let $z = x + iy$, $x, y \in \mathbb{R}$

then $e^{z} = e^{x+iy}$
$= e^{x} \cdot e^{iy}$
$= e^{x}(\cos y + i \sin y)$

* Properties of e^{z} :-

$\boxed{1}$ e^{z} is entire function. → analytic دائماً

→ Analytic $\forall z \in \mathbb{C}$

$\boxed{2}$ $\frac{d}{dz}(e^{z}) = e^{z}$ → مشتقته = نفسه

Subject Date No.

$\boxed{3}$ $|e^z| = e^x$

proof :- $e^z = e^{x+iy}$

$= e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$= e^x\cos y + ie^x\sin y$

$\Rightarrow |e^z| = \sqrt{(e^x\cos y)^2 + (e^x\sin y)^2}$

$= \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}$

$= \sqrt{e^{2x}(\underbrace{\cos^2 y + \sin^2 y}_{=1})}$

$= \sqrt{e^{2x}} = e^x$ ← e^x دائماً موجبة

$\boxed{4}$ $\arg(e^z) = y + 2\pi n \quad , n \in \mathbb{N}$

$\boxed{5}$ $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$ → في حالة الضرب الاسس تجمع

$\boxed{6}$ $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$ → في حالة القسمة نطرح اس البسط من اس المقام

$\boxed{7}$ $\frac{1}{e^z} = e^{-z}$

$\boxed{8}$ $e^z \neq 0$ → for any complex number z

$\boxed{9}$ $(e^z)^n = e^{nz}$ ← اس للاس ضرب

NOTEBOOK

<u>Ex</u> solve the following equation :-

① $e^{z} = -3$ → السؤال يتكون من جزئيتين (i) (ii)

(i) (ii)

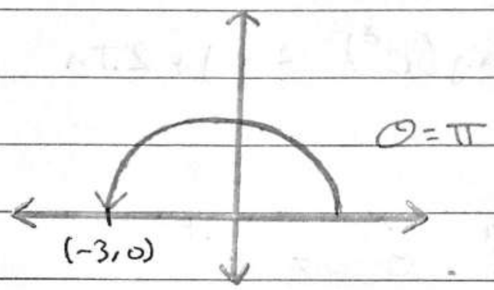
→ (i) $e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$

→ (ii) $-3 = r e^{i\theta}$ → بدنا نكتبها بال صيغة polar coordinate

يمثل

$\underbrace{-3}_{x} + \underbrace{0i}_{y} = r e^{i\theta}$

$r = \sqrt{(-3)^2 + (0)^2} = 3$



$\theta = \pi + 2k\pi$

$\therefore -3 = r e^{i\theta}$

$= 3 e^{(\pi + 2k\pi) i}$

$\therefore e^{z} = -3$

→ $e^{x} \cdot e^{iy} = 3 e^{(\pi + 2k\pi) i}$

نكمل

→ $e^{x} = 3 \xrightarrow[\text{للطرفين}]{\text{اخذ } \ln} x = \ln 3$

اخذنا ln للطرفين لأنها اعداد حقيقية

→ $e^{iy} = e^{i(\pi + 2k\pi)}$

→ $y = \pi + 2k\pi$

بالمقابلة لا يجوز اخذ ln للطرفين

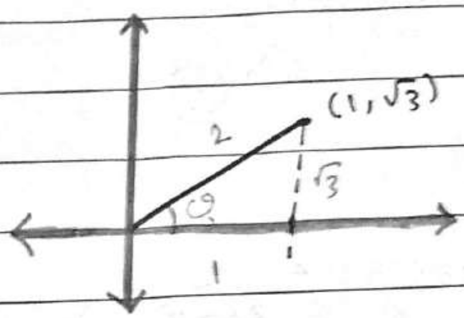
$\therefore z = x + iy$

$= \ln 3 + i(\pi + 2k\pi)$ ⋇

(2) $e^{z} = 1+\sqrt{3}\, i$

(i) (ii)

