م معتصم المصري م

△1

الثاني 1
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near Algebra 1

1, —

Matrices :-

Def : A matrices is a rectangular array of numbers. The number in the array are called the enties in the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

* A matrix with $\boxed{m}$ rows and $\boxed{n}$ columns is called matrix of size $\boxed{m \times n}$

* If rows $= m =$ colums $= n$ then A is called a square matrix of order (n)

ex: Some of matrices are :-

$$\underset{2\times2}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \quad \underset{1\times4}{\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}} \quad \underset{3\times1}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} \quad \underset{3\times3}{\begin{bmatrix} \sqrt{2} & \pi & e \\ 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad \underset{1\times1}{\begin{bmatrix} 4 \end{bmatrix}}$$

* Types of matrices :-

أنواع المصفوفة

مصفوفة صفرية

△1 Zero matrix

$$A = \begin{bmatrix} 0 & 0 \cdots & 0 \\ \vdots & & \\ 0 & \cdots 0 & 0 \end{bmatrix}$$

مصفوفة مربعة

△2 Square matrix عدد الصفوف = عدد الأعمدة

ex: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times2}$

rows = colums $\Rightarrow m = n$

صفوف المصفوفات

3) row matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}$$

عامود المصفوفات

4) column matrix

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

5) & 6) upper & lower triangular matrix

$$A = \begin{bmatrix} 1 & 7 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular matrix * إذا الأرقام في الأعلى

upper إذا كان الاصفار تحت تسمى

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Lower triangular matrix : إذا الأرقام في الأسفل

lower إذا كان الاصفار فوق

7) I dentity matrix

العنصر الحالي يكون قطر واحدات

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8) Diagonal matrix

قطري

العنصر القطري يكون أرقام مختلفة

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

سؤال إذا كانت ~~كل~~ الارقام متساوية (نفسها)

* M = عدد الصفوف
N = عدد الأعمدة

* Operation of matrix :-

1 matrix addition

* عند جمع اي مصفوفتين يجب ان تكون لهم نفس الرتبة.

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

ex: $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$

find : 1 $A + B = \begin{bmatrix} 3 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix}$

2 $A + C$ — Not possible

2 Subraction

* عند طرح اي مصفوفتين يجب ان تكون لهم نفس الرتبة

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix}$

ex: $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A - B = \begin{bmatrix} 1 & -4 & 4 \\ -4 & -1 & 1 \end{bmatrix}$

multiplication

د) multiplication

← عند ضرب مصفوفة ب عدد ثابت يضرب جميع مدخلات المصفوفة بالثابت.

ex: $A = \begin{bmatrix} 5 & 5 \\ 2 & 10 \end{bmatrix}$ find $5A$?

$$5A = 5 \begin{bmatrix} 5 & 5 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 10 & 50 \end{bmatrix}$$

4 Transpose

← تحويل الصفوف الى اعمدة

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

ex: $A = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$ find A^T? $A^T = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

ex: $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ find A^T? $A^T = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 1 \\ 3 & 2 & 5 \end{bmatrix}$

$$* \ (A^T)^T = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$* \ (A^T)^T = A$$

قاعدة

A is symmetric matrices If $A^T = A$

ex: $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ -1 & 8 & 5 & 6 \\ 3 & 5 & 4 & -2 \\ 4 & 6 & -2 & 0 \end{bmatrix}$ find A^T ?

$A^T = \begin{bmatrix} 1 & -1 & 3 & 4 \\ -1 & 8 & 5 & 6 \\ 3 & 5 & 4 & -2 \\ 4 & 6 & -2 & 0 \end{bmatrix} \Rightarrow A^T = A$

* Diagonal matrics and Identity matrixs are symmetric

ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = A^T$

ex: $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ $B = B^T$

6 skew symmetric If $A^T = -A$

ex: $A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$

$-A = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$

$\Rightarrow \boxed{A^T = -A}$

7 multiplication

مهم *

* عند ضرب اي مصفوفتين يجب ان يكون عدد اعمدة المصفوفة الاولى تساوي عدد الصفوف المصفوفة الثانية.

ex: $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

find AB ?

$$AB = \begin{bmatrix} (-1)(1)+(0)(3) & (-1)(2)+(0)(0) \\ (2)(1)+(3)(3) & (2)(2)+(3)(0) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

ex: $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 9 & 5 \\ 1 & 2 \end{bmatrix}$ find AB ?

$3*3$ $3*2$

$$AB = \begin{bmatrix} 33 & 41 \\ 14 & 50 \\ 28 & 10 \end{bmatrix}$$

8 conjugate matrix

مرافق

فقط التغير على إشارة الجزء الذي مع i ⇐ $a + bi$

ex: $A = \begin{bmatrix} 3+5i & 5 \\ 2i & 1-i \\ -1 & 0 \end{bmatrix}$ find $\bar{A}$?

$$\bar{A} = \begin{bmatrix} 3-5i & 5 \\ -2i & 1+i \\ -1 & 0 \end{bmatrix}$$

9 orthongnal matrix

A is orthongnal matrix Iff $\boxed{AA^T = I_{n \times n}}$

عمودي

ex: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10 + 11
ايحتمال

Hermition matrix ايحتمال

$A^H = (\bar{A})^T$

$i = \sqrt{-1}$

$i^2 = -1$

ex: $A = \begin{bmatrix} 3 & 1+2i & i \\ 1-2i & 5 & 7 \\ -i & 7 & 0 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 3 & 1-2i & -i \\ 1+2i & 5 & 7 \\ -i & 7 & 0 \end{bmatrix}$ $(\bar{A})^T = \begin{bmatrix} 3 & 1+2i & i \\ 1-2i & 5 & 7 \\ -i & 7 & 0 \end{bmatrix}$ A is Hermition

11 unitary matrix:

A is unitary marrix Iff $AA^H = I$, $A^H A = I$

H.w

12 Normal matrix

A is Normal Iff $AA^T = A^T A$

ex: $A = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$ is normal ? Why

سؤال
RS
Gaysl

Sol $\Longrightarrow$

$A^T = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix}$ $AA^T = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix}$

$= \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$

$A^T A = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$

$AA^T = A^T A \Rightarrow A$ is normal.

Rules of matrix Arithmetic p.g 38

(a) $A + B = B + A$ — commutative Law

(b) $A + (B + C) = (A + B) + C$ — associative Law

(c) $A(BC) = (AB)C$

(d) $A(B + C) = AB + AC$

(e) $(B + C)A = BA + CA$

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

Theorem (1.4.2) p.g 38

(a) $A + 0 = 0 + A = A$

(b) $A - A = 0$

(c) $0 - A = -A$

(d) $A0 = 0A = 0$

ex: consider the matrices

$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

$$AB \neq BA$$

A

ex: $A = \begin{bmatrix} 5 & -6 \\ 7 & -8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$B + A = \begin{bmatrix} 6 & 4 \\ 9 & 9 \end{bmatrix}$ $A + B = \begin{bmatrix} 6 & 4 \\ 9 & 9 \end{bmatrix}$ $A + B = B + A$

Inverse of A matrix: معكوس A

Def ⇒ If A is a square matrix and iff a matrix B of the same size can be found s.t $AB = I = BA$ then A is said to be invertible & B is called an inverse of $A \Rightarrow A^{-1}$ معكوس

ex: $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is an invers of $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$?

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is iveritible If $ad - bc \neq 0$ in which case the inverse is given by the formula.

نطبقها اذا كانت 2×2

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem: If A & B are invertible matrices of the same size then $(AB)^{-1} = B^{-1}A^{-1}$ $(AB)^{-1}$

ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ find A^{-1}?

$$A^{-1} = \frac{1}{(2)(5)-(3)(4)} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Theorem: If A, B are invertible

مهمه كتير

$\triangle 1$ $(A^{-1})^{-1} = A$

$\triangle 2$ $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ $\alpha =$ رقم

$\triangle 3$ $(AB)^{-1} = B^{-1} A^{-1}$

$\triangle 4$ $(A^T)^{-1} = (A^{-1})^T$

× $\triangle 5$ $(A+B)^{-1} \neq A^{-1} + B^{-1}$

⇒ proof: $\triangle 1$ since A, B are inveratible A^{-1}, B^{-1} exist & $AA^{-1} = A^{-1}A = I$ HW

$B^{-1}B = BB^{-1} = I$

$(A^{-1})(A^{-1})^{-1} = I$

⇒ $\triangle 2$ $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

$(\alpha A)(\alpha A)^{-1} = \alpha A \frac{1}{\alpha} A^{-1}$ نضرب الطرفين

$$\boxed{I = I}$$

proof

3) $(AB)^{-1} = B^{-1}A^{-1}$ نأخذ الطرف الأيسر

$$(AB)(B^{-1}A^{-1}) = I$$
$$A(BB^{-1})A^{-1} = I$$
$$AA^{-1} = I$$
$$I = I$$

ex: If A is a square matrix s.t $A^3 = O_{n\times m}$ show that $(I-A)^{-1} = I + A + A^2$

نضرب بالطرفين

sol $\Rightarrow (I-A)(I-A)^{-1} = (I-A)(I+A+A^2)$

$$I = I + A + A^2 - IA - A^2 - A^3$$
$$= I + A - A - A^3$$
$$= I - A^3$$
$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$$

$* \; IA = A$

$* \; I^n = I$

ex: Let $A = \begin{bmatrix}1&0\\2&3\end{bmatrix}$ find :-

a) show that A is root of $f(x) = x^2 - 4x + 3$

$$\Rightarrow f(A) = \begin{bmatrix}1&0\\2&3\end{bmatrix}^2 - 4\begin{bmatrix}1&0\\2&3\end{bmatrix} + 3\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

الرقم الذي لا يوجد جنبه متغير نضع جنبه I

$$= \begin{bmatrix}1&0\\8&9\end{bmatrix} - \begin{bmatrix}4&0\\8&12\end{bmatrix} + \begin{bmatrix}3&0\\0&3\end{bmatrix}$$
$$= \begin{bmatrix}-3&0\\0&-3\end{bmatrix} + \begin{bmatrix}3&0\\0&3\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$A = \begin{bmatrix}2&3\\4&5\end{bmatrix}$ $\quad B = \begin{bmatrix}3&2\\1&10\end{bmatrix}$

find:

$AB \quad BA \quad A^{-1} \quad B^{-1}$

نعوض A بدل x + f(x)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det A = ad - cb$

سؤال ع جاي

b) use part (a) to find A^{-1}

$$A^2 - 4A + 3I = 0$$

$$\frac{A^2 - 4A}{-3} = \frac{-3I}{-3}$$

$$-\frac{1}{3}(A^2 - 4A) = I$$

$$A\left(-\frac{1}{3}(A - 4I)\right) = I$$

$$A^{-1} = -\frac{1}{3}(A - 4I)$$

$$= -\frac{1}{3}\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= -\frac{1}{3}\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)$$

$$= -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & +\frac{1}{3} \end{bmatrix}$$

$\boxed{Hw}$ $x^2 + 3x - 10 = 0$

$\boxed{H.w}$ suppose A is an invertible matrix satisfies $A^2 + 3A - 10I = 0$ find A^{-1}?

$$\Rightarrow \frac{A^2 + 3A}{10} = \frac{10I}{10}$$

$$\frac{1}{10}(A^2 + 3A) = I$$

$$A\left(\frac{1}{10}(A + 3I)\right) = I$$

$$A\left(\frac{A}{10} + \frac{3}{10}I\right) = I \Rightarrow A^{-1} = \frac{A}{10} + \frac{3}{10}I$$

To find the inverse of the matrix A If exist

→ step 1 : write the matrix $[A_{n\times n} | I_n]$

→ step 2 : $[A | I] \longrightarrow [C | D]$

→ step 3 : ① If C has a zero raw the A is non invertible & A^{-1} does not exist

② If $C = I_n$ then $D = A^{-1}$

⇒ ex: $A = \begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$ find A^{-1} if it exist?

$$A^{-1} = \frac{1}{4-(-2)(4)} \begin{bmatrix} 2 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

⇒ طريقة اخرى

* المدخل الاول يجب ان يكون ①

* تحت الواحد يجب ان يكون صفر

x الاوتار

$$\begin{array}{cc} A & A^{-1} \\ \end{array}$$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \longrightarrow \frac{1}{2}R_1 = R_1$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \longrightarrow 2R_1 + R_2 = R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 6 & 1 & 1 \end{array}\right] \longrightarrow \frac{1}{6}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \end{array}\right] \longrightarrow -2R_2 + R_1 = R_1$$

$-2 \quad 0 \quad 1$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{-1}{3} \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \end{array}\right] = \frac{-2}{6} \quad 2(1) + 2$$

A $\qquad$ A^{-1}

$A \cdot A^{-1} = I$

$$\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{2}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{6} + \frac{4}{6} & \frac{-4}{6} + \frac{4}{6} \\ \frac{-2}{6} + \frac{2}{6} & \frac{4}{6} + \frac{2}{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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ex: $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ find A^{-1} if it exist?

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow R_1 \Rightarrow R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow -2R_1 + R_3 = R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right] \Rightarrow \text{has zero row so } A^{-1} \text{ doesn't exist.}$$

ex: $A = \begin{bmatrix} 5 & 0 & 0 \\ -6 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ find A^{-1}?

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ -6 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \frac{1}{5}R_1 = R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ -6 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow 6R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{6}{5} & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow 3R_1 + R_3 = R_3$$

التحقق

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{6}{5} & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{5} & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 5 & 0 & 0 \\ -6 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{6}{5} & 1 & 0 \\ \frac{3}{5} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$-R_2+R_1=R_2$

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ex: $A=\begin{bmatrix}1&0&3\\-2&2&0\\0&1&1\end{bmatrix}$ find A^{-1}?

$$\left[\begin{array}{ccc|ccc}1&0&3&1&0&0\\-2&2&0&0&1&0\\0&1&1&0&0&1\end{array}\right]\longrightarrow 2R_1+R_2=R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&2&6&2&1&0\\0&1&1&0&0&1\end{array}\right]\longrightarrow \frac{1}{2}R_2=R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&3&1&\frac{1}{2}&0\\0&1&1&0&0&1\end{array}\right]\longrightarrow -1R_2+R_3=R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&3&1&\frac{1}{2}&0\\0&0&-2&-1&-\frac{1}{2}&1\end{array}\right]\longrightarrow -\frac{1}{2}R_3=R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&3&1&0&0\\0&1&3&1&\frac{1}{2}&0\\0&0&1&\frac{1}{2}&\frac{1}{4}&-\frac{1}{2}\end{array}\right]\longrightarrow -3R_3+R_1=R_1$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&-\frac{3}{4}&\frac{3}{2}\\0&1&3&1&\frac{1}{2}&0\\0&0&1&\frac{1}{2}&\frac{1}{4}&-\frac{1}{2}\end{array}\right]\longrightarrow -3R_3+R_2=R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&-\frac{3}{4}&\frac{3}{2}\\0&1&0&-\frac{1}{2}&-\frac{1}{4}&\frac{3}{2}\\0&0&1&\frac{1}{2}&\frac{1}{4}&-\frac{1}{2}\end{array}\right]$$

ex: $A=\begin{bmatrix}4&0&1\\0&1&0\\1&0&-2\end{bmatrix}$ find A^{-1}?

$$\left[\begin{array}{ccc|ccc}4&0&1&1&0&0\\0&1&0&0&1&0\\1&0&-2&0&0&1\end{array}\right]\longrightarrow R_3=R_1$$

$$\left[\begin{array}{ccc|ccc}1&0&-2&0&0&1\\0&1&0&0&1&0\\4&0&1&1&0&0\end{array}\right]\longrightarrow -4R_1+R_3=R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&-2&0&0&1\\0&1&0&0&1&0\\0&0&9&1&0&-4\end{array}\right]\longrightarrow \frac{1}{9}R_3=R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&-2&0&0&1\\0&1&0&0&1&0\\0&0&1&\frac{1}{9}&0&-\frac{4}{9}\end{array}\right]\longrightarrow 2R_3+R_1=R_1$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{2}{9}&0&\frac{1}{9}\\0&1&0&0&1&0\\0&0&1&\frac{1}{9}&0&-\frac{4}{9}\end{array}\right]$$

$$\underbrace{\begin{bmatrix}\frac{2}{9}&0&\frac{1}{9}\\0&1&0\\\frac{1}{9}&0&-\frac{4}{9}\end{bmatrix}}_{A^{-1}}$$

ex: $A^T = \begin{bmatrix} 3 & 4 & 1 \\ 6 & -2 & 5 \end{bmatrix}$ find $4A$?

$$A = \begin{bmatrix} 3 & 6 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \Rightarrow 4A = \begin{bmatrix} 12 & 24 \\ 16 & -8 \\ 4 & 20 \end{bmatrix}$$

ex: Solve the matrix equation

$$\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$$ find a, b, c, d?

$\boxed{a = 4}$

$a + b = -2$

$4 + b = -2 \Rightarrow \boxed{b = -6}$

$d - 2c = 3$

$d + 2c = -1$

$\frac{2d}{2} = \frac{2}{2} \Rightarrow \boxed{d = 1} \Rightarrow 1 - 2c = 3$

$-2c = 2$

$\boxed{c = -1}$

ex: for the

ex: for the following ? HW:

$$\begin{bmatrix}1 & a & 2\\ d & 2 & -1\end{bmatrix}\begin{bmatrix}4 & 1 & c\\ 6 & 2 & -2\\ -1 & b & 5\end{bmatrix} = \begin{bmatrix}8 & a & -3\\ 9 & e & f\end{bmatrix}$$

find the a, b, c, d, e, f ?

$(1)(4) + (a)(6) + (2)(-1) = 8$

$4 + 6a + -2 = 8$

$6a + 2 = 8$

$6a = 8 - 2$

$6a = 6 \Rightarrow \boxed{a = 1}$

$(1)(1) + (a)(2) + (2)(b) = a$

$1 + 2a + 2b = a$

$1 + 2 + 2b = 1$

$2b = 1 - 3 \Rightarrow 2b = -2 \Rightarrow \boxed{b = -1}$

$(1)(c) + (a)(-2) + (2)(5) = -3$

$c + -2 + 10 = -3 \Rightarrow c + 8 = -3 \Rightarrow \boxed{c = -11}$

$(4)(d) + (2)(6) + (-1)(-1) = 9$

$4d + 12 + 1 = 9 \Rightarrow 4d = 9 - 13 \Rightarrow 4d = -4 \Rightarrow \boxed{d = -1}$

$d + 4 + -b = e$

$-1 + 4 + 1 = e \Rightarrow \boxed{e = 4}$

$dc + (2)(-2) + (-1)(5) = f$

$11 + -4 + -5 = f \Rightarrow 7 + -5 = f \Rightarrow \boxed{f = 2}$

ex: $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ find A^{-3} ?

$$A^3 = AAA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$(A^3)^{-1} = \frac{1}{8-0} \begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

ex: Let $(2A+I)^{-1} = \begin{bmatrix} 9 & 8 \\ 4 & 4 \end{bmatrix}$ find A ?

$$\left((2A+I)^{-1}\right)^{-1} = \begin{bmatrix} 9 & 8 \\ 4 & 4 \end{bmatrix}^{-1}$$

نأخذ المعكوس للطرفين

① $[\ ]_{3\times 3}$ find A^{-1}

② $(\ )^{-1} = [\ ]$

③ fx

ⓐ

ⓑ

④

$$2A + I = \frac{1}{4} \begin{bmatrix} 4 & -8 \\ -4 & 4 \end{bmatrix}$$

$$2A + I = \begin{bmatrix} 1 & -2 \\ -1 & +\frac{4}{4} \end{bmatrix}$$

$$2A = \begin{bmatrix} 1 & -2 \\ -1 & \frac{9}{4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2A = \begin{bmatrix} 0 & -2 \\ -1 & \frac{5}{4} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & \frac{5}{8} \end{bmatrix}$$

المحدد

Def: Let A be a square matrix, then determint function is denoted by (det) or | | and we define det(A) to be the sum of all singned elematray product from A
The number det(A) is called the detertiour

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

ex: $A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}$ find det(A)?

$$\det(A) = |A| = (3)(-2) - (1)(4) = -6 - 4 = -10$$

$$\Rightarrow A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$ find det(A)?

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

ex: $A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 0 & 3 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \end{array}$ find det(A)?

$$|A| = 0\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 0 - 3\left((1)(1) - (2)(2)\right) + 5\left((1)(1) - (-1)(2)\right)$$

$$= 0 - 3(-3) + 5(3)$$

$$= 0 - 3(-3) + 5(3) = 15 + 9 = 24$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $\det A^t$

$\det A^t = \det A$
$= (1)(4) - (2)(3) = -2$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\Rightarrow \det(A) \Rightarrow \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$

$\det|A| = (a_{11}a_{22}a_{33}) + (a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31})$
$- (a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33}) = \boxed{} = \det$

Thm:- Let A be a square matrix:-

(a) If A has a row of zero or a column of zero then $\det A = 0$.