$e^{x} \cdot e^{iy} = re^{i\theta}$



(i) $e^{z} = e^{x} \cdot e^{iy}$

$r = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

(ii) $1+\sqrt{3}\, i = re^{i\theta}$

$\hookrightarrow (1,\sqrt{3})$

$\sin\theta = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{3} + 2K\pi$

$= 2e^{(\frac{\pi}{3}+2K\pi)i}$

$\Rightarrow e^{z} = 1+\sqrt{3}\, i$

$e^{x} \cdot e^{iy} = 2e^{(\frac{\pi}{3}+2K\pi)i}$

$\rightarrow e^{x} = 2$

$\ln e^{x} = \ln 2$

$x = \ln 2$

$\rightarrow e^{iy} = e^{(\frac{\pi}{3}+2K\pi)i}$

$y = (\frac{\pi}{3} + 2K\pi)$

مساواة الأس بالأس

$\therefore Z = x + iy$

$= \ln 2 + (\frac{\pi}{3} + 2K\pi)i$

اخذنا ln للطرفين لأنهما اعداد حقيقية $\mathbb{R}$

لايجوز أخذ ln هنا للطرفين لأنه complex

Subject Date No.

③ $e^{2z-1} = 1$

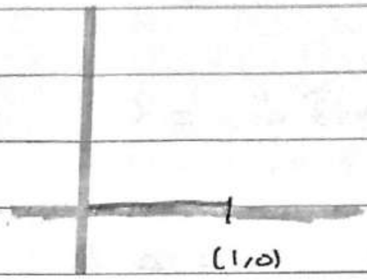
(i) (ii)

(i) $e^{2z-1} = e^{2(x+iy)-1}$

$= e^{2x+2iy-1} = e^{2x-1} \cdot e^{2iy}$

(ii) $1 = re^{i\theta}$

$1 + 0i = re^{i\theta}$



$\Rightarrow 1 = e^{i(2k\pi)}$

$\theta = 0 + 2k\pi$

$r = \sqrt{1^2 + 0^2} = 1$

$\Rightarrow e^{2z-1} = 1$

$e^{2x-1} \cdot e^{2iy} = e^{i(2k\pi)}$

$e^{2x-1} = 1$

$\ln e^{2x-1} = \ln 1 = 0$

$2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$

$e^{2iy} = e^{i(2k\pi)}$

$2y = 2k\pi$

$y = k\pi$

$\therefore z = x + iy$

$= \frac{1}{2} + (k\pi)i$

Subject Date No.

⇒ Triganometric functions :- الاقترانات المثلثية

* $e^{i\theta} = \cos\theta + i\sin\theta$

⇒ $e^{iz} = \cos z + i\sin z$ ---①

* $e^{-iz} = \cos(-z) + i\sin(-z)$

$= \cos z - i\sin z$ ---②

Remark :-

* $\cos(-z) = \cos z$

* $\sin(-z) = -\sin z$

① ... $e^{iz} = \cos z + i\sin z$

② ... $e^{-iz} = \cos z - i\sin z$

→ ① + ② ⇒ $e^{iz} + e^{-iz} = 2\cos z$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}}$$

① ... $e^{iz} = \cos z + i\sin z$

② ... $e^{-iz} = \cos z - i\sin z$ ← ضرب المعادلة ② بـ سالب

→ ① + ② ⇒ $e^{iz} - e^{-iz} = 2i\sin z$

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

*Properties :-

$\boxed{1}$ $\cos(-z) = \cos z$

$\sin(-z) = -\sin z$

$\boxed{2}$ $\frac{d}{dz}(\sin z) = \cos z$

proof :-
$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$

$$= \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{\cancel{i}(e^{iz}+e^{-iz})}{2\cancel{i}}$$

$$= \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

$\boxed{3}$ $\frac{d}{dz}(\cos z) = -\sin z$

نفس اثبات الفرع السابق

proof
$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right)$$

$$= \frac{ie^{iz}-ie^{-iz}}{2} = \frac{i(e^{iz}-e^{-iz})}{2} \cdot \frac{i}{i} = -\frac{(e^{iz}-e^{-iz})}{2i}$$

$$= -\sin z$$ *

$\boxed{4}$ $\sin^2 z + \cos^2 z = 1$

proof :-

$$\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \stackrel{?}{=} 1$$

$$\frac{(e^{iz} - e^{-iz})^2}{(2i)^2} + \frac{(e^{iz} + e^{-iz})^2}{(2)^2} \stackrel{?}{=} 1$$

$$\frac{e^{2iz} - 2e^{iz} \cdot e^{-iz} + e^{-2iz}}{-4} + \frac{e^{2iz} + 2e^{iz} \cdot e^{-iz} + e^{-2iz}}{4} \stackrel{?}{=} 1$$

$$\frac{\cancel{e^{2iz}} + 2 + \cancel{e^{-2iz}}}{4} - \frac{(\cancel{e^{2iz}} - 2 + \cancel{e^{-2iz}})}{4} \stackrel{?}{=} 1$$

$$\frac{4}{4} \stackrel{?}{=} 1 \rightarrow 1 = 1 \checkmark$$

$\boxed{5}$ $\sin 2z = 2 \sin z \cos z$

i ii

i $\sin 2z = \dfrac{e^{2iz} - e^{-2iz}}{2i}$

ii $2 \sin z \cos z = \cancel{2}\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right) \cdot \left(\dfrac{e^{iz} + e^{-iz}}{\cancel{2}}\right)$ مكمل

$$= \frac{(e^{iz} - e^{-iz})(e^{iz} + e^{iz})}{2i} = \frac{e^{2iz} - e^{-2iz}}{2i} = \sin 2z$$

فوق بنعوض بيهم

$\boxed{6}$ $\cos 2z = \cos^2 z - \sin^2 z$

$\boxed{7}$ $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

$\boxed{8}$ $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\boxed{9}$ $\sin iy = i \sinh y$

إذا كان هناك i بالزاوية

نخرجها إلى الخارج ونحول $\sin \leftarrow \sinh$

Remark:

$* \sinh y = \dfrac{e^y - e^{-y}}{2}$

$* \cosh y = \dfrac{e^y + e^{-y}}{2}$

proof :-

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i}$$

أخذ عامل مشترك (-1)

$$= -\frac{(e^y - e^{-y})}{2i} = \frac{i^2 (e^y - e^{-y})}{2i}$$

نحولها إلى i^2

أو نقل السالب بضرب بالمرافق $\dfrac{-2i}{-2i}$

$$= i \frac{(e^y - e^{-y})}{2} = i \sinh y$$

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$\boxed{10}$ $\cos(iy) = \cosh y$

proof (H.W)

$$\cosh(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2}$$

$$= \frac{e^{y} + e^{-y}}{2} = \cosh y$$

$\boxed{11}$ $\frac{d}{dz}\tan z = \sec^2 z$

$\frac{d}{dz}\cot z = -\csc^2 z$

$\frac{d}{dz}\sec z = \sec z \tan z$

$\frac{d}{dz}\csc z = -\csc z \cot z$

Subject Date No.

<u>Ex</u> solve :- $\sin Z = 13$ * قيمة sin محصورة بين ١ و -١
لكن بالكومبلكس هذا عادي

$\sin Z = 13 \implies \frac{e^{iZ} - e^{-iZ}}{2i} \times \frac{13}{1}$

ضرب تبادلي

$e^{iZ}(e^{iZ} - e^{-iZ} = 26i) \leftarrow e^{iZ}$ نضرب الطرفين بـ

$e^{2iZ} - 1 = 26i \cdot e^{iZ}$

$(e^{iZ})^2 - 26i\, e^{iZ} - 1 = 0$ "معادلة تربيعية"

let $y = e^{iZ} \implies y^2 - 26iy - 1 = 0 \leftarrow$ لا تحلل مباشرة
فنحلها بالقانون العام

$a=1 \quad b=-26i \quad c=-1$

$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \leftarrow$ القانون العام لكل المعادلات التربيعية

$= \frac{26i \pm \sqrt{(-26i)^2 - 4 * 1 * -1}}{2 * 1} = \frac{26i \pm \sqrt{-676 + 4}}{2}$