مصفوفة مربعة وكان احد الصفوف او
* اذا كانت اي ... المحددة تساوي صفر
الاعمدة اصفار تكون

ex: $A = \begin{bmatrix} 5 & 4 & -7 & 2 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 \\ 3 & 5 & 1 & 6 \end{bmatrix} \Rightarrow \det(A) = 0$

(b) $\det A = \det A^T$

ex: $A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \Rightarrow \det(A) = (3)(-2) - (1)(4) = -10$

$A^T = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix} \Rightarrow \det(A) = 3(-2) - (1)(4) = -10$

$\therefore \det A = \det A^T$

*

Thm: If A is an $n \times m$ triangular matrix (upper triangular and lower triangular) or diagonal then ($\det A$) is the product of the entries on the main diagonal that is; $\det A = a_{11}, a_{22}, \dots a_{nk}$

ex: $A = \begin{bmatrix} 2 & 7 & -3 & 8 & 4 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow \det A = (2)(-3)(6)(4)(4) = -1216$

ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det A = (1)(3)(1) = 3$

Thm: If A is a square matrix with two proportional raws or two proportional columns then $\det A = 0$

ex: $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 3 & -2 \\ 2(1 & 3 & -2) \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow \det(A) = 0$

$[\times]$

ex: $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 6 \\ 3 & 3 & 7 \end{bmatrix} \Rightarrow \det(A) = 0$

3×3

$2 . x$

* Some proprtions of $\det A$:—

خواص

1. $\det A^T = \det A$
2. $\det \alpha A_{n \times n} = \alpha^n \det A$
3. $\det AB = \det A \det B$
4. If A is invertible then $\det A \neq 0$
5. $\det A^n = (\det A)^n$
6. $\det A^{-1} = \dfrac{1}{\det A}$

Invertible

قابل للعكس

Ex: Find the inverse of -

$$A=\begin{bmatrix}1 & 2 & 3\\ & 5 & 3\\ 2 & 0 & 8\end{bmatrix}$$

22

$\det(A+B) \neq \det A + \det B$

* $\det(A+B) = \det A + \det B$

ex: $A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$

$\Rightarrow A+B=\begin{bmatrix}4 & 3\\ 3 & 8\end{bmatrix}$

$\det A = 1$ $\quad \det B = 8$ $\quad \det(A+B) = 23$

$\therefore \det(A+B) \neq \det A + \det B$

ex: consider the matrices

$A=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & 3\\ 5 & 8\end{bmatrix}$

$AB=\begin{bmatrix}2 & 17\\ 3 & 14\end{bmatrix}$

$\det(A) = 1$ $\quad \det(B) = -23$

$\det AB = -23$ $\quad \therefore \det AB = \det A \det B$

ex: If A, B, F be 3×3 matrices s.t $A^T B F^7 = I$ and $\det A = 2$, $\det F = -1$ evaluate $\det B$

$\det A^T B F^7 = \det I$

$\det A^T \cdot \det B \cdot \det F^7 = \det I$

$\det A \cdot \det B \cdot (\det F)^7 = \det I$

$2 \cdot \det B \cdot (-1)^7 = 1$

$-2 \det B = 1$

$\boxed{\det B = -\frac{1}{2}}$

Al-Sugier

1.w] Let A, B be two 4×4 matrices $A^T B^2 A^{-1} = 5B$

find $\det B$?

$\Rightarrow \det A^T B^2 A^{-1} = \det 5B$

$\det A^T \cdot \det B^2 \cdot \det A^{-1} = \det 5B$

$\cancel{\det A} \cdot \det B^2 \cdot \dfrac{1}{\cancel{\det A}} = 5^4 \det B$

$(\det B)^2 = 5^4 \det B$

$\det B = 5^4 = 625$

ex: If A is orthognal matrix show that $\det A = \pm 1$

$A^T A = I$

$\Rightarrow \boxed{AA^T = I}$

$\det AA^T = \det I$

$\det A \det A^T = \det I$

$\det A \det A = \det I$

$(\det A)^2 = 1$

$\det A = \pm 1$

$AA^T = I$

توضيحه لكل الأرقام

التعريف بالمصفوفة الواحدة

matrix :-

The adjoint matrix:

Let A be a square matrix of order n then the adjoint matrix of A is an $n \times n$ matrix give by

$$\operatorname{adj}(A) = \left[(-1)^{i+j} \, |M_{ij}| \right]^T$$

[illegible]

ex: $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$ use the adjoint to find A^{-1}, If exist?

$\det A = 2\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} + -4\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$ $\Rightarrow$

$= -38 + +6 + -4(4) = -46$

$= -36$

$A = \begin{bmatrix} a_{11} & \\ \vdots & \\ & b_{32} \end{bmatrix}$

$$\Rightarrow adj(A) = \begin{bmatrix} +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} \end{bmatrix}^T$$

$$\begin{bmatrix} -18 & 2 & 4 \\ & 14 & 5 \\ -10 & & 8 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} -15 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}^T$$

$$adj(A) = \begin{bmatrix} -15 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{-46} \begin{bmatrix} -18 & 11 & -10 \\ -2 & 14 & 4 \\ 4 & -5 & -8 \end{bmatrix}$$

$\boxed{H.W}$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow det(A) = 2$

$$adj = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{det A} adj(A)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

أصفار

first

تاعـــدة

saewnde.

$$\boxed{X = A^{-1} b}$$

The Linear system

The system of linear equation (linear system) has the following general form.

$$a_{11} X_1 + a_{12} X_2 + \dots + a_{1n} X_n = b_1$$
$$a_{21} X_1 + a_{22} X_2 + \dots + a_{2n} X_n = b_2$$
$$\vdots$$
$$a_{m1} X_1 + a_{m2} X_2 + \dots + a_{mn} X_n = b_m$$

where m number of equation a constant (coefficients) and we can write it as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

ex: solve the equation $X + y = 3$
$2X - y = 6$

$\Rightarrow$ طريقة الحذف $\Rightarrow$

$$\begin{array}{r} X + y = 3 \\ 2X - y = 6 \\ \hline 3X = 9 \end{array} \Rightarrow X = 3 \Rightarrow y = 0$$

$$\Rightarrow \underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} X \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1-2} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} X \\ y \end{bmatrix} = \underset{A^{-1}}{\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}} \underset{b}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$$

$$\begin{bmatrix} X \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

تطبيق القاعدة

ex: Solve the linear system by using the inverse of A:

$$x_1 + x_2 + x_3 = 1$$
$$x_2 + 2x_3 = -5$$
$$x_1 + 2x_2 + 4x_3 = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow \det(A) = 1\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1(0) - 1(-2) + 1(-1) = 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 3 & -1 \\ 1 & 2 & -1 \end{bmatrix}^T \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} 0 & 2 & 1 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ -17 \\ -6 \end{bmatrix}$$

Cramers Rule:

Let $AX = B$ be a linear system s.t $|A| \neq 0$

Then the linear system solution given by

$$X_1 = \frac{|A_1|}{|A|}$$

$$X_2 = \frac{|A_2|}{|A|}$$

$$\vdots$$

$$X_n = \frac{|A_n|}{|A|}$$

when A_i is a matrix obtaied by replasing the i column of A by the constant matrix B.

ex: Solve the linear system using Gramers Rule.

$$2x + 3y - z = 1$$
$$3x + 5y + 2z = 8$$
$$x - 2y - 3z = -1$$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 8 \\ -1 \end{bmatrix}$$

$$|A| = 2\begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} + -1\begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix} = 2(-11) - 3(-11) + -1(-11)$$
$$= 22$$

تعويض قيم B.

$$X = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix}}{22} = \frac{66}{22} = 3$$

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix}}{22} = \frac{-22}{22} = -1$$

$$Z = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix}}{22} = \frac{44}{22} = 2$$

Ex ⇒ Solve the linear system using Gramas Rule

$x_1 + x_2 + x_3 = 5$

$x_1 - 2x_2 - 3x_3 = -1$

$2x_1 + x_2 - x_3 = 3$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

$\det(A) = 1 \begin{vmatrix} -2 & -3 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$

$x_1 = \dfrac{|A_1|}{|A|} = \dfrac{\begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix}}{5} = \dfrac{20}{5} = 4$

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{\begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix}}{5} = \dfrac{-10}{5} = -2$

$x_3 = \dfrac{|A_3|}{|A|} = \dfrac{\begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix}}{5} = \dfrac{15}{5} = 3$

Ex: find det A :-

$3(-4 - 40) - 0(8 - 5) + 0$

1) $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 4 & 4 \end{bmatrix} \Rightarrow \det(A) = -147$

2) $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix} \Rightarrow \det(A) = -40$

3) $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix} \Rightarrow \det(A) = -66$

Linear equation:-

$a_1 X_1 + a_2 X_2 + \ldots + a_n X_n = b$

where

$X_1, \ldots, X_n$ variables

$a_1, \ldots, a_n$ constants

$b \in \mathbb{R}$

ex: which of the following equation is linear and which not:-

1. $3X = 4$ Linear
2. $X_1 + X_2 = 10$ Linear
3. $X_1 + 2X_2 X_1 + 5X_3 = 12$ Not linear
4. $\sin X_1 + X_2$ not linear
5. $\ln X_1 + 10 = 0$ not linear
6. $X_1^2 + X_2^2 = 1$ not linear

* Gaussion elimination & Gauss - Jordan elimination

1. If a row doesn't consist entirely of zero than the first non zero number "we call this a leading 1"

2. If the are any rows that entirely of zero than they are grouped to gether at the bottom of matrix

3. In any two successive rows that do not consist entirely of zero the leading 1 in the lower occurs fether to the right then leading 1 in the high row.

4. Each column that constain a leading 1 has zero any here else.

ex: Solve $2x_1 + 6x_2 + 6x_3 = 2$

$2x_1 + 7x_2 + 6x_3 = 6$

$2x_1 + 7x_2 + 7x_3 = 8$

$$\left[\begin{array}{ccc|c} 2 & 6 & 6 & 2 \\ 2 & 7 & 6 & 6 \\ 2 & 7 & 7 & 8 \end{array}\right] \rightarrow \tfrac{1}{2} R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 7 & 6 & 6 \\ 2 & 7 & 7 & 8 \end{array}\right] \rightarrow -2R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 4 \\ 2 & 7 & 7 & 8 \end{array}\right] \rightarrow -2R_1 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{array}\right] \rightarrow -R_2 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow x_1 + 3x_2 + 3x_3 = 1$$

$x_2 + 0x_3 = 4 \Rightarrow \boxed{x_2 = 4}$

$\boxed{x_3 = 2}$

$x_1 + 3(4) + 3(2) = 1$

$\boxed{x_1 = -17}$

نزوى 31 +1

ex: $-2X_1 - 2X_2 - 4X_3 = -18$

$3X_1 + 6X_2 - 5X_3 = 0$

$4X_1 + 8X_2 - 6X_3 = 2$ by using:

1. Gaussion elimination
2. Gauss - Jordan elimination

1

$$\left[\begin{array}{ccc|c} -2 & -2 & -4 & -18 \\ 3 & 6 & -5 & 0 \\ 4 & 8 & -6 & 2 \end{array}\right] \rightarrow -\tfrac{1}{2}R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \\ 4 & 8 & -6 & 2 \end{array}\right] \rightarrow -3R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 3 & -11 & -27 \\ 4 & 8 & -6 & 2 \end{array}\right] \rightarrow -4R_1 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 3 & -11 & -27 \\ 0 & 4 & -14 & -34 \end{array}\right] \rightarrow \tfrac{1}{3}R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{11}{3} & -9 \\ 0 & 4 & -14 & -34 \end{array}\right] \rightarrow -4R_2 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{11}{3} & -9 \\ 0 & 0 & \frac{2}{3} & 2 \end{array}\right] \rightarrow \tfrac{3}{2}R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{11}{3} & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \text{Gaussion elimination}$$

$\boxed{X_3 = 3}$

$X_2 - \frac{11}{3}X_3 = -9$

$X_2 - \frac{11}{3}(3) = -9$

$\boxed{X_2 = 2}$

$X_1 + 2 + 2(3) = 9$

$\boxed{X_1 = 1}$

2

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{11}{3} & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow -1R_2 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{17}{3} & 18 \\ 0 & 1 & -\frac{11}{3} & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \tfrac{11}{3}R_3 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{17}{3} & 18 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow -\tfrac{17}{3}R_3 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \boxed{\begin{array}{c} X_1 = 1 \\ X_2 = 2 \\ x_3 = 3 \end{array}}$$

x: Solve the following system

$$15X_1 - 6X_2 + 18X_3 = 0$$
$$-4X_1 + 2X_2 + 6X_3 = 2$$

$$\left[\begin{array}{ccc|c} 15 & -6 & 18 & 0 \\ -4 & 2 & 6 & 2 \end{array}\right] \xrightarrow{\frac{1}{15}R_1 = R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ -4 & 2 & 6 & 2 \end{array}\right] \xrightarrow{4R_1 + R_2 = R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & \frac{2}{5} & \frac{54}{5} & 2 \end{array}\right] \xrightarrow{\frac{5}{2}R_2 = R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & 1 & 27 & 5 \end{array}\right] \xrightarrow{\frac{2}{5}R_2 + R_1 = R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right] \longrightarrow \begin{array}{l} X_1 + 0X_2 + 12X_3 = 2 \\ 0X_1 + X_2 + 27X_3 = 5 \end{array}$$