$= \frac{26i \pm \sqrt{-672}}{2} = \frac{26i \pm i\sqrt{672}}{2}$

but $e^{iZ} = y$

تكملة →

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$$e^{iz} = \frac{26i \pm i\sqrt{672}}{2}$$ أخذ ln للطرفين

$$\rightarrow \ln e^{iz} = \ln\left|\frac{26i \pm i\sqrt{672}}{2}\right|$$

$$\frac{1}{i} \cdot iz = \ln\left(\frac{26i \pm i\sqrt{672}}{2}\right) \cdot \frac{1}{i}$$

$$z = \frac{1}{i}\ln\left(\frac{26i \pm i\sqrt{672}}{2}\right)$$

⟹ Hyperbolic functions :-

$$\boxed{\sinh z = \frac{e^{z} - e^{-z}}{2}} \qquad \boxed{\cosh z = \frac{e^{z} + e^{-z}}{2}}$$

*Properties :-

[1] $\sinh(-z) = -\sinh z$

$\cosh(-z) = \cosh z$

[2] $\frac{d}{dz}\sinh z = \cosh z$

$\frac{d}{dz}\cosh z = \sinh z$

[3] $\cosh^2 z - \sinh^2 z = 1$

Ex solve :- $\cosh z = \frac{1}{2}$

$$\frac{e^{z} + e^{-z}}{2} \cancel{=} \frac{1}{2}$$

ضرب تبادلي

$$2(e^{z} + e^{-z}) = 2 \quad \rightarrow e^{z}(e^{z} + e^{-z} = 1)$$ ضرب الطرفين بـ e^{z}

$$e^{2z} + 1 = e^{z} \Rightarrow e^{2z} - e^{z} + 1 = 0$$

$$\text{let } y = e^{z} \Rightarrow y^2 - y + 1 = 0$$

(باستخدام القانون العام) $$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\text{but } y = e^{z} \Rightarrow e^{z} = \frac{1 \pm i\sqrt{3}}{2}$$ ← أخذ ln للطرفين

$$\Rightarrow \ln e^{z} = \ln\left|\frac{1 \pm i\sqrt{3}}{2}\right|$$

$$\therefore z = \ln\left|\frac{1 \pm i\sqrt{3}}{2}\right|$$ ※

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The logarithem functions :-

Def :- The complex logarithmic function $\log Z$ is defined by :-

$$\boxed{\begin{aligned} \operatorname{Log} Z &= \ln r + i\theta \\ &= \ln|Z| + i \arg\theta \end{aligned}}$$

Remark :-

$\operatorname{Log} Z$ is multiple-value function.

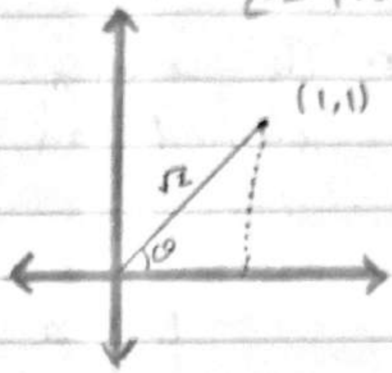
له عدة أجوبة

راجا بقواحدة
→ Singular - Value.

Ex Find the following :

لازم نطلع r , θ

① $\text{Log}(1+i)$



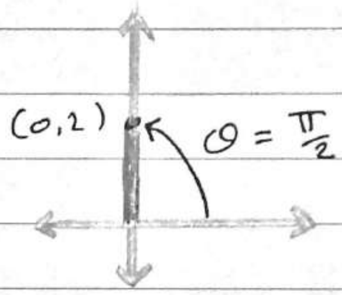
$Z = 1+i$

$\rightarrow r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$\rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} + 2k\pi$

$$\text{Log}(1+i) = \ln(r) + i\theta$$
$$= \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

② $\text{Log}(2i)$



$Z = 2i \rightarrow (0,2)$

$r = \sqrt{0^2+2^2} = \sqrt{2^2} = 2$

$\rightarrow \log(2i) = \ln(2) + i\left(\frac{\pi}{2} + 2k\pi\right)$

Subject Date: No.

The Principal value of Log Z is :-

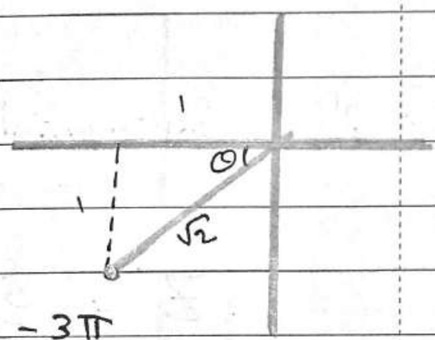
$$\rightarrow \text{Log } Z = \ln r + i \, \text{Arg}(\theta)$$

$$-\pi < \text{Arg}(\theta) < \pi$$

* اذا وجدت كلمة principal تكون Arg

<u>Ex</u> Find principal value of $\text{Log}(-1-i)$

$Z = -1 - i \rightarrow (-1, -1)$



$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\sin\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4} \rightarrow \theta = -\frac{3\pi}{4}$

بناخذ Arg

$$\therefore \text{Log}(-1-i) = \ln\sqrt{2} + i\left(-\frac{3\pi}{4}\right)$$

Properties of Log Z :-

$\boxed{1}$ $\frac{d}{dz} \text{Log } z = \frac{1}{z}$

$\boxed{2}$ $e^{\log z} = z$

$\boxed{3}$ $\text{Log } e^z = z + 2\pi k i$

Complex Exponantial :-

$$\boxed{z^c = e^{c(\ln|z| + i \arg z)}}$$

→ Principal branch of z^c :-

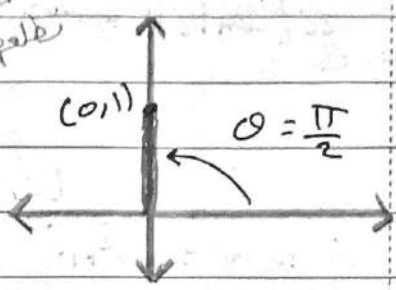
$$z^c = e^{c(\ln|z| + i \,\text{Arg}\, z)}$$

Ex Find the following :-

① $(i)^{2i}$

لازم طلعها

$$= e^{2i[\ln|i| + i \arg(i)]}$$



$Z = i \rightarrow (0,1)$

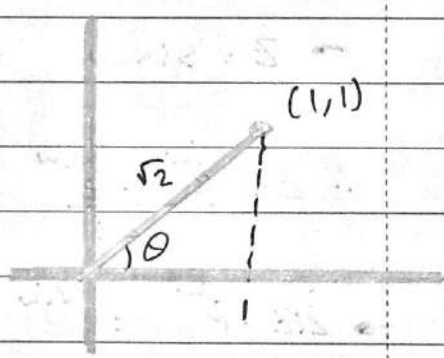
$r = \sqrt{0^2+1^2} = 1$

$$= e^{2i[\cancel{\ln 1} + i(\frac{\pi}{2} + 2k\pi)]}$$

$$= e^{2i(i(\frac{\pi}{2}+2k\pi))}$$

② $(1+i)^{3i}$

$$= e^{3i[\ln(1+i) + i \arg(1+i)]}$$



$(1+i) \rightarrow (1,1)$

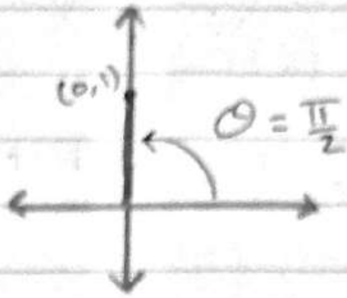
$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\sin\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4} + 2k\pi$

$$= e^{3i(\ln|\sqrt{2}| + i(\frac{\pi}{4}+2k\pi))}$$

Ex Find the Principal branch of i^{2i}

$$= e^{2i[\ln|i| + \operatorname{Arg}(i)]}$$



$i \rightarrow (0,1)$

$\therefore r = \sqrt{0^2+1^2} = 1$

$$= e^{2i[\ln 1 + i(\frac{\pi}{2}))} = e^{-2(\frac{\pi}{2})} = e^{-\pi}$$

Inverse trigonometric function and hyperbolic :-

$\boxed{1}$ $\sin^{-1} z = -i \operatorname{Log}[iz \pm \sqrt{1-z^2}]$

Proof :

$\sin^{-1} z = w \rightarrow \sin(\sin^{-1} z) = \sin w$

$\rightarrow z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$\therefore z = \dfrac{e^{iw} - e^{-iw}}{2i} \rightarrow (2iz = e^{iw} - e^{-iw}) \cdot e^{iw}$ ضرب طرفي المعادلة بـ