$X_3 = t \;,\; t \in R$

$X_2 = 5 - 27t$

$X_1 = 2 - 12t$

The system has infinity may solution

ex: solve the following system:-

$$x_1 + 2x_2 - x_3 = -\frac{2}{3}$$
$$x_2 - \frac{3}{2}x_3 = -\frac{1}{2}$$
$$-6x_2 + 4x_3 = 4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 4 & 4 \end{array}\right] \rightarrow -2R_2 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 4 & 4 \end{array}\right] \rightarrow 6R_2 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right] \rightarrow$$

The system does not have a solution "No solution"

ex: solve the following system:-

$$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = -3$$
$$\frac{2}{x} + \frac{3}{y} + \frac{8}{z} = 3$$
$$-\frac{1}{x} + \frac{4}{y} + \frac{10}{z} = 5$$

بالفرض

$$\Rightarrow x_1 = \frac{1}{x}, \quad x_2 = \frac{1}{y}, \quad x_3 = \frac{1}{z}$$

$$\Rightarrow x_1 + 2x_2 - 4x_3 = -3$$
$$2x_1 + 3x_2 + 8x_3 = 3$$
$$-x_1 + 4x_2 + 10x_3 = 5$$

$\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 2 & 3 & 8 & 3 \\ -1 & 4 & 10 & 5 \end{array}\right] \longrightarrow -2R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & -1 & 16 & 9 \\ -1 & 4 & 10 & 5 \end{array}\right] \longrightarrow 1R_1 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & -1 & 16 & 9 \\ 0 & 11 & 6 & 2 \end{array}\right] \longrightarrow -1R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{13} \\ 0 & 1 & 0 & -\frac{11}{91} \\ 0 & 0 & 1 & \frac{101}{182} \end{array}\right]$$

$x_1 = -\frac{7}{13}$, $x_2 = -\frac{11}{91}$, $x_3 = \frac{101}{182}$

$\Rightarrow x = -\frac{13}{7}$, $y = -\frac{91}{11}$, $z = \frac{182}{101}$

W

ex: Solve the following system

$$x^2 + y^2 + z^2 = 6$$
$$x^2 - y^2 + 2z^2 = 2$$
$$2x^2 + y^2 - z^2 = 3$$

$\Rightarrow X_1 = x^2 \quad , \quad X_2 = y^2 \quad , \quad X_3 = z^2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{array}\right] \rightarrow -1R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 2 & 1 & -1 & 3 \end{array}\right] \rightarrow -2R_1 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{array}\right] \rightarrow -\frac{1}{2} R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & -1 & -3 & 9 \end{array}\right] \rightarrow 1R_2 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{7}{2} & -7 \end{array}\right] \rightarrow -1R_2 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 4 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{7}{2} & -7 \end{array}\right] \rightarrow -\frac{2}{7} R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 4 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \frac{1}{2} R_3 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow -\frac{3}{2} R_3 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

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ex: Solve the following system:

$2\sin X - \cos y + 3\tan z = 3$ $\quad y \in [0, 2\pi]$

$4\sin X + 2\cos y - 2\tan z = 2$

$6\sin X - 3\cos y + \tan z = 9$ $\quad z \in [0, \pi]$

$X_1 = \sin X$

$X_2 = \cos y$

$X_3 = \tan z$

$\Rightarrow 2X_1 - X_2 + 3X_3 = 3$

$4X_1 + 2X_2 - 2X_3 = 2$

$6X_1 - 3X_2 + X_3 = 4$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 4 \end{array}\right] \longrightarrow \frac{1}{2}R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 4 \end{array}\right] \longrightarrow -4R_1 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -8 & -4 \\ 6 & -3 & 1 & 4 \end{array}\right] \longrightarrow -6R_1 + R_3 = R_3$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{array}\right] \longrightarrow \frac{1}{4}R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -8 & 0 \end{array}\right] \longrightarrow \frac{1}{2}R_2 + R_1 = R_1$$

$-\frac{1}{8}R_3 = R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow 2R_3 + R_2 = R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow -\frac{1}{2}R_3 + R_1 = R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$X_3 = 0$

$X_2 = -1$

$X_1 = \frac{1}{2}$

$X_1 = 30^\circ \quad \frac{\pi}{2}$

$y = \pi$

$z = 0$

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How 1

$$10X_1 - 10X_2 - 6X_3 + 6X_4 = 70$$
$$0X_1 - 12X_2 - 4X_3 + 2X_4 = 22$$
$$5X_1 - 18X_2 - 6X_3 + 3X_4 = 48$$
$$5X_1 - 14X_2 - 6X_3 + 3X_4 = 44$$

$$\left[\begin{array}{cccc|c} 10 & -10 & -6 & 6 & 70 \\ 0 & -12 & -4 & 2 & 22 \\ 5 & -18 & -6 & 3 & 48 \\ 5 & -14 & -6 & 3 & 44 \end{array}\right] \rightarrow \frac{1}{10}R_1 = R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & -12 & -4 & 2 & 22 \\ 5 & -18 & -6 & 3 & 48 \\ 5 & -14 & -6 & 3 & 44 \end{array}\right] \rightarrow -5R_1 + R_3 = R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & -12 & -4 & 2 & 22 \\ 0 & -13 & -3 & 0 & 13 \\ 5 & -14 & -6 & 3 & 44 \end{array}\right] \rightarrow -5R_1 + R_4 = R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & -12 & -4 & 2 & 22 \\ 0 & -13 & -3 & 0 & 13 \\ 0 & -4 & -3 & 0 & 4 \end{array}\right] \rightarrow \frac{-1}{12}R_2 = R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{6} & \frac{-11}{6} \\ 0 & -13 & -3 & 0 & 13 \\ 0 & -4 & -3 & 0 & 4 \end{array}\right] \rightarrow 13R_2 + R_3 = R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{6} & \frac{-11}{6} \\ 0 & 0 & \frac{4}{3} & \frac{-13}{6} & \frac{-65}{6} \\ 0 & -4 & -3 & 0 & 4 \end{array}\right] \rightarrow 4R_2 + R_4 = R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{6} & \frac{-11}{6} \\ 0 & 0 & \frac{4}{3} & \frac{-13}{6} & \frac{-65}{6} \\ 0 & 0 & 0 & \frac{-3}{2} & \frac{-15}{2} \end{array}\right] \rightarrow \frac{3}{4}R_3 = R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{6} & \frac{-11}{6} \\ 0 & 0 & 1 & \frac{-13}{8} & \frac{-65}{8} \\ 0 & 0 & 0 & \frac{-3}{2} & \frac{-15}{2} \end{array}\right] \rightarrow \frac{-2}{3}R_4 = R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -\frac{3}{5} & \frac{3}{5} & 7 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{6} & \frac{-11}{6} \\ 0 & 0 & 1 & \frac{-13}{8} & \frac{-65}{8} \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$X_4 = 5$

$X_3 = 0$

$X_2 = -1$

$X_1 = 3$

* The trace of a matrix : $\equiv tr\, A$

* trace $A \equiv tr\, A$: * نجمع هنا عناصر القطر في المصفوفة

ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find $tr\,A$?

$tr\,A = 1 + 5 + 9 = 15$

ex: $A = \begin{bmatrix} 5 & 6 & 10 & -1 \\ 3 & -5 & 9 & 20 \\ 7 & 18 & -4 & 5 \\ 6 & 2 & 4 & 8 \end{bmatrix}$ find $tr\,A$?

$tr\,A = 5 + -5 + -4 + 8 = 4$

* properties :-

1. $tr\,A = tr\,A^T$
2. $tr(A \pm B) = tr\,A \pm tr\,B$
3. $tr(\alpha A) = \alpha\, tr\,A$
4. $tr\,AB = tr(AB) \longrightarrow * tr(B.A)$
5. $tr\,AB \neq tr\,A\; tr\,B$

ex: $A = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} \Rightarrow tr\,A = 5 + 1 = 6$

$A^T = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \Rightarrow tr\,A^T = 5 + 1 = 6$

$\therefore tr\,A = tr\,A^T$

ex: $A = \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}$ find $tr(AA^T)$?

قانون

$\Longrightarrow tr(AA^T) = \sum a_{ij}^2$

sol ① $\Longrightarrow tr(A A^T) = (5)^2 + (2)^2 + (-4)^2 + (3)^2$

$= 54$

باستخدام القانون

sol ② $\Longrightarrow A^T = \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}$

$AA^T = \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 29 & -14 \\ -14 & 25 \end{bmatrix}$

$tr(AA^T) = 29 + 25 = 54$

ex: $A = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

find $tr\, 7A$, $tr\, BA^T$, $tr\, BB^T$?

① $tr\, 7A = 7\, tr A = 7(6+1+3) = 7(10) = 70$

② $A^T = \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \Longrightarrow BA^T = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 17 & 8 & 15 \\ -3 & 3 & -1 \\ 32 & 7 & 26 \end{bmatrix}$

$tr\, BA^T = 17 + 3 + 26 = 46$

③ $tr\, BB^T = (1)^2 + (5)^2 + (2)^2 + (-1)^2 + (0)^2 + (1)^2 + (3)^2 + (2)^2 + (4)^2$

$= 61$

exam ر فاينل علاوه

امتحان نهائي

جبر خطي

Eigenvalues & Eigenvectors

تحديد النص للاختبار

⊛ Let A an $(n\times n)$ matrix, then λ is called to be an egenvalues of A if and only if the homog system $(\lambda I - A)x = 0$ has only many solu.

or if and only if $\boxed{\det(\lambda I - A) = 0}$

⊛ Finding the eigenvalues of the matrix :-

$\det(\lambda I - A)$

final

let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

① $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

② $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix}$

③ $\det(\lambda I - A) = (\lambda-3)(\lambda+1) - 0 = (\lambda-3)(\lambda+1) = 0$

$\therefore (\lambda-3)(\lambda+1) = 0 \rightarrow$ characteristic equ of A.

$\lambda = 3$ & $\lambda = -1$

$\therefore$ the eigenvalues of A are $\boxed{\lambda = 3}$ & $\boxed{\lambda = -1}$.

⊛ Find the eigenvalues of $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$?

Solu: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-2 & 3 \\ 0 & \lambda-2 \end{bmatrix}$

$\det(\lambda I - A) = 0 \Rightarrow \det (\lambda-2)(\lambda-2) - 0 = 0$

$\therefore \boxed{\lambda = 2}$

- يمكن كتابة (char. equ) بالطريقة التالية :-

$P(\lambda) = \lambda^2 - 4\lambda + 4$

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Ex: let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ Find the following:

1. characteristic equation of A
2. characteristic polynomial of A
3. The eigenvalues of A.

solu:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ \lambda & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix}$$

$$\det \ (\lambda-1) \begin{vmatrix} \lambda-4 & -1 \\ 2 & \lambda-1 \end{vmatrix} = (\lambda-1)\big((\lambda-4)(\lambda-1)+2\big) = 0$$

$\therefore \ (\lambda-1)(\lambda^2-\lambda-4\lambda+4+2) = (\lambda-1)(\lambda^2-5\lambda+6) = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore \ \boxed{\lambda=1}$ & $\boxed{\lambda=2}$ & $\boxed{\lambda=3}$

1. the charact. equ of A $\longrightarrow$ $\boxed{(\lambda-1)(\lambda-2)(\lambda-3)=0}$
2. the charact. poly. of A $\longrightarrow$ $\boxed{P(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)}$
3. the eigenvalues of A $\longrightarrow$ $\boxed{\lambda=1 \mid \lambda=2 \mid \lambda=3}$

Ex let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ Find the eigenvalues of A ?

solu:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix}$$

$$\det = \lambda \begin{vmatrix} \lambda & -1 \\ 17 & \lambda-8 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -4 & \lambda-8 \end{vmatrix} = \lambda\big(\lambda(\lambda-8)+17\big) + (0-4)\big)$$

$$= \lambda(\lambda^2-8\lambda+17) - 4 = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$\therefore$ charact. equ $= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

charact. poly $= P(\lambda) = \lambda^3 - 8\lambda^2 + 17\lambda - 4$.

Eigenvalues of a diagonal & triangular matrices :-

* If A is diagonal or triangular matrix, then the eigenvalues of A are the entries on its diagonal.

← ذلك يعني أن (eigenvalue) هي عناصر القطر .

Ex let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ find the eigenvalue of A ?

solu: The matrix A is diagonal ∴ The eigenvalue of A are : $\boxed{\lambda = 1}$ $\boxed{\lambda = 2}$ $\boxed{\lambda = 3}$ ← عناصر القطر

Ex: let $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 7 & 6 \\ 0 & 0 & -5 \end{bmatrix}$ Find the eigenvalues of A ?

solu: A is upper triangular

∴ the eigenvalues of A are : $\boxed{\lambda = 3}$ $\boxed{\lambda = 7}$ $\boxed{\lambda = -5}$

Ex let $A = \begin{bmatrix} -6 & 0 & 0 \\ 7 & -2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ Find the eigenvalue of A ?

solu: A is lower triangular

∴ the eigenvalues of A are $\boxed{\lambda = -6}$ $\boxed{\lambda = -2}$ $\boxed{\lambda = 3}$

Ex let $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -7 & 6 & 0 & 0 \\ 9 & 0 & 9 & 0 \\ 4 & 3 & -5 & -2 \end{bmatrix}$ find the following :

λ -

[1] The eigenvalues of A [2] characteristic equ of A

[3] characteristic poly. of A.

solu: [1] The eigen value : $\lambda = 3, \lambda = 6, \lambda = 9, \lambda = -2$

[2] charact. equ :

$(\lambda - 3)(\lambda - 6)(\lambda - 9)(\lambda + 2) = 0$

[3] charact. poly.

$P(\lambda) = (\lambda - 3)(\lambda - 6)(\lambda - 9)(\lambda + 2)$

$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -7 & 6 & 0 & 0 \\ 9 & 0 & 9 & 0 \\ 4 & 3 & -5 & -2 \end{bmatrix}$

A is lower triangular

③ If A is invertible, then $\lambda = 0$ is not an eigenvalue

Ex: let $A = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$ Find ① eigenvalue of A.

② show that A is not invertible

Solu:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -6 \\ 1 & \lambda+3 \end{bmatrix}$$

$$\det (\lambda-2)(\lambda+3)+6 = \lambda^2+\lambda-6+6 = \lambda^2+\lambda = 0$$

$$\lambda^2+\lambda = 0 \Rightarrow \lambda(\lambda+1) = 0 \rightarrow \lambda = 0 \ \& \ \lambda = -1$$

① the eigenvalues of A are $\boxed{\lambda = 0 \mid \lambda = -1}$

② Because $\boxed{\lambda = 0}$ is an eigenvalue of A, then A is not invertible

Ex: let $A = \begin{bmatrix} 3 & 2 \\ -3 & -4 \end{bmatrix}$ ① Find the eigenvalue of A.

② show that A is invertible.

Solu:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \lambda-3 & -2 \\ 3 & \lambda+4 \end{bmatrix}$$

$$\det = (\lambda-3)(\lambda+4)+6 = \lambda^2+\lambda-12+6 = \lambda^2+\lambda-6 = 0$$

$$\lambda^2+\lambda-6 = 0 \Rightarrow (\lambda-2)(\lambda+3) = 0 \quad \therefore \lambda = 2, \ \lambda = -3$$

① the eigenvalues of A are: $\boxed{\lambda = 2 \mid \lambda = -3}$

② neither of eigenvalues $= 0 \Rightarrow \lambda \neq 0 \quad \therefore A$ is invertible #

Ex: let $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ using the eigenvalues to show that A is not invertible?

Solu:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 0 & -4 \\ 0 & \lambda & 0 \\ -2 & -3 & \lambda-1 \end{bmatrix}$$

$$\det = \lambda \begin{vmatrix} \lambda-1 & -4 \\ -2 & \lambda-1 \end{vmatrix} = \lambda((\lambda-1)(\lambda-1)-8) = 0$$

$$\lambda(\lambda^2-2\lambda-A) = 0 \rightarrow \boxed{\lambda = 0}$$

because $\boxed{\lambda = 0}$ is an eigenvalue of A, then

A is not invertible.

اذا كانت المصفوفة lower, upper نجد قيمة λ مباشرة

" " عادية نجد λ عن طريق ← $(\lambda I - A)$.

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• the characteristic poly of adigonal matrix A
$P(\lambda) = (\lambda-2)(\lambda+3)(\lambda+1)$, then:

1. what's the size of A ?
2. Find the eigenvalue of A ?
3. Find $\det(A)$.
4. Is A invertible ?
5. Find A ?

Solu: $P(\lambda) = (\lambda-2)(\lambda+3)(\lambda+1)$ $\therefore$ A is diagonal

1. the size of A is (3×3) because the degree of $\boxed{P(\lambda) = 3}$

2. To find the eigenvalues of A:

$$(\lambda-2)(\lambda+3)(\lambda+1) = 0 \quad \therefore \boxed{\lambda = 2} \; \boxed{\lambda = -3} \; \boxed{\lambda = -1}$$

3. To Find $\det(A)$:

because A is digonal, then the entrics on its diagonal are the eigenvalues of A.

$$\det(A) = (2)(-3)(-1) = 6$$

لـ : (eigenvalues) هي عناصر القطر الرئيسي

لـ (digonal) هو ضرب عناصر القطر.

4. A is invertible because all $\lambda \neq 0$

5. To find A

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ or } A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ or } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\text{or } A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ or } A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

] let $P(\lambda) = (\lambda+2)(\lambda-3)^2(\lambda+1)^5$ is the character polynomial of atriagular matrix A.

1 find the size of A 2 Find the eigen values of A *
3 Is A invertible? Justify 4 Find det (A).

sol: $P(\lambda) = (\lambda+2)(\lambda-3)^2(\lambda+1)^5$, ∴ A is triagonal

1 The size of $A = (8 \times 8)$

* حجم المصفوفة = مجموع الأس في حالة وجودهم
أما في عدم وجود الأس عدد الأقواس.

2 To find eigenvalues of A:

$(\lambda+2)(\lambda-3)^2(\lambda+1)^5 = 0$

$\boxed{\lambda = -2 \mid \lambda = 3 \mid \lambda = -1}$ → distinct eigenvalues بدون تكرار

3 A is invertible because all $\lambda \neq 0$ #

4 To find det(A): because A is triangular, then the eigenvalue of A are the entires of its diagonal

عناصر القطر → $-2, 3, 3, -1, -1, -1, -1, -1$

$\det(A) = (-2)(3)^2(-1)^5 = (-2)(9)(-1) = \boxed{18}$.

Ex] let $P(\lambda) = \lambda^3 - 4\lambda^2 + 2\lambda + 4$ is the charactristic Polynomial of A.

نعي مثل هذه الحالة يكون حجم مصفوفة أعلى أس

1 Find the size of A 2 Find det(A) 3 Is A invertible?

sol: $P(\lambda) = \lambda^3 - 4\lambda^2 + 2\lambda + 4$.

1 the size of A is (3×3) → أعلى أس في $P(\lambda)$

2 to find det (A):

$P(\lambda) = \det(\lambda I - A) = \lambda^3 - 4\lambda^2 + 2\lambda + 4$

$P(0) = \det(-A) = (0)^3 - 4(0)^2 + 2(0) + 4$

حجم مصفوفة

$\det(-A) = 4 \Rightarrow (-1)^{3} \cdot \det(A) = 4$ $\boxed{\det(A) = -4}$

Ex] let $P(\lambda) = \lambda^4 - \lambda^3 + 9$ is the charact. poly of A, show that (A) is $\boxed{\text{invertible}}$?

roof: The size of A is (4×4)

$\det(\lambda I - A) = P(\lambda) = \lambda^4 - \lambda^3 + 9$

حجم مصفوفة

if $\lambda = 0 \Rightarrow \det(-A) = 9 \Rightarrow (-1)^{4} \det(A) = 9$ ∴ $\boxed{\det(A) = 9}$

because $\det(A) \neq 0$, then A is invertible #

D ... that the characteristic equation of a (2×2) ...ix A can be expressed as $\lambda^2 - tr(A)\lambda + det(A) = 0$

proof let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad tr(A) = a+d$

$\quad det(A) = ad - bc$

To find charact. equation.

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - b \end{bmatrix}.$$

$det = (\lambda - a)(\lambda - d) - bc = 0$

$\lambda^2 - d\lambda - a\lambda + ad - bc = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$

$\therefore \boxed{\lambda^2 - tr(A)\lambda + det(A) = 0}$ ##

Ex let $P(\lambda) = \delta^5 + 5\lambda^4 - 2\lambda^3 + 7$ is the characteristic polynomial of A. Find :

[1] The size of A [2] $tr(A)$ [3] $det(A)$.

solu $P(\lambda) = \lambda^5 + \boxed{5}\lambda^4 - 2\lambda^3 + 7.$

[1] A is (5×5) matrix

الحل / 5 = tr يكون معامل الـ ... بالسالب ⇐ حالة

[2] $tr(A) = -5$

[3] $P(\lambda) = det(\lambda I - A) = \lambda^5 + 5\lambda^4 - 2\lambda^3 + 7.$

$P(0) = det(-A) = 7 \Rightarrow (-1)^5 det(A) = 7.$

$\therefore det(A) = (-1)^5(7) = \boxed{-7}$

Ex let $P(\lambda) = (\lambda - 3)(\lambda + 2)^2$ is the charac poly of A & find $tr(A)$?

solu: $P(\lambda) = (\lambda - 3)(\lambda + 2)^2$

$= (\lambda - 3)(\lambda^2 + 4\lambda + 4)$

$= (\lambda^3 + 4\lambda^2 + 4\lambda - 3\lambda^2 - 12\lambda - 12)$

$= \lambda^3 + \lambda^2 - 8\lambda - 12$

$\therefore \boxed{tr(A) = -1}$ سالب معامل λ^2

③ If $p(0) = 0$, then A is not invertible. **

If $p(0) \neq 0$, then A is invertible. **

④ to find $A^{-1} \Rightarrow p(A) = 0$. **

$\boxed{\text{Ex}}$ let $P(\lambda) = \lambda^2 - 2\lambda + 4$ is the charac. poly. of A

$\boxed{1}$ Is A invertible? Justify $\boxed{2}$ Find A^{-1}?

soly.