$\rightarrow 2iz \cdot e^{iw} = e^{2iw} - 1 \rightarrow e^{2iw} - 2iz\, e^{iw} - 1 = 0$

let $y = e^{iw} \rightarrow y^2 - 2izy - 1 = 0$

تابع

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<u>(تكملة)</u> $y^2 - 2iZy - 1 = 0$

$a = 1 \,/\, b = -2iZ \,/\, c = -1$

$$\therefore y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2iZ \pm \sqrt{-4Z^2 + 4}}{2}$$

$$= \frac{2iZ \pm \sqrt{4(1 - Z^2)}}{2} = \frac{2iZ \pm 2\sqrt{1 - Z^2}}{2}$$

$$= iZ \pm \sqrt{1 - Z^2}$$ but $y = e^{iw}$

$$e^{iw} = iZ \pm \sqrt{1 - Z^2}$$

$$\operatorname{Log} e^{iw} = \operatorname{Log}(iZ \pm \sqrt{1 - Z^2})$$

$$\frac{\cancel{i}w}{\cancel{i}} = \frac{\operatorname{Log}(iZ \pm \sqrt{1 - Z^2})}{i}$$

$$w = \frac{\operatorname{Log}(iZ \pm \sqrt{1 - Z^2})}{i} * \frac{i}{i}$$

$$w = -i \log(iZ \pm \sqrt{1 - Z^2})$$ ※

$\boxed{2}$ $\cos^{-1} z = -i \log(z \pm i\sqrt{1-z^2})$

↳ proof : (H.W)

$\cos^{-1} z = w \rightarrow z = \cos w$

$z = \dfrac{e^{iw} + e^{-iw}}{2}$

$(2z = e^{iw} + e^{-iw}) \cdot e^{iw}$

$2z e^{iw} = (e^{iw})^2 + 1 \rightarrow (e^{iw})^2 - 2z e^{iw} + 1 = 0$

let $y = e^{iw} \rightarrow y^2 - 2zy + 1 = 0$

$y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2z \pm \sqrt{4z^2 - 4}}{2}$

$= \dfrac{2z \pm 2\sqrt{-1}\sqrt{1-z^2}}{2} = \dfrac{2z \pm 2i\sqrt{1-z^2}}{2}$

$= z \pm i\sqrt{1-z^2}$

but $y = e^{iw} \rightarrow e^{iw} = z \pm i\sqrt{1-z^2}$

$\mathrm{Log}\, e^{iw} = \mathrm{Log}(z \pm i\sqrt{1-z^2})$

$\dfrac{iw}{i} = \dfrac{\mathrm{Log}(z \pm i\sqrt{1-z^2})}{i} \cdot \dfrac{i}{i}$

$w = -i\,\mathrm{Log}(z \pm i\sqrt{1-z^2})$

[3] $\tan^{-1} Z = \frac{i}{2} \operatorname{Log}\left(\frac{i+Z}{i-Z}\right)$

[4] $\sinh^{-1} Z = \operatorname{Log}\left(Z \pm \sqrt{Z^2+1}\right)$

[5] $\cosh^{-1} Z = \operatorname{Log}\left(Z \pm \sqrt{Z^2-1}\right)$

[6] $\tanh^{-1} Z = \frac{1}{2} \operatorname{Log}\left[\frac{1+Z}{1-Z}\right]$

Ex solve for Z if $\sin Z = 2$

نستطيع أن نحل
مثل هذا السؤال على
الدروس السابقة

لكن هنا نحله بالـinverse
بنطوننا وينتهي الحل

$\sin Z = 2$

نكتب محل كل Z
= 2

$Z = \sin^{-1}(2)$

$= -i \operatorname{Log}\left(2i \pm \sqrt{1-(2)^2}\right)$

$= -i \log\left(2i \pm \sqrt{3}\, i\right)$

Ex Find for Z if $\cosh Z = 3$

$\cosh Z = 3$

$Z = \cosh^{-1} 3$

$= \operatorname{Log}\left(3 \pm \sqrt{9-1}\right) = \operatorname{Log}\left(3 \pm 2\sqrt{2}\right)$