$\boxed{1}$ $p(0) = 0^2 - 2(0) + 4 \Rightarrow p(0) = 4 \neq 0$

because $p(0) \neq 0$, then A is invertible

$\boxed{2}$ To find $A^{-1} \rightarrow p(A) = 0$

$$P(A) = A^2 - 2A + 4I = 0$$

$A \cdot A^{-1} = I \Rightarrow \frac{I}{A} = A^{-1}$

$$A^2 - 2A = -4I$$

$$A(A - 2I) = -4I$$

$$-4A^{-1} = A - 2I$$

$$\therefore \boxed{A^{-1} = \frac{-1}{4}(A - 2I)}$$

$\boxed{\text{Ex}}$ let $P(\lambda) = (\lambda+2)(\lambda-1)^2$

$\boxed{1}$ Is A invertible? $\boxed{2}$ Find A^{-1}?

soly.: $\boxed{1}$ $P(0) = (2)(-1)^2 = 2 \neq 0$ $\therefore$ A is invertible

$\boxed{2}$ $P(\lambda) = (\lambda+2)(\lambda^2 - 2\lambda + 1) = \lambda^3 - 2\lambda^2 + \lambda + 2\lambda^2 - 4\lambda + 2$

$$P(\lambda) = \lambda^3 - 3\lambda + 2$$

$$P(A) = A^3 - 3A + 2I = 0$$

$$A^3 - 3A = -2I$$

$$A(A^2 - 3I) = -2I$$

$$\boxed{A^{-1} = \frac{-1}{2}(A^2 - 3I)}$$

.ow that if A is a square matrix, then A & A^T e the same eigenvalues ?

oof ⊛ The characteristic equation of A is:

$\det(\lambda I - A) = 0$

⊛ The charac. equation of A^T is: $\det(\lambda I - A^T) = 0$

⊛ $\det(\ldots)^T = \det(\ldots) \longrightarrow$ ex $\det(A^T) = \det(A)$.

$\det(\lambda I - A)^T = \det(\lambda I - A) \ldots\ldots \boxed{1}$

* $\det(\lambda I - A)^T = \det((\lambda I)^T - A^T)$

because (λI) is diagonal, then $(\lambda I)^T = \lambda I$.

$\det(\lambda I - A)^T = \det(\lambda I - A^T) \ldots\ldots \boxed{2}$

from $\boxed{1}$ & $\boxed{2}$

$\det(\lambda I - A) = \det(\lambda I - A^T) = 0$

∴ the charact. equation of A is the same charact. equation of A^T.

* the eigenvalues are the roots of the charact. equ

∴ (A) and (A^T) have the same eigenvalues #

⊠ **Eigenvectors and Bases for Eigenspace** :-

⊛ The basis for the eigenspace of a matrix A is the basis for the solution space of the homog. system

$(\lambda I - A)x = 0$.

⊛ find bases for the eigenspace of the matrix :

الخطوات

$\boxed{1}$ نجد (eigen value) $\Leftarrow \lambda_1 = (\ ), \lambda_2 = (\ )$

$\boxed{2}$ نعوض قيم λ في المصفوفة $(\lambda I - A)$ كلًّ على حدة

(٣) نحل كل نظام ناتج من النقطة (٢)

(٤) نوجد الـ (basis) لـ (solu. space).

1) let $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ Find bases for eigenspace

sol:

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} \quad \text{---- } (*)$$

$$\det = (\lambda+1)(\lambda) - 6 = \lambda^2 + \lambda - 6 \Rightarrow \boxed{(\lambda+3)(\lambda-2) = 0} \text{ characteristic}$$

The eigenvalues of A are : $\boxed{\lambda = -3 \;|\; \lambda = 2}$

نعوض قيم λ في المصفوفة $(\lambda I - A)$ السابقة $[*]$

$$\boxed{\lambda = -3} \Rightarrow \begin{bmatrix} -3+1 & -3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}$$

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -2 & -3 & 0 \\ -2 & -3 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free variable $\Rightarrow x_2 = t$

$$x_1 = \frac{-3}{2}t \Longrightarrow \text{sol. set is } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

The basis for the eigenspace corresponding to space

$\lambda = 3$ is $\boxed{P_1 = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}}$

$$\boxed{\lambda = 2} \Rightarrow \begin{bmatrix} 2+1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free variable $\Rightarrow x_2 = t$

$$x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the basis for eigenspace corresponding to $\lambda = 2$

is $\boxed{P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

4) $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ Find bases for the eigenspace of A

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \text{ ---- } (*)$$

$$\det \; (\lambda-2) \begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda(\lambda-3)+2)$$

$$= (\lambda-2)(\lambda^2-3\lambda+2) = 0$$

$$(\lambda-2)(\lambda-2)(\lambda-1) = (\lambda-2)^2(\lambda-1)=0$$

The eigen values of A are :- $\boxed{\lambda = 2} \; \boxed{\lambda = 1}$

$\boxed{\lambda = 2}$ $\rightarrow \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2-2 & -1 \\ -1 & 0 & 2-3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}$ & $(\lambda I - A)x = 0$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 & x_3 are free variables, $\underline{x_3 = t}$, $\underline{x_2 = S}$ $\Rightarrow$ $\underline{x_1 = -x_3 \; , \; x_1 = -t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ S \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ the basis for the eigen space corresponding to $\boxed{\lambda = 2}$ is $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\lambda = 1}$ $\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1-2 & -1 \\ -1 & 0 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free $\Rightarrow x_3 = t$, $x_2 = x_3 \Rightarrow x_2 = t$

$x_1 = -2x_3 = -2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ the basis for eigenspace corresponding to $\underline{\lambda = 1}$

$\boxed{11}$

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x] let $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 9 \end{bmatrix}$ Find bases for eigen sp..

<u>solu</u>:

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 0 & 3 \\ 0 & \lambda & 0 \\ 3 & 0 & \lambda-9 \end{bmatrix}$$

$$\det = \lambda \begin{vmatrix} \lambda-1 & 3 \\ 3 & \lambda-9 \end{vmatrix} = \lambda\big((\lambda-1)(\lambda-9)-9\big)$$

$$= \lambda(\lambda^2 - 10\lambda + 9) - 9 = \lambda(\lambda^2 - 10\lambda) = 0$$

$$\lambda^2(\lambda - 10) = 0 \quad \therefore \boxed{\lambda = 0} \boxed{\lambda = 10}$$

$\boxed{\lambda = 0}$

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -9 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

<u>x_2, x_3</u> are free $\Rightarrow x_3 = t$, $x_2 = S$, $x_1 = 3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ S \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{basis } \boxed{P_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}, \boxed{P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\boxed{\lambda = 10}$

$$\begin{bmatrix} 9 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 9 & 0 & 3 & 0 \\ 0 & 10 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R.E.F} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/3 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free $\Rightarrow x_3 = t$, $x_2 = 0$, $x_1 = -\frac{1}{3}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \qquad \therefore P_3 = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \text{ basis}$$

Let λ_1 and x_1 ... A ... is an eigen... eigenvector. Proof

= ملاحظه

+ نفرض   لا يوجد نفرض

Number of $\boxed{\text{eigenspace}}$ = number of $\boxed{\text{eigen value}}$

<u>Ex</u>: let $p(\lambda) = (\lambda-2)(\lambda+4)^2(\lambda-3)^3$ is the charact. poly of A. How many eigenspace does A have?

<u>Solu</u>: The eigenvalues of A are: $\lambda = 2$, $\lambda = -4$, $\lambda = 3$

number of eigen values = 3 = number of eigen space.

Show that if λ is an eigenvalue of an invertible matrix A and x is corresponding eigenvector, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and x is a corresponding eigenvector.

Proof

$$Ax = \lambda x \xrightarrow{\cdot A^{-1}} A^{-1}Ax = A^{-1}\lambda x$$

$$Ix = \lambda A^{-1}x \longrightarrow x = \lambda A^{-1}x \quad \text{but } \lambda \text{ is constant}$$

$\div \lambda$ $\quad \therefore \frac{1}{\lambda} x = A^{-1}x \rightarrow \boxed{A^{-1}x = \frac{1}{\lambda} x}$

$\therefore \frac{1}{\lambda}$ is an eigenvalue of A^{-1} and x is a corresponding eigenvector ※.

ملاحظات

$A^T = A$

* $\boxed{A \,\&\, A^T}$ have same eigenvalues.
* If λ is an eigenvalue of A, then
* $\boxed{\lambda^k}$ is an eigenvalue of $\boxed{A^k}$
* $\boxed{\lambda - S}$ is an eigenvalue of $\boxed{A - SI}$ S: scalar
* $\boxed{S\lambda}$ is an eigenvalue of $\boxed{SA}$.
* $\boxed{\frac{1}{\lambda}}$ is an eigenvalue of $\boxed{A^{-1}}$

$\boxed{\text{Ex}}$: let $A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 2 & 0 \\ 6 & 3 & -5 \end{bmatrix}$ Find the following:

$\boxed{1}$ The eigenvalues of A? $2, 2, -5$

$\boxed{2}$ " " " A^3? $2^3, 2^3, (-5)^3$

$\boxed{3}$ " " " A^{-1}? $\frac{1}{2}, \frac{1}{2}, \frac{-1}{5}$

$\boxed{4}$ " " " $5A$? $10, 10, -25$

$\boxed{5}$ " " " $A - 3I$? $(2-3), (2-3), (-5-3)$

$\boxed{-1, \; -1, \; -8}$

$\boxed{13}$ The End

Final ١٠ –

الأول $\boxed{1}$ سي

لحد هنا ١٨
د. عبد الله لحساباته
حوط (١١)

★ Real Vector spaces ★

★ let V be a set of objects on which two operations are defined : (addition and scalar multiplication) and let u, v, w are objects in V

★ V is called a Vector space , and the objects u, v, w called a Vectors if all the following axioms are satisfied

$\boxed{1}$ $u+v$ is in V or $u+v \in V$ (closure under addition)

$\boxed{2}$ $\boxed{u+v = v+u}$

$\boxed{3}$ $u+(v+w) = (u+v)+w$

$\boxed{4}$ $u+\vec{0} = \vec{0}+u = u$

$\boxed{5}$ $u+(-u) = (-u)+u = \vec{0}$

$\boxed{6}$ ku is in V or $ku \in V$: k is scalar

(closure under scalar multiplication)

$\boxed{7}$ $k(u+v) = ku+kv$

$\boxed{8}$ $(k+m)u = ku+mu$ k, m are scalar

$\boxed{9}$ $k(mu) = (km)u$

$\boxed{10}$ $\boxed{1 \cdot u = u}$

☆ إذا تحققت الشروط الـ ١١ السابقة جميعها ⟸ يكون V (vector space)

☆ وإذا شرط واحد أو أكثر لم يتحقق ⟸ يكون V (not vector space)

EX 1: let $V = R^2$ with the standerd operations of addition and scalar multiplications. show that the set V is a vector space or not?

Sol: Let $u = (u_1, u_2)$, $v = (v_1, v_2)$

$w = (w_1, w_2)$

☆ يجب أن نفحص الشروط الـ ١١ السابقة جميعها واحدة واحدة :-

$\boxed{1}$ $u + v = (u_1, u_2) + (v_1, v_2) = (\underbrace{u_1 + v_1}_{z_1}, \underbrace{u_2 + v_2}_{z_2})$

$= (z_1, z_2)$ حيث

$(u_1 + v_1, u_2 + v_2) = (z_1, z_2) \longrightarrow$ 2-tuples $\therefore u + v \in R^2 \Rightarrow \in V$

$\therefore$ Closed under addition ✓

$\boxed{2}$ $u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

$v + u = (v_1, v_2) + (u_1, u_2) = (v_1 + u_1, v_2 + u_2)$

☆ علمنا سابقاً أن ما داخل الأقواس أرقام عادية v_1, u_1 أرقام عادية والجمع تبديلي

$\therefore u + v = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2)$

$= v + u$

$u + v = v + u$ ✓

3 $u + (v + w) = (u_1, u_2) + ((v_1, v_2) + (w_1, w_2))$

$= (u_1, u_2) + (v_1 + w_1, v_2 + w_2)$

$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2)$

$((u + v) + w = ((u_1, u_2) + (v_1, v_2)) + (w_1, w_2)$

$= (u_1 + v_2, u_2 + v_2) + (w_1, w_2)$

$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2)$

$= u + (v + w)$

$\therefore u + (v + w) = (u + v) + w$

4 $\vec{0} = (0, 0)$

$u + \vec{0} = (u_1, u_2) + (0, 0) = (u_1 + 0), u_2 + 0)$

$= (u_1, u_2) = u$

$\vec{0} + u = (0, 0) + (u_1, u_2) = (0 + u_1, 0 + u_2)$

$= (u_1, u_2 = u$

$\therefore u + \vec{0} = \vec{0} + u = u$

5 $(-u) = (-u_1, -u_2)$

$u + -u = (u_1, u_2) + (-u_1, -u_2) = (u_1 - u_1, u_2 - u_2) = (0, 0$

$= \vec{0}$

$(-u) + u = (-u_1, -u_2) + (u_1, u_2) = (-u_1 + u_1, -u_2 + u_2)$

$= (0, 0) = \vec{0}$

$\therefore u + (-u) = (-u) + u = \vec{0}$

$\boxed{6}$ $ku = k(u_1, u_2) = (\underbrace{ku_1}_{z_1}, \underbrace{ku_2}_{z_2}) \longrightarrow$ 2-tuples $\in R^2$

$\therefore ku \in V$ closed under scalar multiplication ✓

$\boxed{7}$ $\boxed{k(u+v)} = k((u_1, u_2)+(v_1, v_2)) = k(u_1+v_1, u_2+v_2)$

$= (k(u_1+v_1), k(u_2+v_2))$

$= (ku_1 + kv_1, ku_2 + kv_2)$

$\boxed{ku+kv} = k(u_1, u_2) + k(v_1, v_2) = (ku_1, ku_2) + (kv_1, kv_2)$

$= (ku_1 + kv_1, ku_2 + kv_2)$

$= k(u+v)$

$\therefore k(u+v) = ku + kv$ ✓

$\boxed{8}$ $\boxed{k(mu)} = k(m(u_1, u_2)) = k(mu_1, mu_2)$

$= (kmu_1, kmu_2)$

$\boxed{(km)u} = \underbrace{(km)}_{\text{ثابت}}(u_1, u_2) = (kmu_1, kmu_2) = k(mu)$

$\therefore k(mu) = (km)u$ ✓

$\boxed{9}$ $\boxed{(k+m)u} = \underbrace{(k+m)}_{\text{ثابت}}(u_1, u_2) = ((k+m)u_1, (k+m)u_2)$

$= (ku_1 + mu_1, ku_2 + mu_2)$

$\boxed{ku+mu} = k(u_1, u_2) + m(u_1, u_2) = (ku_1, ku_2) + (mu_1, mu_2)$

$= (ku_1 + mu_1, ku_2 + mu_2)$

$\therefore (k+m)u = ku + mu$ ✓

$\boxed{10}$ $1 \cdot u = 1 \cdot (u_1 , u_2) = (1 \cdot u_1 , 1 \cdot u_2) = (u_1 , u_2) = u$

because all axioms are satisfied , then

V is Vector space

EX 2: Let V the set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.
IS the set V is a vector space ? Justify.

Sol: ★ يوجد شرط في السؤال: يجب أن تكون جميع المتجهات على الصيغة التالية: $(x,0)$

يعني: $u = (u, 0)$, $y = (y, 0)$, $z = (z, 0)$

ويجب أن يتحقق الشرط في الشروط العشر السابقة.

Let $x = (x, 0)$, $y = (y, 0)$, $z = (z, 0)$

$\boxed{1}$ $x + y = (x, 0) + (y, 0) = (x + y, 0) \equiv (v, 0)$

$\therefore x + y \in V \Rightarrow$ closed under addition. ✓

$\boxed{2}$ $\boxed{x+y} = (x, 0) + (y, 0) = (x + y, 0)$

$\boxed{y+x} = (y, 0) + (x, 0) = (y + x, 0) = (x + y, 0)$

$\therefore x + y = y + x$ ✓

$\boxed{3}$ $\boxed{x+(y+z)} = (x,0) + ((y,0)+(z,0)) = (x,0) + (y+z$

$\boxed{(x+y)+z} = ((x,0)+(y+0)) + (z,0) = (x+y,0) + (z,0)$

$= (x+y+z$

$= (x+y+z,0)$

$\therefore x+(y+z) = (x+y)+z$ ✓

$\boxed{4}$ $(\vec{0})$ (Zero vector)

Let $x = (x,0)$

let $\vec{0} = (w,0)$

$x+\vec{0} = x \Rightarrow (x,0) + (w,0) = (x,0)$

$(x+w,0) = (x,0) \Rightarrow \cancel{x}+w = \cancel{x} \Rightarrow w = 0$

$\therefore \vec{0} = (0,0)$

$\boxed{x+\vec{0}} = (x,0) + (0,0) = (x,0) = \vec{x}$

$\boxed{\vec{0}+x} = (0,0) + (x,0) = (x,0) = \vec{x}$

$\therefore x+\vec{0} = \vec{0}+x = x$ ✓

$\boxed{5}$ Let $x = (x,0)$, Let $(-x) = (u,0)$ $\vec{0} = (0,0)$

$x+(-x) = \vec{0} \Rightarrow (x,0) + (u,0) = (0,0)$

$((x+u),0) = (0,0)$

$x+u = 0 \Rightarrow u = -x \Rightarrow (-x) = (-x,0)$

Note / top Page.

يتبع $\Rightarrow$

$\vec{0} = (0,0)$, $(-X) = (-X, 0)$ $\quad X = (X, 0)$

$\boxed{X + -X} = (X, 0) + (-X, 0) = (X - X, 0) = (0,0) = \vec{0}$

$\boxed{(-X) + X} = (-X, 0) + (X, 0) = (-X + X, 0) = (0,0) = \vec{0}$

$\therefore \; X + (-X) = (-X) + X = \vec{0}$

$\boxed{6}$ $KX = K(X, 0) = (KX, 0)$

$\therefore \; KX = (KX, 0) \in V$

closed under scalar multiplication

$\boxed{7}$ $\boxed{K(X+y)} = K((X,0) + (y,0)) = K(X+y, 0) = (K(X+y), 0)$

$= (KX + Ky, 0)$

$\boxed{KX + Ky} = K(X,0) + K(y,0) = (KX, 0) + (Ky, 0)$

$= (KX + Ky, 0)$

$\therefore \; K(X+y) = KX + Ky$

$\boxed{8}$ $\boxed{(K+m)X} = (K+m)(X, 0) = ((K+m)X, 0) = (KX + mX, 0)$

$\boxed{KX + mX} = K(X,0) + m(X,0) = (KX, 0) + (mX, 0) = (KX + Km, ...$

$\therefore \; (K+m)X = Km + mX$

$\boxed{9}$ $\boxed{K(m\vec{X})} = K(m(X,0)) = K(mX, 0) = (KmX, 0)$

$\boxed{(Km)\vec{X}} = (Km)(X, 0) = (KmX, 0)$

$\therefore \; K(mX) = (Km)X$

$\boxed{10}$ $1 \cdot \vec{X} = 1 \cdot (X, 0) = (1 \cdot X, 0) = (X, 0) = \vec{X} \Rightarrow 1 \cdot X = ...$

$\therefore$ V is a vector space

Ex3 Let V the set of all pairs of real numbers of from $(1, x)$ with the following operations:-

$$(1, x) + (1, y) = (1, x+y) \quad , \quad k(1+x) = (1, kx)$$

Is V a vector space? Justify.

Sol: يوجد شرط بالسؤال :- يجب أن تكون جميع المتجهات على صيغة $(1, x)$

لـ نفرض :- $x = (1, x) \quad y = (1, y) \quad z = (1, z)$

ولازم هذا الشرط يتحقق على الجمع $\boxed{\text{axiom}_1}$ والضرب بثابت $\boxed{\text{axiom } 6}$

سوف يتم فحص الشرط الأول $\boxed{\text{axiom}_1}$ والسادس $\boxed{\text{axiom}_6}$ وأنت افحص الباقي.

$\boxed{1}$ $x + y = (1, x) + (1, y) = (1, x+y) = (1, w)$

$\therefore x + y = (1, x+y) \in V$

closed under addition.

$\boxed{2}$ $kx = k(1, x) = (1, kx) \in V$

closed under scalar multiplication

لـ جميع الشروط تتحقق V is a vector space

Ex 4: Let V is the set of all pairs of real numbers of the form (x,y), where $x \geq 0$, with standard operations on R^2.

why V is not vector space?

يوجد شرط في السؤال وهو $x \geq 0$ ويجب أن يتحقق على الشروط .

Sol:

V is not vector space because axiom 6 is failed.

$\boxed{6}$ $k(x,y) = (kx, ky)$

إذا ضربت المتجهة (x,y) بأي ثابت سالب تتغير إشارة x

let $k = -2$, then $k(x,y) = -2(x,y) = (-2x, -2y)$

ولكن الشرط في السؤال $x \geq 0 \Leftarrow -2x \leq 0$

يصبح الطرف الأيسر في المتجهة سالب .

$\therefore$ not closed under scalar multiplications

$\therefore$ not vector space #

Ex 5: Let V the set of all 2×2 matrices of the form:-

$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ with standard matrix operations on $M_{2\times 2}$

Show that V is a vector space?

Sol:

1) $A+B = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} (a_1+b_1) & 0 \\ 0 & (a_2+b_2) \end{bmatrix}$

$\therefore A+B \in V \Rightarrow$ closed under addition.

2) $\boxed{A+B} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & 0 \\ 0 & a_2+b_2 \end{bmatrix}$

$\boxed{B+A} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1+a_1 & 0 \\ 0 & b_2+a_2 \end{bmatrix}$

$= \begin{bmatrix} a_1+b_1 & 0 \\ 0 & b_2+a_2 \end{bmatrix}$ $\therefore A+B = B+A$

3) $A+(B+C) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \right)$

$= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1+c_1 & 0 \\ 0 & b_2+c_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1+c_1 & 0 \\ 0 & a_2+b_2+c_2 \end{bmatrix}$

$(A+B)+C = \left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \right) + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$

$= \begin{bmatrix} a_1+b_1 & 0 \\ 0 & a_2+b_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1+c_1 & 0 \\ 0 & a_2+b_2+c_2 \end{bmatrix}$

$\therefore A+(B+C) = (A+B)+C$ ✓

4 $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ —

$A + 0 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = A$

$0 + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = A$

$\therefore \quad A + \vec{0} = \vec{0} + A = A$

5 $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ (-A) = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}$

$A + (-A) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}$

$(-A) + A = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}$

$\therefore \quad A + (-A) = (-A) + A$ ✓

6 $KA = K \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} Ka_1 & 0 \\ 0 & Ka_2 \end{bmatrix} \in M_{2 \times 2}$

$KA \in M_{2\times 2} \Longrightarrow$ closed under scalar mult. ✓

7 $\boxed{K(A+B)} = K\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \right)$

$= K \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}$

يتبع $\Longrightarrow$

$$= \begin{bmatrix} k(a_1+b_1) & 0 \\ 0 & k(a_2+b_2) \end{bmatrix} = \begin{bmatrix} ka_1+kb_1 & 0 \\ 0 & ka_2+kb_2 \end{bmatrix}$$

$$\boxed{kA+kB} = k\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + k\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

$$= \begin{bmatrix} ka_1 & 0 \\ 0 & ka_2 \end{bmatrix} + \begin{bmatrix} kb_1 & 0 \\ 0 & kb_2 \end{bmatrix} = \begin{bmatrix} ka_1+kb_1 & 0 \\ 0 & ka_2+kb_2 \end{bmatrix}$$

$\therefore k(A+B) = kA+kB$

8) $\boxed{(k+m)A} = (k+m)\begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} = \begin{bmatrix} (k+m)a_1 & 0 \\ 0 & (k+m)a_2 \end{bmatrix}$

$$= \begin{bmatrix} ka_1+ma_1 & 0 \\ 0 & ka_2+ma_2 \end{bmatrix}$$

$$\boxed{kA+mA} = k\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + m\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} ka_1 & 0 \\ 0 & ka_2 \end{bmatrix}$$

$$+ \begin{bmatrix} ma_1 & 0 \\ 0 & ma_2 \end{bmatrix} = \begin{bmatrix} ka_1+ma_1 & 0 \\ 0 & ka_2+ma_2 \end{bmatrix}.$$

$\therefore (k+m)A = kA+mA$

9) $\boxed{k(mA)} = k\left(m\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\right) = k\begin{bmatrix} ma_1 & 0 \\ 0 & ma_2 \end{bmatrix} = \begin{bmatrix} kma_1 & 0 \\ 0 & kma_2 \end{bmatrix}$

$\boxed{(km)A} = (km)\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} kma_1 & 0 \\ 0 & kma_2 \end{bmatrix}$ $\therefore k(mA) = (km)A$

10) $1 \cdot A = 1\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = A$ V is a vector space.

Ex6: let $V = R^2$ with standard addition, but with scalar multiplication defined by :-

$k(x_1, x_2) = (k^2 x_1, 0)$. show that V is not vectorspace?

Sol: because axiom 8 is not satisfied :-
let k, m any scalars :-

$\boxed{(k+m)\,x} = (k+m)(x_1, x_2) = ((k+m)^2 x_1, 0)$

but $\boxed{kx+mx} = k(x_1, x_2) + m(x_1, x_2)$

$= (k^2 x_1, 0) + (m^2 x_1, 0)$

$= (k^2 x_1 + m^2 x_1, 0) = (k^2 + m^2 x_1, 0)$

$(k+m)\,x \neq kx+mx \Rightarrow \therefore V$ is not vector space

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Ex 7: Let $V = R^2$ define the addition by : $u+v = (u_1 + v_1, u_2 + v_2)$ and define the scalar multiplication by : $ku = (0, ku_2)$.
why V is not vector space?

Sol: Let $u = (u_1, u_2)$

axiom 10 : $1.u = 1.(u_1, u_2) = (0, u_2) \neq u$

$\therefore$ axiom 10 is not satisfied (not hold)

$\therefore V$ is not vector space.

Subspaces :-

مفتاح مهم جداً

★ If V be a vector space and W is a set in V, then W is said to be a supspace of V if the following conditions are satisfied :-

1. If u, v are two vectors in W, then $u+v$ in W closure under addition $(u+v \in W)$

2. If u a vector in W, then ku in W Closure under scalar multiplications $(ku \in W)$

Ex 1 :- Let W be the set of all vectors of the form $(a, 0, 0)$. Is W is a subspace of R^3? Justify.

Sol:

1. let $u = (u, 0, 0)$, $v = (v, 0, 0)$

$u+v = (u, 0, 0) + (v, 0, 0) = (u+v, 0, 0) \in W$

$\therefore$ Closed under addition

2. $ku = k(u, 0, 0) = (ku, 0, 0)$

$\therefore ku = (ku, 0, 0) \in W \rightarrow$ closed under scalar multiplic ✓

$\therefore W$ is a subspace of R^3

Ex 2: Let W be the set of all vectors of the form $(a, 1, 1)$. Is W a subspace of R^3? Justify

يمكن أن يكتب السؤال بصيغة أخرى كما يلي :-

or Ex: Let $W = \{(a, 1, 1) : a \in R\}$

Is W subspace of R^3?

Sol: Let $u = (u, 1, 1)$, $v = (v, 1, 1)$:

$\boxed{1}$ $u + v = (u, 1, 1) + (v, 1, 1) = (u+v, 2, 2) \notin W$

not closed under addition

$\boxed{2}$ $ku = k(u, 1, 1) = (ku, k, k) \notin W$

not closed under scalar multiplication

$\therefore$ W is not subspace of R^3

Ex 3: Let W be the set of all the vectors of the form $(a, b, 0)$. I W a subspace of R^3?

or let $W = \{(a, b, 0) : a, b \in R\}$ Is W a subspace of R^3?

Sol: Let $u = (u_1, u_2, 0)$

Let $v = (v_1, v_2, 0)$

$\boxed{1}$ $u+v = (u_1, u_2, 0) + (v_1, v_2, 0)$
$= (u_1 + v_1, u_2 + v_2, 0) \in W$

$\therefore$ Closed under addition

$\boxed{2}$ $ku = k(u_1, u_2, 0) = (ku_1, ku_2, 0) \in W$

$\therefore$ Closed under scalar multip
$\therefore$ W is a subspace of R^3

Ex 4 :- Let W be the set of all vectors of the form (x_1, x_2, x_3), where $x_3 = x_1 + x_2$. Is W a subspace of R^3 ?

صيغة أخرى للسؤال OR Let $W = \{(x_1, x_2, x_3) : x_3 = x_1 + x_2\}$
Is W a subspace of R^3?

Sol: let $x = (x_1, x_2, x_3) \Rightarrow x_3 = x_1 + x_2$
Let $y = (y_1, y_2, y_3) \Rightarrow y_3 = y_1 + y_2$

$\boxed{1}$ $x+y = (x_1, x_2, x_3) + (y_1, y_2, y_3)$
$= (\underbrace{x_1+y_1}_{z_1}, \underbrace{x_2+y_2}_{z_2}, \underbrace{x_3+y_3}_{z_3})$

$z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$ $z_3 = x_3 + y_3$
(but $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$)

$z_3 = x_1 + x_2 + y_1 + y_2 = \underbrace{(x_1 + y_1)}_{z_1} + \underbrace{(x_2 + y_2)}_{z_2}$

$z_3 = z_1 + z_2$ ✓

$\therefore$ Closed under addition

يتبع

$\boxed{2}$ $Kx = K(x_1, x_2, x_3) = (\underbrace{Kx_1}_{Z_1}, \underbrace{Kx_2}_{Z_2}, \underbrace{Kx_3}_{Z_3})$

$Z_3 = Kx_3$ (but $x_3 = x_1 + x_2$)

$= K(x_1 + x_2) = \underbrace{Kx_1}_{Z_1} + \underbrace{Kx_2}_{Z_2} = Z_1 + Z_2$

$\therefore Z_3 = Z_1 + Z_2$

$\therefore$ closed under scalar multip

W is a subspace of R^3

Ex 5 :- Let $W = \{(u_1, u_2, u_3) : u_2 = u_1 + u_3 + 1\}$

Is W a subspace of R^3?

Sol: let $u = (u_1, u_2, u_3) \longrightarrow u_2 = u_1 + u_3 + 1$

let $v = (v_1, v_2, v_3) \longrightarrow v_2 = v_1 + v_3 + 1$

$\boxed{1}$ $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (\underbrace{u_1 + v_1}_{w_1}, \underbrace{u_2 + v_2}_{w_2}, \underbrace{u_3 + v_3}_{w_3})$

$w_1 = u_1 + v_1 \qquad w_2 = u_2 + v_2 \qquad w_3 = u_3 + v_3$

$w_2 = u_2 + v_2$ (but $u_2 = u_1 + v_3 + 1$, $v_2 = v_1 + v_3 + 1$)

$= u_1 + u_3 + 1 + v_1 + v_3 + 1$

$= \underbrace{u_1 + v_1}_{w_1} + \underbrace{u_3 + v_3}_{w_3} + 2$

$w_2 = w_1 + w_3 + \longrightarrow \notin W$

not closed under addition $\longrightarrow$ W is not subspace of R^3

Ex 6 : let $W = \{(x_1, x_2, 0) : x_1 = x_2 + 1\}$
Is W a subspace of R^3 ?

sol: let $x = (x_1, x_2, 0) \longrightarrow x_1 = x_2 + 1$
let $y = (y_1, y_2, 0) \longrightarrow y_1 = y_2 + 1$

$\boxed{1}$ $x + y = (x_1, x_2, 0) + (y_1, y_2, 0)$

$$= (\underbrace{x_1 + y_1}_{z_1}, \underbrace{x_2 + y_2}_{z_2}, \underbrace{0}_{z_3})$$

$z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, $z_3 = 0$

$z_1 = x_1 + y_1$, but $(x_1 = x_2 + 1 , y_1 = y_2 + 1)$

$z_1 = x_2 + 1 + y_2 + 1 = x_2 + y_2 + 2 = z_2 + 2 \notin W$

$\therefore$ not closed under addition
$\therefore$ W is not subspace of R^3

Ex 7 :- Let $W = \{(v, 1, v, 1, v, 1, \ldots) : v \in R\}$
Is W a subspace of R^∞ ?

sol: Let $v = (v, 1, v, 1, v, 1, \ldots)$, $u = (u, 1, u, 1, u, 1, \ldots)$

$\boxed{1}$ $u + v = (u, 1, u, 1, u, 1, \ldots) + (v, 1, v, 1, v, 1, \ldots)$

$$= (\underbrace{u + v}_{z}, 2, \underbrace{u + v}_{z}, 2, \underbrace{u + v}_{z}, \ldots)$$

$= (z, 2, z, 2, z, 2, \ldots) \notin W$
not closed under addition
W is not subspace of R^∞